A second clue as to what's going on is that there are lines for the Thomson solutions which are uniform polyhedra (e.g. tetrahedron), but there seem to be split lines where we'd expect to see non-uniform solutions (e.g. instead of a ⅗ line for triangular bipyramids we see a co-occurence of points at ⅔ for triangles and points at ½ for a antipodes). In a uniform polyhedron, all vertices have the same geometry, and so if we embed features as them each feature has the same dimensionality. But if we embed features as a non-uniform polyhedron, different features will have more or less interference with others.

In particular, many of the Thomson solutions can be understood as tegum products (an operation which constructs polytopes by embedding two polytopes in orthogonal subspaces) of smaller uniform polytopes. (In the earlier graph visualizations of feature geometry, two subgraphs are disconnected if and only if they are in different tegum factors.) As a result, we should expect their dimensionality to actually correspond to the underlying factor uniform polytopes.

A triangular bipyramid is the tegum product of a triangle and an antipode. As a result, we observe 3×2/3 features and 2×1/2 features, rather than 6×3/5 featurs.

A pentagonal bipyramid is the tegum product of a pentagon and an antipode. As a result, we observe 5×2/5 features and 2×1/2 features, rather than 7×3/7 features.

An octahedron is the tegum product of three antipodes. This doesn’t change the observed lines since 3/6=1/2.

This also suggests a possible reason why we observe 3D Thomson problem solutions, despite the fact that we're actually studying a higher dimensional version of the problem. Just as many 3D Thomson solutions are tegum products of 2D and 1D solutions, perhaps higher dimensional solutions are often tegum products of 1D, 2D, and 3D solutions.

The orthogonality of factors in tegum products has interesting implications. For the purposes of superposition, it means that there can't be any "interference" across tegum-factors. This may be preferred by the toy model: having many features interfere simultaneously could be really bad for it. (See related discussion in our earlier mathematical analysis.)

Aside: Polytopes and Low-Rank Matrices

At this point, it's worth making explicit that there's a correspondence between *polytopes* and *symmetric, positive-definite, low-rank matrices* (i.e. matrices of the form W^TW). This correspondence underlies the results we saw in the previous section, and is generally useful for thinking about superposition.

In some ways, the correspondence is trivial. If one has a rank-m $n\times n$ -matrix of the form W^TW, then W is a $n\times m$ -matrix. We can interpret the columns of W as n points in a m-dimensional space. The place where this starts to become interesting is that it makes it clear that W^TW is driven by the geometry. In particular, we can see how the off-diagonal terms are driven by the geometry of the points.

Put another way, there's an exact correspondence between polytopes and strategies for superposition. For example, every strategy for putting three features in superposition in a 2-dimensional space corresponds to a triangle, and every triangle corresponds to such a strategy. From this perspective, it doesn't seem surprising that if we have three equally important and equally sparse features, the optimal strategy is an equilateral triangle.