Components and codimension of mixed and $\mathscr{A}$-discriminants for square polynomial systems

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Abstract

The discriminant of a multivariate polynomial with indeterminate coefficients is not necessarily a hypersurface, and characterizing its codimension was an open problem for quite a while. We resolve this problem for discriminants of square and overdetermined systems of equations. This version is more involved, in the sense that the discriminant may have several components of different dimension.

We enumerate all components and find their dimension and degree, for each of the three conventional ways to formalize the notion of the discriminant in this setting (namely, for mixed, Cayley and A-discriminants).

1 Introduction

Given an algebraic torus $T \simeq (\mathbb{C}^\times)^n$ with the character lattice $M \simeq \mathbb{Z}^n$, every finite set of monomials $A \subset M$ generates the vector space of Laurent polynomials, denoted by $\mathbb{C}_A$. Starting from [GKZ94], lots of attention is paid to the *$\mathscr{A}$-discriminant* $D_\mathscr{A} \subset \mathbb{C}_\mathscr{A} = \mathbb{C}_{A_1} \oplus \cdots \oplus \mathbb{C}_{A_k}$, the closure of all tuples of polynomials $(f_1, \dots, f_k)$ such that the the system of equations $f_1 = \cdots = f_k = 0$ has a *degenerate root* (i.e. a point $x \in T$ at which $f_1(x) = \cdots = f_k(x) = 0$ and $df_1 \wedge \ldots \wedge df_k(x) = 0$). The motivation varies from algebraic geometry and PDEs to mathematical physics and symbolic algebra.

A question of particular interest is the classification of tuples $\mathscr{A} = (A_1, \dots, A_k) \subset M$, for which the discriminant is not a hypersurface. For instance, the case $k = 1$ is equivalent to the classical problem of dual defective varieties (whose projectively dual is not a hypersurface), for the special case of toric varieties. This problem was resolved in [DR06, CC07, DFS07, Est10, MT11, FI21, CD22]. At the other extreme, for $n = k$, the discriminant was shown to have a hypersurface component, once the *support sets* $A_1, \dots, A_n$ can not be shifted to an affine plane or the tuple of standard simplexes by an automorphism of the lattice. This was proved in [Est19, BN20], motivated by applications in Galois theory and lattice polytope geometry respectively.

It's possible to define two more types of discriminants. For a tuple $\mathscr{A}$, the *mixed discriminant* is the closure of all polynomial systems in $\mathbb{C}_\mathscr{A}$ having a *non-degenerate multiple root* (i.e. a degenerate root $x \in T$ such that no proper subtuple of $df_1, ..., df_k(x)$ is linearly dependent). Mixed discriminants were introduced in [CCD+13] and were investigated in [DEK14, Est19, DDRM23].

For a subset $I \subseteq \{1, \dots, k\}$, the *Cayley trick* for the subtuple $\mathscr{B} = (A_i,\ i \in I)$ is the map sending a polynomial system $f \in \mathbb{C}_\mathscr{A}$ to the polynomial $\sum_{i \in I} \lambda_i f_i(x)$ of variables x and λ. The support of this polynomial is the *Cayley set* $cay(\mathscr{B}) = \cup_{i \in I} A_i \times \{i\} \subset M \times \mathbb{Z}^{|I|}$. The *Cayley discriminant* of a subtuple $\mathscr{B}$ is the preimage of the discriminant for the Cayley set $cay(\mathscr{B})$ under the Cayley trick [Est10, Est19]. The Cayley discriminant was introduced as an intermediate object allowing to reduce the study of $\mathscr{A}$-discriminants of systems of equations to A-discriminants of one polynomial (which is much simpler and well-understood).

In case $n = k$, if the Cayley discriminant is a hypersurface, then the mixed discriminant is the same hypersurface [CCD+13]. For tuples, called irreducible, Esterov showed that the three types of discriminants have the same hypersurface component and conjectured the lack of other components for $\mathscr{A}$-discriminants [Est19]. The paper [Poka] proves the conjecture, and the three types of discriminants are the same hypersurface for an irreducible tuple.

In case $1 < k < n$, the study of discriminants is substantially more complicated than the classical one $k = 1$ and our $n \le k$: see [Est10, Est13, DDRM23] for some partial results.

We study the case $n \le k$ more comprehensively: for arbitrary support sets $\mathscr{A} = (A_1, \ldots, A_k) \subset M$, we give the complete list of the irreducible components of the $\mathscr{A}$-discriminant, the Cayley discriminant, the mixed discriminant in $\mathbb{C}_{\mathscr{A}}$, specify their codimensions and degrees.

The answer is stated in terms of the following fundamental quantity: the *defect* of a subtuple $\mathscr{B}$ is the number $\delta(\mathscr{B}) = \dim(\text{affine span of the Minkowski sum } \sum_{i \in I} A_i) - |I|$. A tuple is *linearly dependent* if it contains a subtuple of negative defect.

We first describe discriminants for linearly dependent tuples: this simplest case reduces to the sparse resultant as introduced in [Stu94].

Theorem 1.1. *For a linearly dependent tuple $\mathscr{A}$ with the minimal (by inclusion) subtuple $\mathscr{M}$ of minimal defect,*

1) the $\mathscr{A}$-discriminant is the sparse resultant $R_{\mathscr{M}}$ of codimension $-\delta(\mathscr{M})$ (Theorem 5.19*);*

2) the mixed discriminant is empty if the defect of the subtuple $\mathscr{M}$ is less than -1; otherwise, the mixed discriminant is the sparse resultant $R_{\mathscr{M}}$, and it's a hypersurface (Theorem 7.5).

The case of linearly independent tuples is the essence of the matter, and the answer requires the following notions.

Definition 1.2. Linearly independent tuples of the zero defect are called *BK-tuples*.

A BK-tuple is *simple* if it is not a union of its proper BK-subtuples. A simple BK-subtuple is *maximal* if it is not contained in another simple BK-subtuple.

A simple BK-tuple is *prelinear* if, for every projection of lattices, sending each set from its maximal (by inclusion) proper BK-subtuple to zero, there is a lattice automorphism, mapping the projection of every other set into the standard unit simplex.

Some combinatorial properties of these objects, which we use throughout this text, are proved in a separate paper [Pokb], which is in preparation now. All such results are cited with exact statements and do not look surprising, so we hope this does not reduce readability or reliability of this text.

Theorem 1.3. *For a BK-tuple $\mathscr{A}$, the $\mathscr{A}$-discriminant has the following distinct components:*

- the Cayley discriminants of non-prelinear simple BK-subtuples of codimension one;

- the Cayley discriminants of prelinear simple BK-subtuples of codimension two, unless it's contained in a non-prelinear one (see Theorem 5.15 and Proposition 6.3).

Remark 1.4. If a prelinear BK-subtuple is contained in a non-prelinear BK-subtuple, then the Cayley discriminant of the former subtuple lies in the Cayley discriminant of the latter.

Theorem 1.5. *For a BK-tuple, the mixed discriminant is empty/a hypersurface/a variety of codimension 2 if the tuple is non-simple/non-prelinear simple/prelinear simple. For a simple BK-tuple, the mixed discriminant equals the Cayley discriminant* (Theorem 7.3; [CCD+13]).

The known results [PS93, GKZ94, DFS07, Est07, MT11] allow us to write degrees for discriminants. For a linearly dependent tuple/BK-tuple $\mathscr{A}$, see Corollary 8.4/8.6 and Corollary 8.1/Remark 8.7 for degrees of the $\mathscr{A}$-discriminant and the mixed discriminant correspondingly.

Despite Cayley discriminants being well-known A-discriminants, the current paper provides an alternative view to Cayley discriminants for linearly dependent and BK-tuples in Theorems 6.6 and 6.5. For a BK-tuple, Theorem 6.5 shows the Cayley discriminant equals the complete intersection of Cayley discriminants of all maximal simple BK-subtuples. This theorem leads to a slight simplification of the Matsui-Takeuchi degree formula in Corollary 8.9.

The structure of the paper is as follows. Sections 2 and 3 remind some combinatorial results and general facts. In Section 4, we construct a special multiplication of varieties necessary to describe discriminants for BK-tuples. Section 5 characterizes $\mathscr{A}$-discriminants, Section 6 - Cayley discriminants, and Section 7 - mixed discriminants. For each type of discriminant, we enumerate components and compute their codimensions first for BK-tuples and then for linearly dependent tuples. All computations of degrees are collected in Section 8.

2 Combinatorial Review

This part provides an overview of combinatorial results concerning sublattice configurations, tuples of finite sets, and the mixed volume. For a sublattice tuple $\mathsf{n} = (S_1, ..., S_k)$ from a lattice M, there is a notion of irreducibility motivated by questions from algebraic geometry. For a simple case of a tuple with n sublattices in $\mathbb{Z}^n$, there is a partition of a reducible tuple on some subtuples corresponding to irreducible ones. This combinatorial decomposition is a shadow of the decomposition of an $\mathscr{A}$-discriminant for polynomial systems into irreducible components.

Definition 2.1. The *saturation* of a sublattice N is the maximal sublattice $\overline{N}$ containing N with the same dimension as N.

The *linear span* $\langle \mathsf{n} \rangle$ of a sublattice tuple n is the minimal sublattice containing all sublattices from the tuple. The *cardinality* $\mathfrak{c}(\mathsf{n})$ is the number of sets in the tuple n, and the *defect* of n is the difference $\delta(\mathsf{n}) = dim \langle \mathsf{n} \rangle - \mathfrak{c}(\mathsf{n})$.

A tuple n is *essential* if every subtuple of cardinality no more than $dim \langle \mathsf{n} \rangle$ has a non-negative defect (see [DFS07]). A tuple is *linearly independent* if the defects of all subtuples are non-negative. A linearly independent tuple is *irreducible* if the defects of all proper subtuples are positive. A *BK-tuple* is a linearly independent tuple with zero defect.

For tuples n and k in a lattice M, the *quotient tuple* is the projection of the complement subtuple $\mathsf{n}/\mathsf{k} = \pi(\mathsf{n} \backslash \mathsf{k})$, where $\pi : M \to M/\overline{\langle \mathsf{k} \rangle}$.

The notion of a BK-tuple arose from the Kouchnirenko-Bernstein theorem [Ber75].
Essential tuples of zero defect are automatically BK-tuples.

Proposition 2.2. [Pokb] *For a reducible BK-tuple* n *and its BK-subtuple* k*,*

1) linear spans of irreducible BK-subtuples don't intersect each other except for the origin;
2) the quotient tuple n/k *is a BK-tuple;*
3) BK-subtuples of the quotient n/k *are in bijection with BK-subtuples of* n *containing* k*;*
4) there is the identity for quotients $\mathsf{n}/\mathsf{k} \cong \frac{\mathsf{n}/\mathsf{l}}{\mathsf{k}/\mathsf{l}}$*, where* l *is a BK-subtuple of* k.

Definition 2.3. A *filtration* on a tuple n is an increasing family of subtuples $F_0\mathsf{n} \hookrightarrow F_1\mathsf{n} \hookrightarrow ... \hookrightarrow F_m\mathsf{n} = \mathsf{n}$. A filtration is a *BK-filtration* if all quotients $F_j\mathsf{n}/F_{j-1}\mathsf{n}$ are BK. A BK-filtration is *maximal* if all quotients $F_j\mathsf{n}/F_{j-1}\mathsf{n}$ are irreducible.

Proposition 2.4. *A reducible BK-tuple* n *admits a maximal BK-filtration. Moreover, there are linear isomorphisms between successive quotients* $F_j\mathsf{n}/F_{j-1}\mathsf{n}$ *and* $F'_j\mathsf{n}/F'_{j-1}\mathsf{n}$*, for given maximal BK-filatrations* $F_\bullet\mathsf{n}$ *and* $F'_\bullet\mathsf{n}$.

Definition 2.5. An *order ideal* (or down-set) of a poset P is a subposet I of P such that if $\beta \in I$ and $\alpha \leq \beta$, then $\alpha \in I$. For an element β from a poset, the *principal order ideal* (β) is the order ideal of all elements that are not greater than β.

The dual notions are *order filter* and *principal order filter* $[\beta]$ (inverse all $\leq$).

The *length* of a chain C is the number $\ell(C)$ that is one less than the number of elements in the chain. The *height* of an element α from a poset is the maximal length of a chain from the order ideal (α) and is denoted by $h(\alpha)$.

Since the intersection and union of BK-subtuples are BK-subtuples in a BK-tuple n, the set of BK-subtuples forms a distributive lattice L_{n} by inclusion. By fundamental theorem for distributive lattices [Sta11], there is a poset P_{n}, whose lattice of order ideals is isomorphic to L_{n}. This poset P_{n} defines a partition of the BK-tuple n: every element α corresponds to some subtuple k_α of n, and every order ideal I of P_{n} corresponds to some BK-subtuple $\mathsf{k}_I = \bigsqcup_{\alpha \in I} \mathsf{k}_\alpha$.

Theorem 2.6. (Poset partition of a reducible BK-tuple, [Pokb]) *A reducible BK-tuple n admits the unique partition $\mathsf{n} = \bigsqcup_{\alpha \in P_{\mathsf{n}}} \mathsf{k}_\alpha$ such that the subtuples $\hat{\mathsf{k}}_\alpha = \mathsf{k}_{(\alpha)}/\mathsf{k}_{(\alpha)\setminus\alpha}$ are irreducible BK-tuples for every element α from the poset P_{n}.*

We use these results to construct partitions of tuples of finite sets. The *affine linear span* of a finite set A from a lattice is the affine sublattice $\langle A \rangle$ generated by this set. Using a shift, we can always ensure that the finite set A contains the point zero, and its linear span is a sublattice. Then, a tuple of finite sets $\mathscr{A} = (A_1, ..., A_m)$ with a common point 0 provides a sublattice tuple. We characterize a tuple of finite sets via the generated sublattice tuple.

By the mixed volume $\mathrm{MV}_M(\mathscr{A})$ of a tuple of finite sets $\mathscr{A}$ in a lattice M, we imply the mixed volume of the corresponding convex hulls of finite sets. We highlight that a BK-tuple $\mathscr{A}$ has a positive mixed volume in the sublattice $\langle \mathscr{A} \rangle$ by Minkowski's theorem (see Theorem 8 [Kho16]). Another significant result is the decomposition of the mixed volume:

Theorem 2.7. *For tuples $\mathscr{B} \subset \mathscr{A}$ with zero defect in a lattice M, the mixed volume decomposes*

$$\mathrm{MV}_M(\mathscr{A}) = \mathrm{MV}_{\overline{\langle \mathscr{B} \rangle}}(\mathscr{B})\, \mathrm{MV}_{M/\overline{\langle \mathscr{B} \rangle}}(\mathscr{A}/\mathscr{B}).$$

Proof. For instance, see [ST10] for a geometric proof and [Est19] for an algebraic proof. ☐

Corollary 2.8. *A reducible BK-tuple $\mathscr{A}$ admits the unique partition $\mathscr{A} = \bigsqcup_{\alpha \in P_{\mathscr{A}}} \mathscr{B}_\alpha$ such that tuples $\hat{\mathscr{B}}_\alpha = \mathscr{B}_{(\alpha)}/\mathscr{B}_{(\alpha)\setminus\alpha}$ are irreducible BK-tuples for every element α from the poset $P_{\mathscr{A}}$.*

3 Discriminants of polynomial systems

For a group lattice N, the *dual lattice* is $M = N^\vee = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and the *algebraic torus* is $T(N) = N \otimes_{\mathbb{Z}} \mathbb{C}^\times$. The *character group* is $\mathrm{Hom}_{\mathbb{Z}}(T(N), \mathbb{C}^\times)$, and its elements are called *characters* χ_a [Ful93, CLS11]. The character group for N is isomorphic to the dual lattice M. For a tuple of finite sets $\mathscr{A} \subset M$, we have defined the space of polynomial systems $\mathbb{C}_{\mathscr{A}}$, and every polynomial system Φ has solutions in the torus $T(N)$.

For a splitting short exact sequence (s.e.s.) of lattices $0 \to N' \to N \xrightarrow{\pi} N'' \to 0$, we have splitting s.e.s. for dual lattices $0 \to M'' \xrightarrow{i} M \to M' \to 0$, character groups and tori. The projection of lattices π induces the *pushforward* $\pi^* = \pi \otimes \mathbb{C}^\times$ of tori and the *pullback* $\pi_* = \mathrm{Hom}(\pi^*, \mathbb{C}^\times)$ of character groups, $\pi_*(\chi_a) = \chi_a \circ \pi^* = \chi_{i(a)}$, for a character χ_a for N''.

The injective map of dual lattices $M'' \xrightarrow{i} M$ provides the equality for discriminants, $\pi_*(D_{\mathscr{A}}) = D_{i(\mathscr{A})}$. Hence, an isomorphism of dual lattices $M'' \xrightarrow{i} M$ leads to an isomorphism for discriminants, $D_{\mathscr{A}} \cong D_{i(\mathscr{A})}$. In particular, the discriminants $D_{\mathscr{A}} \cong D_{g\mathscr{A}}$ are isomorphic for any element g of the affine general linear group $\mathsf{AGL}(n,\mathbb{Z})$. Also, if there is a one-to-one correspondence between sets from tuples $\mathscr{A}$ and $\mathscr{B}$ such that each set $A \in \mathscr{A}$ is a translation of the set $B \in \mathscr{B}$, then the discriminants are equal, $D_{\mathscr{A}} = D_{\mathscr{B}}$.

These observations allow us to choose convenient tuples for subsequent proofs and use the results of Section 2. In the sequel, we suppose that the linear span of the tuple $\mathscr{A}$ equals the dual lattice, $\langle\mathscr{A}\rangle = M$. Also, we will use the notation $T(\mathscr{A})$ for the torus $T(N)$.

Consider $0 \to M'' \to M \to M' \to 0$ a splitting s.e.s. of dual lattices. Lattices are reflexive $\mathbb{Z}$-modules (i.e. there is an isomorphism $N \cong \operatorname{Hom}(\operatorname{Hom}(N,\mathbb{Z}),\mathbb{Z})$) as finitely generated free $\mathbb{Z}$-modules. It means the lattice $\operatorname{Hom}(M'',\mathbb{Z})$ is isomorphic to some lattice N'' such that $M'' = \operatorname{Hom}(N'',\mathbb{Z})$. Let's compute the lattice N''. Notice that the dual sublattice M'' naturally corresponds to the sublattice $N' = M''^{\perp} = \{n \in N \mid m(n) = 0\ \forall m \in M''\} \subseteq N$. From splitting, the lattice N'' can be defined as the quotient $N/N' = N/M''^{\perp}$.

For a subtuple $\mathscr{B}$ from a tuple $\mathscr{A}$, we search solutions for a subsystem from $\mathbb{C}_{\mathscr{B}}$ in the special torus $T(N/\overline{\langle\mathscr{B}\rangle}^{\perp})$, denoted just $T(\mathscr{B})$.

A *cofiltration* $G_\bullet N$ is a sequence of quotients $N = G_k N \twoheadrightarrow G_{k-1}N \twoheadrightarrow \dots \twoheadrightarrow G_0 N \twoheadrightarrow 0$. There is a bijection between cofiltrations of a lattice N and filtrations of the dual lattice M.

Every BK-tuple $\mathscr{A}$ admits a maximal BK-filtration $F_\bullet$. Saturated linear spans of the tuples from the BK-filtration form a filtration of the dual lattice $F_\bullet M$, $F_i M = \overline{\langle F_i\mathscr{A}\rangle}$. Then there is a cofilatration $G_\bullet N$, and, hence, the cofiltration for the torus $G_\bullet T(N) = T(G_\bullet N)$. Therefore, every polynomial system $\Phi \in \mathbb{C}_{\mathscr{A}}$ admits a BK-filtration $F_\bullet\Phi$, and its solutions admit a cofiltration $G_\bullet V(\Phi)$. For every subsystem $F_i(\Phi_{\mathscr{A}}) = \Phi_{\mathscr{B}}$, we will search for solutions in the torus $T(G_i N) = T(\mathscr{B})$.

4 BK-multiplication

To compute discriminants for BK-tuples, we build a specific multiplication between varieties.

Consider a splitting s.e.s. of dual lattices $0 \to M'' \to M \xrightarrow{\tau} M' \to 0$ and a finite set $A \subset M$, $B = \tau(A)$. Then the substitution $f(x,y) \xrightarrow{ev_{x_0}} f(x_0,y)$ of a point x_0 from the algebraic torus $T(N'')$ into polynomials from $\mathbb{C}_A$ is a linear projection on $\mathbb{C}_B$, $\mathbb{C}_A \xrightarrow{ev_{x_0}} \mathbb{C}_B$. For a tuple of finite sets $\mathscr{A} \subset M$, $\mathscr{B} = \tau(\mathscr{A})$, the substitution $\Phi(x,y) \xrightarrow{ev_{x_0}} \Phi(x_0,y)$ of the point x_0 corresponds to a linear projection $\mathbb{C}_{\mathscr{A}} \xrightarrow{ev_{x_0}} \mathbb{C}_{\mathscr{B}}$. Then we get

Lemma 4.1. *For any quasi-affine algebraic set $Y \subset \mathbb{C}_{\mathscr{B}}$ and $E = ev_x^{-1}(Y)$, the triple (E, ev_x, Y) is a trivial vector bundle over Y of the rank $\sum_{A\in\mathscr{A}} |A| - |\tau(A)|$.*

Remark 4.2. The total space E is a variety in $\mathbb{C}_{\mathscr{A}\setminus\mathscr{B}}$ if and only if Y is a variety in $\mathbb{C}_{\mathscr{A}/\mathscr{B}}$. We call E an *evaluation bundle* over Y and denote it by $E^x_{\mathscr{A}}(Y) = E$.

Lemma 4.3. *For a chain of BK-tuples $\mathscr{C} \subset \mathscr{B} \subset \mathscr{A}$, a quasi-affine algebraic set $Y \subset \mathbb{C}_{\mathscr{B}/\mathscr{C}}$ and a point $x \in T(\mathscr{C})$, the following holds:*

$$E^x_{\mathscr{A}\setminus\mathscr{C}}\left(Y \times \mathbb{C}_{(\mathscr{A}\setminus\mathscr{B})/\mathscr{C}}\right) = E^x_{\mathscr{B}\setminus\mathscr{C}}(Y) \times \mathbb{C}_{\mathscr{A}\setminus\mathscr{B}}.$$

Proof. Notice that the evaluation map $ev_x : \mathbb{C}_{\mathscr{A}\setminus\mathscr{C}} \to \mathbb{C}_{\mathscr{A}/\mathscr{C}}$ splits on the two

$u_x : \mathbb{C}_{\mathscr{B}\setminus\mathscr{C}} \to \mathbb{C}_{\mathscr{B}/\mathscr{C}}$ and $v_x : \mathbb{C}_{\mathscr{A}\setminus\mathscr{B}} \to \mathbb{C}_{(\mathscr{A}\setminus\mathscr{B})/\mathscr{C}}$, $ev_x = u_x \oplus v_x$.

Then we have $ev_x^{-1}(Y \times \mathbb{C}_{(\mathscr{A}\setminus\mathscr{B})/\mathscr{C}}) = u_x^{-1}(Y) \times v_x^{-1}(\mathbb{C}_{(\mathscr{A}\setminus\mathscr{B})/\mathscr{C}})$. □

Theorem 4.4. (Kouchnirenko-Bernstein, [Ber75]) *For a tuple $\mathscr{A}$ of n finite sets in a n-dimensional lattice M, there is an open subset U in $\mathbb{C}_{\mathscr{A}}$ such that the set of solutions for every polynomial system $\Phi \in U$ consists of exactly $\mathrm{MV}_M(\mathscr{A})$-points.*

The complement $\mathbb{C}_{\mathscr{A}} \backslash U$ is a *bifurcation divisor* according to Esterov work [Est13]. Discriminants of different type lie in the bifurcation divisor.

Definition 4.5. For BK-tuples $\mathscr{B} \subset \mathscr{A}$, the *BK-multiplication* $X \circ Y$ of algebraic sets $X \subset \mathbb{C}_{\mathscr{B}}$ and $Y \subset \mathbb{C}_{\mathscr{A}/\mathscr{B}}$ is called the quasi-affine set $\{\Phi \times \bigcup_{x \in V(\Phi)} ev_x^{-1}(Y) \mid \Phi \in X\} \subset \mathbb{C}_{\mathscr{A}}$. Denote by $X \bullet Y$ the algebraic closure of $X \circ Y$.

Remark 4.6. 1) By Kouchnirenko-Bernstein Theorem 4.4, the generic fiber in the BK-multiplication is a union of $\mathrm{MV}_{\overline{\langle \mathscr{B} \rangle}}(\mathscr{B})$-number different trivial evaluation bundles over Y.

2) Denote by $Z_{\mathscr{B}}$ the set of polynomial systems in $\mathbb{C}_{\mathscr{B}}$ with the empty set of solutions. Then the contribution $\Phi \times \bigcup_{x \in V(\Phi)} ev_x^{-1}(Y)$ is empty for every polynomial system $\Phi \in Z_{\mathscr{B}}$. That's why we take the algebraic closure of the multiplication.

Lemma 4.7. *The following holds: $\mathbb{C}_{\mathscr{A}} = \mathbb{C}_{\mathscr{B}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}}$.*

Proof. Every system Φ from $\mathbb{C}_{\mathscr{B}} \backslash Z_{\mathscr{B}}$ has at least one solution x. Hence, every fiber equals the same vector space $\bigcup_{x \in V(\Phi)} ev_x^{-1}(\mathbb{C}_{\mathscr{A}/\mathscr{B}}) = \mathbb{C}_{\mathscr{A} \backslash \mathscr{B}}$. Therefore, the algebraic closure of the set $\{\Phi \times \mathbb{C}_{\mathscr{A} \backslash \mathscr{B}} \mid \Phi \in \mathbb{C}_{\mathscr{B}} \backslash Z_{\mathscr{B}}\}$ coincides with the space $\mathbb{C}_{\mathscr{A}}$. This lemma is a shadow of the mixed volume decomposition Theorem 2.7. □

Corollary 4.8. *Every variety X from $\mathbb{C}_{\mathscr{B}} \backslash Z_{\mathscr{B}}$ satisfies the equality: $X \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}} = \overline{X} \times \mathbb{C}_{\mathscr{A} \backslash \mathscr{B}}$.*

Lemma 4.9. *For a tuple $\mathscr{A}$, the set of solutions $V(\Phi_{\mathscr{A}}(x))$ is a variety in $T(\mathscr{A}) \times \mathbb{C}_{\mathscr{A}}$.*

Proof. Let $\Pi = \bigcap_{A \in \mathscr{A}} \Pi_A$ be the intersection of hyperplanes Π_A in $\mathbb{C}_{\mathscr{A}}$, defined by equations $\sum_{a \in A} c_a = 0$. Consider the map $T(\mathscr{A}) \times \mathbb{C}_{\mathscr{A}} \overset{\sigma_{\mathscr{A}}}{\twoheadrightarrow} \mathbb{C}_{\mathscr{A}}$ such that $\sigma_{\mathscr{A}}(x, c) = (c_a x^a)_{a \in A \in \mathscr{A}}$. Notice the preimage $W = \sigma_{\mathscr{A}}^{-1}(\Pi)$ is a variety isomorphic to the product $T(\mathscr{A}) \times \Pi$. Indeed, the isomorphism is defined by the regular maps $W \underset{\iota}{\overset{\sigma}{\rightleftarrows}} T(\mathscr{A}) \times \Pi$ such that $\iota(c_a) = (\frac{c_a}{x^a})$ and $\sigma(c_a) = (c_a x^a)$ for every $a \in A \in \mathscr{A}$ and $x \in T(\mathscr{A})$ (ι and σ are identity on other coordinates). Moreover, the preimage of each hyperplane $\sigma_{\mathscr{A}}^{-1}(\Pi_A)$ equals the zero locus of the equation $f_A(x) = 0$ in $T(\mathscr{A}) \times \mathbb{C}_{\mathscr{A}}$. Hence, the variety W equals the set of solutions $V(\Phi_{\mathscr{A}}(x))$. □

Theorem 4.10. *For a variety $Y \subset \mathbb{C}_{\mathscr{A}/\mathscr{B}}$, the BK-multiplication $\mathbb{C}_{\mathscr{B}} \circ Y$ is a variety.*

Proof. By Lemma 4.9, the set of solutions $V(\Phi_{\mathscr{B}}(x))$ is a variety W in $T(\mathscr{B}) \times \mathbb{C}_{\mathscr{B}}$ for a BK-subtuple $\mathscr{B}$. Consider the evaluation map

$$T(\mathscr{B}) \times \mathbb{C}_{\mathscr{A}} \overset{ev}{\twoheadrightarrow} T(\mathscr{B}) \times \mathbb{C}_{\mathscr{B}} \times \mathbb{C}_{\mathscr{A}/\mathscr{B}}$$

such that $ev(x, \Phi_{\mathscr{B}}, \Phi_{\mathscr{A} \backslash \mathscr{B}}) = (x, \Phi_{\mathscr{B}}, ev_x(\Phi_{\mathscr{A} \backslash \mathscr{B}}))$. Notice that the product $W \times Y$ is a variety, and the preimage $ev^{-1}(W \times Y)$ is a trivial vector bundle over $W \times Y$ with the rank $\sum_{A \in \mathscr{A} \backslash \mathscr{B}} |A| - |\tau(A)|$. Indeed, for a fixed $x \in T(\mathscr{B})$, the preimage is defined by linear equations $\sum_{a \in A_b} c_a x^a = c_b$ for every $b \in B \in \mathscr{A}/\mathscr{B}$ (every A set in $\mathscr{A} \backslash \mathscr{B}$ admits the partition $A = \bigsqcup_{b \in B} A_b$ for the corresponding $B \in \mathscr{A}/\mathscr{B}$). Then the projection of the variety $ev^{-1}(W \times Y)$ on the space $\mathbb{C}_{\mathscr{A}}$ is a variety equal to the BK-multiplication $\mathbb{C}_{\mathscr{B}} \circ Y$. □

Remark 4.11. The closed BK-multiplication is associative: $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for $X \in \mathbb{C}_{\mathscr{C}}$, $Y \in \mathbb{C}_{\mathscr{B}/\mathscr{C}}$, $Z \in \mathbb{C}_{\mathscr{A}/\mathscr{B}}$, and a chain of BK-tuples $\mathscr{C} \subset \mathscr{B} \subset \mathscr{A}$.

Corollary 4.12. (Coherency Relations) *For a chain of BK-tuples $\mathscr{C} \subset \mathscr{B} \subset \mathscr{A}$ and quasi-affine algebraic sets $Y_{\mathscr{B}/\mathscr{C}} \subseteq \mathbb{C}_{\mathscr{B}/\mathscr{C}} \backslash Z_{\mathscr{B}/\mathscr{C}}$ and $Y_{\mathscr{A}/\mathscr{B}} \subseteq \mathbb{C}_{\mathscr{A}/\mathscr{B}}$, the equalities hold:*

$$\mathbb{C}_{\mathscr{C}} \bullet (Y_{\mathscr{B}/\mathscr{C}} \times \mathbb{C}_{(\mathscr{A}\backslash\mathscr{B})/\mathscr{C}}) = \mathbb{C}_{\mathscr{C}} \bullet Y_{\mathscr{B}/\mathscr{C}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}},$$
$$\mathbb{C}_{\mathscr{B}} \bullet Y_{\mathscr{A}/\mathscr{B}} = \mathbb{C}_{\mathscr{C}} \bullet \mathbb{C}_{\mathscr{B}/\mathscr{C}} \bullet Y_{\mathscr{A}/\mathscr{B}}.$$

Proof. By the associativity, Lemma 4.7 and Corollary 4.8. □

5 $\mathscr{A}$-discriminants

Lemma 5.1. *Consider a linearly dependent set of vectors $\{v_i\}_{i\in I}$ in a vector space V. Suppose a subset $\{v_j\}_{j\in J}$ is linearly independent, $J \subset I$. Then the set $\{\pi(v_k)\}_{k\in I\backslash J}$ is linearly dependent in U, where $\pi : V \to U = V/\overline{\langle v_j\rangle_{j\in J}}$.*

Proof. If we denote the kernel of π by W, then the vector space V is isomorphic to the direct sum $V \cong U \oplus W$. Since the set $\{v_j\}_{j\in J}$ is linearly independent, there are constants c_i such that $\sum\limits_{i\in I} c_i v_i = 0$. From the isomorphism, we can decompose every vector $v_i = u_i + w_i$. Notice that vectors v_j lie in the subspace W for $j \in J$, and $u_j = 0$. Therefore, the projected subset is linearly dependent: $\sum\limits_{k\in I\backslash J} c_k u_k = 0$, $u_k = \pi(v_k)$. □

Theorem 5.2. *For BK-tuples $\mathscr{B} \subset \mathscr{A}$, the discriminant $D_{\mathscr{A}}$ equals the union*

$$D_{\mathscr{A}} = D_{\mathscr{B}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}} + \mathbb{C}_{\mathscr{B}} \bullet D_{\mathscr{A}/\mathscr{B}}.$$

Proof. For a generic system $\Phi \in D_{\mathscr{A}}$, there is a singular point $(x, y) \in T(\mathscr{A})$ such that vectors $\{df_A(x, y)\}_{A\in\mathscr{A}}$ are linearly dependent, and $x \in T(\mathscr{B})$. If the set $\{df_B(x)\}_{B\in\mathscr{B}}$ is linearly dependent, then the point Φ belongs to $D_{\mathscr{B}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}}$. Otherwise, the set $\{d(ev_x f_C)(y)\}_{C\in\mathscr{A}/\mathscr{B}}$ is linearly dependent by Lemma 5.1, and the substitution of the root x to the system $\Phi_{\mathscr{A}\backslash\mathscr{B}}$ gives us the singular system $ev_x\Phi_{\mathscr{A}\backslash\mathscr{B}}$ from the discriminant $D_{\mathscr{A}/\mathscr{B}}$. Then, the system Φ lies in $\mathbb{C}_{\mathscr{B}} \bullet D_{\mathscr{A}/\mathscr{B}}$. □

The theorem above doesn't state that one part (collection of components) of the union doesn't lie in another. The theorem shows the possibility of having more than one component. Nevertheless, the result is powerful since, with every BK-subtuple $\mathscr{B}$, we can associate some part $C(\mathscr{B})$ in the discriminant $D_{\mathscr{A}}$ by a sequence of splittings and usage of coherency relations for BK-multiplications. We name by *stratum* an irreducible part.

Proposition 5.3. *The part $\mathbb{C}_{\mathscr{B}} \bullet D_{\mathscr{A}/\mathscr{B}}$ doesn't lie in another $D_{\mathscr{B}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}}$.*

Proof. The part $\mathbb{C}_{\mathscr{B}} \bullet D_{\mathscr{A}/\mathscr{B}}$ contains the dense subset $(\mathbb{C}_{\mathscr{B}} \backslash B_{\mathscr{B}}) \circ D_{\mathscr{A}/\mathscr{B}}$ that doesn't lie in $D_{\mathscr{B}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}}$. □

Corollary 5.4. *Suppose the poset $P_{\mathscr{A}}$ (see Corollary 2.8) has a few components, and denote by I one of them. Then the parts $C(\mathscr{B}_I)$ and $C(\mathscr{A}\backslash\mathscr{B}_I)$ are not contained one in other.*

Proof. Notice that I and $P_{\mathscr{A}} \backslash I$ are order ideals corresponding to BK-subtuples and use Proposition 5.3 twice. □

Lemma 5.5. (Separation Lemma) *For BK-tuples $\mathscr{C} \subset \mathscr{B} \subset \mathscr{A}$, the corresponding part $C(\mathscr{B}\backslash\mathscr{C})$ in the discriminant $D_{\mathscr{A}}$ is $\mathbb{C}_{\mathscr{C}} \bullet D_{\mathscr{B}/\mathscr{C}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}}$.*

Proof. Direct computations with use of Theorem 5.2 and Corollary 4.12:

$$D_{\mathscr{A}} = D_{\mathscr{C}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{C}} + \mathbb{C}_{\mathscr{C}} \bullet (D_{\mathscr{B}/\mathscr{C}} \bullet \mathbb{C}_{\frac{\mathscr{A}/\mathscr{C}}{\mathscr{B}/\mathscr{C}}} + \mathbb{C}_{\mathscr{B}/\mathscr{C}} \bullet D_{\frac{\mathscr{A}/\mathscr{C}}{\mathscr{B}/\mathscr{C}}}) =$$
$$= D_{\mathscr{C}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{C}} + \mathbb{C}_{\mathscr{C}} \bullet D_{\mathscr{B}/\mathscr{C}} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}} + \mathbb{C}_{\mathscr{B}} \bullet D_{\mathscr{A}/\mathscr{B}}.$$

□

Definition 5.6. A tuple is *linear* if it can be mapped to the tuple of standard simplexes by an automorphism of the lattice. Linear/nonlinear irreducible BK-tuples are called *lir/nir*.

Remark 5.7. 1) Linear BK-tuples are the only tuples with unit mixed volume [CCD+13, EG15]. Discriminants for linear tuples were considered in works [Est19, BN20].

2) For a reducible BK-tuple $\mathscr{A}$ and an element α from the poset $\mathscr{A}$, the simple BK-subtuple $\mathscr{B}_{(\alpha)}$ is prelinear if and only if the tuple $\hat{\mathscr{B}}_\alpha = \mathscr{B}_{(\alpha)}/\mathscr{B}_{(\alpha)\backslash\alpha}$ is linear.

In 2018, Esterov provided an important conjecture [Est19], proved in [Poka]:

Theorem 5.8. *For a lir/nir $\mathscr{A}$, the discriminant $D_{\mathscr{A}}$ is a variety of codimension 2/1 in $\mathbb{C}_{\mathscr{A}}$.*

Corollary 5.9. *Every element from the poset $P_{\mathscr{A}}$ corresponds to some stratum of $D_{\mathscr{A}}$.*

Proof. Corollary 2.8 provides the unique decomposition of the tuple $\mathscr{A}$, encoded by a poset $P_{\mathscr{A}}$. Every element α from the poset $P_{\mathscr{A}}$ corresponds to a subtuple $\mathscr{B}_\alpha$ such that the BK-tuple $\hat{\mathscr{B}}_\alpha$ is irreducible. The corresponding part in the discriminant $D_{\mathscr{A}}$ looks $\mathbb{C}_{\mathscr{B}_{(\alpha)\backslash\alpha}} \bullet D_{\hat{\mathscr{B}}_\alpha} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}_{(\alpha)}}$ by Separation Lemma 5.5. This part is irreducible and forms a stratum by Theorems 4.10 and 5.8 about the irreducibility of BK-multiplication and the discriminant $D_{\hat{\mathscr{B}}_\alpha}$. □

The main goal now is to figure out which strata don't contain one in another.

Lemma 5.10. *For incomparable elements α, β from the poset $P_{\mathscr{A}}$, the corresponding strata $C(\mathscr{B}_\alpha)$ and $C(\mathscr{B}_\beta)$ are not contained in each other.*

Proof. Let's choose the order ideal corresponding to the union of principal order ideals $(\alpha)\cup(\beta)$ from the poset $P_{\mathscr{A}}$ and the BK-subtuple $\mathscr{B} = \mathscr{B}_{(\alpha)\cup(\beta)}$. By Proposition 5.3, we can explore the strata $C(\mathscr{B}_\alpha)$ and $C(\mathscr{B}_\beta)$ in the discriminant $D_{\mathscr{B}}$. Consider the intersection of principal order ideals $(\alpha)\cap(\beta)$ and the BK-subtuple $\mathscr{C} = \mathscr{B}_{(\alpha)\cap(\beta)}$. By Theorem 5.2, we can simplify the task by taking the new tuple $\mathscr{B}/\mathscr{C}$ and the discriminant $D_{\mathscr{B}/\mathscr{C}}$. However, the elements α, β are not connected in the poset $(\alpha)\cup(\beta)\backslash((\alpha)\cap(\beta))$. Therefore, the corresponding strata $C(\mathscr{B}_\alpha/\mathscr{C})$ and $C(\mathscr{B}_\beta/\mathscr{C})$ don't contain one another by Corollary 5.4. Consequently, the strata $C(\mathscr{B}_\alpha)$ and $C(\mathscr{B}_\beta)$ are not contained in each other. □

Theorem 5.11. (Height Theorem) *If the height inequality $h(\alpha) < h(\beta)$ holds for elements α, β from the poset $P_{\mathscr{A}}$, then the stratum $C(\mathscr{B}_\beta)$ doesn't lie in the stratum $C(\mathscr{B}_\alpha)$.*

Proof. For incomparable elements α, β, use Lemma 5.10. For comparable elements $\alpha < \beta$, we reduce the case to the subposet $[\alpha, \beta]$ - closed interval, and the tuple $\tilde{\mathscr{A}} = \mathscr{B}_{(\beta)}/\mathscr{B}_{(\beta)\backslash[\alpha,\beta]}$ by Separation Lemma (5.5). Then we apply calculations from Separation Lemma 5.5 to the tuple $\mathscr{A} = \tilde{\mathscr{A}}$ and its subtuples $\mathscr{C} = \tilde{\mathscr{B}}_\alpha$, $\mathscr{B} = \tilde{\mathscr{B}}_{[\alpha,\beta]\backslash\beta}$, and use Proposition (5.3). □

Lemma 5.12. *If the BK-tuple $\hat{\mathscr{B}}_\alpha = \mathscr{B}_{(\alpha)}/\mathscr{B}_{(\alpha)\backslash\alpha}$ is a nir, for some element α from the poset $P_{\mathscr{A}}$, then the stratum $C(\mathscr{B}_\alpha)$ is a hypersurface component in $D_{\mathscr{A}}$.*

Proof. By Height Theorem 5.11 and Separation Lemma 5.5, we can suppose that the element α has zero height. The corresponding stratum looks $C(\mathscr{B}_\alpha) = D_{\mathscr{B}_\alpha} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}_\alpha}$, and it has codimension 1 by Theorem 5.8. Also, the stratum $C(\mathscr{B}_\alpha)$ has the empty intersection with $(\mathbb{C}_{\mathscr{B}_\alpha} \backslash B_{\mathscr{B}_\alpha}) \circ D_{\mathscr{A}/\mathscr{B}_\alpha}$ - a dense open subset of $C(\mathscr{B}_{P_{\mathscr{A}} \backslash \alpha})$ of a codimension no less than 1. Therefore, the intersection $C(\mathscr{B}_\alpha) \cap C(\mathscr{B}_{P_{\mathscr{A}} \backslash \alpha})$ has codimension at least 2, and $C(\mathscr{B}_\alpha) \not\subset C(\mathscr{B}_{P_{\mathscr{A}} \backslash \alpha})$. ☐

Lemma 5.13. *If BK-tuples $\hat{\mathscr{B}}_\beta$ are lir for every element β from the principal order filter $[\alpha]$ then the stratum $C(\mathscr{B}_\alpha)$ is a component of codimension 2 in $D_{\mathscr{A}}$.*

Proof. For the principal order filter $[\alpha]$, its complement $P_{\mathscr{A}} \backslash [\alpha]$ is an order ideal. By Separation Lemma 5.5 and Proposition 5.2, we can explore the discriminant for the new tuple $\tilde{\mathscr{A}} = \mathscr{A}/\mathscr{B}_{P_{\mathscr{A}} \backslash [\alpha]}$ with the new poset $[\alpha]$. Hence, without loss of generality, we suppose that $P_{\mathscr{A}} = [\alpha]$. By Height Theorem 5.11, there are no inclusions $C(\mathscr{B}_\alpha) \not\supset C(\mathscr{B}_\beta)$ for every $\beta > \alpha$.

By Separation Lemma 5.5, the corresponding strata looks $C(\mathscr{B}_\alpha) = D_{\mathscr{B}_\alpha} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}_\alpha}$, and it has codimension 2 by Theorem 5.8. Also, the strata $C(\mathscr{B}_\alpha)$ has the empty intersection with $(\mathbb{C}_{\mathscr{B}_\alpha} \backslash B_{\mathscr{B}_\alpha}) \circ D_{\mathscr{A}/\mathscr{B}_\alpha}$ - a dense open subset of $C(\mathscr{B}_{[\alpha] \backslash \alpha})$ of codimension 2. Therefore, the intersection $C(\mathscr{B}_\alpha) \cap C(\mathscr{B}_{[\alpha] \backslash \alpha})$ has codimension at least 3, and $C(\mathscr{B}_\alpha) \not\subset C(\mathscr{B}_{P_{\mathscr{A}} \backslash \alpha})$. ☐

Lemma 5.14. *Consider a chain of BK-tuples $\mathscr{C} \subset \mathscr{B} \subset \mathscr{A}$ such that $\mathscr{C}$ is a linear BK-subtuple, and $\mathscr{B}/\mathscr{C}$ is a nir. Then there is the inclusion for the strata $C(\mathscr{C}) \subset C(\mathscr{B} \backslash \mathscr{C})$.*

Proof. By Theorem 5.8, the discriminant $D_{\mathscr{B}/\mathscr{C}}$ is defined by one irreducible polynomial $P(\Phi_{\mathscr{B}/\mathscr{C}})$ since $\mathscr{B}/\mathscr{C}$ is a nir. By Remark 4.2, the evaluation bundle $E^x_{\mathscr{B}\backslash\mathscr{C}}(D_{\mathscr{B}/\mathscr{C}})$ is irreducible too, and it is defined by the polynomial $P_x(\Phi_{\mathscr{B}\backslash\mathscr{C}})$, obtained from the polynomial $P(\Phi_{\mathscr{B}/\mathscr{C}})$ by changing every coefficient c_b on the linear polynomial $ev_x^{-1}(c_b)$ for every $b \in C \in \mathscr{B}/\mathscr{C}$ and some fixed point $x \in T(\mathscr{C})$. Denote its degree by $d = deg\, P(\Phi_{\mathscr{B}/\mathscr{C}})$.

Consider a generic point $(\Phi^\circ_{\mathscr{C}}, \Phi_{\mathscr{B}\backslash\mathscr{C}}) \in D_{\mathscr{C}} \times \mathbb{C}_{\mathscr{B}\backslash\mathscr{C}}$. Since $\Phi_{\mathscr{C}}$ is a degenerate linear system of more than one variable, the zero locus $V(\Phi^\circ_{\mathscr{C}})$ is a vector subspace in $\mathbb{C}(\mathscr{C}) = \langle\mathscr{C}\rangle^\vee \otimes \mathbb{C}$ of a positive dimension. The dimension of $V(\Phi^\circ_{\mathscr{C}})$ usually equals one. However, if it's not the case, choose an arbitrary linear subspace $U \subseteq V(\Phi^\circ_{\mathscr{C}})$. By definition of the discriminant, in the general case, it is possible to choose the subspace U, intersecting the torus $T(\mathscr{C})$. If we parameterize U in the natural way, $U = \{at + b, \text{ for some } a, b \in \mathbb{C}(\mathscr{C}) \mid t \in \mathbb{C}\}$, and open brackets in the polynomial $P_{at+b}(\Phi_{\mathscr{B}\backslash\mathscr{C}})$, we get:

$$P_{at+b}(\Phi_{\mathscr{B}\backslash\mathscr{C}}) = P_a(\Phi_{\mathscr{B}\backslash\mathscr{C}})t^d + Q_{d-1}(\Phi_{\mathscr{B}\backslash\mathscr{C}})t^{d-1} + \dots + Q_1(\Phi_{\mathscr{B}\backslash\mathscr{C}})t + P_b(\Phi_{\mathscr{B}\backslash\mathscr{C}}),$$

where $Q_j(\Phi_{\mathscr{B}\backslash\mathscr{C}})$ are some homogeneous polynomials of degree d. For a fixed point $\Phi^\circ_{\mathscr{B}\backslash\mathscr{C}}$, the polynomial $P_{at+b}(\Phi^\circ_{\mathscr{B}\backslash\mathscr{C}})$ of one-variable in t has d roots with multiplicities if $P_a(\Phi^\circ_{\mathscr{B}\backslash\mathscr{C}}) \neq 0$ and $P_b(\Phi^\circ_{\mathscr{B}\backslash\mathscr{C}}) \neq 0$. Denote by $Y_x = V(P_x(\Phi_{\mathscr{B}\backslash\mathscr{C}})) \subset \mathbb{C}_{\mathscr{B}\backslash\mathscr{C}}$ for a fixed $x \in T(\mathscr{C})$. Hence, there are d nonzero solutions with multiplicities of the equation $P_{at+b}(\Phi_{\mathscr{B}\backslash\mathscr{C}}) = 0$ for every $\Phi_{\mathscr{B}\backslash\mathscr{C}} \in \mathbb{C}_{\mathscr{B}\backslash\mathscr{C}} \backslash (Y_a \cup Y_b)$. These roots correspond to the evaluations sending the system $\Phi_{\mathscr{B}\backslash\mathscr{C}}$ to the discriminant $D_{\mathscr{B}/\mathscr{C}}$. Therefore, $\Phi^\circ_{\mathscr{C}} \times (\mathbb{C}_{\mathscr{B}\backslash\mathscr{C}} \backslash (Y_a \cup Y_b)) \subset \mathbb{C}_{\mathscr{C}} \circ D_{\mathscr{B}/\mathscr{C}}$ for any point $\Phi^\circ_{\mathscr{C}} \in D_{\mathscr{C}}$, and there is the inclusion for the closure $D_{\mathscr{C}} \bullet \mathbb{C}_{\mathscr{B}/\mathscr{C}} \subset \mathbb{C}_{\mathscr{C}} \bullet D_{\mathscr{B}/\mathscr{C}}$. ☐

Theorem 5.15. *For a BK-tuple $\mathscr{A}$, the discriminant $D_{\mathscr{A}}$ is the union $D_{\mathscr{A}} = \bigcup_{\alpha \in P_{\mathscr{A}}} C(\mathscr{B}_\alpha)$ of strata, enumerated by elements of the poset $P_{\mathscr{A}}$. More precisely, the discriminant $D_{\mathscr{A}}$ is a union of the components of codimension 1 or 2: every hypersurface component corresponds to a stratum $C(\mathscr{B}_\alpha)$ such that the tuple $\hat{\mathscr{B}}_\alpha$ is a nir; every component of codimension 2 corresponds to a stratum $C(\mathscr{B}_\alpha)$ such that tuples $\hat{\mathscr{B}}_\beta$ are lir for all elements β from the order filter $[\alpha]$.*

Proof. By Corollary 5.9, every element α from the poset $P_{\mathscr{A}}$ corresponds to a stratum $C(\mathscr{B}_\alpha)$. By Separation Lemma 5.5 and Definition 4.5 of BK-multiplication, the codimension of the stratum $C(\mathscr{B}_\alpha)$ equals the codimension of the corresponding discriminant $D_{\hat{\mathscr{B}}_\alpha}$ in $\mathbb{C}_{\hat{\mathscr{B}}_\alpha}$. By Theorem 5.8, the codimension of $D_{\hat{\mathscr{B}}_\alpha}$ equals $2/1$ if the tuple $\hat{\mathscr{B}}_\alpha$ is lir/nir.

If an element α is covered by an element β such that $\hat{\mathscr{B}}_\alpha$ is a lir and $\hat{\mathscr{B}}_\beta$ is a nir, then we have the inclusion of strata $C(\mathscr{B}_\alpha) \subset C(\mathscr{B}_\beta)$ by Lemma 5.14. Otherwise, the strata don't contain one another and form components by Lemma 5.12 and Lemma 5.13. □

Remark 5.16. Denote by $Q_{\mathscr{A}}$ the order ideal in $P_{\mathscr{A}}$, generated by elements α with nir $\hat{\mathscr{B}}_\alpha$, and its BK-subtuple by $\mathscr{N} = \bigsqcup_{\alpha \in Q_{\mathscr{A}}} \mathscr{B}_\alpha$. By Theorem 5.15, the discriminant $D_{\mathscr{N}}$ is a union of hypersurfaces, and the discriminant $D_{\mathscr{A}/\mathscr{N}}$ is a union of components of codimension 2 in the space of linear systems $\mathbb{C}_{\mathscr{A}/\mathscr{N}}$.

Let's review $\mathscr{A}$-discriminants for linearly dependent tuples. The work [Pokb] suggests a purely combinatorial proof of the following statement

Proposition 5.17. *Every linearly dependent tuple $\mathscr{A}$ contains the unique minimal by inclusion subtuple $\mathscr{M}$ with minimal defect. Moreover, the tuple $\mathscr{A}/\mathscr{M}$ is linearly independent.*

Remark 5.18. Notice that an essential linearly dependent tuple $\mathscr{A}$ coincides with its minimal subtuble $\mathscr{M}$ of the minimal defect by definition, $\mathscr{A} = \mathscr{M}$.

Theorem 5.19. *For a linearly dependent tuple $\mathscr{A}$, the discriminant $D_{\mathscr{A}}$ is the sparse resultant $R_{\mathscr{M}}$ of codimension $-\delta(\mathscr{M})$ for the minimal subtuple $\mathscr{M}$ with the minimal defect.*

Proof. $\boxed{\supseteq}$ Since the subtuple $\mathscr{M}$ has a negative defect, the vectors $\{df_B(x)\}_{B\in\mathscr{M}}$ are linearly dependent, in case x is a solution for the system $\Phi_{\mathscr{M}}$ in the torus $T(\mathscr{M})$. By Proposition 5.17, the tuple $\mathscr{A}/\mathscr{M}$ is linearly independent. Hence, a generic system $\Phi_{\mathscr{A}/\mathscr{M}}$ has a solution y in the torus $T(\mathscr{A}/\mathscr{M})$, and the sparse resultants coincide $R_{\mathscr{A}} = R_{\mathscr{M}}$. Therefore, the point (x, y) is a singular point for a system $\Phi_{\mathscr{A}}$ from the sparse resultant $R_{\mathscr{M}}$, and the system $\Phi_{\mathscr{A}}$ belongs to the discriminant $D_{\mathscr{A}}$. $\boxed{\subseteq}$ It's clear.

The codimension of the resultant $R_{\mathscr{M}}$ equals $-\delta(\mathscr{M})$ by algebro-geometric proofs in Theorem 1.1 [Stu94] and Theorem 2.15 [Est07], and by a tropical proof in Theorem 2.23 [JY13]. □

6 Cayley discriminants

The *Cayley discriminant* $D_{cay(\mathscr{A})}$ is the discriminant for the Cayley set $cay(\mathscr{A})$. If a BK-tuple $\mathscr{A}$ is irreducible, then the $\mathscr{A}$-discriminant equals the Cayley discriminant [Est19, Poka].

Notice the isomorphism of vector spaces $\mathbb{C}_{\mathscr{A}} \cong \mathbb{C}_{cay(\mathscr{A})}$.

Lemma 6.1. *If the poset $P_{\mathscr{B}}$ is a connected component of the poset $P_{\mathscr{A}}$ for BK-tuples $\mathscr{B} \subset \mathscr{A}$, then the Cayley discriminant represents via the intersection*

$$D_{cay(\mathscr{A})} \cong D_{cay(\mathscr{B})} \bullet \mathbb{C}_{\mathscr{A}/\mathscr{B}} \cap \mathbb{C}_{\mathscr{B}} \bullet D_{cay(\mathscr{A}/\mathscr{B})}.$$

Proof. Since the poset $P_{\mathscr{B}}$ is a connected component, the tuple $\mathscr{A}\backslash\mathscr{B}$ is a BK-tuple. Hence, the linear span of the BK-tuple $\mathscr{A}$ decomposes on the direct sum $\langle\mathscr{A}\rangle = \langle\mathscr{B}\rangle \oplus \langle\mathscr{A}\backslash\mathscr{B}\rangle$, and the torus splits $T(\mathscr{A}) = T(\mathscr{B}) \times T(\mathscr{A}\backslash\mathscr{B})$. Then every system $\Phi_{\mathscr{A}}(x, y)$ from $\mathbb{C}_{\mathscr{A}}$ splits on two independent systems $\Phi_{\mathscr{B}}(x)$ and $\Phi_{\mathscr{A}\backslash\mathscr{B}}(y)$, and every polynomial $f \in \mathbb{C}_{cay(\mathscr{A})}$ can be written

$$f(x, y, \lambda) = \sum_{A\in\mathscr{A}} \lambda_A f_A(x, y) = \sum_{B\in\mathscr{B}} \lambda_B f_B(x) + \sum_{C\in\mathscr{A}\backslash\mathscr{B}} \lambda_C f_C(y).$$

It means that a point $(x^\circ, y^\circ, \lambda^\circ)$ is a singular point for the polynomial $f_{cay(\mathscr{A})}$ if and only if $(x^\circ, \lambda^\circ_{\mathscr{B}})$ and $(y^\circ, \lambda^\circ_{\mathscr{A}\setminus\mathscr{B}})$ are singular points for polynomials $f_{cay(\mathscr{B})}$ and $f_{cay(\mathscr{A}\setminus\mathscr{B})}$. ☐

Lemma 6.2. *Consider an irreducible BK-subtuple $\mathscr{B}$ of a BK-tuple $\mathscr{A}$ with a connected poset $P_{\mathscr{A}}$. Then the Cayley discriminant is isomorphic to $D_{cay(\mathscr{A})} \cong \mathbb{C}_{\mathscr{B}} \bullet D_{cay(\mathscr{A}/\mathscr{B})}$.*

Proof. Without loss of generality, let systems from $\mathbb{C}_{\mathscr{B}}$ depend on variables $x \in T(\mathscr{B})$ and systems from $\mathbb{C}_{\mathscr{A}\setminus\mathscr{B}}$ depend on variables $(x, y) \in T(\mathscr{A})$. Then every polynomial $f \in \mathbb{C}_{cay(\mathscr{A})}$ can be written $f(x, y, \lambda) = \sum_{B\in\mathscr{B}} \lambda_B f_B(x) + \sum_{C\in\mathscr{A}\setminus\mathscr{B}} \lambda_C f_C(x, y)$.

$\boxed{\subseteq}$ A generic polynomial $f \in D_{cay(\mathscr{A})}$, $f(x, y, \lambda) = \sum_{A\in\mathscr{A}} \lambda_A f_A(x, y)$, has a singular point $(x^\circ, y^\circ, \lambda^\circ)$ in the torus $T(cay(\mathscr{A}))$. Since $\frac{\partial f}{\partial \lambda_A} = f_A(x, y)$, the point (x°, y°) is the solution for the polynomial system $\Phi_{\mathscr{A}}$. Notice that $\nabla_y f_B(x) = 0$ for every set $B \in \mathscr{B}$. Then, the equation for the root $(x^\circ, y^\circ, \lambda^\circ)$ of the polynomial f to be a singular point looks

$$\nabla_y f(x^\circ, y^\circ, \lambda^\circ) = \sum_{B\in\mathscr{A}\setminus\mathscr{B}} \lambda^\circ_B \nabla_y f_B(x^\circ, y^\circ) = \sum_{B\in\mathscr{A}\setminus\mathscr{B}} \lambda^\circ_B \nabla_y (ev_{x^\circ} f_B)(y^\circ) = 0.$$

Hence, the point $(y^\circ, \lambda^\circ_{\mathscr{A}\setminus\mathscr{B}})$ is singular for the polynomial $f_{cay(\mathscr{A}/\mathscr{B})} = \sum_{B\in\mathscr{A}\setminus\mathscr{B}} \lambda^\circ_B (ev_{x^\circ} f_B)(y)$, and for the corresponding system $(\Phi_{\mathscr{B}}, f_{cay(\mathscr{A}\setminus\mathscr{B})}) \in \mathbb{C}_{\mathscr{B}} \bullet D_{cay(\mathscr{A}/\mathscr{B})}$.

$\boxed{\supseteq}$ For a generic system $\Phi_{\mathscr{A}} = (\Phi_{\mathscr{B}}, f_{cay(\mathscr{A}\setminus\mathscr{B})}) \in \mathbb{C}_{\mathscr{B}} \circ D_{cay(\mathscr{A}/\mathscr{B})}$, there is a solution $(x^\circ, y^\circ, \lambda^\circ_{\mathscr{A}\setminus\mathscr{B}})$ such that the system of vectors $\sum_{B\in\mathscr{A}\setminus\mathscr{B}} \lambda^\circ_B \nabla_y (ev_{x^\circ} f_B)(y^\circ) = 0$ is linearly dependent. We need to show the existence of nonzero constants $\lambda_{\mathscr{A}}$ such that the system of vectors $\{\nabla f_A(x^\circ, y^\circ)\}_{A\in\mathscr{A}}$ is linearly dependent, and $\sum_{A\in\mathscr{A}} \lambda_A \nabla f_A(x^\circ, y^\circ) = 0$.

Notice that the intersection of sublattices $\langle \mathscr{B}_{\mathscr{A}\setminus\mathscr{B}}\rangle$ and $\langle \mathscr{B}_{\mathscr{B}}\rangle$ has a positive dimension since the poset $P_{\mathscr{A}}$ is connected. Hence, in a generic case, there is at least one variable x_i such that $\sum_{B\in\mathscr{A}\setminus\mathscr{B}} \lambda^\circ_B \frac{\partial f_B}{\partial x_i}(x^\circ, y^\circ) \neq 0$. Then the vectors $\nabla_x f_B(x^\circ)$, $B \in \mathscr{B}$ and $\sum_{B\in\mathscr{A}\setminus\mathscr{B}} \lambda^\circ_B \nabla_x (ev_{y^\circ} f_B)(x^\circ)$ are linearly dependent, and there is their nontrivial linear combination

$$\sum_{B\in\mathscr{B}} \lambda_B \nabla_x f_B(x^\circ) + \mu \sum_{B\in\mathscr{A}\setminus\mathscr{B}} \lambda^\circ_B \nabla_x (ev_{y^\circ} f_B)(x^\circ) = 0,$$

for some constants $\{\mu_C\}_{C\in\mathscr{C}}$ and μ. These constants are non-zero because the BK-tuple $\mathscr{B}$ is irreducible. Therefore, the corresponding Cayley polynomial $f_{cay(\mathscr{A})}$ has the singular point $(x^\circ, y^\circ, \lambda_{\mathscr{B}}, \mu\lambda^\circ_{\mathscr{A}\setminus\mathscr{B}})$, and $f_{cay(\mathscr{A})} \in D_{cay(\mathscr{B})}$. ☐

Proposition 6.3. *For an element α from the poset $P_{\mathscr{A}}$, the stratum $C(\mathscr{B}_\alpha)$ of the $\mathscr{A}$-discriminant expresses via the Cayley discriminant: $D_{cay(\mathscr{B}_{(\alpha)})} = \mathbb{C}_{\mathscr{B}_{(\alpha)\setminus\alpha}} \bullet D_{\hat{\mathscr{B}}_\alpha}$.*

Proof. The order ideal (α) has the unique maximal element α. Apply Lemma 6.2 iteratively and use the coherency relations 4.12. In the last step, we get the Cayley discriminant of the irreducible BK-tuple $\hat{\mathscr{B}}_\alpha$, which coincides with the $\mathscr{A}$-discriminant by Theorem 5.8. ☐

Remark 6.4. This proposition sets up the consistency of Theorem 5.15 about components of the $\mathscr{A}$-discriminant for a BK-tuple with Esterov's Theorem 2.31 [Est10] about components of the reduced discriminant, expressed as Cayley discriminants.

Theorem 6.5. *For a BK-tuple $\mathscr{A}$, the Cayley discriminant $D_{cay(\mathscr{A})}$ equals the intersection of $\mathscr{A}$-discriminant's components such that numerated by maximal elements of the poset $P_{\mathscr{A}}$*

$$D_{cay(\mathscr{A})} = \bigcap_{\alpha\in max\, P_{\mathscr{A}}} \mathbb{C}_{\mathscr{A}\setminus\mathscr{B}_\alpha} \bullet D_{\hat{\mathscr{B}}_\alpha}.$$

The intersection is complete of codimension $2l_{\mathscr{A}} + n_{\mathscr{A}}$, where $l_{\mathscr{A}}/n_{\mathscr{A}}$ is the number of all maximal elements α in the poset $P_{\mathscr{A}}$ such that the BK-tuple $\hat{\mathscr{B}}_\alpha$ is lir/nir.

Proof. By Lemma 6.2 and coherency relations from Corollary 4.12, we can simplify the Cayley discriminant for the BK-tuple $\mathscr{A}$ to the Cayley discriminant for the BK-tuple $\mathscr{A}/\mathscr{C}$, where $\mathscr{C} = \mathscr{A} \setminus \bigcup_{\alpha \in max\, P_{\mathscr{A}}} \mathscr{B}_\alpha$. Then, the poset $P_{\mathscr{A}/\mathscr{C}}$ is a disjoint union of $|max\, P_{\mathscr{A}}|$ incomparable elements. It means that the BK-tuple $\mathscr{A}/\mathscr{C}$ is semi-irreducible, and $\mathbb{C}_{\mathscr{A}/\mathscr{C}} = \prod_{\alpha \in max\, P_{\mathscr{A}}} \mathbb{C}_{\hat{\mathscr{B}}_\alpha}$. Hence, the Cayley discriminant $D_{cay(\mathscr{A}/\mathscr{C})}$ equals the direct product $\prod_{\alpha \in max\, P_{\mathscr{A}}} D_{\hat{\mathscr{B}}_\alpha}$ by Lemma 6.1, and the isomorphism of discriminants $D_{cay(\hat{\mathscr{B}}_\alpha)} = D_{\hat{\mathscr{B}}_\alpha}$ for irreducible BK-tuples $\hat{\mathscr{B}}_\alpha$. Therefore, the intersection $\bigcap_{\alpha \in max\, P_{\mathscr{A}}} \mathbb{C}_{\mathscr{A}\setminus\mathscr{B}_\alpha} \bullet D_{\hat{\mathscr{B}}_\alpha}$ is complete, and the codimension is clear. ☐

Following the proof of Lemma 6.2, it's possible to show (notice Proposition 5.17)

Theorem 6.6. *For a linearly dependent tuple $\mathscr{A}$, the Cayley discriminant equals the multiplication $R_{\mathscr{M}} \bullet D_{cay(\mathscr{A}/\mathscr{M})}$, where $R_{\mathscr{M}}$ is the resultant of the minimal subtuple with minimal defect $\mathscr{M}$.*

Proposition 6.7. *For an essential linearly dependent tuple $\mathscr{A}$, the Cayley discriminant equals the sparse resultant $R_{\mathscr{A}}$ of codimension $-\delta(\mathscr{A})$* (see Proposition 6.1 and Lemma 6.3 [DFS07]).

7 Mixed discriminants

Lemma 7.1. *If the poset $P_{\mathscr{A}}$ has more than one connectivity component for a BK-tuple $\mathscr{A}$, then the mixed discriminant is empty.*

Proof. Every connected component corresponds to a BK-subtuple of $\mathscr{A}$, and the tuple $\mathscr{A}$ decomposes into a disjoint union of BK-subtuples. Then, we can split variables and polynomial systems into independent subsystems with BK-supports. This observation implies that a polynomial system with the support $\mathscr{A}$ cannot have a non-degenerate multiple root. ☐

Lemma 7.2. *For BK-tuples $\mathscr{B} \subset \mathscr{A}$, the mixed discriminant $\mathring{D}_{\mathscr{A}}$ equals $\mathbb{C}_{\mathscr{B}} \bullet \mathring{D}_{\mathscr{A}/\mathscr{B}}$.*

Proof. $\boxed{\subseteq}$ If the mixed discriminant $\mathring{D}_{\mathscr{A}}$ is not empty, then its generic system Φ has a non-degenerate multiple root $(x, y) \in T(\mathscr{A})$ such that vectors $\{df_A(x, y)\}_{A \in \mathscr{A}}$ are linearly dependent, and $x \in T(\mathscr{B})$. If the point x is singular for the subsystem $\{df_B(x)\}_{B \in \mathscr{B}}$, then the point (x, y) can not be a non-degenerate multiple root for the system Φ. Hence, the set $\{d(ev_x f_C)(y)\}_{C \in \mathscr{A}/\mathscr{B}}$ is linearly dependent by Lemma 5.1. Moreover, the point y is a non-degenerate multiple root for the system $ev_x \Phi_{\mathscr{A}\setminus\mathscr{B}}$, and we get the inclusion for discriminants.
$\boxed{\supseteq}$ It's clear. ☐

Theorem 7.3. *For a BK-tuple $\mathscr{A}$, the mixed discriminant equals the Cayley discriminant if the poset $P_{\mathscr{A}}$ has only one maximal element. Otherwise, the mixed discriminant is empty.*

Proof. By Lemma 7.2 and coherency relations from Corollary 4.12, we can simplify the mixed discriminant for the BK-tuple $\mathscr{A}$ to the mixed discriminant for the BK-tuple $\mathscr{A}/\mathscr{C}$, where $\mathscr{C} = \mathscr{A} \setminus \bigcup_{\alpha \in max\, P_{\mathscr{A}}} \mathscr{B}_\alpha$. Then, the poset $P_{\mathscr{A}/\mathscr{C}}$ is a disjoint union of $|max\, P_{\mathscr{A}}|$ incomparable elements. The mixed discriminant $\mathring{D}_{\mathscr{A}/\mathscr{C}}$ is not empty only if the poset $P_{\mathscr{A}/\mathscr{C}}$ consists of one element α by Lemma 7.1. In that case, the BK-tuple $\mathscr{A}/\mathscr{C}$ is irreducible, and the three types of discriminants coincide. Therefore, using Proposition 6.3 for the BK-tuple $\mathscr{A}$, the mixed discriminant and the Cayley discriminant are equal $\mathbb{C}_{\mathscr{C}} \bullet D_{\mathscr{A}/\mathscr{C}}$. ☐

Remark 7.4. The paper [CCD+13] shows that if the Cayley discriminant for a BK-tuple is a hypersurface, then the mixed discriminant is the same hypersurface.

A *circuit* is a minimal linearly dependent tuple. Circuits form a matroid and always have the defect -1 [Pokb]. Circuits in their linear spans are Sturmfel's essential tuples [Stu94].

Theorem 7.5. *The mixed discriminant of a linearly dependent tuple isn't empty if and only if the tuple contains only one circuit $\mathscr{C}$. Then the mixed discriminant is the resultant $R_{\mathscr{C}}$.*

Proof. If a tuple $\mathscr{A}$ contains more than one circuit, then the tuples $\mathscr{A}\backslash A$ are linearly dependent for every set $A \in \mathscr{A}$. It means that systems from the space $\mathbb{C}_{\mathscr{A}}$ can not have a non-degenerate multiple root, and the mixed discriminant is empty.

If the tuple $\mathscr{A}$ contains only one circuit $\mathscr{C}$, then every tuple $\mathscr{A}\backslash C$ is linearly independent for every set $C \in \mathscr{C}$. Therefore, every root for a polynomial system from $\mathbb{C}_{\mathscr{A}}$ is a non-degenerate multiple root. Hence, the mixed discriminant coincides with the resultant $R_{\mathscr{C}}$ and equals the $\mathscr{A}$-discriminant. $\square$

8 Degrees of discriminants

Corollary 8.1. [PS93] *The degree of the mixed discriminant for a linearly dependent tuple with the unique circuit $\mathscr{C}$ equals $\sum\limits_{C \in \mathscr{C}} \mathrm{MV}_{\overline{\langle \mathscr{C} \rangle}}(\mathscr{C}\backslash C)$.*

Corollary 8.2. *For an essential linearly dependent tuple $\mathscr{A}$, the $\mathscr{A}$-discriminant and the Cayley discriminant are equal to the sparse resultant $R_{\mathscr{A}}$ and have the degree*

$$\sum_{\mathscr{B} \subset \mathscr{A} :\, \mathfrak{c}(\mathscr{B}) = dim\, \langle \mathscr{A} \rangle} \mathrm{MV}_{\overline{\langle \mathscr{A} \rangle}}(\mathscr{B}).$$

Proof. By Theorem 5.19, Proposition 6.7 and Corollary 6.5 from [DFS07]. $\square$

Let's describe the degree of resultants in a general case.

For a tuple $\mathscr{M}$ of a negative defect $-\delta$, construct the lattice $L = \overline{\langle \mathscr{M} \rangle} \times \mathbb{Z}^{\delta}$ and the new tuple $\mathscr{M}^{\flat} = (A \times \triangle_{\delta},\, A \in \mathscr{M})$, where $\triangle_{\delta}$ is the standard simplex in $\mathbb{Z}^{\delta}$, and each set $A \times \triangle_{\delta}$ is considered in the lattice L. Notice the tuple $\mathscr{M}^{\flat}$ has zero defect.

Proposition 8.3. (Esterov) *For a linearly dependent tuple $\mathscr{A}$ with the minimal subtuple of the minimal defect $\mathscr{M}$, the sparse resultant $R_{\mathscr{A}}$ is the sparse resultant $R_{\mathscr{M}}$ of degree $\mathrm{MV}_L(\mathscr{M}^{\flat})$.*

Proof. The tuple $\mathscr{A}/\mathscr{M}$ is linearly independent, and a generic system from $\mathbb{C}_{\mathscr{A}/\mathscr{M}}$ always has a solution. Hence, the sparse resultants coincide, $R_{\mathscr{A}} = R_{\mathscr{M}}$.

The resultant $R_{\mathscr{M}}$ is a variety of codimension $\delta = -\delta(\mathscr{M})$. By definition, the degree of a variety of codimension δ equals the number of points of an intersection with a generic δ-dimensional vector subspace. Choose a δ-dimensional affine vector subspace Π in $\mathbb{C}_{\mathscr{M}}$ and choose its parametrization: $\Phi = \Phi^0 + \sum\limits_{i=1}^{\delta} y_i \Phi^i$ for some fixed choice of points $\Phi^0, \ldots, \Phi^{\delta}$ from $\mathbb{C}_{\mathscr{M}}$ and new variables $y_1, \ldots, y_{\delta}$. Each coefficient of the system Φ is a linear function from the new variables y. The parametrization $(\Phi^0, \Phi^1, \ldots, \Phi^{\delta})$ of the subspace Π defines a point in $\mathbb{C}_{\mathscr{M}^{\flat}}$. By Kouchnirenko-Bernstein theorem, a generic system from $\mathbb{C}_{\mathscr{M}^{\flat}}$ has $\mathrm{MV}_L(\mathscr{M}^{\flat})$ solutions. Each solution corresponds to an intersection of the hyperplane Π with the resultant.

For a minimal subtuple with minimal defect $\mathscr{M}$, the tuple $\mathscr{M}^{\flat}$ is an irreducible BK-tuple because every proper subtuple $\mathscr{B}^{\flat}$ has a positive defect: $\delta(\mathscr{B}^{\flat}) = \delta(\mathscr{B}) - \delta(\mathscr{M}) > 0$. This observation ensures that the mixed volume $\mathrm{MV}_L(\mathscr{M}^{\flat})$ is always positive. $\square$

Corollary 8.4. *For a linearly dependent tuple $\mathscr{A}$ with the minimal sublattice $\mathscr{M}$ of minimal defect, the degree of $\mathscr{A}$-discriminant equals the mixed volume $\mathrm{MV}_L(\mathscr{M}^\flat)$.*

Denote by $e^{A',A}$ the Euler obstructions of the set A at its face A' (see [Est10]).

Theorem 8.5. [MT11] For a finite set A, the A-discriminant of codimension δ has the degree

$$deg\, D_A = \sum_{A'\subseteq A} e^{A',A}\left(\binom{dim\,\langle A'\rangle - 1}{\delta} + (-1)^{\delta+1}(\delta+1)\right)\mathrm{Vol}(A').$$

The formula for codimension one coincides with known results [GKZ94, DFS07]. This theorem describes degrees of Cayley discriminants for the Cayley set $cay(\mathscr{A})$ of a tuple $\mathscr{A}$. If the tuple $\mathscr{A}$ is linearly dependent with minimal linearly dependent tuple $\mathscr{M}$, then we use the formula for $\delta = -\delta(\mathscr{M})$. If the tuple $\mathscr{A}$ is BK, then we use the formula for the codimension $\delta = 2l_{\mathscr{A}} + n_{\mathscr{A}}$ by Theorem 8.6. Moreover, we can use Theorem 8.5 to compute degrees for components of $\mathscr{A}$-discriminants for a BK-tuple $\mathscr{A}$.

Corollary 8.6. *For a BK-tuple $\mathscr{A}$ and $\alpha\in P_{\mathscr{A}}$, the component $C(\mathscr{B}_\alpha)$ has the degree*

$$deg\, C(\mathscr{B}_\alpha) = \begin{cases} \sum\limits_{A\subseteq cay(\mathscr{B}_{(\alpha)})} e^{A,cay(\mathscr{B}_{(\alpha)})}\,(dim\,\langle A\rangle + 1)\,\mathrm{Vol}(A), & \text{if } \hat{\mathscr{B}}_\alpha \text{ is nir,}\\ \frac{1}{2}\sum\limits_{A\subseteq cay(\mathscr{B}_{(\alpha)})} e^{A,cay(\mathscr{B}_{(\alpha)})}\,(dim\,\langle A\rangle + 1)\,(dim\,\langle A\rangle - 4)\,\mathrm{Vol}(A), & \text{if } \hat{\mathscr{B}}_\alpha \text{ is lir.}\end{cases}$$

Remark 8.7. 1) Corollary 8.6 describes degrees of mixed discriminants for BK-tuples $\mathscr{A}$ such that the poset $P_{\mathscr{A}}$ has a unique maximal element α, $\mathscr{A} = \mathscr{B}_{(\alpha)}$.

2) For a nir $\hat{\mathscr{B}}_\alpha$ and its face A, the number $(dim\,\langle A\rangle + 1)\,\mathrm{Vol}(A)$ is the total degree of the A-Euler discriminant (see Corollary 1.9 [Est13]).

Every face of a Cayley set is a Cayley set for some collection of faces. Esterov expressed volumes of Cayley sets (see Lemma 1.7 [Est10]) and mixed volumes for tuples of Cayley sets (see [Est12]) via mixed volumes of its generating sets. The computation of volumes simplifies the Matsui-Takeuchi degree formula (see Corollary 1.14 [Est13]).

Denote the mixed volume of n finite sets by the monomial $A_1\cdot\ldots\cdot A_n$ and the interger simplex $\{a\in\mathbb{Z}^k_{\geq 0}\,|\,a_1+\ldots+a_k = m\}$ by $\triangle_k(m)$.

Corollary 8.8. [Est10] *For a tuple of finite sets $\mathscr{A} = (A_1,\ldots,A_k)$, the volume of the Cayley set $cay(\mathscr{A})$ in its n-dimensional linear span equals*

$$\mathrm{Vol}(cay(\mathscr{A})) = \sum_{a\in\triangle_k(n)} A_1^{a_1}\cdot\ldots\cdot A_k^{a_k}.$$

Corollary 8.9. *For a BK-tuple $\mathscr{A}$, the Cayley discriminant has the degree* $\prod\limits_{\alpha\in max\,P_{\mathscr{A}}} deg\, C(\mathscr{B}_\alpha)$.

Proof. We have a complete intersection by Theorem 6.5. □

Proposition 8.10. *For a lir $\mathscr{A}$, the $\mathscr{A}$-discriminant has the degree $deg\, D_{\mathscr{A}} = \frac{c\,(c+1)}{2}$, $c = \mathfrak{c}(\mathscr{A})$.*

Proof. For a linear BK-tuple $\mathscr{A}$, the discriminant $D_{\mathscr{A}}$ is a determinantal variety. One of the proofs is written in Example 19.10 [Har92]. □

Problem. The description of components and degrees of mixed and $\mathscr{A}$-discriminants is still open for underdetermined polynomial systems consisting of more than one equation.

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