

HARMONIC 3-FORMS ON COMPACT HOMOGENEOUS SPACES

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ABSTRACT. The third real de Rham cohomology of compact homogeneous spaces is studied. Given $M = G/K$ with G compact semisimple, we first show that each bi-invariant symmetric bilinear form Q on \mathfrak{g} such that $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$ naturally defines a G -invariant closed 3-form H_Q on M , which plays the role of the so called Cartan 3-form $Q([\cdot, \cdot], \cdot)$ on the compact Lie group G . Indeed, every class in $H^3(G/K)$ has a unique representative H_Q . Secondly, focusing on the class of homogeneous spaces with the richest third cohomology (other than Lie groups), i.e., $b_3(G/K) = s - 1$ if G has s simple factors, we give the conditions to be fulfilled by Q and a given G -invariant metric g in order for H_Q to be g -harmonic, in terms of algebraic invariants of G/K . As an application, we obtain that any 3-form H_Q is harmonic with respect to the standard metric, although for any other normal metric, there is only one H_Q up to scaling which is harmonic. Furthermore, among a suitable $(2s - 1)$ -parameter family of G -invariant metrics, we prove that the same behavior occurs if \mathfrak{k} is abelian: either every H_Q is g -harmonic (this family of metrics depends on s parameters) or there is a unique g -harmonic 3-form H_Q (up to scaling). In the case when \mathfrak{k} is not abelian, the special metrics for which every H_Q is g -harmonic depend on 3 parameters.

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1. INTRODUCTION

The understanding of the de Rham cohomology $H(M)$ of a compact homogeneous space $M = G/K$ in terms of algebraic invariants of the groups G , K and the embedding $K \subset G$ is a classical problem which has been studied by many renowned mathematicians, including E. Cartan, Chevalley, Eilenberg, Koszul, A. Weil, H. Cartan and Borel (see [B]). In this paper, we are interested in $H^3(M)$ over \mathbb{R} , with a particular emphasis on finding

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an explicit description of 3-forms which are harmonic with respect to a given G -invariant Riemannian metric. Our interest comes from potential applications to the study of many well-known classes of geometric structures on M involving closed, coclosed or harmonic 3-forms.

In the case when $M = G$ is a compact connected semisimple Lie group, it is well known that every class in $H^3(G)$ has a unique bi-invariant representative, which is necessarily of the form $\overline{Q} := Q([\cdot, \cdot], \cdot) \in \Lambda^3 \mathfrak{g}^*$ for some bi-invariant symmetric bilinear form Q on the Lie algebra \mathfrak{g} of G . The third Betti number $b_3(G)$ is therefore the number of simple factors of G . Moreover, any \overline{Q} is harmonic with respect to any bi-invariant metric on G . We show in §3 that, beyond the bi-invariant context, the harmonicity of \overline{Q} relative to a left-invariant metric on G depends on tricky conditions in terms of the structural constants of \mathfrak{g} .

Let $M = G/K$ be a homogeneous space, where G is a compact, connected and semisimple Lie group and K is a connected closed subgroup. It is easy to see that $b_1(G/K) = 0$ and $b_2(G/K) = \dim \mathfrak{z}(\mathfrak{k})$, where $\mathfrak{z}(\mathfrak{k})$ is the center of the Lie algebra \mathfrak{k} of K (see §2). Concerning third cohomology, we found that there is also a canonical G -invariant closed 3-form H_Q attached to each bi-invariant symmetric bilinear form Q on \mathfrak{g} such that $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$, given by

$$H_Q(X, Y, Z) := 4Q([X, Y], Z) - Q([X, Y]_{\mathfrak{p}}, Z) + Q([X, Z]_{\mathfrak{p}}, Y) - Q([Y, Z]_{\mathfrak{p}}, X),$$

for all $X, Y, Z \in \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is any reductive decomposition (see §4). An equivalent way to define H_Q is by $\pi^* H_Q := \overline{Q} + d\alpha_Q$, where $\pi : G \rightarrow G/K$ is the usual projection and $\alpha_Q \in \Lambda^2 \mathfrak{g}^*$ is given by

$$\alpha_Q|_{\mathfrak{k} \times \mathfrak{k}} = 0, \quad \alpha_Q|_{\mathfrak{p} \times \mathfrak{p}} = 0, \quad \alpha_Q(X, Z) = Q(X, Z), \quad \forall X \in \mathfrak{p}, \quad Z \in \mathfrak{k}.$$

Moreover, we prove that every class in $H^3(G/K)$ has a unique representative of the form H_Q and so the third Betti number is given by

$$(1) \quad b_3(G/K) = s - d_{G/K},$$

where $d_{G/K} := \dim \{Q|_{\mathfrak{k} \times \mathfrak{k}} : Q \text{ bi-invariant on } \mathfrak{g}\}$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ is a decomposition of \mathfrak{g} into simple ideals. In particular, $b_3(G/K) = s$ if and only if K is trivial. The authors believe that formula (1) must be known, although they were not able to find it in the literature. In any case, it only takes one self-contained half page to prove it.

Our aim in this paper is the study of the following natural problem:

Find the conditions to be fulfilled by a given bi-invariant symmetric bilinear form Q (with $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$) and a given G -invariant metric g on $M = G/K$ in order for H_Q to be g -harmonic, in terms of algebraic invariants of G/K .

For simplicity, we consider the class of homogeneous spaces with the richest third cohomology (other than Lie groups), i.e., $b_3(G/K) = s - 1$. We also fix a decomposition $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_t$ in simple ideals. We first show that such a class is characterized by the following algebraic condition on G/K in the irreducible case (see Proposition 4.10), which we have called *aligned*: there exist $c_1, \dots, c_s > 0$ such that

$$B_{\mathfrak{g}_i}(Z_i, W_i) = \frac{1}{c_i} B_{\mathfrak{g}}(Z, W), \quad \forall Z, W \in \mathfrak{k}, \quad i = 1, \dots, s.$$

In other words, the ideals \mathfrak{k}_j 's and the center are uniformly embedded on each \mathfrak{g}_i in some sense. Here $B_{\mathfrak{h}}$ denotes the Killing form of a Lie algebra \mathfrak{h} . It is easy to see that the aligned condition implies that $B_{\mathfrak{k}_j} = \lambda_j B_{\mathfrak{g}}|_{\mathfrak{k}_j \times \mathfrak{k}_j}$ for some positive number λ_j for all $j = 1, \dots, t$ and $\pi_i(\mathfrak{k}) \simeq \mathfrak{k}$ for all $i = 1, \dots, s$ (see Definition 4.7 for further details).

Note that G/K is aligned and consequently $b_3(G/K) = s - 1$ as soon as \mathfrak{k} is simple (or one-dimensional) and $\pi_i(\mathfrak{k}) \neq 0$ for all $i = 1, \dots, s$, as well as when $G = H \times \cdots \times H$ (s -times, $s \geq 2$) and $K = \Delta L$ for any subgroup $L \subset H$ (see also Example 4.12). In the

aligned case, the condition $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$ for a bi-invariant Q , say $Q = y_1 B_{\mathfrak{g}_1} + \cdots + y_s B_{\mathfrak{g}_s}$, is simply given by $\frac{y_1}{c_1} + \cdots + \frac{y_s}{c_s} = 0$.

1.1. Main results. Let $M = G/K$ be an aligned homogeneous space with positive constants c_1, \dots, c_s and $\lambda_1, \dots, \lambda_t$. For any given bi-invariant metric

$$g_b = z_1(-B_{\mathfrak{g}_1}) + \cdots + z_s(-B_{\mathfrak{g}_s}), \quad z_1, \dots, z_s > 0,$$

we consider the g_b -orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the normal metric on M determined by $g_b|_{\mathfrak{p} \times \mathfrak{p}}$ and a suitable g_b -orthogonal $\text{Ad}(K)$ -invariant decomposition

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s \oplus \mathfrak{p}_{s+1} \oplus \cdots \oplus \mathfrak{p}_{2s-1},$$

see Proposition 5.1 for more details. As $\text{Ad}(K)$ -representations, \mathfrak{p}_i is equivalent to the isotropy representation of the homogeneous space $G_i/\pi_i(K)$ for all $i = 1, \dots, s$ and \mathfrak{p}_j is equivalent to the adjoint representation \mathfrak{k} for all $j = s+1, \dots, 2s-1$ (in particular, an aligned G/K is never multiplicity-free for $s \geq 3$, which makes computations much more difficult).

We assume that none of the irreducible components of $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ is equivalent to any of the simple factors of \mathfrak{k} as $\text{Ad}(K)$ -representations and that either $\mathfrak{z}(\mathfrak{k}) = 0$ or the trivial representation is not contained in any of $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. This implies that $\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$ and $\mathfrak{p}_{s+1} \oplus \cdots \oplus \mathfrak{p}_{2s-1}$ are ω -orthogonal for any $\omega \in (\Lambda^2 \mathfrak{p}^*)^K$ (see the paragraph containing (29) for more details on this assumption).

Theorem 1.1. *Given a G -invariant metric of the form*

$$g = x_1 g_b|_{\mathfrak{p}_1 \times \mathfrak{p}_1} + \cdots + x_{2s-1} g_b|_{\mathfrak{p}_{2s-1} \times \mathfrak{p}_{2s-1}}, \quad x_1, \dots, x_{2s-1} > 0,$$

we set

$$\begin{aligned} a_j &:= \dim \mathfrak{k} \frac{1}{c_{j+1} x_{j+1}^2} + \text{Cas}_0 \left(\frac{1}{x_{s+j}^2} - \frac{1}{x_{j+1}^2} \right), \quad j = 1, \dots, s-1, \\ b_j &:= \dim \mathfrak{k} \left(\frac{1}{c_1 x_1^2} + \cdots + \frac{1}{c_j x_j^2} \right) + \text{Cas}_0 \left(\frac{1}{x_{s+1}^2} - \frac{1}{x_1^2} + \cdots + \frac{1}{x_{s+j}^2} - \frac{1}{x_j^2} \right), \end{aligned}$$

where $\text{Cas}_0 := \lambda_1 \dim \mathfrak{k}_1 + \cdots + \lambda_t \dim \mathfrak{k}_t$. Then the closed 3-form H_Q , where

$$Q = y_1 B_{\mathfrak{g}_1} + \cdots + y_s B_{\mathfrak{g}_s}, \quad \frac{y_1}{c_1} + \cdots + \frac{y_s}{c_s} = 0,$$

is g -harmonic if and only if

$$(2) \quad x_{s+k} \left(a_k A_k + b_k + 2 \text{Cas}_0 \left(\frac{1}{x_{s+j}^2} - \frac{1}{x_{s+k}^2} \right) \right) C_j = x_{s+j} (a_j A_j + b_j) C_k,$$

for all $1 \leq j < k \leq s-1$, where

$$A_j := -\frac{c_{j+1}}{z_{j+1}} \left(\frac{z_1}{c_1} + \cdots + \frac{z_j}{c_j} \right), \quad C_j := \frac{y_1}{c_1} + \cdots + \frac{y_j}{c_j} + A_j \frac{y_{j+1}}{c_{j+1}}.$$

Note that condition (2) trivially holds if $s = 2$ or $\mathfrak{k} = 0$ and that for the normal metric $g = g_b$, the constants a_j and b_j represent just algebraic invariants of G/K . We do not know whether all the G -invariant metrics are covered or not by the above theorem in the case when all the spaces $G_i/\pi_i(K)$ are isotropy irreducible and K is either simple or one-dimensional (see Remark 7.10; note that otherwise it is clear that not every G -invariant metric is covered).

Corollary 1.2. *Let $M = G/K$ be an aligned homogeneous space and let g be a G -invariant metric as in the above theorem.*

- (i) *If \mathfrak{k} is abelian, then either every closed 3-form H_Q is g -harmonic (this family of metrics depends on s parameters), or there is a unique g -harmonic 3-form H_Q (up to scaling).*

- (ii) If \mathfrak{k} is not abelian, then the family of metrics such that every closed 3-form H_Q is g -harmonic depends on 3 parameters x_1, x_s, x_{s+1} and can be described as follows:

$$g = (x_1, x_2, \dots, x_{s-1}, x_s, x_{s+1}, \dots, x_{s+1}, x_{2s-1})_{g_b},$$

where x_2, \dots, x_{s-1} are determined by x_1, x_{s+1} and x_{2s-1} by x_1, x_s, x_{s-1} .

- (iii) The standard metric g_B is the unique normal metric on G/K (up to scaling) satisfying that every H_Q is harmonic.
- (iv) For any normal metric $g_b \neq g_B$, there exists a unique g_b -harmonic 3-form H_Q (up to scaling).
- (v) For any nonzero closed 3-form H_Q , there exists a one-parameter family (up to scaling) of normal metrics $g_b(t) \neq g_B$ such that H_Q is $g_b(t)$ -harmonic for all t .

See Remarks 7.5-7.10 and Corollaries 7.11-7.13 for more detailed statements of the results.

1.2. Geometric applications. We have applied these results in [LW2] to the existence problem of G -invariant generalized metrics on $M = G/K$ which are *Bismut Ricci flat*, i.e., precisely the fixed points of the generalized Ricci flow (see the recent book [GS] and [PR1, PR2] for further information). The structure consists of a pair (g, H) , where g is a Riemannian metric and H is a closed 3-form, and the corresponding *Bismut connection* ∇^B is defined by

$$g(\nabla_X^B Y, Z) = g(\nabla_X^g Y, Z) + \frac{1}{2}H(X, Y, Z), \quad \forall X, Y, Z \in \chi(M),$$

where ∇^g is the Levi Civita connection of (M, g) . It turns out that ∇^B is Ricci flat if and only if,

$$H \text{ is } g\text{-harmonic} \quad \text{and} \quad \text{Rc}(g) = \frac{1}{4}H_g^2,$$

where $\text{Rc}(g)$ is the Ricci tensor of g and $H_g^2(X, Y) := g(\iota_X H, \iota_Y H)$ for all $X, Y \in \chi(M)$.

The results of the present paper can also be applied to a number of problems in differential geometry in the context of compact homogeneous spaces, including the study of Einstein manifolds with skew torsion (see [AF, AFF]) and of Killing 2-forms and 3-forms (see [S, BMS]).

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2. PRELIMINARIES

Let M^n be a compact connected differentiable manifold. We will assume throughout the paper that M is homogeneous and fix an almost-effective transitive action of a compact connected Lie group G on M . The G -action determines a presentation $M = G/K$ of M as a homogeneous space, where $K \subset G$ is the isotropy subgroup at some point $o \in M$. We also assume that the compact Lie group K is connected.

Since G is compact, the real de Rham cohomology of $M = G/K$ can be computed within G -invariant forms. Given any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (i.e., \mathfrak{p} is $\text{Ad}(K)$ -invariant), the space of all G -invariant k -forms is identified with $(\Lambda^k \mathfrak{p}^*)^K$, the space of $\text{Ad}(K)$ -invariant k -forms on \mathfrak{p} , i.e.,

$$\alpha([Z, \cdot], \cdot, \dots, \cdot) + \dots + \alpha(\cdot, \dots, \cdot, [Z, \cdot]) = 0, \quad \forall Z \in \mathfrak{k},$$

and the differential $d := d_M$ of forms on the manifold M is given by $d : \Lambda^k \mathfrak{p}^* \rightarrow \Lambda^{k+1} \mathfrak{p}^*$,

$$d\alpha(X_1, \dots, X_{k+1}) := \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j]_{\mathfrak{p}}, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}),$$

giving rise to $H^k(G/K) = \text{Ker } d / \text{Im } d$. The k th Betti number is given by $b_k(G/K) := \dim H^k(G/K)$.

Alternatively, the isomorphism

$$(3) \quad (\Lambda^k \mathfrak{p}^*)^K \longrightarrow \Lambda^k(\mathfrak{g}, K) := \left\{ \beta \in (\Lambda^k \mathfrak{g}^*)^K : \iota_{\mathfrak{k}} \beta = 0 \right\}, \quad \alpha \mapsto \hat{\alpha} := \pi^* \alpha,$$

where $\pi : G \rightarrow G/K$ is the usual projection map and $\iota_Z \beta := \beta(Z, \cdot, \dots, \cdot)$, can be used to compute $H^k(G/K)$ upstairs within left-invariant forms on the Lie group G . Indeed, the corresponding differential of forms on the Lie group G , which will be denoted by \hat{d} , satisfies that $\hat{d}\Lambda^k(\mathfrak{g}, K) \subset \Lambda^{k+1}(\mathfrak{g}, K)$ and $d\alpha = 0$ if and only if $\hat{d}\hat{\alpha} = 0$ (note that $\hat{d}\pi^* = \pi^*d$), so $H^k(G/K) = \text{Ker } \hat{d} / \text{Im } \hat{d}$.

Any G -invariant metric g on $M = G/K$, which will always be identified with an $\text{Ad}(K)$ -invariant inner product on \mathfrak{p} , determines an inner product on each $\Lambda^k \mathfrak{p}^*$ given by

$$g(\alpha, \beta) := \sum_{i_1, \dots, i_k} \alpha(X_{i_1}, \dots, X_{i_k}) \beta(X_{i_1}, \dots, X_{i_k}),$$

where $\{X_i\}$ is any g -orthonormal basis of \mathfrak{p} . Note that $\{\frac{1}{\sqrt{k!}} X^{i_1} \wedge \dots \wedge X^{i_k}\}$ is therefore a g -orthonormal basis of $\Lambda^k \mathfrak{p}^*$, where $\{X^i\}$ is the basis of \mathfrak{p}^* g -dual to $\{X_i\}$. If

$$d_g^* : (\Lambda^{k+1} \mathfrak{p}^*)^K \longrightarrow (\Lambda^k \mathfrak{p}^*)^K$$

is the adjoint of d with respect to g (i.e., $g(d_g^* \cdot, \cdot) = g(\cdot, d \cdot)$), then a k -form α is closed and coclosed (i.e., $d_g^* \alpha = 0$) if and only if α is in the kernel of the Hodge Laplacian

$$\Delta_g := dd_g^* + d_g^* d : (\Lambda^k \mathfrak{p}^*)^K \longrightarrow (\Lambda^k \mathfrak{p}^*)^K,$$

and it is called g -harmonic in that case. Since

$$(\Lambda^k \mathfrak{p}^*)^K = \underbrace{\text{Im } d \oplus \text{Ker } \Delta_g}_{\text{Ker } d} \oplus \overbrace{\text{Ker } d_g^*}^{\text{Ker } d_g^*} \oplus \text{Im } d_g^*,$$

we have that $H^k(G/K) \simeq \text{Ker } \Delta_g$, that is, each class has a unique g -harmonic representative.

We consider an $\text{Ad}(K)$ -invariant left-invariant metric \hat{g} on G such that $\hat{g}(\mathfrak{k}, \mathfrak{p}) = 0$ and $\hat{g}|_{\mathfrak{p} \times \mathfrak{p}} = g$. It is easy to check that

$$\hat{g}(\hat{\alpha}, \hat{\beta}) = g(\alpha, \beta), \quad \forall \alpha, \beta \in (\Lambda^k \mathfrak{p}^*)^K,$$

so for any $\alpha \in (\Lambda^k \mathfrak{p}^*)^K$ and $\beta \in (\Lambda^{k-1} \mathfrak{p}^*)^K$,

$$g(d_g^* \alpha, \beta) = g(\alpha, d\beta) = \hat{g}(\hat{\alpha}, \pi^* d\beta) = \hat{g}(\hat{\alpha}, \hat{d}\hat{\beta}) = \hat{g}(\hat{d}_g^* \hat{\alpha}, \hat{\beta}).$$

Note that $\hat{d}_g^* \hat{\alpha} \in (\Lambda^{k-1} \mathfrak{g}^*)^K$, but not necessarily in $\Lambda^{k-1}(\mathfrak{g}, K)$. Thus a k -form $\alpha \in (\Lambda^k \mathfrak{p}^*)^K$ is g -coclosed if and only if $\hat{d}_g^* \hat{\alpha}$ is \hat{g} -orthogonal to $\Lambda^{k-1}(\mathfrak{g}, K)$.

A G -invariant metric is called *normal* when it is determined by $g_b|_{\mathfrak{p} \times \mathfrak{p}}$ for some bi-invariant metric g_b on \mathfrak{g} and if $g_b = -B_{\mathfrak{g}}$, where $B_{\mathfrak{g}}$ denotes the Killing form of \mathfrak{g} , then it is called *standard* and denoted by g_B . We fix a normal metric g_b on $M = G/K$ and consider the g_b -orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the g_b -orthogonal decomposition

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1, \quad \text{where } \mathfrak{p}_0 := \{X \in \mathfrak{p} : [\mathfrak{k}, X] = 0\},$$

that is, \mathfrak{p}_0 is the trivial K -representation isotypic component of the isotropy representation of G/K .

2.1. Harmonic 1-forms. Any 1-form on \mathfrak{p} is of the form $\theta_X := g_b(\cdot, X)$ for some $X \in \mathfrak{p}$ and since

$$\theta_X([Z, \cdot]) = g_b([Z, \cdot], X) = -g_b(\cdot, [Z, X]), \quad \forall Z \in \mathfrak{k},$$

θ_X is $\text{Ad}(K)$ -invariant if and only if $X \in \mathfrak{p}_0$, so $(\mathfrak{p}^*)^K = \mathfrak{p}_0^*$. It follows from $d\theta_X = -g_b([\cdot, \cdot], X)$ that θ_X ($X \in \mathfrak{p}_0$) is closed if and only if X is g_b -orthogonal to $[\mathfrak{p}, \mathfrak{p}]$. This is equivalent to $X \in \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ is the center of \mathfrak{g} (indeed, $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{k}, \mathfrak{k}] + [\mathfrak{k}, \mathfrak{p}] + [\mathfrak{p}, \mathfrak{p}]$ and X is already g_b -orthogonal to both \mathfrak{k} and $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}_1$). We therefore obtain that

$$H^1(G/K) = \{[\theta_X] : X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p}_0\}, \quad b_1(G/K) = \dim \mathfrak{z}(\mathfrak{k}) \cap \mathfrak{p}_0.$$

In particular, $b_1(G/K) = 0$ if G is semisimple. Note that $[\theta_X] = \{\theta_X\}$ and so θ_X is g -harmonic for all $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p}_0$ and any G -invariant metric g .

2.2. Harmonic 2-forms. Assume that G is semisimple. Any closed 2-form on G is exact and so it is necessarily of the form

$$\hat{\omega}_X := g_b([\cdot, \cdot], X), \quad \text{for some } X \in \mathfrak{g}.$$

Using that for all $Z \in \mathfrak{k}$ and $Y, W \in \mathfrak{g}$,

$$\hat{\omega}_X([Z, Y], W) + \hat{\omega}_X(Y, [Z, W]) = g_b([Z, [Y, W]], X) = -g_b([Y, W], [Z, X]),$$

and

$$(\iota_Z \hat{\omega}_X)(Y) = \hat{\omega}_X(Z, Y) = g_b([Z, Y], X) = -g_b(Y, [Z, X]),$$

we obtain that $\hat{\omega}_X$ is $\text{Ad}(K)$ -invariant if and only if $\iota_{\mathfrak{k}} \hat{\omega}_X = 0$, if and only if $[\mathfrak{k}, X] = 0$, so

$$\{\hat{\omega} \in \Lambda^2(\mathfrak{g}, K) : \hat{d}\hat{\omega} = 0\} = \{\hat{\omega}_X : X \in \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{p}_0\}.$$

This implies that

$$\{\omega \in (\Lambda^2 \mathfrak{p}^*)^K : d\omega = 0\} = \{\omega_X : X \in \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{p}_0\}, \quad \text{where } \omega_X := g_b([\cdot, \cdot]_{\mathfrak{p}}, X),$$

and since $\omega_X = -d\theta_X$ for any $X \in \mathfrak{p}_0$, we obtain that

$$H^2(G/K) = \{[\omega_Z] : Z \in \mathfrak{z}(\mathfrak{k})\} \simeq \mathfrak{z}(\mathfrak{k}), \quad b_2(G/K) = \dim \mathfrak{z}(\mathfrak{k}).$$

Note that $[\omega_Z] = \{\omega_{Z+X} : X \in \mathfrak{p}_0\}$ for any $Z \in \mathfrak{z}(\mathfrak{k})$, so if $\mathfrak{p}_0 = 0$ then ω_Z is g -harmonic for every G -invariant metric g .

If $\{e_i\}$ is a g_b -orthonormal basis of \mathfrak{p} , then for any $Z \in \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{p}_0$ and $X \in \mathfrak{p}_0$,

$$\begin{aligned} g_b(\omega_Z, d\theta_X) &= \sum_{i,j} \omega_Z(e_i, e_j) d\theta_X(e_i, e_j) = - \sum_{i,j} g_b([e_i, e_j], Z) g_b([e_i, e_j], X) \\ &= - \text{tr ad } Z|_{\mathfrak{p}} \text{ ad } X|_{\mathfrak{p}} = -B_{\mathfrak{g}}(Z, X). \end{aligned}$$

Thus the closed 2-form ω_Z is g_b -harmonic if and only if $B_{\mathfrak{g}}(Z, \mathfrak{p}_0) = 0$. The g_b -harmonic representative inside a class $[\omega_Z]$, $Z \in \mathfrak{z}(\mathfrak{k})$ is therefore given by ω_{Z+X_Z} , where X_Z is the unique vector in \mathfrak{p}_0 such that $B_{\mathfrak{g}}(X, X_Z) = -B_{\mathfrak{g}}(X, Z)$ for all $X \in \mathfrak{p}_0$. Note that \mathfrak{p}_0 and so X_Z depend on g_b .

In particular, ω_Z is g_B -harmonic for any $Z \in \mathfrak{z}(\mathfrak{k})$.

3. HARMONIC 3-FORMS ON COMPACT LIE GROUPS

Let $M^n = G$ be a compact connected Lie group. It is well known that every class in $H^3(G)$ has a unique bi-invariant representative (see [Br, Chapter V, Corollary 12.7]), which in the semisimple case, is necessarily of the form

$$\overline{Q}(X, Y, Z) := Q([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{g},$$

for some bi-invariant symmetric bilinear form Q on \mathfrak{g} (see [Br, Chapter V, Theorem 12.10]). These closed 3-forms are often called *Cartan 3-forms*. Thus the third Betti number $b_3(G) := \dim H^3(G)$ is precisely the number of simple factors if G is semisimple,

and it is well known that $H^1(G) = 0$ and $H^2(G) = 0$ in that case (see §2.1 and §2.2). Note that if G is simple, then $H^3(G) = \mathbb{R}[H_B]$, where $H_B := \overline{B_{\mathfrak{g}}}$ and $B_{\mathfrak{g}}$ is the Killing form of \mathfrak{g} . It is also well known that Cartan 3-forms are all harmonic with respect to any bi-invariant metric on G (see e.g. [V, Theorem 3.4.7]). We study in this section the harmonicity condition for a Cartan 3-form with respect to any left-invariant metric on G .

We fix from now on a bi-invariant metric g_b on G . For any left-invariant metric g on G , there exists a g_b -orthonormal basis $\{e_1, \dots, e_n\}$ of \mathfrak{g} such that $g(e_i, e_j) = x_i \delta_{ij}$ for some $x_1, \dots, x_n > 0$, which will be denoted by $g = (x_1, \dots, x_n)_{g_b}$. Note that $\{e_i/\sqrt{x_i}\}$ is a g -orthonormal basis of \mathfrak{g} with dual basis $\{\sqrt{x_i}e_i\}$, where e_i also denotes the dual basis defined by $e_i(e_j) := \delta_{ij}$. The ordered basis $\{e_1, \dots, e_n\}$ determines structural constants given by

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \quad \text{or equivalently,} \quad c_{ij}^k := g_b([e_i, e_j], e_k).$$

Lemma 3.1. *For any metric $g = (x_1, \dots, x_n)_{g_b}$ and $\beta \in \Lambda^3 \mathfrak{g}^*$,*

$$d_g^* \beta = -\frac{3}{2} \sum_{k < l} \left(x_k \sum_{i,j} \frac{c_{ij}^k \beta(e_i, e_j, e_l)}{x_i x_j} - x_l \sum_{i,j} \frac{c_{ij}^l \beta(e_i, e_j, e_k)}{x_i x_j} \right) e_k \wedge e_l.$$

Proof. For any 2-form α ,

$$\begin{aligned} g(d_g^* \beta, \alpha) &= g(\beta, d\alpha) = \sum_{i,j,k} \frac{\beta(e_i, e_j, e_k) d\alpha(e_i, e_j, e_k)}{x_i x_j x_k} \\ &= \sum_{i,j,k} \frac{\beta(e_i, e_j, e_k) (-\alpha([e_i, e_j], e_k) + \alpha([e_i, e_k], e_j) - \alpha([e_j, e_k], e_i))}{x_i x_j x_k} \\ &= -3 \sum_{i,j,k} \frac{\beta(e_i, e_j, e_k) \alpha([e_i, e_j], e_k)}{x_i x_j x_k} = -3 \sum_{i,j,k,l} \frac{\beta(e_i, e_j, e_k) c_{ij}^l \alpha(e_l, e_k)}{x_i x_j x_k}, \end{aligned}$$

so for $\alpha = e_r \wedge e_s$, one obtains that

$$g(d_g^* \beta, \alpha) = -3 \sum_{i,j} \frac{\beta(e_i, e_j, e_s) c_{ij}^r}{x_i x_j x_s} - \frac{\beta(e_i, e_j, e_r) c_{ij}^s}{x_i x_j x_r}.$$

The fact that $\{\sqrt{\frac{x_k x_l}{2}} e_k \wedge e_l\}$ is a g -orthonormal basis of $\Lambda^2 \mathfrak{g}^*$ concludes the proof. \square

We fix a decomposition

$$(4) \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s,$$

where the \mathfrak{g}_i 's are simple ideals of \mathfrak{g} and \mathfrak{g}_0 is the center of \mathfrak{g} . Thus

$$g_b = g_0 + z_1(-B_{\mathfrak{g}_1}) + \dots + z_s(-B_{\mathfrak{g}_s}), \quad \text{for some } z_1, \dots, z_s > 0,$$

and some inner product g_0 on \mathfrak{g}_0 . Since the set of Ricci eigenvalues of g_b is $\{\frac{1}{4}z_1^{-1}, \dots, \frac{1}{4}z_s^{-1}\}$, the moduli space of all bi-invariant metrics on G up to isometry and scaling depends on $s-1$ parameters. Note that g_b is Einstein if and only if $\mathfrak{g}_0 = 0$ (i.e., G semisimple) and $z_1 = \dots = z_s$, that is, a single point in the moduli space.

The following corollary of Lemma 3.1 shows that in general, for a given left-invariant metric g , even the g -harmonicity of the Cartan 3-form H_B depends on tricky conditions in terms of the structural constants.

Corollary 3.2. *For any metric $g = (x_1, \dots, x_n)_{g_b}$ as above, the following holds.*

(i)

$$d_g^* H_B = -\frac{3}{2} \sum_{1 \leq k < l \leq n} (x_k - x_l) \left(\sum_{1 \leq i, j \leq n} \frac{c_{ij}^k c_{ij}^l}{x_i x_j} \right) e_k \wedge e_l.$$

(ii) H_B is g -harmonic if and only if,

$$\sum_{1 \leq i, j \leq n} \frac{c_{ij}^k c_{ij}^l}{x_i x_j} = 0, \quad \forall k, l \text{ such that } x_k \neq x_l.$$

In particular, H_B is g_b -harmonic for any bi-invariant metric g_b .(iii) $H_B + d\alpha$, where $\alpha \in \Lambda^2 \mathfrak{g}^*$, is g -harmonic if and only if

$$\begin{aligned} (x_k - x_l) \sum_{1 \leq i, j \leq n} \frac{c_{ij}^k c_{ij}^l}{x_i x_j} &= -x_k \sum_{1 \leq i, j \leq n} \frac{c_{ij}^k d\alpha(e_i, e_j, e_l)}{x_i x_j} \\ &\quad + x_l \sum_{1 \leq i, j \leq n} \frac{c_{ij}^l d\alpha(e_i, e_j, e_k)}{x_i x_j}, \end{aligned}$$

for all $1 \leq k < l \leq n$.

Example 3.3. It is easy to see that the usual basis $\{e_{rs} := E_{rs} - E_{sr}\}$ of $\mathfrak{so}(n)$ satisfies that $c_{ij}^k c_{ij}^l = 0$ for all $k \neq l$. Thus H_B is g -harmonic for any metric g on $\mathrm{SO}(n)$ which is diagonal with respect to $\{e_{rs}\}$ (i.e., $\{e_{rs}\}$ is g -orthogonal) by Corollary 3.5. A basis of a Lie algebra is called *nice* when it satisfies the above condition together with $c_{ij}^k c_{rs}^k = 0$ as soon as $\{i, j\} \cap \{r, s\} \neq \emptyset$ (which follows from the former one if the basis is orthogonal with respect to a bi-invariant metric). On a nilpotent Lie group, a basis satisfies that any metric which is diagonal has also a diagonal Ricci tensor if and only if it is nice (see [LW1]), and the same holds for bases which are orthogonal with respect to a bi-invariant metric on compact Lie groups (see [K]). However, the above is the only nice basis known so far on a compact simple Lie group, their existence on the other compact simple Lie groups is still open.

Example 3.4. For the standard g_B -orthogonal basis $\{e_1, \dots, e_8\}$ of $\mathfrak{su}(3)$, the only nonzero products of the form $c_{ij}^k c_{ij}^l$ are $c_{36}^1 c_{36}^2$ and $c_{47}^1 c_{47}^2$, where $c_{36}^1 = \frac{\sqrt{3}}{6} = c_{47}^1$ and $c_{36}^2 = -\frac{1}{2} = -c_{47}^2$. Thus H_B is g -harmonic for a diagonal metric $g = (x_1, \dots, x_8)_{g_B}$ on $\mathrm{SU}(3)$ if and only if $x_1 = x_2$ or $x_3 x_6 = x_4 x_7$ (see Corollary 3.2). Moreover, using that the only nonzero brackets involving e_1 or e_2 are $[e_1, e_2] = [e_2, e_5] = [e_2, e_8] = 0$ and

$$[e_3, e_6] = \frac{\sqrt{3}}{6} e_1 - \frac{1}{2} e_2, \quad [e_4, e_7] = \frac{\sqrt{3}}{6} e_1 + \frac{1}{2} e_2, \quad [e_5, e_8] = \frac{\sqrt{3}}{3} e_1,$$

it is straightforward to see that the g -harmonic 3-form is given by $H_B + td(e_1 \wedge e_2)$, where

$$t := \frac{-\sqrt{3}(x_1 - x_2) \left(\frac{1}{x_3 x_6} - \frac{1}{x_4 x_7} \right)}{(x_1 + 3x_2) \left(\frac{1}{x_3 x_6} + \frac{1}{x_4 x_7} \right) + 12 \frac{x_1}{x_5 x_8}}.$$

Any bi-invariant symmetric bilinear form is of the form

$$Q = Q_0 + y_1 B_{\mathfrak{g}_1} + \dots + y_s B_{\mathfrak{g}_s}, \quad \text{for some } y_1, \dots, y_s \in \mathbb{R},$$

where Q_0 is any symmetric bilinear form on \mathfrak{g}_0 .

We assume that the g_b -orthonormal basis $\{e_i\}$ considered above is adapted to decomposition (4), in the sense that it is the union of bases $\{e_\alpha^i : \alpha = 1, \dots, n_i\}$ of each \mathfrak{g}_i , $i = 0, 1, \dots, s$, where $n_i := \dim \mathfrak{g}_i$, and we denote by $c_{i\alpha, i\beta}^{i\gamma} := g_b([e_\alpha^i, e_\beta^i], e_\gamma^i)$ the corresponding structural constants. Any metric g such that decomposition (4) is g -orthogonal

can be written on each \mathfrak{g}_i as $(x_1^i, \dots, x_{n_i}^i)_{g_b}$. Note that g is bi-invariant if and only if $x_1^i = \dots = x_{n_i}^i$ for all $i = 1, \dots, s$.

The following corollary of Lemma 3.1 implies the well-known fact that any bi-invariant 3-form on a compact Lie group is harmonic with respect to any bi-invariant left-invariant metric.

Corollary 3.5. *For any g_b , Q and g as above (assume that decomposition (4) is g -orthogonal), the following holds.*

(i)

$$d_g^* \bar{Q} = -\frac{3}{2} \sum_{\substack{1 \leq \gamma < \delta \leq n_i \\ 1 \leq i \leq s}} \frac{y_i}{z_i} (x_\gamma^i - x_\delta^i) \left(\sum_{1 \leq \alpha, \beta \leq n_i} \frac{c_{i\alpha, i\beta}^{i\gamma} c_{i\alpha, i\beta}^{i\delta}}{x_\alpha^i x_\beta^i} \right) e_\gamma^i \wedge e_\delta^i.$$

(ii) \bar{Q} is g -harmonic if and only if for all $i = 1, \dots, s$,

$$\sum_{1 \leq \alpha, \beta \leq n_i} \frac{c_{i\alpha, i\beta}^{i\gamma} c_{i\alpha, i\beta}^{i\delta}}{x_\alpha^i x_\beta^i} = 0, \quad \forall \gamma, \delta \text{ such that } x_\gamma^i \neq x_\delta^i.$$

In particular, \bar{Q} is g -harmonic for any bi-invariant metric g_b .

(iii) $\bar{Q} + d\omega$, where $\omega \in \Lambda^2 \mathfrak{g}^*$, is g -harmonic if and only if

$$\begin{aligned} (x_\gamma^i - x_\delta^i) \sum_{1 \leq \alpha, \beta \leq n_i} \frac{c_{i\alpha, i\beta}^{i\gamma} c_{i\alpha, i\beta}^{i\delta}}{x_\alpha^i x_\beta^i} &= -x_\gamma^i \sum_{1 \leq \alpha, \beta \leq n_i} \frac{c_{i\alpha, i\beta}^{i\gamma} d\omega(e_\alpha^i, e_\beta^i, e_\delta^i)}{x_\alpha^i x_\beta^i} \\ &\quad + x_\delta^i \sum_{1 \leq \alpha, \beta \leq n_i} \frac{c_{i\alpha, i\beta}^{i\delta} d\omega(e_\alpha^i, e_\beta^i, e_\gamma^i)}{x_\alpha^i x_\beta^i}, \end{aligned}$$

for all $1 \leq \gamma < \delta \leq n_i$ and $i = 1, \dots, s$.

4. THIRD COHOMOLOGY OF COMPACT HOMOGENEOUS SPACES

In this section, we study the real de Rham cohomology of 3-forms on a homogeneous space $M = G/K$ as in §2 with G semisimple. As in the Lie group case studied in §3, each bi-invariant symmetric bilinear form Q on \mathfrak{g} defines a G -invariant 3-form on G/K given by $\tilde{Q} := Q([\cdot, \cdot], \cdot) \in (\Lambda^3 \mathfrak{p}^*)^K$. However, \tilde{Q} is never closed if nonzero (it is however g_b -coclosed with respect to any normal metric g_b on G/K , see (22) below) and so the following question arises:

Does any bi-invariant symmetric bilinear form Q on \mathfrak{g} naturally determine a G -invariant closed 3-form on G/K ?

We consider fixed decompositions,

$$(5) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s, \quad \mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \dots \oplus \mathfrak{k}_t,$$

where the \mathfrak{g}_i 's and \mathfrak{k}_j 's are simple ideals of \mathfrak{g} and \mathfrak{k} , respectively, and \mathfrak{k}_0 is the center of \mathfrak{k} of dimension d_0 . The following result answers the above question in a satisfactory way.

Proposition 4.1. *Let $M = G/K$ be a homogeneous space as in §2 with G semisimple and consider any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.*

(i) *Each bi-invariant symmetric bilinear form Q on \mathfrak{g} such that $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$ defines a closed 3-form $H_Q \in (\Lambda^3 \mathfrak{p}^*)^K$ by*

$$\begin{aligned} (6) \quad H_Q(X, Y, Z) &:= 4Q([X, Y], Z) - Q([X, Y]_{\mathfrak{p}}, Z) + Q([X, Z]_{\mathfrak{p}}, Y) - Q([Y, Z]_{\mathfrak{p}}, X) \\ &= Q([X, Y], Z) + Q([X, Y]_{\mathfrak{k}}, Z) - Q([X, Z]_{\mathfrak{k}}, Y) + Q([Y, Z]_{\mathfrak{k}}, X), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{p}$. Equivalently, $\hat{H}_Q := \overline{Q} + \hat{d}\alpha_Q$, where $\alpha_Q \in \Lambda^2 \mathfrak{g}^*$ is defined by

$$\alpha_Q|_{\mathfrak{k} \times \mathfrak{k}} = 0, \quad \alpha_Q|_{\mathfrak{p} \times \mathfrak{p}} = 0, \quad \alpha_Q(X, Z) = Q(X, Z), \quad \forall X \in \mathfrak{p}, \quad Z \in \mathfrak{k}.$$

- (ii) Every class in $H^3(G/K)$ has a unique representative of the form H_Q as in part (i), that is,

$$H^3(G/K) \simeq \{Q \in \text{sym}^2(\mathfrak{g})^G : Q|_{\mathfrak{k} \times \mathfrak{k}} = 0\} \quad \text{and} \quad b_3(G/K) = s - d_{G/K},$$

where

$$d_{G/K} := \dim \{Q|_{\mathfrak{k} \times \mathfrak{k}} : Q \text{ is a bi-invariant symmetric bilinear form on } \mathfrak{g}\}.$$

Remark 4.2. The definition of the 3-form H_Q given in (6) depends on the reductive complement \mathfrak{p} .

Proof. Given a closed 3-form $H \in (\Lambda^3 \mathfrak{p}^*)^K$, there exist a bi-invariant symmetric bilinear form Q on \mathfrak{g} and $\alpha \in (\Lambda^2 \mathfrak{g}^*)^K$ such that $\hat{H} = \overline{Q} + \hat{d}\alpha$. Thus

$$Q([\cdot, \cdot], Z) = -\iota_Z \hat{d}\alpha = \hat{d}_Z \alpha - \mathcal{L}_Z \alpha = \alpha([\cdot, \cdot], Z), \quad \forall Z \in \mathfrak{k},$$

which implies that $Q(X, Z) = \alpha(X, Z)$ for all $X \in \mathfrak{g}$, $Z \in \mathfrak{k}$. In particular, $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$ and for all $X, Y, W \in \mathfrak{p}$,

$$\begin{aligned} H(X, Y, W) &= \hat{H}(X, Y, W) = Q([X, Y], W) + \hat{d}\alpha(X, Y, W) \\ &= Q([X, Y], W) - \alpha([X, Y], W) + \alpha([X, W], Y) - \alpha([Y, W], X) \\ &= Q([X, Y], W) - \alpha([X, Y]_{\mathfrak{k}}, W) + \alpha([X, W]_{\mathfrak{k}}, Y) \\ &\quad - \alpha([Y, W]_{\mathfrak{k}}, X) - \alpha([X, Y]_{\mathfrak{p}}, W) + \alpha([X, W]_{\mathfrak{p}}, Y) - \alpha([Y, W]_{\mathfrak{p}}, X) \\ &= Q([X, Y], W) + Q([X, Y]_{\mathfrak{k}}, W) - Q([X, W]_{\mathfrak{k}}, Y) + Q([Y, W]_{\mathfrak{k}}, X) \\ &\quad - \alpha([X, Y]_{\mathfrak{p}}, W) + \alpha([X, W]_{\mathfrak{p}}, Y) - \alpha([Y, W]_{\mathfrak{p}}, X) \\ &= H_Q(X, Y, W) + d\beta(X, Y, W), \end{aligned}$$

where $\beta := \alpha|_{\mathfrak{p} \times \mathfrak{p}} \in (\Lambda^2 \mathfrak{p}^*)^K$, that is, $H = H_Q + d\beta$. Thus $H_Q \in (\Lambda^3 \mathfrak{p}^*)^K$, $dH_Q = 0$ and so the first statement of part (i) and part (ii) follow. Conversely, given a bi-invariant Q such that $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$, we consider the \hat{d} -closed 3-form $\overline{Q} + \hat{d}\alpha_Q \in \Lambda^3(\mathfrak{g}, K)$. This 3-form therefore corresponds to a closed 3-form in $(\Lambda^3 \mathfrak{p}^*)^K$, which by the above computation coincides with H_Q , that is, $\hat{H}_Q = \overline{Q} + \hat{d}\alpha_Q$, concluding the proof of part (i). \square

Given $Z \in \mathfrak{k}$, there exist unique vectors $Z_i \in \mathfrak{g}_i$ such that $Z = (Z_1, \dots, Z_s)$, where $Z_i = \pi_i(Z)$ and $\pi_i : \mathfrak{g} \rightarrow \mathfrak{g}_i$ are the projections relative to the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s$. Thus π_i is a Lie algebra homomorphism and $\pi_i : \mathfrak{k}_j \rightarrow \pi_i(\mathfrak{k}_j)$ is an isomorphism as soon as $\pi_i(\mathfrak{k}_j) \neq 0$, in which case $\pi_i(\mathfrak{k}_j)$ is a simple Lie subalgebra of \mathfrak{g}_i for any $j \geq 1$. For any pair i, j , $1 \leq i \leq s$ and $0 \leq j \leq t$, there exists a constant $0 \leq c_{ij} \leq 1$ such that

$$(7) \quad B_{\pi_i(\mathfrak{k}_j)} = c_{ij} B_{\mathfrak{g}_i}|_{\pi_i(\mathfrak{k}_j) \times \pi_i(\mathfrak{k}_j)},$$

where $B_{\pi_i(\mathfrak{k}_j)}$ and $B_{\mathfrak{g}_i}$ are respectively the Killing forms of $\pi_i(\mathfrak{k}_j)$ and \mathfrak{g}_i . We have that $c_{ij} = 0$ if and only if $j = 0$ or $\pi_i(\mathfrak{k}_j) = 0$, and $c_{ij} = 1$ if and only if $\pi_i(\mathfrak{k}_j) = \mathfrak{g}_i$. Note that if $\pi_i(\mathfrak{k}_j) \neq 0$, then

$$B_{\mathfrak{k}}(Z, W) = B_{\mathfrak{k}_j}(Z, W) = B_{\pi_i(\mathfrak{k}_j)}(Z_i, W_i) = c_{ij} B_{\mathfrak{g}_i}(Z_i, W_i), \quad \forall Z, W \in \mathfrak{k}_j.$$

A homogeneous space G/K is determined by the Lie groups G and K and last but not least, by the way K is embedded in G . Note that each Killing constant c_{ij} is an invariant of how is \mathfrak{k}_j embedded in \mathfrak{g}_i .

The following bounds and equalities on $b_3(G/K)$ are actually immediate consequences of the spectral sequence for the fibration $G \rightarrow G/K \rightarrow B_K$. We prove them for self-containedness.

Proposition 4.3. *Let $M = G/K$ be a homogeneous space as above.*

(i) *If \mathfrak{g} has s simple factors and \mathfrak{k} has t simple factors and d_0 -dimensional center, then*

$$s - t - \frac{d_0(d_0+1)}{2} \leq b_3(G/K) \leq s.$$

(ii) *If K is semisimple, then a bi-invariant $Q = y_1 B_{\mathfrak{g}_1} + \cdots + y_s B_{\mathfrak{g}_s}$, $y_i \in \mathbb{R}$, satisfies that $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$ if and only if the following t conditions hold:*

$$(8) \quad \sum_{i: \pi_i(\mathfrak{k}_j) \neq 0} \frac{1}{c_{ij}} y_i = 0, \quad \forall j = 1, \dots, t, \quad \text{where } B_{\pi_i(\mathfrak{k}_j)} = c_{ij} B_{\mathfrak{g}_i}|_{\pi_i(\mathfrak{k}_j) \times \pi_i(\mathfrak{k}_j)}.$$

(iii) *Equality $b_3(G/K) = s$ holds if and only if $\mathfrak{k} = 0$ (i.e., K is the trivial subgroup).*

(iv) *$b_3(G/K) = s - 1$ if \mathfrak{k} is simple or one-dimensional.*

Remark 4.4. Generically,

$$b_3(G/K) = \begin{cases} s - t - \frac{d_0(d_0+1)}{2}, & \text{if } s \geq t + \frac{d_0(d_0+1)}{2}, \\ 0, & \text{if } s < t + \frac{d_0(d_0+1)}{2}. \end{cases}$$

See Examples 4.5, 4.6 and 4.12 below for exceptions.

Proof. It follows from Proposition 4.1, (ii) that $b_3(G/K) \leq s$. We first prove part (ii). We have that $Q(\mathfrak{k}_j, \mathfrak{k}_l) = 0$ for all $j \neq l$ and it follows from (7) that

$$Q(Z, Z) = \sum_{i=1}^s y_i B_{\mathfrak{g}_i}(Z_i, Z_i) = \sum_{i: \pi_i(\mathfrak{k}_j) \neq 0} \frac{y_i}{c_{ij}} B_{\mathfrak{k}}(Z, Z), \quad \forall Z \in \mathfrak{k}_j, \quad j = 1, \dots, t.$$

Thus $Q(Z, Z) = 0$ for all $Z \in \mathfrak{k}$ if and only if condition (8) holds.

On the other hand, if $\{Z^1, \dots, Z^{d_0}\}$ is a basis of \mathfrak{k}_0 , then for each $Z = a_1 Z^1 + \cdots + a_{d_0} Z^{d_0} \in \mathfrak{k}_0$, $Q(Z, Z) = 0$ if and only if (y_1, \dots, y_s) is orthogonal with respect to the canonical inner product of \mathbb{R}^s to

$$(9) \quad \begin{aligned} & \sum_{\alpha=1}^{d_0} a_\alpha^2 (B_{\mathfrak{g}_1}(Z_1^\alpha, Z_1^\alpha), \dots, B_{\mathfrak{g}_s}(Z_s^\alpha, Z_s^\alpha)) \\ & + \sum_{1 \leq \alpha < \beta \leq d_0} 2a_\alpha a_\beta (B_{\mathfrak{g}_1}(Z_1^\alpha, Z_1^\beta), \dots, B_{\mathfrak{g}_s}(Z_s^\alpha, Z_s^\beta)), \end{aligned}$$

which, by varying $Z \in \mathfrak{k}_0$, generates a subspace of dimension $\leq \frac{d_0(d_0+1)}{2}$. This, together with (8), gives that $s - t - \frac{d_0(d_0+1)}{2} \leq b_3(G/K)$, so part (i) follows. Moreover, the only way to obtain that $b_3(G/K) = s$ is that $\pi_i(\mathfrak{k}_j) = 0$ for all i, j (i.e., $[\mathfrak{k}, \mathfrak{k}] = 0$) and $B_{\mathfrak{g}_i}(Z_i^\alpha, Z_i^\alpha) = 0$ for all $1 \leq \alpha \leq d_0$, $i = 1, \dots, s$ (i.e., $\mathfrak{k}_0 = 0$), that is, if and only if $\mathfrak{k} = 0$, which proves (iii). Finally, part (iv) follows from (i) and (iii), concluding the proof. \square

Example 4.5. For $M = \mathrm{SU}(2n) \times \mathrm{SU}(2n)/\Delta \mathrm{SU}(2)^n$ with the standard block diagonal embedding, we have $s = 2$, $t = n$, $d_0 = 0$ and $c_{1j} = c_{2j} = \frac{1}{n}$ for all $j = 1, \dots, n$ (see [DZ, pp.37]). It follows from (8) that $Q := B_{\mathfrak{g}_1} - B_{\mathfrak{g}_2}$ produces a nonzero class $[H_Q]$ and so $b_3(G/K) = 1$, in spite of $s - t - \frac{d_0(d_0+1)}{2} = 2 - n$ is way negative for n large.

Example 4.6. If $M = \mathrm{SU}(n) \times \mathrm{SU}(n)/\Delta T^{n-1}$, then $s = 2$, $t = 0$ and $d_0 = n - 1$, and one obtains from (9) that $[H_Q] \neq 0$ for $Q := B_{\mathfrak{g}_1} - B_{\mathfrak{g}_2}$. Thus $b_3(M) = 1$ and the number $s - t - \frac{d_0(d_0+1)}{2} = 2 - \frac{n(n-1)}{2}$ is however very negative.

Conditions (8) and (9) motivate the definition of the following concept guaranteeing a large third Betti number for a homogeneous space.

Definition 4.7. A homogeneous space G/K as above, where G has s simple factors and K has t simple factors and d_0 -dimensional center, is called *aligned* when there exist positive constants c_1, \dots, c_s such that:

(i) The Killing constants defined in (7) satisfy

$$(c_{1j}, \dots, c_{sj}) = \lambda_j(c_1, \dots, c_s), \quad \text{for some } \lambda_j > 0, \quad \forall j = 1, \dots, t.$$

(ii) There exists an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{k}_0 such that

$$B_{\mathfrak{g}_i}(Z_i, W_i) = \frac{-1}{c_i} \langle Z, W \rangle, \quad \forall Z, W \in \mathfrak{k}_0.$$

(iii) $\frac{1}{c_1} + \dots + \frac{1}{c_s} = 1$. In particular, $B_{\mathfrak{k}_j} = \lambda_j B_{\mathfrak{g}}|_{\mathfrak{k}_j \times \mathfrak{k}_j}$ for all $j = 1, \dots, t$ and $\langle \cdot, \cdot \rangle = -B_{\mathfrak{g}}|_{\mathfrak{k}_0 \times \mathfrak{k}_0}$.

If G/K is aligned, then π_i is injective for all i (in particular, $\pi_i(\mathfrak{k}_j) \simeq \mathfrak{k}_j$ and $\pi_i(\mathfrak{k}) \simeq \mathfrak{k}$ for all i, j) and the Killing constants are given by

$$c_{ij} = c_i \lambda_j > 0, \quad \forall i = 1, \dots, s, \quad j = 1, \dots, t.$$

Example 4.8. If $\pi_i(\mathfrak{k}) \neq 0$ for all $i = 1, \dots, s$ and \mathfrak{k} is simple or one-dimensional, then G/K is aligned.

Example 4.9. If $\mathfrak{g}_1 = \dots = \mathfrak{g}_s = \mathfrak{h}$ and $\pi_1 = \dots = \pi_s$, i.e., $G = H \times \dots \times H$ (s -times) and $K = \Delta L$ for some subgroup $L \subset H$, then G/K is aligned with $(c_1, \dots, c_s) = (s, \dots, s)$ and $B_{\mathfrak{l}_j} = s \lambda_j B_{\mathfrak{h}}|_{\mathfrak{l}_j \times \mathfrak{l}_j}$ (or equivalently, $B_{\mathfrak{l}_j} = \lambda_j B_{\mathfrak{g}}|_{\mathfrak{l}_j \times \mathfrak{l}_j}$) for each simple factor \mathfrak{l}_j of \mathfrak{l} . It is easy to see that $M = G/K$ is diffeomorphic to $(H/L) \times H^{s-1}$, where $H^{s-1} := H \times \dots \times H$ ($(s-1)$ -times). In the particular case when $L = H$ these spaces are called *Ledger-Obata* (see [NN]).

Proposition 4.10. *The following conditions on a homogeneous space G/K as above are equivalent:*

- (i) $G/K = G'/K \times H$, where G'/K is aligned and H is semisimple.
- (ii) $b_3(G/K) = s - 1$.
- (iii) *There exists an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{k} such that $Q|_{\mathfrak{k} \times \mathfrak{k}}$ coincides with $\langle \cdot, \cdot \rangle$ up to scaling for any bi-invariant symmetric bilinear form Q on \mathfrak{g} .*

Remark 4.11. The proof of part (iii) assuming (i) given below determines the inner product $\langle \cdot, \cdot \rangle = -B_{\mathfrak{g}}|_{\mathfrak{k} \times \mathfrak{k}}$.

Proof. The equivalence between (ii) and (iii) follows from Proposition 4.1, (ii), so we will only prove the equivalence between (i) and (iii). If G/K is aligned, then there is an *adapted basis* $\{Z^1, \dots, Z^{\dim \mathfrak{k}}\}$ of $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \dots \oplus \mathfrak{k}_t$ (in the sense that it is the union of bases for each \mathfrak{k}_j) such that

$$B_{\mathfrak{g}_i}(Z_i^\alpha, Z_i^\beta) = \delta_{\alpha\beta} \frac{-1}{c_i}, \quad \forall Z^\alpha, Z^\beta \in \mathfrak{k}_0, \quad i = 1, \dots, s, \quad (\text{see Definition 4.7, (ii)}),$$

$$B_{\mathfrak{k}_j}(Z^\alpha, Z^\beta) = \delta_{\alpha\beta} (-\lambda_j), \quad \forall Z^\alpha, Z^\beta \in \mathfrak{k}_j, \quad 1 \leq j \leq t,$$

which implies that

$$B_{\mathfrak{g}_i}(Z_i^\alpha, Z_i^\beta) = \frac{1}{c_i \lambda_j} B_{\mathfrak{k}_j}(Z^\alpha, Z^\beta) = \delta_{\alpha\beta} \frac{-1}{c_i}, \quad \forall Z^\alpha, Z^\beta \in \mathfrak{k}_j, \quad 1 \leq j \leq t.$$

Thus for any $Q = y_1 B_{\mathfrak{g}_1} + \dots + y_s B_{\mathfrak{g}_s}$, $y_i \in \mathbb{R}$, we obtain that for all $i = 1, \dots, s$,

$$(10) \quad Q(Z_i^\alpha, Z_i^\beta) = -\delta_{\alpha\beta} \frac{y_i}{c_i}, \quad \forall Z^\alpha, Z^\beta \in \mathfrak{k}, \quad Q(Z_i, W_i) = -\frac{y_i}{c_i} \langle Z, W \rangle, \quad \forall Z, W \in \mathfrak{k},$$

and hence

$$(11) \quad Q(Z, W) = -\left(\frac{y_1}{c_1} + \cdots + \frac{y_s}{c_s}\right) \langle Z, W \rangle, \quad \forall Z, W \in \mathfrak{k},$$

where $\langle \cdot, \cdot \rangle$ is the inner product on \mathfrak{k} such that $\{Z^\alpha\}$ is $\langle \cdot, \cdot \rangle$ -orthonormal. This proves part (iii). We are using here that $Q(Z_i, W_i) = 0$ for all $Z \in \mathfrak{k}_j$, $W \in \mathfrak{k}_m$, where $j \neq m$, i.e., $\pi(\mathfrak{k}) = \pi_i(\mathfrak{k}_0) \oplus \cdots \oplus \pi_i(\mathfrak{k}_t)$ is a Q -orthogonal decomposition, which follows from the fact that $Q|_{\pi_i(\mathfrak{k}) \times \pi_i(\mathfrak{k})}$ is bi-invariant.

Conversely, assume that there exists an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{k} such that $Q|_{\mathfrak{k} \times \mathfrak{k}} = -f(Q)\langle \cdot, \cdot \rangle$ for any $Q = y_1 B_{\mathfrak{g}_1} + \cdots + y_s B_{\mathfrak{g}_s}$. Thus f is a linear function which is ≥ 0 if all the y_i 's are ≥ 0 , so

$$f(y_1, \dots, y_s) = a_1 y_1 + \cdots + a_s y_s, \quad \text{for some } a_1, \dots, a_s \geq 0.$$

On the other hand, for any nonzero $Z \in \mathfrak{k}$,

$$-f(y_1, \dots, y_s) = \frac{1}{\langle Z, Z \rangle} \sum_{i=1}^s Q(Z_i, Z_i) = \frac{1}{\langle Z, Z \rangle} \sum_{i=1}^s y_i B_{\mathfrak{g}_i}(Z_i, Z_i),$$

from which follows that $a_i = -\frac{B_{\mathfrak{g}_i}(Z_i, Z_i)}{\langle Z, Z \rangle}$ for all $i = 1, \dots, s$. This implies that either $\pi_i(\mathfrak{k}) \simeq \mathfrak{k}$ (if $a_i > 0$, giving rise to G') or $\pi_i(\mathfrak{k}) = 0$ (if $a_i = 0$, giving rise to H). Moreover, $a_i = -\frac{B_{\mathfrak{g}_i}(Z, Z)}{\langle Z, Z \rangle} \frac{1}{c_{ij}}$ if positive for all $Z \in \mathfrak{k}_j$ and any $j = 1, \dots, t$, that is, $c_{ij} = \lambda_j / a_i$ for some $\lambda_j > 0$. Thus G'/K is aligned, concluding the proof. \square

In other words, aligned homogeneous spaces are those with the richest third cohomology among the class of all compact homogeneous spaces with infinite isotropy. It follows from (8) and (9) that if G/K is aligned, then a bi-invariant $Q = y_1 B_{\mathfrak{g}_1} + \cdots + y_s B_{\mathfrak{g}_s}$ satisfies that $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$ if and only if

$$(12) \quad \frac{1}{c_1} y_1 + \cdots + \frac{1}{c_s} y_s = 0,$$

in which case it provides a class $[H_Q] \in H^3(G/K)$.

Example 4.12. Consider $M = \mathrm{SU}(n_1) \times \cdots \times \mathrm{SU}(n_s) / \mathrm{SU}(k_1) \times \cdots \times \mathrm{SU}(k_t)$, where $k_1 + \cdots + k_t \leq n_i$ for all i and the standard block diagonal embedding is always taken. It follows from [DZ, pp.37] that

$$(c_{1j}, \dots, c_{sj}) = k_j \left(\frac{1}{n_1}, \dots, \frac{1}{n_s} \right), \quad \forall j = 1, \dots, t.$$

Thus G/K is aligned and so $b_3(G/K) = s - 1$ by Corollary 4.3. Note that the lower bound $s - t$ provided by Corollary 4.3, (i) can be much smaller.

Example 4.13. For $M = \mathrm{SO}(14) \times E_6 / \mathrm{SU}(6) \times \mathrm{SO}(8)$ with the standard embeddings, it follows from [DZ, pp.38] that

$$(c_{1j}, c_{2j}) = \left(\frac{1}{2}, \frac{1}{2} \right), \quad \forall j = 1, 2.$$

Thus G/K is aligned with $(c_1, c_2) = (2, 2)$ and $\lambda_1 = \lambda_2 = \frac{1}{4}$, which gives $b_3(G/K) = 1$ by Corollary 4.3.

5. ON REDUCTIVE DECOMPOSITIONS

The literature on the geometry of compact homogeneous spaces G/K with G simple is much larger than for G non-simple. A possible explanation for this is that the isotropy representation of G/K is almost never multiplicity-free in the case when G is not simple. We study such behavior in this section, which may be considered of independent interest, by describing some natural reductive decompositions for a given homogeneous space G/K with G non-simple.

Given a homogeneous space $M = G/K$ as in §4 with decompositions of \mathfrak{g} and \mathfrak{k} as in (5), we consider the $B_{\mathfrak{g}_i}$ -orthogonal reductive decomposition for the homogeneous space $G_i/\pi_i(K)$,

$$(13) \quad \mathfrak{g}_i = \pi_i(\mathfrak{k}) \oplus \mathfrak{q}_i, \quad \text{for each } i = 1, \dots, s.$$

We assume from now on that the following irreducibility condition: $\pi_i(\mathfrak{k}) \neq 0$ for all $i = 1, \dots, s$ (cf. Proposition 4.10, (i)).

Any $\text{Ad}(K)$ -invariant subspace $\tilde{\mathfrak{p}}$ such that

$$\tilde{\mathfrak{k}} := \pi_1(\mathfrak{k}) \oplus \dots \oplus \pi_s(\mathfrak{k}) = \mathfrak{k} \oplus \tilde{\mathfrak{p}}$$

therefore determines a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for $M = G/K$ given by

$$(14) \quad \mathfrak{p} := \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_s \oplus \tilde{\mathfrak{p}}, \quad \text{where } \mathfrak{p}_i := (0, \dots, 0, \mathfrak{q}_i, 0, \dots, 0).$$

For any fixed vector $\eta = (a_1, \dots, a_s) \in \mathbb{R}^s$, we consider the subspace of the Lie subalgebra $\tilde{\mathfrak{k}}$ given by

$$(15) \quad \mathfrak{g}_\eta := \{(a_1 Z_1, \dots, a_s Z_s) : Z \in \mathfrak{k}\},$$

Note that $\mathfrak{k} = \mathfrak{g}_{\eta_s}$, where $\eta_s := (1, \dots, 1)$.

It is easy to show that in the case when $\pi_i : \mathfrak{k} \rightarrow \mathfrak{g}_i$ is in addition injective for any $i = 1, \dots, s$, the subspaces $\mathfrak{g}_{\eta_1}, \dots, \mathfrak{g}_{\eta_k}$ are independent as soon as the subset $\{\eta_1, \dots, \eta_k\} \subset \mathbb{R}^s$ is linearly independent. Moreover, for any nonzero η , $\dim \mathfrak{g}_\eta = \dim \mathfrak{k}$ and \mathfrak{g}_η is $\text{Ad}(K)$ -invariant and equivalent to the adjoint representation \mathfrak{k} . All this implies that for any basis of \mathbb{R}^s of the form $\{\eta_1, \dots, \eta_{s-1}, \eta_s\}$, the $\text{Ad}(K)$ -invariant subspace

$$(16) \quad \tilde{\mathfrak{p}} := \mathfrak{g}_{\eta_1} \oplus \dots \oplus \mathfrak{g}_{\eta_{s-1}}$$

satisfies that $\tilde{\mathfrak{k}} = \mathfrak{k} \oplus \tilde{\mathfrak{p}}$, giving rise to a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $M = G/K$ as in (14). We will use

$$\eta_1 := (1, -1, 0, \dots, 0), \quad \eta_2 := (1, 1, -2, 0, \dots, 0), \quad \dots \quad \eta_{s-1} := (1, \dots, 1, -(s-1)),$$

in the rest of this section.

We now study the case when the projections are not necessarily all injective. It is easy to check that

$$\tilde{\mathfrak{k}} = \tilde{\mathfrak{k}}_0 \oplus \dots \oplus \tilde{\mathfrak{k}}_t, \quad \text{where } \tilde{\mathfrak{k}}_j := \pi_1(\mathfrak{k}_j) \oplus \dots \oplus \pi_s(\mathfrak{k}_j), \quad \forall j = 0, \dots, t.$$

For each $j = 1, \dots, t$, consider the nonempty subset

$$\{i_1, \dots, i_{n_j}\} := \{1 \leq i \leq s : \pi_i(\mathfrak{k}_j) \neq 0\}.$$

Any $\eta = (a_1, \dots, a_{n_j}) \in \mathbb{R}^{n_j}$ defines the following $\text{Ad}(K)$ -invariant subspace of $\tilde{\mathfrak{k}}_j$,

$$\mathfrak{g}_\eta^j := \{a_1 Z_{i_1} + \dots + a_{n_j} Z_{i_{n_j}} : Z \in \mathfrak{k}_j\},$$

which is equivalent to \mathfrak{k}_j as an $\text{Ad}(K)$ -representation. Thus for the basis of \mathbb{R}^{n_j} given by $\{(1, \dots, 1), \eta_1, \dots, \eta_{n_j-1}\}$, we have that

$$(17) \quad \tilde{\mathfrak{k}}_j := \mathfrak{k}_j \oplus \mathfrak{g}_{\eta_1}^j \oplus \dots \oplus \mathfrak{g}_{\eta_{n_j-1}}^j.$$

On the other hand, if $\{Z^1, \dots, Z^{d_0}\}$ is any basis of \mathfrak{k}_0 , then we proceed as above for each of the subspaces $\mathbb{R}Z^k$ (instead of \mathfrak{k}_j) to obtain a decomposition $\tilde{\mathfrak{k}}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where

$$(18) \quad \mathfrak{p}_0 := \mathfrak{h}_{\eta_1}^1 \oplus \dots \oplus \mathfrak{h}_{\eta_{m_1-1}}^1 \oplus \dots \oplus \mathfrak{h}_{\eta_1}^{d_0} \oplus \dots \oplus \mathfrak{h}_{\eta_{m_{d_0}-1}}^{d_0},$$

$$\mathfrak{h}_\eta^\alpha := \mathbb{R} \left(a_1 Z_{i_1}^\alpha + \dots + a_{m_\alpha} Z_{i_{m_\alpha}}^\alpha \right) \text{ for any } \eta = (a_1, \dots, a_{m_\alpha}) \in \mathbb{R}^{m_\alpha},$$

$$\{i_1, \dots, i_{m_\alpha}\} := \{1 \leq i \leq s : \pi_i(Z^\alpha) \neq 0\}, \quad \forall \alpha = 1, \dots, d_0.$$

This provides a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the homogeneous space $M = G/K$, where

$$(19) \quad \mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_s \oplus \mathfrak{p}_0 \oplus \mathfrak{g}_{\eta_1}^1 \oplus \dots \oplus \mathfrak{g}_{\eta_{m_1-1}}^1 \oplus \dots \oplus \mathfrak{g}_{\eta_1}^t \oplus \dots \oplus \mathfrak{g}_{\eta_{m_t-1}}^t,$$

where \mathfrak{p}_i is as in (14) and \mathfrak{p}_0 is the $(m_1 + \dots + m_{d_0} - d_0)$ -dimensional subspace given in (18). As $\text{Ad}(K)$ -representations, \mathfrak{p}_i is the isotropy representation of the homogeneous space $G_i/\pi_i(K)$, each $\mathfrak{h}_{\eta_i}^\alpha \subset \mathfrak{p}_0$ is a one-dimensional trivial representation and any $\mathfrak{g}_{\eta_i}^j$ is equivalent to the irreducible representation \mathfrak{k}_j . In particular, G/K is not multiplicity-free under any of the following conditions:

- a) $G_i/\pi_i(K)$ is not multiplicity-free.
- b) $d_0 \geq 2$.
- c) $d_0 = 1$ and $m_\alpha \geq 3$ for some $\alpha = 1, \dots, d_0$.
- d) $n_j \geq 3$ for some $j = 1, \dots, t$.

Note that if all the projections are injective, then $n_1 = \dots = n_t = s$, $m_1 = \dots = m_{d_0} = s$ and each summand \mathfrak{g}_{η_i} , $i = 1, \dots, s-1$ in decomposition (16) can be written as

$$(20) \quad \mathfrak{g}_{\eta_i} = \mathfrak{h}_{\eta_i}^1 \oplus \dots \oplus \mathfrak{h}_{\eta_i}^{d_0} \oplus \mathfrak{g}_{\eta_i}^1 \oplus \dots \oplus \mathfrak{g}_{\eta_i}^t.$$

Another natural choice of a reductive decomposition for G/K is the g_b -orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where g_b is any bi-invariant metric on G , in which case $\tilde{\mathfrak{p}}$ is necessarily the g_b -orthogonal complement of \mathfrak{k} in $\pi_1(\mathfrak{k}) \oplus \dots \oplus \pi_s(\mathfrak{k})$. Note that additionally, $\tilde{\mathfrak{p}}$ can be decomposed as $\tilde{\mathfrak{p}} = \mathfrak{p}_{s+1} \oplus \dots \oplus \mathfrak{p}_r$ in g_b -orthogonal $\text{Ad}(K)$ -invariant subspaces, producing a g_b -orthogonal decomposition of \mathfrak{p} ,

$$(21) \quad \mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_s \oplus \mathfrak{p}_{s+1} \oplus \dots \oplus \mathfrak{p}_r,$$

in $\text{Ad}(K)$ -invariant subspaces.

5.1. Aligned case. The above two choices of reductive decompositions coincide in the aligned case.

Proposition 5.1. *Assume that $M = G/K$ is aligned with positive constants (c_1, \dots, c_s) and consider a bi-invariant metric $g_b = z_1(-B_{\mathfrak{g}_1}) + \dots + z_s(-B_{\mathfrak{g}_s})$, $z_1, \dots, z_s > 0$ and the corresponding g_b -orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the following is a g_b -orthogonal decomposition of \mathfrak{p} in $\text{Ad}(K)$ -invariant subspaces:*

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_s \oplus \mathfrak{p}_{s+1} \oplus \dots \oplus \mathfrak{p}_{2s-1},$$

where $\mathfrak{p}_i = (0, \dots, 0, \mathfrak{q}_i, 0, \dots, 0)$ for all $i = 1, \dots, s$ (see (13)),

$$\mathfrak{p}_{s+1} = \mathfrak{g}_{\eta_1}, \quad \dots \quad \mathfrak{p}_{2s-1} = \mathfrak{g}_{\eta_{s-1}} \quad (\text{see (15)})$$

and

$$\eta_j := \left(1, \dots, 1, -\frac{c_{j+1}}{z_{j+1}} \left(\frac{z_1}{c_1} + \dots + \frac{z_j}{c_j} \right), 0, \dots, 0 \right) \quad (1 \text{ appears } j\text{-times}),$$

for all $j = 1, \dots, s-1$. In turn, each \mathfrak{g}_{η_j} g_b -orthogonally decomposes in $\text{Ad}(K)$ -irreducible invariant subspaces as

$$\mathfrak{g}_{\eta_j} = \mathfrak{g}_{\eta_j}^0 \oplus \mathfrak{g}_{\eta_j}^1 \oplus \dots \oplus \mathfrak{g}_{\eta_j}^t,$$

where $\mathfrak{g}_\eta^l := \{(a_1 Z_1, \dots, a_s Z_s) : Z \in \mathfrak{k}_l\}$ for any $\eta = (a_1, \dots, a_s) \in \mathbb{R}^s$.

Remark 5.2. As $\text{Ad}(K)$ -representations, $\mathfrak{p}_i \simeq \mathfrak{q}_i$, the isotropy representation of the homogeneous space $G_i/\pi_i(K)$, $\mathfrak{g}_{\eta_j} \simeq \mathfrak{k}$ and $\mathfrak{g}_{\eta_i}^j \simeq \mathfrak{k}_j$. In particular, an aligned G/K is never multiplicity-free if $s \geq 3$.

Remark 5.3. If g_b is the standard metric g_B (i.e., $z_1 = \dots = z_s = 1$), then each η_j defines a closed 3-form H_{Q_j} , where Q_j is the linear combination of the $B_{\mathfrak{g}_i}$'s using the coordinates of η_j as the coefficients y_i 's. In particular, $\{[H_{Q_1}], \dots, [H_{Q_{s-1}}]\}$ is a basis for $H^3(G/K)$ (see Theorem 4.1, (iii)).

Proof. If $i < j$ and $A_{s+j} := -\frac{c_{j+1}}{z_{j+1}} \left(\frac{z_1}{c_1} + \dots + \frac{z_j}{c_j} \right)$, then for any $Z, W \in \mathfrak{k}_m$, $1 \leq m \leq s$,

$$\begin{aligned} & g_b((Z_1, \dots, Z_i, A_{s+i}Z_{i+1}, 0, \dots, 0), (W_1, \dots, W_j, A_{s+j}W_{j+1}, 0, \dots, 0)) \\ &= \sum_{l=1}^i g_b(Z_l, W_l) + A_{s+i}g_b(Z_{i+1}, W_{i+1}) = - \sum_{l=1}^i z_l B_{\mathfrak{g}_l}(Z_l, W_l) - A_{s+i}z_{i+1} B_{\mathfrak{g}_{i+1}}(Z_{i+1}, W_{i+1}) \\ &= -B_{\mathfrak{k}}(Z, W) \frac{1}{\lambda_m} \left(\sum_{l=1}^i \frac{z_l}{c_l} + A_{s+i} \frac{z_{i+1}}{c_{i+1}} \right) = 0. \end{aligned}$$

The same holds for any $Z, W \in \mathfrak{k}_0$, which follows in much the same way by using Definition 4.7, (ii). Finally, for $Z \in \mathfrak{k}_m$ and $W \in \mathfrak{k}_n$, $m \neq n$, we use that $\pi_i(\mathfrak{k}) = \pi_i(\mathfrak{k}_0) \oplus \dots \oplus \pi_i(\mathfrak{k}_t)$ is a g_b orthogonal decomposition since $g_b|_{\pi_i(\mathfrak{k}) \times \pi_i(\mathfrak{k})}$ is bi-invariant. We conclude that $g_b(\mathfrak{g}_{\eta_i}, \mathfrak{g}_{\eta_j}) = 0$ for all $i \neq j$, as was to be shown. \square

6. HARMONIC 3-FORMS ON COMPACT HOMOGENEOUS SPACES

Let $M = G/K$ be a homogeneous space, where G is compact, connected and semisimple and K is connected. We have seen in §4 that, given a reductive decomposition, there is a canonical closed 3-form H_Q attached to each bi-invariant symmetric bilinear form Q on \mathfrak{g} such that $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$. For any given bi-invariant metric g_b on G , let us fix the g_b -orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and as a background G -invariant metric the normal metric $g_b|_{\mathfrak{p} \times \mathfrak{p}}$.

The well-known fact that any bi-invariant 3-form on a compact Lie group is harmonic with respect to any bi-invariant metric motivates the following natural problem:

Is any closed 3-form H_Q g_b -harmonic? If not, what are the conditions in terms of algebraic invariants of the groups G , K and the embedding $K \subset G$ to be fulfilled by Q and a normal metric g_b on $M = G/K$ in order for H_Q to be g_b -harmonic? Can some other G -invariant metric g (not necessarily normal) satisfy that H_Q is g -harmonic for every Q ?

We first note that for any bi-invariant symmetric bilinear form Q on \mathfrak{g} ,

$$(22) \quad \text{the 3-form } \tilde{Q} = Q([\cdot, \cdot], \cdot) \in (\Lambda^3 \mathfrak{p}^*)^K \text{ is } g_b\text{-coclosed.}$$

Indeed, for any $\beta \in (\Lambda^2 \mathfrak{p}^*)^K$,

$$g_b(\tilde{Q}, d\beta) = g_b(\tilde{Q}, \hat{d}\hat{\beta}) = g_b(\overline{Q}, \hat{d}\hat{\beta}) = g_b(\hat{d}_{g_b}^* \overline{Q}, \hat{\beta}) = 0,$$

since g_b and \overline{Q} are both bi-invariant on G and so $\hat{d}_{g_b}^* \overline{Q} = 0$.

For any g_b -orthogonal decomposition,

$$(23) \quad \mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_r,$$

in $\text{Ad}(K)$ -invariant subspaces (not necessarily irreducible), we take a g_b -orthonormal basis $\{e_\alpha^i\}$ of \mathfrak{p}_i for each $i = 1, \dots, r$ and consider the corresponding structural constants

$$c_{i\alpha, j\beta}^{k\gamma} := g_b([e_\alpha^i, e_\beta^j], e_\gamma^k).$$

Note that $c_{i\alpha,j\beta}^{k\gamma} = -c_{j\beta,i\alpha}^{k\gamma} = -c_{i\alpha,k\gamma}^{j\beta}$. Given any G -invariant metric g , there exists at least one g_b -orthogonal decomposition as in (23), which is also g -orthogonal, such that g has the form

$$(24) \quad g = x_1 g_b|_{\mathfrak{p}_1 \times \mathfrak{p}_1} + \cdots + x_r g_b|_{\mathfrak{p}_r \times \mathfrak{p}_r}, \quad x_i > 0.$$

This metric will be denoted by

$$g = (x_1, \dots, x_r)_{g_b}.$$

Note that the background normal metric is $g_b = (1, \dots, 1)_{g_b}$ and $\{\frac{1}{\sqrt{x_i}} e_\alpha^i\}$ is a g -orthonormal basis of \mathfrak{p}_i .

We now consider a g_b -orthogonal reductive decomposition as in (21) (recall also the decompositions (5) and (14)), so

$$\mathfrak{p} = \bar{\mathfrak{p}} \oplus \tilde{\mathfrak{p}}, \quad \text{where } \bar{\mathfrak{p}} := \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s, \quad \tilde{\mathfrak{p}} := \mathfrak{p}_{s+1} \oplus \cdots \oplus \mathfrak{p}_r.$$

It is easy to check that for all $1 \leq i \neq j \leq s$ and $s+1 \leq k, l \leq r$,

$$(25) \quad [\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{k} + \mathfrak{p}_i + \tilde{\mathfrak{p}}, \quad [\mathfrak{p}_i, \mathfrak{p}_j] = 0, \quad [\mathfrak{p}_k, \mathfrak{p}_i] \subset \mathfrak{p}_i, \quad [\mathfrak{p}_k, \mathfrak{p}_l] \subset \mathfrak{k} + \tilde{\mathfrak{p}}.$$

We have that the bi-invariant metric is given by

$$g_b = z_1(-B_{\mathfrak{g}_1}) + \cdots + z_s(-B_{\mathfrak{g}_s}), \quad \text{for some } z_1, \dots, z_s > 0,$$

and recall that any bi-invariant symmetric bilinear form Q such that $Q|_{\mathfrak{k} \times \mathfrak{k}} = 0$, say,

$$Q = y_1 B_{\mathfrak{g}_1} + \cdots + y_s B_{\mathfrak{g}_s}, \quad y_1, \dots, y_s \in \mathbb{R},$$

defines a closed 3-form H_Q as in (6). Since

$$(26) \quad Q(\mathfrak{p}_i, \mathfrak{p}_j) = Q(\mathfrak{p}_i, \mathfrak{p}_k) = Q(\mathfrak{p}_i, \mathfrak{k}) = 0, \quad \forall i \neq j \leq s < k,$$

the 3-form H_Q is nonzero (up to permutations) only on

$$\mathfrak{p}_i \times \mathfrak{p}_i \times \mathfrak{p}_i, \quad \mathfrak{p}_i \times \mathfrak{p}_i \times \mathfrak{p}_j, \quad \mathfrak{p}_j \times \mathfrak{p}_k \times \mathfrak{p}_l, \quad i \leq s < j, k, l,$$

and since by (26) the subscript \mathfrak{p} can be deleted in formula (6) when the three vectors are in \mathfrak{p}_i , $i \leq s$, we obtain that

$$(27) \quad H_Q(X, Y, Z) = Q([X, Y], Z) = -\frac{y_i}{z_i} g_b([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{p}_i, \quad i \leq s.$$

We now show that the subspace $(\Lambda^2 \bar{\mathfrak{p}}^*)^K$ imposes no condition on the g -coclosedness of H_Q .

Proposition 6.1. $g(H_Q, d\omega) = 0$ for any $\omega \in (\Lambda^2 \mathfrak{p}^*)^K$ such that $\omega(\tilde{\mathfrak{p}}, \cdot) = 0$ and any metric $g = (x_1, \dots, x_r)_{g_b}$.

Proof. If $H \in (\Lambda^3 \mathfrak{p}^*)^K$ and $\omega \in (\Lambda^2 \mathfrak{p}^*)^K$, then

$$\begin{aligned} g(H, d\omega) &= \sum_{\substack{\alpha, \beta, \gamma \\ i, j, k}} \frac{1}{x_i x_j x_k} H(e_\alpha^i, e_\beta^j, e_\gamma^k) d\omega(e_\alpha^i, e_\beta^j, e_\gamma^k) \\ &= \sum_{\substack{\alpha, \beta, \gamma \\ i, j, k}} \frac{1}{x_i x_j x_k} H(e_\alpha^i, e_\beta^j, e_\gamma^k) \left(-\omega([e_\alpha^i, e_\beta^j]_{\mathfrak{p}}, e_\gamma^k) + \omega([e_\alpha^i, e_\gamma^k]_{\mathfrak{p}}, e_\beta^j) - \omega([e_\beta^j, e_\gamma^k]_{\mathfrak{p}}, e_\alpha^i) \right) \\ (28) \quad &= -3 \sum_{\substack{\alpha, \beta, \gamma \\ i, j, k}} \frac{1}{x_i x_j x_k} H(e_\alpha^i, e_\beta^j, e_\gamma^k) \omega([e_\alpha^i, e_\beta^j]_{\mathfrak{p}}, e_\gamma^k). \end{aligned}$$

It therefore follows from (25) that for $\omega \in (\Lambda^2 \mathfrak{p}^*)^K$ such that $\omega(\tilde{\mathfrak{p}}, \cdot) = 0$,

$$\begin{aligned} g(H_Q, d\omega) = & -3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s}} \frac{1}{x_i^3} H_Q(e_\alpha^i, e_\beta^i, e_\gamma^i) \omega([e_\alpha^i, e_\beta^i]_{\mathfrak{p}}, e_\gamma^i) \\ & -6 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j}} \frac{1}{x_i^2 x_j} H_Q(e_\alpha^j, e_\beta^i, e_\gamma^i) \omega([e_\alpha^j, e_\beta^i]_{\mathfrak{p}}, e_\gamma^i). \end{aligned}$$

The second summand always vanishes since if $\pi_i(e_\alpha^j) = \pi_i(Z)$, $Z \in \mathfrak{k}$, then

$$\begin{aligned} \omega([e_\alpha^j, e_\beta^i]_{\mathfrak{p}}, e_\gamma^i) &= \omega([\pi_i(Z), e_\beta^i], e_\gamma^i) = \omega([Z, e_\beta^i], e_\gamma^i) = \omega([Z, e_\gamma^i], e_\beta^i) \\ &= \omega([\pi_i(Z), e_\gamma^i], e_\beta^i) = \omega([e_\alpha^j, e_\gamma^i]_{\mathfrak{p}}, e_\beta^i), \quad \forall i \leq s, \end{aligned}$$

that is, $\omega([e_\alpha^j, e_\beta^i]_{\mathfrak{p}}, e_\gamma^i)$ is symmetric as a function of (β, γ) . On the other hand, according to (27), the first summand equals

$$3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s}} \frac{y_i}{z_i x_i^3} c_{i\alpha, i\beta}^{i\gamma} \omega([e_\alpha^i, e_\beta^i]_{\mathfrak{p}}, e_\gamma^i) = 3 \sum_{\substack{\alpha, \beta, \gamma, \delta \\ i \leq s}} \frac{y_i}{z_i x_i^3} c_{i\alpha, i\beta}^{i\gamma} c_{i\alpha, i\beta}^{i\delta} \omega(e_\delta^i, e_\gamma^i),$$

which vanishes since $c_{k\alpha, k\beta}^{k\gamma} c_{k\alpha, k\beta}^{k\delta}$ and $\omega(e_\delta^k, e_\gamma^k)$ are respectively symmetric and skew-symmetric as functions of (δ, γ) . \square

For each $i = 1, \dots, s$, we consider a decomposition of the isotropy representation of $G_i/\pi_i(K)$,

$$\mathfrak{q}_i = \mathfrak{q}_i^0 \oplus \mathfrak{q}_i^1 \oplus \dots \oplus \mathfrak{q}_i^{r_i},$$

in $\text{Ad}(K)$ -invariant subspaces, where \mathfrak{q}_i^0 is the trivial representation (i.e., $[\mathfrak{k}, \mathfrak{q}_i^0] = 0$) and \mathfrak{q}_i^k is irreducible and non-trivial for all $k = 1, \dots, r_i$.

If we assume that the following two conditions hold:

- (i) \mathfrak{q}_i^k and \mathfrak{k}_j are not equivalent as K -representations for all $i = 1, \dots, s$, $k = 1, \dots, r_i$ and $j = 1, \dots, t$;
- (ii) either $\mathfrak{q}_i^0 = 0$ for all $i = 1, \dots, s$ or $\mathfrak{k}_0 = 0$,

then the above proposition implies that the conditions on Q and $g = (x_1, \dots, x_r)_{g_b}$ in order for H_Q to be g -coclosed will only come out from the linear system of equations given by

$$(29) \quad g(H_Q, d\omega) = 0, \quad \forall \omega \in (\Lambda^2 \mathfrak{p}^*)^K, \quad \omega(\bar{\mathfrak{p}}, \cdot) = 0.$$

Indeed, conditions (i) and (ii) above imply that $\omega(\bar{\mathfrak{p}}, \tilde{\mathfrak{p}}) = 0$ for any $\omega \in (\Lambda^2 \mathfrak{p}^*)^K$ since any two different isotypic components are necessarily ω -orthogonal.

The following are examples of spaces which can not appear as a $G_i/\pi_i(K)$ in order for G/K to satisfy condition (i) above (they were generously provided to the authors by C. Böhm and W. Ziller, respectively).

Example 6.2. Assume that G_i contains a subgroup of the form $H \times H$ and consider G_i/K , where $K := \Delta H \subset H \times H \subset G$. Thus the adjoint representation $\mathfrak{k} \simeq \mathfrak{h}$ also appears in \mathfrak{q}_i .

Example 6.3. If $G_i/\pi_i(K) = \text{SO}(n)/\text{SO}(3)$, then the 3-dimensional irreducible factors in the standard reductive complement \mathfrak{q}_i are necessarily equivalent to the adjoint representation $\mathfrak{k} = \mathfrak{so}(3)$.

6.1. **Case $s = 2$.** In the case when \mathfrak{g} has only two simple factors, we have that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$, where, as an $\text{Ad}(K)$ -representation, \mathfrak{p}_3 is equivalent to an ideal \mathfrak{h} of \mathfrak{k} , say via an equivalence map $\varphi : \mathfrak{h} \rightarrow \mathfrak{p}_3$. This implies that any $\omega \in (\Lambda^2 \mathfrak{p}_3^*)^K$ vanishes on $\varphi([\mathfrak{h}, \mathfrak{h}]) \cap \mathfrak{p}_3$ and so $d\omega = 0$. It follows from Proposition 6.1 that H_Q is g -harmonic for any $g = (x_1, x_2, x_3)_{g_b}$ and any g_b , provided that conditions (i) and (ii) above hold. Note that if in addition \mathfrak{q}_1 and \mathfrak{q}_2 are $\text{Ad}(K)$ -irreducible and \mathfrak{k} is either simple or one-dimensional, then any G -invariant metric is of the form $g = (x_1, x_2, x_3)_{g_b}$.

7. ALIGNED CASE

In this section, we continue the study of the problem stated at the beginning of §6, under the assumption that $M = G/K$ is aligned (see Definition 4.7). We therefore need to focus on condition (29).

We consider the g_b -orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ provided by Proposition 5.1, which is given by

$$\mathfrak{p} = \bar{\mathfrak{p}} \oplus \tilde{\mathfrak{p}}, \quad \text{where} \quad \bar{\mathfrak{p}} := \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s, \quad \tilde{\mathfrak{p}} := \mathfrak{g}_{\eta_1} \oplus \cdots \oplus \mathfrak{g}_{\eta_{s-1}}.$$

Recall that all these subspaces are pairwise g_b -orthogonal. It is easy to see that in addition to the Lie bracket properties given in (25), we have that

$$(30) \quad \begin{aligned} [\mathfrak{p}_i, \mathfrak{p}_i] &\subset \mathfrak{p}_i + \mathfrak{g}_{\eta_{i-1}} + \cdots + \mathfrak{g}_{\eta_{s-1}} + \mathfrak{k}, & \forall i \leq s, \\ [\mathfrak{g}_{\eta_i}, \mathfrak{g}_{\eta_i}] &\subset \mathfrak{g}_{\eta_i} + \cdots + \mathfrak{g}_{\eta_{s-1}} + \mathfrak{k}, & \forall i \leq s, \\ [\mathfrak{g}_{\eta_i}, \mathfrak{g}_{\eta_j}] &\subset \mathfrak{g}_{\eta_i}, & \forall i < j \leq s. \end{aligned}$$

Recall from (12) that, in order to define a closed 3-form H_Q , a bi-invariant symmetric bilinear form $Q = y_1 B_{\mathfrak{g}_1} + \cdots + y_s B_{\mathfrak{g}_s}$ must satisfy

$$\frac{y_1}{c_1} + \cdots + \frac{y_s}{c_s} = 0.$$

The following notation will be strongly used from now on without any further reference: for $j = 1, \dots, s-1$,

$$\begin{aligned} A_{s+j} &:= -\frac{c_{j+1}}{z_{j+1}} \left(\frac{z_1}{c_1} + \cdots + \frac{z_j}{c_j} \right), & B_{s+j} &:= \frac{z_1}{c_1} + \cdots + \frac{z_j}{c_j} + A_{s+j}^2 \frac{z_{j+1}}{c_{j+1}}, \\ B_{2s} &:= \frac{z_1}{c_1} + \cdots + \frac{z_s}{c_s}, & C_{s+j} &:= \frac{y_1}{c_1} + \cdots + \frac{y_j}{c_j} + A_{s+j} \frac{y_{j+1}}{c_{j+1}}, \\ D_{s+j} &:= \frac{z_1}{c_1} + \cdots + \frac{z_j}{c_j} + A_{s+j}^3 \frac{z_{j+1}}{c_{j+1}}, & D_{2s} &:= B_{2s}, \quad C_{2s} := 0. \end{aligned}$$

We consider the inner product $\langle \cdot, \cdot \rangle = -B_{\mathfrak{g}}|_{\mathfrak{k} \times \mathfrak{k}}$ on \mathfrak{k} (cf. Proposition 4.10, (iii)) together with a $\langle \cdot, \cdot \rangle$ -orthonormal adapted basis $\{Z^1, \dots, Z^{\dim \mathfrak{k}}\}$ of $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_t$. It follows from (10) that

$$(31) \quad g_b(Z_i^\alpha, Z_i^\beta) = \delta_{\alpha\beta} \frac{z_i}{c_i}, \quad \forall Z^\alpha, Z^\beta \in \mathfrak{k}, \quad g_b(Z_i, W_i) = \frac{z_i}{c_i} \langle Z, W \rangle, \quad \forall Z, W \in \mathfrak{k},$$

and from (11) that

$$(32) \quad g_b(Z, W) = B_{2s} \langle Z, W \rangle, \quad \forall Z, W \in \mathfrak{k}.$$

If $\varphi_i : \mathfrak{k} \rightarrow \mathfrak{g}_{\eta_i}$ is the $\text{Ad}(K)$ -equivariant isomorphism given by

$$\varphi_i(Z) := (Z_1, \dots, Z_i, A_{s+i} Z_{i+1}, 0, \dots, 0), \quad \forall Z \in \mathfrak{k}, \quad i = 1, \dots, s,$$

(note that $\varphi_s : \mathfrak{k} \rightarrow \mathfrak{g}_{\eta_s} = \mathfrak{k}$ is the identity map) and we set

$$e_\alpha^{s+i} := \frac{1}{\sqrt{B_{s+i}}} \varphi_i(Z^\alpha),$$

then it follows from (31) that $\{e_\alpha^{s+i} : \alpha = 1, \dots, \dim \mathfrak{k}\}$ is a g_b -orthonormal basis of \mathfrak{g}_{η_i} for $1 \leq i \leq s$ (note that $e_\alpha^{2s} = \frac{1}{\sqrt{B_{2s}}} Z^\alpha$) and so

$$(33) \quad \{e_\alpha^{s+i} : i = 1, \dots, s, \alpha = 1, \dots, \dim \mathfrak{k}\}$$

is a g_b -orthonormal adapted basis of the subalgebra $\tilde{\mathfrak{k}} = \tilde{\mathfrak{p}} \oplus \mathfrak{k} = \mathfrak{g}_{\eta_1} \oplus \dots \oplus \mathfrak{g}_{\eta_{s-1}} \oplus \mathfrak{g}_{\eta_s}$. We also take a g_b -orthonormal adapted basis

$$(34) \quad \{e_\alpha^i : i = 1, \dots, s, \alpha = 1, \dots, \dim \mathfrak{p}_i\}$$

of $\tilde{\mathfrak{p}}$.

Lemma 7.1.

- (i) *The nonzero structural constants of the basis $\{e_\alpha^i\}$ of \mathfrak{g} given in (33) and (34) other than $c_{i\alpha, i\beta}^{i\gamma}$, $i \leq s$, are given by*

$$c_{i\alpha, i\beta}^{j\gamma} = \begin{cases} \frac{\sqrt{B_{2s}}}{\sqrt{B_j}} c_{i\alpha, i\beta}^{2s\gamma}, & i \leq s, i+s \leq j < 2s, \\ \frac{A_{i+s-1}\sqrt{B_{2s}}}{\sqrt{B_j}} c_{i\alpha, i\beta}^{2s\gamma}, & i \leq s, j = i+s-1, \\ 0, & i \leq s < j < i+s-1, \end{cases} \quad c_{i\alpha, i\beta}^{j\gamma} = \begin{cases} \frac{1}{\sqrt{B_j}} \bar{c}_{\alpha\beta}^\gamma, & s < i < j \leq 2s, \\ \frac{D_i}{\sqrt{B_i^3}} \bar{c}_{\alpha\beta}^\gamma, & s < i = j \leq 2s, \end{cases}$$

where $\bar{c}_{\alpha\beta}^\gamma$ are the structural constants of \mathfrak{k} with respect to the basis $\{Z^1, \dots, Z^{\dim \mathfrak{k}}\}$.

- (ii) $Q(e_\alpha^{2s}, e_\beta^j) = \delta_{\alpha\beta} \frac{-C_j}{\sqrt{B_j B_{2s}}}$, for all $j = s+1, \dots, 2s-1$.

Proof. For $i \leq s < j < 2s$ we have that,

$$c_{i\alpha, i\beta}^{j\gamma} = \frac{1}{\sqrt{B_j}} g_b((0, \dots, 0, [e_\alpha^i, e_\beta^i], 0, \dots, 0), (Z_1^\gamma, \dots, Z_{j-s}^\gamma, A_j Z_{j-s+1}^\gamma, 0, \dots, 0)),$$

and thus the formula in part (i) follows from the fact that

$$g_b([e_\alpha^i, e_\beta^i], Z_i^\gamma) = g_b([e_\alpha^i, e_\beta^i], Z^\gamma) = \sqrt{B_{2s}} g_b([e_\alpha^i, e_\beta^i], e_\gamma^{2s}) = \sqrt{B_{2s}} c_{i\alpha, i\beta}^{2s\gamma}.$$

Using (31), for $s < i < j \leq 2s$ we obtain,

$$\begin{aligned} c_{i\alpha, i\beta}^{j\gamma} &= g_b([e_\alpha^i, e_\beta^i], e_\gamma^j) = \frac{1}{B_i \sqrt{B_j}} g_b([\varphi_{i-s}(Z^\alpha), \varphi_{i-s}(Z^\beta)], \varphi_{j-s}(Z^\gamma)) \\ &= \frac{1}{B_i \sqrt{B_j}} g_b\left(\left([Z_1^\alpha, Z_1^\beta], \dots, [Z_{i-s}^\alpha, Z_{i-s}^\beta], A_i^2 [Z_{i-s+1}^\alpha, Z_{i-s+1}^\beta], 0, \dots, 0\right), \varphi_{j-s}(Z^\gamma)\right) \\ &= \frac{1}{B_i \sqrt{B_j}} \left(\sum_{l=1}^{i-s} g_b([Z^\alpha, Z^\beta]_l, Z_l^\gamma) + A_i^2 g_b([Z^\alpha, Z^\beta]_{i-s+1}, Z_{i-s+1}^\gamma) \right) \\ &= \frac{1}{B_i \sqrt{B_j}} \left(\sum_{l=1}^{i-s} \frac{z_l}{c_l} + A_i^2 \frac{z_{i-s+1}}{c_{i-s+1}} \right) \langle [Z^\alpha, Z^\beta], Z^\gamma \rangle = \frac{1}{\sqrt{B_j}} \bar{c}_{\alpha\beta}^\gamma. \end{aligned}$$

In much the same way, if $s < i = j \leq 2s$ then

$$c_{i\alpha, i\beta}^{i\gamma} = \frac{1}{B_i \sqrt{B_i}} \left(\sum_{l=1}^{i-s} \frac{z_l}{c_l} + A_i^3 \frac{z_{i-s+1}}{c_{i-s+1}} \right) \langle [Z^\alpha, Z^\beta], Z^\gamma \rangle = \frac{D_i}{\sqrt{B_i^3}} \bar{c}_{\alpha\beta}^\gamma.$$

Finally, part (ii) follows from the following computation:

$$\begin{aligned}
Q(e_\alpha^{2s}, e_\beta^j) &= \frac{1}{\sqrt{B_{2s}B_j}} Q(Z^\alpha, \varphi_{j-s}(Z^\beta)) \\
&= \frac{1}{\sqrt{B_{2s}B_j}} \left(\sum_{l=1}^{j-s} y_l B_{\mathfrak{g}_l}(Z_l^\alpha, Z_l^\beta) + y_{j-s+1} A_j B_{\mathfrak{g}_{j-s+1}}(Z_{j-s+1}^\alpha, Z_{j-s+1}^\beta) \right) \\
&= \frac{1}{\sqrt{B_{2s}B_j}} \left(\sum_{l=1}^{j-s} y_l \delta_{\alpha\beta} \frac{-1}{c_l} + y_{j-s+1} A_j \delta_{\alpha\beta} \frac{-1}{c_{j-s+1}} \right) = \delta_{\alpha\beta} \frac{-C_j}{\sqrt{B_{2s}B_j}},
\end{aligned}$$

concluding the proof. \square

According to Proposition 5.1, if we set

$$\tilde{\mathfrak{p}}_0 := \mathfrak{g}_{\eta_1}^0 + \cdots + \mathfrak{g}_{\eta_{s-1}}^0, \quad \tilde{\mathfrak{p}}_{\geq 1} := \sum_{j=1}^t \mathfrak{g}_{\eta_1}^j + \cdots + \mathfrak{g}_{\eta_{s-1}}^j,$$

then $\tilde{\mathfrak{p}} = \tilde{\mathfrak{p}}_0 \oplus \tilde{\mathfrak{p}}_{\geq 1}$ and it is easy to check that

$$(35) \quad [\tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}] \subset \tilde{\mathfrak{p}}_{\geq 1} + \mathfrak{k}.$$

Lemma 7.2.

- (i) Any skew-symmetric $(s-1) \times (s-1)$ matrix $[\omega_{ij}]$, $s < i, j \leq 2s-1$, defines a 2-form $\omega \in (\Lambda^2 \mathfrak{p}^*)^K$ such that $\omega(\tilde{\mathfrak{p}}, \cdot) = 0$ and which on $\tilde{\mathfrak{p}} \times \tilde{\mathfrak{p}}$ is given by,

$$\omega(\varphi_{i-s}(Z), \varphi_{j-s}(W)) := \omega_{ij} g_b(Z, W), \quad \forall Z, W \in \mathfrak{k}, \quad s < i, j \leq 2s-1.$$

- (ii) The above 2-form satisfies that

$$\omega(e_\alpha^i, e_\beta^j) = \delta_{\alpha\beta} \omega_{ij} \frac{B_{2s}}{\sqrt{B_i B_j}}, \quad \forall s < i, j \leq 2s-1.$$

- (iii) Any $\omega \in (\Lambda^2 \mathfrak{p}^*)^K$ such that $\omega(\tilde{\mathfrak{p}}, \cdot) = 0$ is of the form $\omega = \omega_1 + \omega_2$, where ω_1 is as above, $\omega_2(\tilde{\mathfrak{p}}_{\geq 1}, \cdot) = 0$ and $d\omega_2 = 0$.

Proof. It follows from the $\text{Ad}(K)$ -equivariance of each φ_{i-s} that for any $U, Z, W \in \mathfrak{k}$,

$$\begin{aligned}
\omega([U, \varphi_{i-s}(Z)], \varphi_{j-s}(W)) &= \omega(\varphi_{i-s}([U, Z]), \varphi_{j-s}(W)) = \omega_{ij} g_b([U, Z], W) \\
&= -\omega_{ij} g_b(Z, [U, W]) = -\omega(\varphi_{i-s}(Z), \varphi_{j-s}([U, W])) \\
&= -\omega(\varphi_{i-s}(Z), [U, \varphi_{j-s}(W)]),
\end{aligned}$$

which shows that ω is $\text{Ad}(K)$ -invariant and so part (i) holds.

Part (ii) follows from (32) and part (iii) from the fact that $(\Lambda^2 \mathfrak{g}_{\eta_i}^j)^K = 0$ for all i and $j \geq 1$ (recall that the K -representation $\mathfrak{g}_{\eta_i}^j$ is equivalent to the adjoint representation of the simple Lie algebra \mathfrak{k}_j) and so $(\Lambda^2(\mathfrak{g}_{\eta_1}^1 + \cdots + \mathfrak{g}_{\eta_{s-1}}^{j_{s-1}}))^K = 0$, as they are pairwise non-equivalent as K -representations. Note that the closedness of ω_2 is a consequence of (35), concluding the proof. \square

We consider the Casimir operator of the \mathfrak{k} -representation \mathfrak{p}_i relative to the bi-invariant inner product $\langle \cdot, \cdot \rangle = -B_{\mathfrak{g}}|_{\mathfrak{k} \times \mathfrak{k}} = \frac{1}{B_{2s}} g_b|_{\mathfrak{k} \times \mathfrak{k}}$ on \mathfrak{k} , given by

$$C_{\mathfrak{p}_i, \langle \cdot, \cdot \rangle} = - \sum_{\alpha=1}^{\dim \mathfrak{p}_i} (\text{ad } Z^\alpha|_{\mathfrak{p}_i})^2, \quad i = 1, \dots, s,$$

and the Casimir operator $C_{\text{ad } \mathfrak{k}, \langle \cdot, \cdot \rangle}$ of the adjoint representation. Recall that \mathfrak{p}_i is equivalent to the isotropy representation \mathfrak{q}_i of the homogeneous space $G_i/\pi_i(K)$ (see (13)). The following algebraic invariants of G/K will appear in the computations below:

$$(36) \quad \begin{aligned} Cas_i &:= \text{tr } C_{\mathfrak{p}_i, \langle \cdot, \cdot \rangle} = B_{2s} \sum_{\alpha, \beta, \gamma} (c_{i\alpha, i\beta}^{2s\gamma})^2, \quad 1 \leq i \leq s, \\ Cas_0 &:= \text{tr } C_{\text{ad } \mathfrak{k}, \langle \cdot, \cdot \rangle} = \sum_{\alpha, \beta, \gamma} (\bar{c}_{\alpha\beta}^\gamma)^2 = \lambda_1 \dim \mathfrak{k}_1 + \cdots + \lambda_t \dim \mathfrak{k}_t. \end{aligned}$$

We are using that $B_{\mathfrak{k}_j} = \lambda_j B_{\mathfrak{g}}|_{\mathfrak{k} \times \mathfrak{k}}$ and that $\text{tr } C_{\text{ad } \mathfrak{k}_j, -B_{\mathfrak{k}_j}} = \dim \mathfrak{k}_j$ for any $j = 1, \dots, t$ in the last equality. On the other hand, it follows from [LL, Lemma 3.1, (iii)] that

$$\text{tr } C_{\mathfrak{q}_i, -B_{\mathfrak{g}_i}|_{\pi_i(\mathfrak{k})}} + \text{tr } C_{\text{ad } \pi_i(\mathfrak{k}), -B_{\mathfrak{g}_i}|_{\pi_i(\mathfrak{k})}} = \dim \pi_i(\mathfrak{k}) = \dim \mathfrak{k},$$

and since $\langle \cdot, \cdot \rangle = c_i(-B_{\mathfrak{g}_i})(\pi_i \cdot, \pi_i \cdot)$ and consequently,

$$C_{\mathfrak{p}_i, \langle \cdot, \cdot \rangle} = C_{\mathfrak{q}_i, c_i(-B_{\mathfrak{g}_i})|_{\pi_i(\mathfrak{k})}} = \frac{1}{c_i} C_{\mathfrak{q}_i, -B_{\mathfrak{g}_i}|_{\pi_i(\mathfrak{k})}}, \quad C_{\text{ad } \mathfrak{k}, \langle \cdot, \cdot \rangle} = \frac{1}{c_i} C_{\text{ad } \pi_i(\mathfrak{k}), -B_{\mathfrak{g}_i}|_{\pi_i(\mathfrak{k})}},$$

we obtain that

$$(37) \quad Cas_i + Cas_0 = \frac{1}{c_i} \dim \mathfrak{k}, \quad \forall i = 1, \dots, s.$$

We are now ready to perform the main computation of this paper.

Proposition 7.3. *Consider any G -invariant metric $g = (x_1, \dots, x_{2s-1})_{g_b}$ as in (24). Then $g(H_Q, d\omega) = 0$ for any $\omega \in (\Lambda^2 \mathfrak{p}^*)^K$ such that $\omega(\bar{\mathfrak{p}}, \cdot) = 0$ if and only if for all $s+1 \leq j < k \leq 2s-1$,*

$$(38) \quad x_k \left(E_k + Cas_0 F_k + 2Cas_0 \left(\frac{1}{x_j^2} - \frac{1}{x_k^2} \right) \right) C_j = x_j (E_j + Cas_0 F_j) C_k,$$

where

$$\begin{aligned} E_{s+j} &:= \frac{Cas_1}{x_1^2} + \cdots + \frac{Cas_j}{x_j^2} + A_{s+j} \frac{Cas_{j+1}}{x_{j+1}^2}, \quad \forall j = 1, \dots, s-1, \\ F_k &:= \frac{1}{x_{s+1}^2} + \cdots + \frac{1}{x_{k-1}^2} + \frac{D_k}{B_k} \frac{1}{x_k^2}, \quad \forall k = s+1, \dots, 2s-1. \end{aligned}$$

Proof. We consider the g_b -orthonormal basis $\{e_\alpha^i : 1 \leq i \leq 2s-1, 1 \leq \alpha \leq \dim \mathfrak{p}_i\}$ of \mathfrak{p} given in (33) and (34). It follows from Lemma 7.2, (iii) that ω can be assumed to be as in parts (i) and (ii) of the same lemma. Recall from (22) that the 3-form \tilde{Q} is g_b -coclosed. It therefore follows from (6), (28), (25) and (30) (which in particular implies that $Q(\mathfrak{k}, e_\alpha^i) = 0$

for all $i \leq s$, $[e_\alpha^i, e_\beta^j] = 0$ for all $i \neq j \leq s$ and $[e_\alpha^i, e_\beta^j]_{\mathfrak{k}} = 0$ for all $s < i \neq j$) that

$$\begin{aligned}
g_b(H_Q, d\omega) &= g_b(d\alpha_Q, d\omega) \\
&= -3 \sum_{\substack{\alpha, \beta, \gamma \\ i, j; s < k}} \frac{1}{x_i x_j x_k} \left(Q([e_\alpha^i, e_\beta^j]_{\mathfrak{k}}, e_\gamma^k) - Q([e_\alpha^i, e_\gamma^k]_{\mathfrak{k}}, e_\beta^j) + Q([e_\beta^j, e_\gamma^k]_{\mathfrak{k}}, e_\alpha^i) \right) \omega([e_\alpha^i, e_\beta^j]_{\mathfrak{p}}, e_\gamma^k) \\
&= -3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j}} \frac{1}{x_i^2 x_j} Q([e_\alpha^i, e_\beta^i]_{\mathfrak{k}}, e_\gamma^j) \omega([e_\alpha^i, e_\beta^i]_{\mathfrak{p}}, e_\gamma^j) \\
&\quad - 3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i}} \frac{1}{x_i^3} (Q([e_\alpha^i, e_\beta^i]_{\mathfrak{k}}, e_\gamma^i) - Q([e_\alpha^i, e_\gamma^i]_{\mathfrak{k}}, e_\beta^i) + Q([e_\beta^i, e_\gamma^i]_{\mathfrak{k}}, e_\alpha^i)) \omega([e_\alpha^i, e_\beta^i]_{\mathfrak{p}}, e_\gamma^i) \\
&\quad - 3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i \neq j}} \frac{1}{x_i^2 x_j} Q([e_\alpha^i, e_\beta^i]_{\mathfrak{k}}, e_\gamma^j) \omega([e_\alpha^i, e_\beta^i]_{\mathfrak{p}}, e_\gamma^j) \\
&\quad + 3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i \neq j}} \frac{1}{x_i^2 x_j} Q([e_\alpha^i, e_\gamma^i]_{\mathfrak{k}}, e_\beta^j) \omega([e_\alpha^i, e_\beta^j]_{\mathfrak{p}}, e_\gamma^i) - 3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i \neq j}} \frac{1}{x_i^2 x_j} Q([e_\beta^i, e_\gamma^i]_{\mathfrak{k}}, e_\alpha^j) \omega([e_\alpha^j, e_\beta^i]_{\mathfrak{p}}, e_\gamma^i) \\
&= -3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j}} \frac{1}{x_i^2 x_j} Q([e_\alpha^i, e_\beta^i]_{\mathfrak{k}}, e_\gamma^j) \omega([e_\alpha^i, e_\beta^i]_{\mathfrak{p}}, e_\gamma^j) + 6 \sum_{\substack{\alpha, \beta, \gamma \\ s < i}} \frac{1}{x_i^3} Q([e_\alpha^i, e_\gamma^i]_{\mathfrak{k}}, e_\beta^i) \omega([e_\alpha^i, e_\beta^i]_{\mathfrak{p}}, e_\gamma^i) \\
&\quad - 3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i, j}} \frac{1}{x_i^2 x_j} Q([e_\alpha^i, e_\beta^i]_{\mathfrak{k}}, e_\gamma^j) \omega([e_\alpha^i, e_\beta^i]_{\mathfrak{p}}, e_\gamma^j) + 6 \sum_{\substack{\alpha, \beta, \gamma \\ s < j < i}} \frac{1}{x_i^2 x_j} Q([e_\alpha^i, e_\beta^i]_{\mathfrak{k}}, e_\gamma^j) \omega([e_\alpha^i, e_\beta^j]_{\mathfrak{p}}, e_\gamma^i).
\end{aligned}$$

The fourth line in the above computation consists of three summands, the first one is contained in the left summand of the last line and the other two form the right summand of the penultimate line. We also note that the sixth line becomes the right summand of the last line (recall that $[e_\alpha^i, e_\beta^j] \in \mathfrak{p}_i$ and so $\omega([e_\alpha^i, e_\beta^j], e_\gamma^i) = 0$ for any $s < i < j$).

Using Lemma 7.1 and the fact that $\omega(e_\alpha^i, e_\beta^j) = 0$ for all $\alpha \neq \beta$ (see Lemma 7.2, (ii)), we write each $[e_\alpha^i, e_\beta^j]_{\mathfrak{k}}$ and each $[e_\alpha^i, e_\beta^j]_{\mathfrak{p}}$ as a linear combination of the g_b -orthonormal bases $\{e_\delta^{2s}\}$ and $\{e_\delta^k\}$, respectively, to obtain that

$$\begin{aligned}
g_b(H_Q, d\omega) &= -3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j; i+s-1 \leq k}} \frac{1}{x_i^2 x_j} c_{i\alpha, i\beta}^{2s\gamma} Q(e_\gamma^{2s}, e_\gamma^j) c_{i\alpha, i\beta}^{k\gamma} \omega(e_\gamma^k, e_\gamma^j) \\
&\quad + 6 \sum_{\substack{\alpha, \beta, \gamma \\ s < i < k}} \frac{1}{x_i^3} c_{i\alpha, i\gamma}^{2s\beta} Q(e_\beta^{2s}, e_\beta^i) c_{i\alpha, i\beta}^{k\gamma} \omega(e_\gamma^k, e_\gamma^i) \\
&\quad - 3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i, j; i \leq k}} \frac{1}{x_i^2 x_j} c_{i\alpha, i\beta}^{2s\gamma} Q(e_\gamma^{2s}, e_\gamma^j) c_{i\alpha, i\beta}^{k\gamma} \omega(e_\gamma^k, e_\gamma^j) \\
&\quad + 6 \sum_{\substack{\alpha, \beta, \gamma \\ s < j < i}} \frac{1}{x_i^2 x_j} c_{i\alpha, i\beta}^{2s\gamma} Q(e_\gamma^{2s}, e_\gamma^j) c_{i\alpha, j\gamma}^{j\beta} \omega(e_\beta^j, e_\beta^i),
\end{aligned}$$

which by Lemma 7.1, (i) gives that ,

$$\begin{aligned}
g_b(H_Q, d\omega) = & -3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j; i+s \leq k}} \frac{1}{x_i^2 x_j} c_{i\alpha, i\beta}^{2s\gamma} Q(e_\gamma^{2s}, e_\gamma^j) \frac{\sqrt{B_{2s}}}{\sqrt{B_k}} c_{i\alpha, i\beta}^{2s\gamma} \omega(e_\gamma^k, e_\gamma^j) \\
& -3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j}} \frac{1}{x_i^2 x_j} c_{i\alpha, i\beta}^{2s\gamma} Q(e_\gamma^{2s}, e_\gamma^j) \frac{A_{i+s-1} \sqrt{B_{2s}}}{\sqrt{B_{i+s-1}}} c_{i\alpha, i\beta}^{2s\gamma} \omega(e_\gamma^{i+s-1}, e_\gamma^j) \\
& +6 \sum_{\substack{\alpha, \beta, \gamma \\ s < i < k}} \frac{1}{x_i^3} \frac{1}{\sqrt{B_{2s}}} \bar{c}_{\alpha\gamma}^\beta Q(e_\beta^{2s}, e_\beta^i) \frac{1}{\sqrt{B_k}} \bar{c}_{\alpha\beta}^\gamma \omega(e_\gamma^k, e_\gamma^i) \\
& -3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i, j; i < k}} \frac{1}{x_i^2 x_j} \frac{1}{\sqrt{B_{2s}}} \bar{c}_{\alpha\beta}^\gamma Q(e_\gamma^{2s}, e_\gamma^j) \frac{1}{\sqrt{B_k}} \bar{c}_{\alpha\beta}^\gamma \omega(e_\gamma^k, e_\gamma^j) \\
& -3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i, j}} \frac{1}{x_i^2 x_j} \frac{1}{\sqrt{B_{2s}}} \bar{c}_{\alpha\beta}^\gamma Q(e_\gamma^{2s}, e_\gamma^j) \frac{D_i}{\sqrt{B_i^3}} \bar{c}_{\alpha\beta}^\gamma \omega(e_\gamma^i, e_\gamma^j) \\
& +6 \sum_{\substack{\alpha, \beta, \gamma \\ s < j < i}} \frac{1}{x_i^2 x_j} \frac{1}{\sqrt{B_{2s}}} \bar{c}_{\alpha\beta}^\gamma Q(e_\gamma^{2s}, e_\gamma^j) \frac{-1}{\sqrt{B_i}} \bar{c}_{\beta\gamma}^\alpha \omega(e_\beta^j, e_\beta^i).
\end{aligned}$$

We now use Lemma 7.1, (ii) and Lemma 7.2, (ii) to obtain that

$$\begin{aligned}
& g_b(H_Q, d\omega) \\
= & -3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j; i+s \leq k}} \frac{1}{x_i^2 x_j} (c_{i\alpha, i\beta}^{2s\gamma})^2 \frac{-C_j}{\sqrt{B_{2s} B_j}} \frac{\sqrt{B_{2s}}}{\sqrt{B_k}} \frac{B_{2s}}{\sqrt{B_k B_j}} \omega_{kj} \\
& -3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j}} \frac{1}{x_i^2 x_j} (c_{i\alpha, i\beta}^{2s\gamma})^2 \frac{-C_j}{\sqrt{B_{2s} B_j}} \frac{A_{i+s-1} \sqrt{B_{2s}}}{\sqrt{B_{i+s-1}}} \frac{B_{2s}}{\sqrt{B_{i+s-1} B_j}} \omega_{(i+s-1)j} \\
& -6 \sum_{\substack{\alpha, \beta, \gamma \\ s < i < k}} \frac{1}{x_i^3} \frac{1}{\sqrt{B_{2s}}} (\bar{c}_{\alpha\beta}^\gamma)^2 \frac{-C_i}{\sqrt{B_{2s} B_i}} \frac{1}{\sqrt{B_k}} \frac{B_{2s}}{\sqrt{B_k B_i}} \omega_{ki} \\
& -3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i, j; i < k}} \frac{1}{x_i^2 x_j} \frac{1}{\sqrt{B_{2s}}} (\bar{c}_{\alpha\beta}^\gamma)^2 \frac{-C_j}{\sqrt{B_{2s} B_j}} \frac{1}{\sqrt{B_k}} \frac{B_{2s}}{\sqrt{B_k B_j}} \omega_{kj} \\
& -3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i, j}} \frac{1}{x_i^2 x_j} \frac{1}{\sqrt{B_{2s}}} (\bar{c}_{\alpha\beta}^\gamma)^2 \frac{-C_j}{\sqrt{B_{2s} B_j}} \frac{D_i}{\sqrt{B_i^3}} \frac{B_{2s}}{\sqrt{B_i B_j}} \omega_{ij} \\
& -6 \sum_{\substack{\alpha, \beta, \gamma \\ s < j < i}} \frac{1}{x_i^2 x_j} \frac{1}{\sqrt{B_{2s}}} (\bar{c}_{\alpha\beta}^\gamma)^2 \frac{-C_j}{\sqrt{B_{2s} B_j}} \frac{1}{\sqrt{B_i}} \frac{B_{2s}}{\sqrt{B_j B_i}} \omega_{ji} \\
= & 3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j; i+s \leq k}} \frac{1}{x_i^2 x_j} B_{2s} (c_{i\alpha, i\beta}^{2s\gamma})^2 \frac{C_j}{B_j B_k} \omega_{kj} + 3 \sum_{\substack{\alpha, \beta, \gamma \\ i \leq s < j}} \frac{1}{x_i^2 x_j} B_{2s} (c_{i\alpha, i\beta}^{2s\gamma})^2 \frac{C_j A_{i+s-1}}{B_j B_{i+s-1}} \omega_{(i+s-1)j} \\
& +3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i \neq j; i < k}} \frac{1}{x_i^2 x_j} \frac{1}{B_k B_j} (\bar{c}_{\alpha\beta}^\gamma)^2 C_j \omega_{kj} + 3 \sum_{\substack{\alpha, \beta, \gamma \\ s < i \neq j}} \frac{1}{x_i^2 x_j} \frac{D_i}{B_i^2 B_j} (\bar{c}_{\alpha\beta}^\gamma)^2 C_j \omega_{ij} \\
& +6 \sum_{\substack{\alpha, \beta, \gamma \\ s < i < k}} \frac{1}{x_i^3} (\bar{c}_{\alpha\beta}^\gamma)^2 \frac{C_i}{B_k B_i} \omega_{ki} + 6 \sum_{\substack{\alpha, \beta, \gamma \\ s < j < i}} \frac{1}{x_i^2 x_j} (\bar{c}_{\alpha\beta}^\gamma)^2 \frac{C_j}{B_j B_i} \omega_{ji}.
\end{aligned}$$

The fourth line above goes to the left summand of the last line and the fifth and sixth lines form the penultimate line.

It now follows from (36) that

$$\begin{aligned}
g_b(H_Q, d\omega) &= \\
&= 3 \sum_{s < j, k} \left(\frac{Cas_1}{x_1^2} + \cdots + \frac{Cas_{k-s}}{x_{k-s}^2} \right) \frac{C_j}{B_j B_k x_j} \omega_{kj} + 3 \sum_{s < j, k} \frac{Cas_{k-s+1}}{x_{k-s+1}^2} A_k \frac{C_j}{B_j B_k x_j} \omega_{kj} \\
&\quad + 3Cas_0 \left(\sum_{s < i \neq j; i < k} \frac{1}{x_i^2} \frac{C_j}{B_k B_j x_j} \omega_{kj} + \sum_{s < k \neq j} \frac{1}{x_k^2} \frac{D_k C_j}{B_k^2 B_j x_j} \omega_{kj} \right) \\
&\quad + 6Cas_0 \sum_{s < j < k} \frac{1}{x_j^2} \frac{C_j}{B_k B_j} \omega_{kj} - 6Cas_0 \left(\sum_{s < j < k} \frac{1}{x_k^2 x_j} \frac{C_j}{B_j B_k} \omega_{kj} \right) \\
&= 3 \sum_{s < k, j} E_k \frac{C_j}{B_k B_j x_j} \omega_{kj} + 3Cas_0 \sum_{s < k, j} F_k \frac{C_j}{B_k B_j x_j} \omega_{kj} + 6Cas_0 \sum_{s < j < k} \left(\frac{1}{x_j^2} - \frac{1}{x_k^2} \right) \frac{C_j}{B_k B_j x_j} \omega_{kj}.
\end{aligned}$$

This implies that $g_b(H_Q, d\omega) = 0$ for any skew-symmetric matrix $[\omega_{kj}]$ if and only if

$$(E_k + Cas_0 F_k) \frac{C_j}{x_j} + 2Cas_0 \left(\frac{1}{x_j^2} - \frac{1}{x_k^2} \right) \frac{C_j}{x_j} = (E_j + Cas_0 F_j) \frac{C_k}{x_k}, \quad \forall j < k,$$

concluding the proof. \square

We are finally in a position to state and prove our main result.

Theorem 7.4. *Let $M = G/K$ be a homogeneous space, where G is compact, connected and semisimple and K is connected, and fix decompositions $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ and $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_t$ in simple factors. We assume that conditions (i) and (ii) below (29) hold and that $M = G/K$ is aligned with positive constants c_1, \dots, c_s and $\lambda_1, \dots, \lambda_t$ (see Definition 4.7). Consider any G -invariant metric of the form*

$$g = x_1 g_b|_{\mathfrak{p}_1 \times \mathfrak{p}_1} + \cdots + x_{2s-1} g_b|_{\mathfrak{p}_{2s-1} \times \mathfrak{p}_{2s-1}}, \quad x_1, \dots, x_{2s-1} > 0,$$

where g_b is the normal metric given by

$$g_b = z_1(-B_{\mathfrak{g}_1}) + \cdots + z_s(-B_{\mathfrak{g}_s}), \quad z_1, \dots, z_s > 0,$$

and $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_{2s-1}$ is the g_b -orthogonal reductive decomposition of G/K given by Proposition 5.1. If we set

$$\begin{aligned}
a_j &:= \dim \mathfrak{k} \frac{1}{c_{j+1} x_{j+1}^2} + Cas_0 \left(\frac{1}{x_{s+j}^2} - \frac{1}{x_{j+1}^2} \right), \quad j = 1, \dots, s-1, \\
b_j &:= \dim \mathfrak{k} \left(\frac{1}{c_1 x_1^2} + \cdots + \frac{1}{c_j x_j^2} \right) + Cas_0 \left(\frac{1}{x_{s+1}^2} - \frac{1}{x_1^2} + \cdots + \frac{1}{x_{s+j}^2} - \frac{1}{x_j^2} \right),
\end{aligned}$$

where $Cas_0 := \lambda_1 \dim \mathfrak{k}_1 + \cdots + \lambda_t \dim \mathfrak{k}_t$, then $a_j, b_j \geq 0$ for all j and the closed 3-form H_Q given in (6), where

$$Q = y_1 B_{\mathfrak{g}_1} + \cdots + y_s B_{\mathfrak{g}_s}, \quad \frac{y_1}{c_1} + \cdots + \frac{y_s}{c_s} = 0,$$

is g -harmonic if and only if

$$(39) \quad x_{s+k} \left(a_k A_k + b_k + 2Cas_0 \left(\frac{1}{x_{s+j}^2} - \frac{1}{x_{s+k}^2} \right) \right) C_j = x_{s+j} (a_j A_j + b_j) C_k,$$

for all $1 \leq j < k \leq s-1$, where

$$A_j := -\frac{c_{j+1}}{z_{j+1}} \left(\frac{z_1}{c_1} + \cdots + \frac{z_j}{c_j} \right), \quad C_j := \frac{y_1}{c_1} + \cdots + \frac{y_j}{c_j} + A_j \frac{y_{j+1}}{c_{j+1}}.$$

Remark 7.5. Condition (39) trivially holds if $s = 2$ (cf. §6.1) or $\mathfrak{k} = 0$ (cf. §3).

Remark 7.6. In the case when $g = g_b$ is a normal metric (i.e., $x_1 = \cdots = x_{2s-1} = 1$), the constants a_j and b_j become

$$a_j = \dim \mathfrak{k} \frac{1}{c_{j+1}}, \quad b_j = \dim \mathfrak{k} \left(\frac{1}{c_1} + \cdots + \frac{1}{c_j} \right), \quad \forall j = 1, \dots, s-1,$$

that is, they are just algebraic invariants of G/K , and condition (39) becomes

$$x_{s+k}(a_k A_k + b_k)C_j = x_{s+j}(a_j A_j + b_j)C_k, \quad \forall 1 \leq j < k \leq s-1.$$

Remark 7.7. Using (37), we can rewrite the non-negative constants a_j and b_j as

$$a_j = \frac{Cas_{j+1}}{x_{j+1}^2} + \frac{Cas_0}{x_{s+j}^2}, \quad j = 1, \dots, s-1,$$

$$b_j = \frac{Cas_1}{x_1^2} + \cdots + \frac{Cas_j}{x_j^2} + Cas_0 \left(\frac{1}{x_{s+1}^2} + \cdots + \frac{1}{x_{s+j}^2} \right),$$

where $Cas_i := \frac{1}{c_i} \operatorname{tr} C_{\mathfrak{q}_i, -B_{\mathfrak{g}_i} |_{\pi_i(\mathfrak{k})}}$ and $C_{\mathfrak{q}_i, -B_{\mathfrak{g}_i} |_{\pi_i(\mathfrak{k})}}$ is the Casimir operator of the isotropy representation \mathfrak{q}_i of the homogeneous space $G_i/\pi_i(K)$ (see (36)).

Remark 7.8. Since Cas_0 is the trace of the Casimir operator of the adjoint representation of K , we have that $Cas_0 = 0$ if and only if \mathfrak{k} is abelian. We also note that $Cas_i = 0$ if and only if $\mathfrak{q}_i = 0$ (i.e., $\pi_i(\mathfrak{k}) = \mathfrak{g}_i$) for $i = 1, \dots, s$; in particular, if at least one is zero then \mathfrak{k} must be simple. In the case when $Cas_i = 0$ for all i , one obtains the Ledger-Obata space $M = K \times \cdots \times K/\Delta K$.

Remark 7.9. The tuple (A_1, \dots, A_{s-1}) determines the metric g_b up to scaling and for any given g_b , the tuple (C_1, \dots, C_{s-1}) determines the closed 3-form H_Q (see (40) below). In particular, the standard metric g_B (i.e., $z_1 = \cdots = z_s = 1$) corresponds to $A_j = -\frac{b_j}{a_j}$ for all $j = 1, \dots, s-1$ and so any H_Q is g_B -harmonic.

Remark 7.10. Let \mathcal{M}^G denote the manifold of all G -invariant metrics on $M = G/K$. If all the spaces $G_i/\pi_i(K)$, $i = 1, \dots, s$ are isotropy irreducible and K is either simple or one-dimensional, then the subspaces $\mathfrak{p}_1, \dots, \mathfrak{p}_{2s-1}$ are all $\operatorname{Ad}(K)$ -irreducible and so

$$\dim \mathcal{M}^G \geq s + \frac{s(s-1)}{2} = \frac{s(s+1)}{2},$$

where equality holds if and only if $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are pairwise inequivalent (recall that $\mathfrak{p}_{s+1} \simeq \cdots \simeq \mathfrak{p}_{2s-1} \simeq \mathfrak{k}$ and that the first s \mathfrak{p}_i 's are pairwise inequivalent to the last $s-1$ ones by assumption). Since each g_b provides the submanifold $\{(x_1, \dots, x_{2s-1})_{g_b}\}$ of metrics covered by Theorem 7.4, the subset \mathcal{T} of all G -invariant metrics covered by the theorem can be described as the union over the s -parametric space of all g_b 's of $(2s-1)$ -dimensional submanifolds. We do not know whether \mathcal{T} is equal to \mathcal{M}^G or not in the case when $\dim \mathcal{M}^G = \frac{s(s+1)}{2}$.

Proof. It follows from Propositions 6.1, 7.3 and (29) that H_Q is g -harmonic if and only if condition (38) holds. On the other hand, it is easy to see that $\frac{D_{s+j}}{B_{s+j}} = 1 + A_j$, which implies by using Remark 7.7 that

$$E_{s+j} + Cas_0 F_{s+j} = a_j A_j + b_j, \quad \forall 1 \leq j \leq s-1.$$

Thus conditions (39) and (38) are equivalent, concluding the proof. \square

The following corollaries of Theorem 7.4 answer all the questions made at the beginning of §6 on the harmonicity of the closed 3-forms H_Q 's.

Corollary 7.11. *Let $M = G/K$ be an aligned homogeneous space as in the above theorem.*

- (i) The standard metric g_B is the unique (up to scaling) normal metric on G/K such that any closed 3-form H_Q is g_B -harmonic.
- (ii) For any normal metric $g_b \neq g_B$ on G/K , there is a unique (up to scaling) closed 3-form H_Q which is g_b -harmonic.
- (iii) For any nonzero closed 3-form H_Q there exists a one-parameter family (up to scaling) of normal metrics $g_b(t) \neq g_B$ on G/K such that H_Q is $g_b(t)$ -harmonic for all t .

Proof. Consider the tuple (C_1, \dots, C_{s-1}) determined by Q and g_b . Since

$$(40) \quad A \begin{bmatrix} y_1/c_1 \\ \vdots \\ y_s/c_s \end{bmatrix} = \begin{bmatrix} C_1 \\ \vdots \\ C_{s-1} \\ 0 \end{bmatrix}, \quad \text{where} \quad A := \begin{bmatrix} 1 & A_1 & 0 & \cdots & 0 \\ 1 & 1 & A_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & A_{s-1} \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

is an invertible $s \times s$ matrix (note that $\det A = (1 - A_1) \dots (1 - A_{s-1})$), we obtain that, conversely, given the metric g_b , the tuple (C_1, \dots, C_{s-1}) determines Q . Part (i) therefore follows from (39) and Remark 7.9.

We now prove part (ii). Given $g_b \neq g_B$, we consider the indexes k_1, \dots, k_r such that $A_{k_i} \neq -\frac{b_{k_i}}{a_{k_i}}$, so $1 \leq r \leq s-1$ and $a_j A_j + b_j = 0$ for all $j \neq k_1, \dots, k_r$. Recall from Remark 7.6 the simpler forms of condition (39) and the constants a_j and b_j in this case. Thus the g_b -harmonicity condition (39) for a given H_Q is equivalent to $C_j = 0$ for all $j \neq k_1, \dots, k_r$ and

$$(41) \quad (a_{k_1} A_{k_1} + b_{k_1}) C_{k_i} = (a_{k_i} A_{k_i} + b_{k_i}) C_{k_1}, \quad \forall i = 2, \dots, r.$$

This implies that C_{k_2}, \dots, C_{k_r} and consequently Q , are determined by one parameter C_{k_1} and the metric g_b , that is, Q is determined up to scaling by g_b .

Finally, we prove part (iii). Given (y_1, \dots, y_s) such that $\sum \frac{y_i}{c_i} = 0$, we need to find a negative solution (A_1, \dots, A_{s-1}) to the system

$$(42) \quad (a_j A_j + b_j) C_i = (a_i A_i + b_i) C_j, \quad i, j = 1, \dots, s-1,$$

where $C_i = C_i(A_i) := \frac{y_1}{c_1} + \dots + \frac{y_i}{c_i} + A_i \frac{y_{i+1}}{c_{i+1}}$ (see (39)). It is therefore enough to consider only the equations with indexes k_1, \dots, k_r such that $C_{k_i} \left(-\frac{b_{k_i}}{a_{k_i}}\right) \neq 0$, as all the remaining equalities hold by setting $A_j := -\frac{b_j}{a_j}$ if $C_j \left(-\frac{b_j}{a_j}\right) = 0$. It is easy to see that $r \geq 1$ unless $Q = 0$. If we assume that each unknown A_{k_i} is sufficiently close to $-\frac{b_{k_i}}{a_{k_i}}$ in order to have $C_{k_i}(A_{k_i}) \neq 0$, then system (42) is equivalent to

$$(43) \quad f_i(A_{k_i}) := \frac{a_{k_i} A_{k_i} + b_{k_i}}{C_{k_i}(A_{k_i})} = \frac{a_{k_1} A_{k_1} + b_{k_1}}{C_{k_1}(A_{k_1})}, \quad i = 1, \dots, r.$$

Since each function f_i satisfies that $f_i\left(-\frac{b_{k_i}}{a_{k_i}}\right) = 0$ and $f'_i\left(-\frac{b_{k_i}}{a_{k_i}}\right) = \frac{a_{k_i}}{C_{k_i}\left(-\frac{b_{k_i}}{a_{k_i}}\right)} \neq 0$, for any sufficiently small value $t = \frac{a_{k_1} A_{k_1} + b_{k_1}}{C_{k_1}(A_{k_1})}$, there exist negative numbers A_{k_2}, \dots, A_{k_r} such that (43) holds, concluding the proof. \square

Corollary 7.12. *Let $M = G/K$ be an aligned homogeneous space as in the above theorem such that $s \geq 3$ and \mathfrak{k} is abelian (i.e., $Ca s_0 = 0$ and $Ca s_1, \dots, Ca s_s > 0$) and let g_b be any normal metric.*

- (i) The set of all metrics of the form $g = (x_1, \dots, x_{2s-1})_{g_b}$ such that every closed 3-form H_Q is g -harmonic can be described as follows:

a) The numbers x_2, \dots, x_s are recursively determined by x_1 in the following way:

$$(44) \quad (-A_j) \frac{Cas_{j+1}}{x_{j+1}^2} = \frac{Cas_1}{x_1^2} + \dots + \frac{Cas_j}{x_j^2}, \quad \forall j = 1, \dots, s-1.$$

b) The s numbers $x_1, x_{s+1}, \dots, x_{2s-1}$ are independent positive parameters.

(ii) For any other G -invariant metric $g = (x_1, \dots, x_{2s-1})_{g_b}$, there is a unique (up to scaling) 3-form H_Q which is g -harmonic.

Proof. We observe that according to Theorem 7.4 and Remark 7.7, any H_Q is g -coclosed for a metric $g = (x_1, \dots, x_{2s-1})_{g_b}$ if and only if $a_j A_j + b_j = 0$ for any $j = 1, \dots, s-1$, which is equivalent to condition (44) and so part (i) follows.

Since the metrics involved in part (ii) are precisely those for which there is at least one j such that $a_j A_j + b_j \neq 0$, the proof of this part follows in much the same way as the proof of part (ii) of Corollary 7.11. \square

Corollary 7.13. *Let $M = G/K$ be an aligned homogeneous space as in the above theorem such that $s \geq 3$, $Cas_1 > 0$ and \mathfrak{k} is not abelian (i.e., $Cas_0 > 0$) and let g_b be any normal metric. Then the set of all metrics of the form $g = (x_1, \dots, x_{2s-1})_{g_b}$ such that every closed 3-form H_Q is g -harmonic depends on the three parameters x_1, x_s, x_{s+1} as follows:*

(i) a) $x_{s+1} = x_{s+2} = \dots = x_{2s-2}$.

b) The numbers x_2, \dots, x_{s-1} are recursively determined by x_1 and x_{s+1} in the following way:

$$(45) \quad (-A_j) \frac{Cas_{j+1}}{x_{j+1}^2} = \frac{Cas_1}{x_1^2} + \dots + \frac{Cas_j}{x_j^2} + Cas_0(j + A_j) \frac{1}{x_{s+1}^2}, \quad \forall j = 1, \dots, s-2,$$

which is positive if and only if $x_{s+1} > u$ for a certain positive number u depending on x_1 . Note that only the x_j 's such that $Cas_j > 0$ are involved in the above equations, if $Cas_j = 0$ then $\mathfrak{p}_j = 0$ and so x_j does not appear in g .

c) The number x_{2s-1} is determined by x_1, x_s and x_{s+1} as follows:

$$(46) \quad \frac{1}{x_{2s-1}^2} Cas_0(1 - A_{s-1}) = \frac{Cas_1}{x_1^2} + \dots + \frac{Cas_{s-1}}{x_{s-1}^2} + A_{s-1} \frac{Cas_s}{x_s^2} + s Cas_0 \frac{1}{x_{s+1}^2},$$

which is positive if and only if $x_s > v$ for a certain positive number v depending on x_1 and x_{s+1} .

(ii) For any other G -invariant metric $g = (x_1, \dots, x_{2s-1})_{g_b}$, there exists at least one 3-form H_Q which is g -harmonic.

Proof. According to (39), if any H_Q is g -coclosed, then

$$a_k A_k + b_k + 2Cas_0 \left(\frac{1}{x_{s+1}^2} - \frac{1}{x_{s+k}^2} \right) = 0, \quad \forall 1 < k,$$

and $a_j A_j + b_j = 0$ for any $j = 1, \dots, s-2$. This implies that $x_{s+1} = \dots = x_{2s-2}$ (i.e., part a)), part b) follows and the only remaining equation to imply (39) is:

$$(47) \quad a_{s-1} A_{s-1} + b_{s-1} + 2Cas_0 \left(\frac{1}{x_{s+1}^2} - \frac{1}{x_{2s-1}^2} \right) = 0,$$

from which part c) easily follows.

Since the metrics involved in part (ii) are precisely those for which either there is at least one $1 \leq j \leq s-2$ such that $a_j A_j + b_j \neq 0$ or the left hand side of (47) is non-zero, the proof of this part follows in much the same way as the proof of existence in part (ii) of Corollary 7.11, the uniqueness may not hold. \square

Example 7.14. Let $M = G/K$ be an aligned homogeneous space as in the above theorem. If $Cas_1 = \dots = Cas_s = 0$, then $M = K \times \dots \times K/\Delta K$ is a Ledger-Obata space (K simple)

and for any normal metric g_b , the unique metric $g = (x_{s+1}, \dots, x_{2s-1})_{g_b}$ satisfying that all the closed 3-forms H_Q are g -harmonic is given, up to scaling, by

$$g = (1, \dots, 1, t)_{g_b}, \quad \text{where } t := \sqrt{\frac{1-A_{s-1}}{s}}, \quad A_{s-1} = -\frac{z_1 + \dots + z_{s-1}}{z_s}.$$

Indeed, it follows from Corollary 7.13 and Remark 7.7 that $x_{s+1} = \dots = x_{2s-2}$ and

$$\frac{1}{x_{2s-1}^2}(1 - A_{s-1}) = s \frac{1}{x_{s+1}^2},$$

and the formula for A_{s-1} follows from the fact that $c_1 = \dots = c_s = s$. In the case when $g_b = g_B$, $A_{s-1} = 1 - s$ and so $t = 1$ and $g = g_B$.

Example 7.15. Consider the aligned homogeneous space $M = H \times \dots \times H / \Delta K$ as in Example 4.9, where H is a simple Lie group and $K \subset H$ is a proper closed simple subgroup, together with a $B_{\mathfrak{h}}$ -orthogonal reductive decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{q}$. If $B_{\mathfrak{k}} = c B_{\mathfrak{h}}|_{\mathfrak{k}}$, then

$$c_1 = \dots = c_s = s, \quad Cas_0 = \frac{c \dim \mathfrak{k}}{s}, \quad Cas_1 = \dots = Cas_s = \frac{(1-c) \dim \mathfrak{k}}{s}.$$

According to Corollary 7.13, if we fix g_B as the background metric (in particular, $A_{s-1} = 1 - s$), then every H_Q is g -harmonic with respect to a given metric $g = (x_1, \dots, x_{2s-1})_{g_B}$ if and only if $x_{s+1} = \dots = x_{2s-2}$,

$$\frac{1}{x_{j+1}^2} = \frac{1}{j} \left(\frac{1}{x_1^2} + \dots + \frac{1}{x_j^2} \right), \quad \forall j = 1, \dots, s-2,$$

which implies that $x_1 = \dots = x_{s-1}$, and so

$$\frac{1}{x_{2s-1}^2} = \frac{(1-c)(s-1)}{cs} \left(\frac{1}{x_1^2} - \frac{1}{x_s^2} \right) + \frac{1}{x_{s+1}^2}.$$

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