Appendices

A Solving for Λ Algebraically

Here we present the detailed derivation of Eqn. 2, including the linear system that is used to calculate Λ from v_*, C, and k_*. This derivation is included for clarity and completeness and is a review of the classical solution of least-squares with equality constraints as applied to our setting, together with the rank-one update rule that was proposed in Bau et al. (2020).

We assume that W is the optimal least-squares solution for memorizing a mapping from a previous set of keys K to values V; this solution can be written using the normal equations as follows.

$$\text{the } W \text{ that minimizes} \quad ||WK - V||_F^2 \tag{5}$$

$$\text{solves} \quad WKK^T = VK^T \tag{6}$$

Here the Frobenius norm is used to write the total square error since the variable being optimized is a matrix W rather than a vector x as in the classical textbook presentation of least squares.

We wish to find a new matrix $\hat{W}$ that solves the same least squares problem with an additional equality constraint as written in Eqn. 2:

$$\hat{W} k_* = v_* \tag{7}$$

This is the well-studied problem of least squares with a linear equality constraint. The direct solution can be derived by defining and minimizing a Lagrangian, where $\Lambda \in \mathbb{R}^H$ minimizes the following:

$$\text{define} \quad L(\hat{W}, \Lambda) = \frac{1}{2}||\hat{W}K - V||_F^2 - \Lambda^T(\hat{W}k_* - v_*) \tag{8}$$

$$= \frac{1}{2}(\hat{W}K)(\hat{W}K)^T - V(\hat{W}K)^T + \frac{1}{2}VV^T - \Lambda^T(\hat{W}k_* - v_*) \tag{9}$$

$$\text{setting} \quad 0 = \frac{\partial L}{\partial \hat{W}} = \hat{W}(KK^T) - VK^T - \Lambda k_*^T \tag{10}$$

$$\hat{W}KK^T = VK^T + \Lambda k_*^T \tag{11}$$

Subtracting Eqn. 6 from Eqn. 11, most terms cancel, and we obtain the update rule:

$$(\hat{W} - W)KK^T = \Lambda k_*^T \tag{12}$$

$$\hat{W} = W + \Lambda(C^{-1}k_*)^T \tag{13}$$

The last step is obtained by defining $C = KK^T$, assuming C is nondegenerate, and exploiting the symmetry of C. Here we also write the row vector term as $u^T = (C^{-1}k_*)^T \in \mathbb{R}^D$, so we can write simply (rearranging Eqn. 2 and Eqn. 13):

$$\hat{W}I - \Lambda u^T = W \tag{14}$$

To solve for Λ, we note that Eqn. 14 and Eqn. 7 form a linear system that allows both $\hat{W}$ and Λ to be solved simultaneously if written together in block form.

$$\left[\begin{array}{c|c} \hat{W} & \Lambda \end{array}\right] \left[\begin{array}{c|c} I & k_* \\ \hline -u^T & 0 \end{array}\right] = \left[\begin{array}{c|c} W & v_* \end{array}\right] \tag{15}$$

That is equivalent to substituting Eqn. 13 into Eqn. 7 and calculating the following:

$$\hat{W}k_* = (W + \Lambda u^T)k_* = Wk_* + \Lambda(u^T k_*) = v_* \tag{16}$$

$$\Lambda = \frac{v_* - Wk_*}{u^T k_*} = \frac{v_* - Wk_*}{(C^{-1}k_*)^T k_*} \tag{17}$$