Doubled Hilbert space in double-scaled SYK

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ABSTRACT: We consider matter correlators in the double-scaled SYK (DSSYK) model. It turns out that matter correlators have a simple expression in terms of the doubled Hilbert space $\mathcal{H}\otimes\mathcal{H}$, where $\mathcal{H}$ is the Fock space of q-deformed oscillator (also known as the chord Hilbert space). In this formalism, we find that the operator which counts the intersection of chords should be conjugated by certain "entangler" and "disentangler". We explicitly demonstrate this structure for the two- and four-point functions of matter operators in DSSYK.

Contents

1 Introduction

To describe a black hole in AdS, it is useful to consider the doubled (two-sided) Hilbert space of boundary CFT. In particular, the eternal black hole in AdS corresponds to the thermo-field double state [1] which is closely related to the idea of ER=EPR [2, 3]. Recently, the doubled Hilbert space in JT gravity and the double-scaled SYK (DSSYK) model has been extensively studied in the literature (see e.g. [4–8] and references therein).

In this paper, we consider matter correlators of DSSYK in the doubled Hilbert space formalism. As shown in [9], the correlators of DSSYK reduce to the counting problem of chord diagrams, which is exactly solved in terms of the q-deformed oscillator $A_\pm$. The Fock space $\mathcal{H}$ of the q-deformed oscillator, also known as the chord Hilbert space, can be thought of as the Hilbert space of bulk gravity theory [10]. It turns out that matter correlators of DSSYK have a simple expression in the doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}$. We find that the operator which counts the intersection of chords is conjugated by the "entangler" $\mathcal{E}$ and the "disentangler" $\mathcal{E}^{-1}$ (see (4.4) and (4.11)). This structure is reminiscent of the tensor network of MERA [11, 12].

This paper is organized as follows. In section 2, we briefly review the known result of matter correlators in DSSYK. In section 3, we define a mapping of the operator X on $\mathcal{H}$ to the state $|X\rangle$ in the doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}$ and rewrite the matter correlators as the overlap $\langle 0, 0|X\rangle$. In section 4, we perform this rewriting explicitly for the two- and four-point functions of matter operators. We find that the intersection-counting operator is conjugated by the entangler and the disentangler as in (4.4) and (4.11). Finally we conclude in section 5 with some discussion on the future problems. In appendix A we summarize some useful formulae used in the main text. In appendix B we explain the derivation of (4.8). In appendix C we prove the crossing symmetry of the R-matrix of $U_q(\mathfrak{su}(1,1))$.

2 Review of DSSYK

In this section we briefly review the result of DSSYK in [9]. SYK model is defined by the Hamiltonian for N Majorana fermions ψ_i $(i = 1, \cdots, N)$ obeying $\{\psi_i, \psi_j\} = 2\delta_{i,j}$ with all-to-all p-body interaction

$$H = \mathrm{i}^{p/2} \sum_{1\le i_1<\cdots<i_p\le N} J_{i_1\cdots i_p}\psi_{i_1}\cdots\psi_{i_p}, \tag{2.1}$$

where $J_{i_1\cdots i_p}$ is a random coupling drawn from the Gaussian distribution. DSSYK is defined by the scaling limit

$$N, p \to \infty \quad \text{with} \quad \lambda = \frac{2p^2}{N} : \text{fixed}. \tag{2.2}$$

As shown in [9], the ensemble average of the moment $\mathrm{Tr}\, H^k$ reduces to a counting problem of the intersection number of chord diagrams

$$\langle \mathrm{Tr}\, H^k\rangle_J = \sum_{\text{chord diagrams}} q^{\#(\text{intersections})} \tag{2.3}$$

with $q = e^{-\lambda}$. This counting problem is solved by introducing the transfer matrix T

$$T = \frac{A_+ + A_-}{\sqrt{1-q}}, \tag{2.4}$$

where $A_\pm$ denote the q-deformed oscillator acting on the chord number state $|n\rangle$

$$A_+|n\rangle = \sqrt{1-q^{n+1}}|n+1\rangle, \qquad A_-|n\rangle = \sqrt{1-q^n}|n-1\rangle. \tag{2.5}$$

Note that $A_\pm$ satisfy the q-deformed commutation relations

$$\begin{aligned} A_-A_+ - qA_+A_- &= 1-q, \\ A_-A_+ - A_+A_- &= (1-q)q^{\widehat{N}}, \end{aligned} \tag{2.6}$$

where $\widehat{N}$ denotes the number operator

$$\widehat{N}|n\rangle = n|n\rangle. \tag{2.7}$$

Then the moment in (2.3) is written as

$$\langle \mathrm{Tr}\, H^k\rangle_J = \langle 0|T^k|0\rangle. \tag{2.8}$$

The transfer matrix T becomes diagonal in the θ-basis

$$T|\theta\rangle = E(\theta)|\theta\rangle, \qquad E(\theta) = \frac{2\cos\theta}{\sqrt{1-q}}, \tag{2.9}$$

and the overlap of $\langle n|$ and $|\theta\rangle$ is given by the q-Hermite polynomial $H_n(\cos\theta|q)$

$$\langle n|\theta\rangle = \frac{H_n(\cos\theta|q)}{\sqrt{(q;q)_n}}, \tag{2.10}$$

where $(q;q)_n$ denotes the q-Pochhammer symbol (see appendix A for the definition). $|\theta\rangle$ and $|n\rangle$ are normalized as

$$\begin{aligned} \langle\theta|\theta'\rangle &= \frac{2\pi}{\mu(\theta)}\delta(\theta-\theta'), \quad \langle n|m\rangle = \delta_{n,m}, \\ \mathbb{1} &= \int_0^\pi \frac{d\theta}{2\pi}\mu(\theta)|\theta\rangle\langle\theta| = \sum_{n=0}^{\infty}|n\rangle\langle n|, \end{aligned} \tag{2.11}$$

and the measure factor $\mu(\theta)$ is given by

$$\mu(\theta) = (q, e^{\pm 2\mathrm{i}\theta}; q)_\infty. \tag{2.12}$$

As discussed in [9], we can also consider the matter operator $\mathcal{O}_\Delta$

$$\mathcal{O}_\Delta = \mathrm{i}^{s/2} \sum_{1\le i_1<\cdots<i_s\le N} K_{i_1\cdots i_s}\psi_{i_1}\cdots\psi_{i_s} \tag{2.13}$$

with a Gaussian random coefficient $K_{i_1\cdots i_s}$ which is drawn independently from the random coupling $J_{i_1\cdots i_p}$ in the SYK Hamiltonian. In the double scaling limit (2.2), the effect of this operator can be made finite by taking the limit $s\to\infty$ with $\Delta = s/p$ held fixed. Then the correlator of $\mathcal{O}_\Delta$'s is also written as a counting problem of the chord diagrams

$$\sum_{\text{chord diagrams}} q^{\#(H\text{-}H \text{ intersections})} q^{\Delta_i \#(H\text{-}\mathcal{O}_{\Delta_i} \text{ intersections})} q^{\Delta_i\Delta_j \#(\mathcal{O}_{\Delta_i}\text{-}\mathcal{O}_{\Delta_j} \text{ intersections})}. \tag{2.14}$$

Note that there appear two types of chords in this computation: H-chords and $\mathcal{O}$-chords coming from the Wick contraction of random couplings $J_{i_1\cdots i_p}$ and $K_{i_1\cdots i_s}$, respectively. The $\mathcal{O}$-chord is also called matter chord.

Let us consider the bi-local operator $\overline{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}$, where the overline denotes the Wick contraction of random coupling $K_{i_1\cdots i_s}$. As shown in [9], this operator is given by (see also [13])

$$\overline{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta} = \sum_{n,m,\ell=0}^{\infty} \frac{(q^{2\Delta};q)_\ell}{(q;q)_\ell}\sqrt{\frac{(q;q)_{m+\ell}(q;q)_{n+\ell}}{(q;q)_m(q;q)_n}}|m+\ell\rangle\langle m|q^{\Delta\widehat{N}}e^{-\beta T}q^{\Delta\widehat{N}}|n\rangle\langle n+\ell|. \tag{2.15}$$

Using the relation

$$|n\rangle = \frac{A_+^n}{\sqrt{(q;q)_n}}|0\rangle, \qquad A_+^\ell|n\rangle = \sqrt{\frac{(q;q)_{n+\ell}}{(q;q)_n}}|n+\ell\rangle, \tag{2.16}$$

(2.15) is rewritten as

$$\overline{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta} = \sum_{\ell=0}^{\infty}\frac{(q^{2\Delta};q)_\ell}{(q;q)_\ell}A_+^\ell q^{\Delta\widehat{N}}e^{-\beta T}q^{\Delta\widehat{N}}A_-^\ell. \tag{2.17}$$

As shown in [9], this bi-local operator commutes with T

$$\Big[T, \overbracket{\mathcal{O}_\Delta e^{-\beta H} \mathcal{O}_\Delta}\Big] = 0. \tag{2.18}$$

The two-point function of matter operator $\mathcal{O}_\Delta$ is given by

$$\langle 0|e^{-\beta_2 T}\overbracket{\mathcal{O}_\Delta e^{-\beta_1 H}\mathcal{O}_\Delta}|0\rangle = \langle 0|e^{-\beta_2 T} q^{\Delta\widehat{N}} e^{-\beta_1 T}|0\rangle. \tag{2.19}$$

Note that only the $\ell = 0$ term in (2.17) contributes to the two-point function since $A_-^\ell|0\rangle = 0$ for $\ell \geq 1$.

Similarly, the uncrossed four-point function is given by

$$\begin{aligned}
&\langle 0|e^{-\beta_4 T}\overbracket{\mathcal{O}_{\Delta_2} e^{-\beta_3 H}\mathcal{O}_{\Delta_2}} e^{-\beta_2 T}\overbracket{\mathcal{O}_{\Delta_1} e^{-\beta_1 H}\mathcal{O}_{\Delta_1}}|0\rangle \\
=&\langle 0|\overbracket{\mathcal{O}_{\Delta_2} e^{-\beta_3 H}\mathcal{O}_{\Delta_2}} e^{-(\beta_2+\beta_4) T}\overbracket{\mathcal{O}_{\Delta_1} e^{-\beta_1 H}\mathcal{O}_{\Delta_1}}|0\rangle \\
=&\langle 0|e^{-\beta_3 T} q^{\Delta_2\widehat{N}} e^{-(\beta_2+\beta_4)T} q^{\Delta_1\widehat{N}} e^{-\beta_1 T}|0\rangle.
\end{aligned} \tag{2.20}$$

In the first equality we have used the relation (2.18) and the last equality follows from the fact that only the $\ell = 0$ term in (2.17) contributes in this computation when sandwiched between $\langle 0|$ and $|0\rangle$.

The crossed four-point function is given by [9]

$$\begin{aligned}
&\langle 0|e^{-\beta_4 T}\mathcal{O}_{\Delta_2} e^{-\beta_3 H}\mathcal{O}_{\Delta_1} e^{-\beta_2 H}\mathcal{O}_{\Delta_2} e^{-\beta_1 H}\mathcal{O}_{\Delta_1}|0\rangle \\
=&\sum_{\ell=0}^{\infty}\frac{(q^{2\Delta_2};q)_\ell}{(q;q)_\ell} q^{\Delta_1 \ell}\langle 0|e^{-\beta_4 T} A_+^\ell q^{\Delta_2\widehat{N}} e^{-\beta_3 T} q^{\Delta_1\widehat{N}} e^{-\beta_2 T} q^{\Delta_2\widehat{N}} A_-^\ell e^{-\beta_1 T}|0\rangle.
\end{aligned} \tag{2.21}$$

Here we have suppressed the overall factor $q^{\Delta_1\Delta_2}$ coming from the intersection of the $\mathcal{O}_{\Delta_1}$-chord and the $\mathcal{O}_{\Delta_2}$-chord.

Let us take a closer look at the two-point function (2.19). Inserting the complete set $\{|n\rangle\}_{n=0,1,\cdots}$ in (2.19), the two-point function becomes

$$\langle 0|e^{-\beta_2 T} q^{\Delta\widehat{N}} e^{-\beta_1 T}|0\rangle = \sum_{n=0}^{\infty} q^{\Delta n}\langle 0|e^{-\beta_2 T}|n\rangle\langle n|e^{-\beta_1 T}|0\rangle. \tag{2.22}$$

As discussed in [9, 10], $|n\rangle$ represents the state at a constant time-slice of the bulk geometry with n H-chords threading that slice. The factor $q^{\Delta n}$ comes from the intersection of matter chord and n H-chords. Thus $q^{\Delta\widehat{N}}$ in (2.19) can be thought of as the operator counting the intersection of $\mathcal{O}_\Delta$-chord and H-chords. This operator $q^{\Delta\widehat{N}}$ plays an important role in what follows.

3 Doubled Hilbert space

As we reviewed in the previous section, the matter correlator of DSSYK takes the form $\langle 0|X|0\rangle$, where X is a linear operator on the chord Hilbert space $\mathcal{H}$ spanned by the chord number states $|n\rangle$ $(n = 0, 1, \cdots)$

$$\mathcal{H} = \bigoplus_{n=0}^{\infty}\mathbb{C}|n\rangle. \tag{3.1}$$

In order to study the matter correlators in DSSYK, it is useful to consider the doubled Hilbert space $\mathcal{H}\otimes\mathcal{H}$ and regard the operator X as a state $|X\rangle$ in $\mathcal{H}\otimes\mathcal{H}$

$$X \in \mathrm{End}(\mathcal{H}) \quad \mapsto \quad |X\rangle \in \mathcal{H}\otimes\mathcal{H}. \tag{3.2}$$

In terms of the basis $\{|n\rangle\}_{n=0,1,\ldots}$, this mapping (3.2) is given by

$$X=\sum_{n,m=0}^{\infty}|n\rangle\langle n|X|m\rangle\langle m| \quad \mapsto \quad |X\rangle=\sum_{n,m=0}^{\infty}|n,m\rangle\langle n|X|m\rangle, \tag{3.3}$$

where $|n,m\rangle$ is the natural basis of $\mathcal{H}\otimes\mathcal{H}$

$$|n,m\rangle := |n\rangle\otimes|m\rangle. \tag{3.4}$$

In particular, the identity operator $\mathbb{1}$ corresponds to the state

$$|\mathbb{1}\rangle=\sum_{n=0}^{\infty}|n,n\rangle=\mathcal{E}|0,0\rangle, \tag{3.5}$$

where $\mathcal{E}$ is given by (see (2.16) and (A.2))

$$\mathcal{E}=\sum_{n=0}^{\infty}\frac{A_+^n\otimes A_+^n}{(q;q)_n}=\frac{1}{(A_+\otimes A_+;q)_\infty}. \tag{3.6}$$

Note that the state $|\mathbb{1}\rangle$ is the maximally entangled state and the operator $\mathcal{E}$ generates the entanglement when acting on the pure state $|0,0\rangle$. Similarly, the operator $q^{\Delta\widehat{N}}$ corresponds to the state $|q^{\Delta\widehat{N}}\rangle$

$$\left|q^{\Delta\widehat{N}}\right\rangle=\sum_{n=0}^{\infty}q^{\Delta n}|n,n\rangle=\mathcal{E}_\Delta|0,0\rangle \tag{3.7}$$

with

$$\mathcal{E}_\Delta=\frac{1}{(q^\Delta A_+\otimes A_+;q)_\infty}. \tag{3.8}$$

Note that we can append and/or prepend strings of operators as [1]

$$|XYZ\rangle=(X\otimes{}^tZ)|Y\rangle \tag{3.9}$$

where $X,Y,Z\in\mathrm{End}(\mathcal{H})$ and ${}^tZ$ denotes the transpose of Z

$$\langle n|{}^tZ|m\rangle=\langle m|Z|n\rangle. \tag{3.10}$$

We should stress that we do not take the complex conjugation of Z on the right hand side of (3.9); we simply reverse the order of multiplication and take the transpose of Z in (3.9).

As an example of (3.9), let us consider the relation

$$A_-q^{\Delta\widehat{N}}=q^{\Delta\widehat{N}}q^\Delta A_-. \tag{3.11}$$

[1] A similar construction is discussed in [14].

Using ${}^tA_\pm = A_\mp$ and (3.9), we find [2]

$$(A_- \otimes \mathbb{1})\big|q^{\Delta\widehat{N}}\big\rangle = (\mathbb{1} \otimes q^\Delta A_+)\big|q^{\Delta\widehat{N}}\big\rangle. \tag{3.14}$$

We can also show that

$$(\mathbb{1} \otimes A_-)\big|q^{\Delta\widehat{N}}\big\rangle = (q^\Delta A_+ \otimes \mathbb{1})\big|q^{\Delta\widehat{N}}\big\rangle. \tag{3.15}$$

From (2.11), the state $|\mathbb{1}\rangle$ is written in terms of the $|\theta\rangle$-basis as

$$|\mathbb{1}\rangle = \int_0^\pi \frac{d\theta}{2\pi}\mu(\theta)|\theta,\theta\rangle, \tag{3.16}$$

and the state corresponding to the operator $e^{-\beta T}$ is given by

$$|e^{-\beta T}\rangle = (e^{-\frac{1}{2}\beta T} \otimes e^{-\frac{1}{2}\beta T})|\mathbb{1}\rangle = \int_0^\pi \frac{d\theta}{2\pi}\mu(\theta)e^{-\beta E(\theta)}|\theta,\theta\rangle. \tag{3.17}$$

This state $|e^{-\beta T}\rangle$ is known as the thermo-field double state.

4 Matter correlators in the doubled Hilbert space formalism

In this section, we consider matter correlators of DSSYK in the doubled Hilbert space formalism. In general, the matter correlator of DSSYK takes the form $\langle 0|X|0\rangle$ with some operator $X \in \mathrm{End}(\mathcal{H})$. In the doubled Hilbert space formalism, $\langle 0|X|0\rangle$ is expressed as

$$\langle 0|X|0\rangle = \langle 0,0|X\rangle. \tag{4.1}$$

4.1 Two-point function

Let us first consider the bi-local operator in (2.17), which is the basic building block of the two-point function and the uncrossed four-point function. The state $|\overbrace{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle$ corresponding to the operator in (2.17) is given by

$$\begin{aligned}|\overbrace{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle &= \sum_{\ell=0}^\infty \frac{(q^{2\Delta};q)_\ell}{(q;q)_\ell}(A_+^\ell \otimes A_+^\ell)(q^{\Delta\widehat{N}} \otimes q^{\Delta\widehat{N}})|e^{-\beta T}\rangle \\ &= \frac{(q^{2\Delta}A_+ \otimes A_+;q)_\infty}{(A_+ \otimes A_+;q)_\infty}(q^{\Delta\widehat{N}} \otimes q^{\Delta\widehat{N}})|e^{-\beta T}\rangle,\end{aligned} \tag{4.2}$$

where we used the summation formula in (A.3). Using the relation

$$q^{\Delta\widehat{N}}A_+ = q^\Delta A_+ q^{\Delta\widehat{N}}, \tag{4.3}$$

[2] The state $\big|q^{\Delta\widehat{N}}\big\rangle$ in (3.7) is reminiscent of the boundary state $|B_a\rangle$ of the end of the world brane [15]

$$|B_a\rangle = \frac{1}{(aA_+;q)_\infty}|0\rangle. \tag{3.12}$$

As shown in [15], the boundary state $|B_a\rangle$ is a coherent state of the q-deformed oscillator

$$A_-|B_a\rangle = a|B_a\rangle, \tag{3.13}$$

where the parameter a is related to the tension of the brane.

(4.2) is rewritten as

$$|\overbracket{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle = \mathcal{E}(q^{\Delta\widehat{N}}\otimes q^{\Delta\widehat{N}})\mathcal{E}^{-1}|e^{-\beta T}\rangle, \tag{4.4}$$

where $\mathcal{E}$ is defined in (3.6). The appearance of the operator $q^{\Delta\widehat{N}}\otimes q^{\Delta\widehat{N}}$ in (4.4) is natural since it counts the number of intersections between the H-chord and the matter chord. The important point is that this operator $q^{\Delta\widehat{N}}\otimes q^{\Delta\widehat{N}}$ should be conjugated by $\mathcal{E}$

$$q^{\Delta\widehat{N}}\otimes q^{\Delta\widehat{N}} \to \mathcal{E}(q^{\Delta\widehat{N}}\otimes q^{\Delta\widehat{N}})\mathcal{E}^{-1}. \tag{4.5}$$

This conjugation guarantees that the $\beta\to 0$ limit of the state (4.4) reduces to $|\mathbb{1}\rangle$ in (3.5)

$$\begin{aligned}\lim_{\beta\to 0}|\overbracket{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle &= \mathcal{E}(q^{\Delta\widehat{N}}\otimes q^{\Delta\widehat{N}})\mathcal{E}^{-1}|\mathbb{1}\rangle\\ &= \mathcal{E}(q^{\Delta\widehat{N}}\otimes q^{\Delta\widehat{N}})|0,0\rangle\\ &= \mathcal{E}|0,0\rangle\\ &= |\mathbb{1}\rangle.\end{aligned} \tag{4.6}$$

In other words, the conjugation (4.5) is necessary for the following operator identity to hold [3]

$$\overbracket{\mathcal{O}_\Delta\mathcal{O}_\Delta} = \mathbb{1}. \tag{4.7}$$

Following the language of tensor networks, we call $\mathcal{E}$ and $\mathcal{E}^{-1}$ as "entangler" and "disentangler", respectively. Our result (4.4) shows that we have to insert the disentangler $\mathcal{E}^{-1}$ before acting the intersection-counting operator $q^{\Delta\widehat{N}}\otimes q^{\Delta\widehat{N}}$. In the context of MERA [11, 12], disentanglers are usually assumed to be unitary operators. However, our $\mathcal{E}$ and $\mathcal{E}^{-1}$ are not unitary. Thus, (4.5) is a similarity transformation, not a unitary transformation.

From the time-translation invariance (2.18) of the bi-local operator $\overbracket{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}$, it follows that the state $|\overbracket{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle$ in (4.4) is diagonal in the $|\theta\rangle$-basis

$$|\overbracket{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle = \int_0^\pi \frac{d\theta}{2\pi}\mu(\theta)|\theta,\theta\rangle\langle\theta|q^{\Delta\widehat{N}}e^{-\beta T}|0\rangle. \tag{4.8}$$

See appendix B for the derivation of this expression.

4.2 Crossed four-point function

Next, let us consider the crossed four-point function (2.21)

$$G_4 := \sum_{\ell=0}^\infty \frac{(q^{2\Delta_2};q)_\ell}{(q;q)_\ell}q^{\Delta_1\ell}\langle 0|e^{-\beta_4 T}A_+^\ell q^{\Delta_2\widehat{N}}e^{-\beta_3 T}q^{\Delta_1\widehat{N}}e^{-\beta_2 T}q^{\Delta_2\widehat{N}}A_-^\ell e^{-\beta_1 T}|0\rangle. \tag{4.9}$$

In the doubled Hilbert space formalism, this is written as

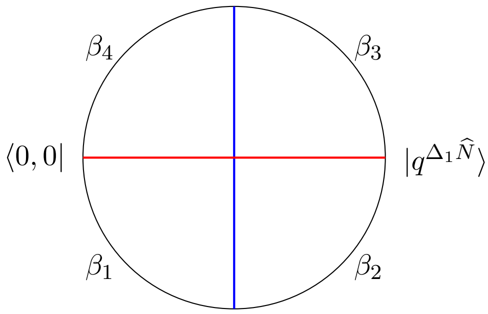
$$G_4 = \sum_{\ell=0}^\infty \frac{(q^{2\Delta_2};q)_\ell}{(q;q)_\ell}q^{\Delta_1\ell}\langle 0,0|(e^{-\beta_4 T}\otimes e^{-\beta_1 T})(A_+^\ell\otimes A_+^\ell)(q^{\Delta_2\widehat{N}}\otimes q^{\Delta_2\widehat{N}})(e^{-\beta_3 T}\otimes e^{-\beta_2 T})|q^{\Delta_1\widehat{N}}\rangle. \tag{4.10}$$

[3] See also footnote 1 in [13].

Using (A.3) and (4.3), one can show that (4.10) is written as

$$G_4 = \langle 0,0|(e^{\beta_4 T} \otimes e^{\beta_1 T})\mathcal{E}_{\Delta_1}(q^{\Delta_2 \widehat{N}} \otimes q^{\Delta_2 \widehat{N}})(\mathcal{E}_{\Delta_1})^{-1}(e^{\beta_3 T} \otimes e^{\beta_2 T})|q^{\Delta_1 \widehat{N}}\rangle, \tag{4.11}$$

where $\mathcal{E}_{\Delta_1}$ is defined in (3.8). Again, the operator $q^{\Delta_2 \widehat{N}} \otimes q^{\Delta_2 \widehat{N}}$ is conjugated by $\mathcal{E}_{\Delta_1}$ in (4.11); $\mathcal{E}_{\Delta_1}$ and $(\mathcal{E}_{\Delta_1})^{-1}$ can be thought of as the entangler and the disentangler associated with the state $|q^{\Delta_1 \widehat{N}}\rangle = \mathcal{E}_{\Delta_1}|0,0\rangle$. G_4 in (4.11) is schematically depicted as



, (4.12)

where the red line and the blue line correspond to the $\mathcal{O}_{\Delta_1}$-chord and the $\mathcal{O}_{\Delta_2}$-chord, respectively. In this picture, the bra and the ket are treated asymmetrically and some of the symmetries of G_4 are not manifest in our representation (4.11). In particular, the crossing symmetry $(12) \leftrightarrow (34)$ of G_4 is not manifest in (4.11).

The crossing symmetry (or the exchange of bra and ket) of G_4 can be seen as follows (see appendix C for the details). Inserting the resolution of identity $\mathbb{1}$ in (2.11), G_4 in (4.11) is written as

$$G_4 = \int \prod_{k=1}^{4} \frac{d\theta_k}{2\pi} \mu(\theta_k) e^{-\beta_k E(\theta_k)} R(\theta_1, \theta_2, \theta_3, \theta_4) \tag{4.13}$$

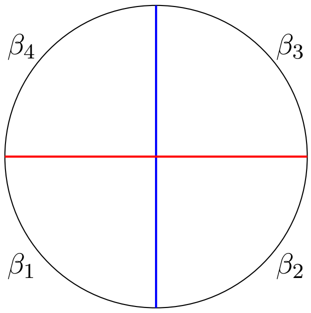
with

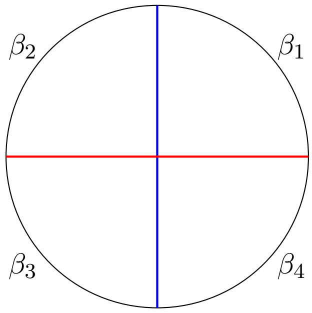
$$R(\theta_1, \theta_2, \theta_3, \theta_4) = \langle \theta_4, \theta_1|\mathcal{E}_{\Delta_1}(q^{\Delta_2 \widehat{N}} \otimes q^{\Delta_2 \widehat{N}})(\mathcal{E}_{\Delta_1})^{-1}|\theta_3, \theta_2\rangle\langle\theta_3|q^{\Delta_1 \widehat{N}}|\theta_2\rangle. \tag{4.14}$$

This $R(\theta_1, \theta_2, \theta_3, \theta_4)$ is proportional to the R-matrix of the quantum group $U_q(\mathfrak{su}(1,1))$, which is written in terms of the basic hypergeometric series ${}_8W_7$. Using the Bailey transformation (C.5) of ${}_8W_7$, one can show that $R(\theta_1, \theta_2, \theta_3, \theta_4)$ in (4.14) is invariant under the crossing symmetry $(12) \leftrightarrow (34)$

$$R(\theta_1, \theta_2, \theta_3, \theta_4) = R(\theta_3, \theta_4, \theta_1, \theta_2), \tag{4.15}$$

which implies that G_4 is invariant under $(\beta_1\beta_2) \leftrightarrow (\beta_3\beta_4)$. This symmetry of G_4 is schematically depicted as

$\langle 0,0|$ 

$|q^{\Delta_1 \widehat{N}}\rangle = \langle 0,0|$ 

$|q^{\Delta_1 \widehat{N}}\rangle$. (4.16)

5 Conclusion and outlook

In this paper we have studied the matter correlators of DSSYK in the doubled Hilbert space formalism. In our formalism, a matter correlator of the form $\langle 0|X|0\rangle$ is expressed as the overlap between $\langle 0,0|$ and the state $|X\rangle \in \mathcal{H}\otimes\mathcal{H}$ corresponding to the operator X, where the relation between X and $|X\rangle$ is given by (3.3). We find that the intersection-counting operator $q^{\Delta\widehat{N}} \otimes q^{\Delta\widehat{N}}$ should be conjugated by the entangler $\mathcal{E}$ and the disentangler $\mathcal{E}^{-1}$ as in (4.4) (or the entangler $\mathcal{E}_{\Delta_1}$ and disentangler $(\mathcal{E}_{\Delta_1})^{-1}$ in the case of crossed four-point function (4.11)). In our representation of a matter correlator $\langle 0,0|X\rangle$ (4.1), the bra and the ket are treated asymmetrically and hence some of the symmetries of the correlators are not manifest. Nevertheless, the bra-ket exchange symmetry (or crossing symmetry) of the four-point function (4.16) can be shown rather non-trivially by using the Bailey transformation (C.5) of ${}_8W_7$.

We should stress that our formalism is different from that in [8]. The authors of [8] introduced the two-sided chord Hilbert space in the presence of the matter operator, spanned by the states $\{|n_L, n_R\rangle\}$ where n_L and n_R denote the number of H-chords to the left and right of the matter chord. Our $|n,m\rangle$ in (3.4) is not equal to $|n_L, n_R\rangle$ in [8]. According to the discussion in [14], our $|n,m\rangle$ can be expanded as a linear combination of $|n_L, n_R\rangle$ in [8]. It would be interesting to find a precise relation between our $|n,m\rangle$ and $|n_L, n_R\rangle$ in [8].

The construction of the two-sided chord Hilbert space in [8] is based on a picture of cutting open the "bulk path integral". On the other hand, our formalism is based on a honest, direct rewriting of the known result of matter correlators in [9]. At present we do not understand clearly how these two approaches are related. In particular, in our formalism we do not need to introduce the co-product of q-deformed oscillator $A_\pm$, which played an important role in the discussion of symmetry algebra in [8]. Perhaps, (3.14) and (3.15) might be a good starting point to consider the relationship between the two approaches. We leave this as an interesting future problem.

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A Useful formulae

In this appendix, we summarize some useful formulae used in the main text. The q-Pochhammer symbol is defined by

$$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\qquad (a_1,\cdots,a_s;q)_n=\prod_{i=1}^{s}(a_i;q)_n. \tag{A.1}$$

The following summation formulae play an important role in this paper:

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}=\frac{1}{(t;q)_\infty}, \tag{A.2}$$

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}t^n=\frac{(ta;q)_\infty}{(t;q)_\infty}. \tag{A.3}$$

The q-Hermite polynomial $H_n(x|q)$ is defined by the recursion relation

$$2xH_n(x|q)=H_{n+1}(x|q)+(1-q^n)H_{n-1}(x|q), \tag{A.4}$$

with the initial condition $H_{-1}=0, H_0=1$. In the computation of matter correlators, we need the following formula for the Poisson kernel of the q-Hermite polynomials

$$\begin{aligned}\langle\theta_1|t^{\widehat{N}}|\theta_2\rangle&=\sum_{n=0}^{\infty}t^n\langle\theta_1|n\rangle\langle n|\theta_2\rangle\\&=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}H_n(\cos\theta_1|q)H_n(\cos\theta_2|q)\\&=\frac{(t^2;q)_\infty}{(te^{\mathrm{i}(\pm\theta_1\pm\theta_2)};q)_\infty}.\end{aligned} \tag{A.5}$$

Al-Salam-Chihara polynomial $Q_n(x|a,b,q)$ is defined by the recursion relation

$$2xQ_n=Q_{n+1}+(a+b)q^nQ_n+(1-q^n)(1-abq^{n-1})Q_{n-1} \tag{A.6}$$

with the initial condition $Q_{-1}=0, Q_0=1$. Using the summation formula in [16], we find that the matrix element of $A_+^\ell q^{\Delta\widehat{N}}$ is given by the Al-Salam-Chihara polynomial

$$\begin{aligned}\langle\theta_1|A_+^\ell t^{\widehat{N}}|\theta_2\rangle&=\sum_{n=0}^{\infty}t^n\langle\theta_1|A_+^\ell|n\rangle\langle n|\theta_2\rangle\\&=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}H_{n+\ell}(\cos\theta_1|q)H_n(\cos\theta_2|q)\\&=\langle\theta_1|t^{\widehat{N}}|\theta_2\rangle\frac{Q_\ell(\cos\theta_1|te^{\pm\mathrm{i}\theta_2},q)}{(t^2;q)_\ell}.\end{aligned} \tag{A.7}$$

B Derivation of (4.8)

In this appendix, we derive the relation (4.8). To this end, let us consider the overlap of $\langle\theta_1,\theta_2|$ and the state $|\overbrace{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle$ in (4.2)

$$\langle\theta_1,\theta_2|\overbrace{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle=\sum_{\ell=0}^{\infty}\frac{(q^{2\Delta};q)_\ell}{(q;q)_\ell}\int_0^\pi\frac{d\theta}{2\pi}\mu(\theta)e^{-\beta E(\theta)}\langle\theta_1|A_+^\ell q^{\Delta\widehat{N}}|\theta\rangle\langle\theta_2|A_+^\ell q^{\Delta\widehat{N}}|\theta\rangle, \tag{B.1}$$

where we used the integral form of the thermo-field double state $|e^{-\beta T}\rangle$ in (3.17). Plugging (A.7) into (B.1), we find

$$\begin{aligned}\langle\theta_1,\theta_2|\overbrace{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle=&\int_0^\pi\frac{d\theta}{2\pi}\mu(\theta)e^{-\beta E(\theta)}\langle\theta_1|q^{\Delta\widehat{N}}|\theta\rangle\langle\theta_2|q^{\Delta\widehat{N}}|\theta\rangle\\&\times\sum_{\ell=0}^{\infty}\frac{Q_\ell(\cos\theta_1|q^\Delta e^{\pm\mathrm{i}\theta},q)Q_\ell(\cos\theta_2|q^\Delta e^{\pm\mathrm{i}\theta},q)}{(q,q^{2\Delta};q)_\ell}.\end{aligned} \tag{B.2}$$

Using the summation formula [17]

$$\sum_{\ell=0}^{\infty}\frac{Q_\ell(\cos\theta_1|q^\Delta e^{\pm\mathrm{i}\theta},q)Q_\ell(\cos\theta_2|q^\Delta e^{\pm\mathrm{i}\theta},q)}{(q,q^{2\Delta};q)_\ell}=\frac{\langle\theta_1|\theta_2\rangle}{\langle\theta_1|q^{\Delta\widehat{N}}|\theta\rangle}, \tag{B.3}$$

(B.2) becomes

$$\begin{aligned}\langle\theta_1,\theta_2|\overbrace{\mathcal{O}_\Delta e^{-\beta H}\mathcal{O}_\Delta}\rangle&=\langle\theta_1|\theta_2\rangle\int_0^\pi\frac{d\theta}{2\pi}\mu(\theta)e^{-\beta E(\theta)}\langle\theta_1|q^{\Delta\widehat{N}}|\theta\rangle\\&=\langle\theta_1|\theta_2\rangle\int_0^\pi\frac{d\theta}{2\pi}\mu(\theta)\langle\theta_1|q^{\Delta\widehat{N}}e^{-\beta T}|\theta\rangle\\&=\langle\theta_1|\theta_2\rangle\langle\theta_1|q^{\Delta\widehat{N}}e^{-\beta T}|0\rangle.\end{aligned} \tag{B.4}$$

In the last equality we used the relation

$$\int_0^\pi\frac{d\theta}{2\pi}\mu(\theta)|\theta\rangle=|0\rangle, \tag{B.5}$$

where $|0\rangle$ on the right hand side stands for $|n=0\rangle$. One can easily see that (B.4) is equivalent to our desired relation (4.8). This completes the proof of (4.8).

C Crossing symmetry of $R(\theta_1,\theta_2,\theta_3,\theta_4)$

In this appendix, we prove the crossing symmetry of $R(\theta_1,\theta_2,\theta_3,\theta_4)$ in (4.15). Using (A.7), $R(\theta_1,\theta_2,\theta_3,\theta_4)$ in (4.14) is written as

$$\begin{aligned}
R(\theta_1,\theta_2,\theta_3,\theta_4) &= \langle\theta_4,\theta_1|\left(\sum_{\ell=0}^{\infty}\frac{(q^{2\Delta_2};q)_\ell}{(q;q)_\ell}q^{\Delta_1\ell}A_+^\ell\otimes A_+^\ell\right)(q^{\Delta_2\widehat{N}}\otimes q^{\Delta_2\widehat{N}})|\theta_3,\theta_2\rangle\langle\theta_3|q^{\Delta_1\widehat{N}}|\theta_2\rangle \\
&= \langle\theta_3|q^{\Delta_1\widehat{N}}|\theta_2\rangle\langle\theta_4|q^{\Delta_2\widehat{N}}|\theta_3\rangle\langle\theta_1|q^{\Delta_2\widehat{N}}|\theta_2\rangle \\
&\quad\times\sum_{\ell=0}^{\infty}\frac{q^{\Delta_1\ell}}{(q^{2\Delta_2},q;q)_\ell}Q_\ell(\cos\theta_4|q^{\Delta_2}e^{\pm\mathrm{i}\theta_3};q)Q_\ell(\cos\theta_1|q^{\Delta_2}e^{\pm\mathrm{i}\theta_2};q) \\
&= \langle\theta_4|q^{\Delta_2\widehat{N}}|\theta_3\rangle\langle\theta_3|q^{\Delta_1\widehat{N}}|\theta_2\rangle\langle\theta_2|q^{\Delta_2\widehat{N}}|\theta_1\rangle\langle\theta_1|q^{\Delta_1\widehat{N}}|\theta_4\rangle \\
&\times\frac{(q^{\Delta_1}e^{-\mathrm{i}(\theta_2+\theta_3)},q^{\Delta_1+\Delta_2}e^{\mathrm{i}(\theta_2\pm\theta_4)},q^{\Delta_1+\Delta_2}e^{\mathrm{i}(\theta_3\pm\theta_1)};q)_\infty}{(q^{2\Delta_1},q^{\Delta_1+2\Delta_2}e^{\mathrm{i}(\theta_2+\theta_3)};q)_\infty} \\
&\times{}_8W_7(q^{-1+\Delta_1+2\Delta_2}e^{\mathrm{i}(\theta_2+\theta_3)};q^{\Delta_1}e^{\mathrm{i}(\theta_2+\theta_3)},q^{\Delta_2}e^{\mathrm{i}(\theta_3\pm\theta_4)},q^{\Delta_2}e^{\mathrm{i}(\theta_2\pm\theta_1)};q,q^{\Delta_1}e^{-\mathrm{i}(\theta_2+\theta_3)}).
\end{aligned} \tag{C.1}$$

In the last step, we have used the Poisson kernel of the Al-Salam-Chihara polynomials [17]

$$\begin{aligned}
&\frac{1}{\langle\theta_1|t^{\widehat{N}}|\theta_4\rangle}\sum_{\ell=0}^{\infty}\frac{t^\ell}{(ab,q;q)_\ell}Q_\ell(\cos\theta_4|a,b;q)Q_\ell(\cos\theta_1|\alpha,\beta;q) \\
=&\frac{(\beta a^{-1}t,\alpha te^{\pm\mathrm{i}\theta_4},ate^{\pm\mathrm{i}\theta_1};q)_\infty}{(t^2,a\alpha t;q)_\infty}{}_8W_7(q^{-1}a\alpha t;\alpha b^{-1}t,ae^{\pm\mathrm{i}\theta_4},\alpha e^{\pm\mathrm{i}\theta_1};q,\beta a^{-1}t)
\end{aligned} \tag{C.2}$$

where $ab=\alpha\beta$ and the well-poised basic hypergeometric series ${}_8W_7$ is defined by [4]

$${}_8W_7(a;b,c,d,e,f;q,z)=\sum_{n=0}^{\infty}\frac{(a,\pm qa^{\frac12},b,c,d,e,f;q)_n}{(q,\pm a^{\frac12},qa/b,qa/c,qa/d,qa/e,qa/f;q)_n}z^n. \tag{C.4}$$

The crossing symmetry of $R(\theta_1,\theta_2,\theta_3,\theta_4)$ in (C.2) can be shown by using the Bailey transform of ${}_8W_7$ (see e.g. [18])

$${}_8W_7\Big(a;b,c,d,e,f;q,\frac{\lambda q}{ef}\Big)=\frac{\Big(aq,\frac{aq}{ef},\frac{\lambda q}{e},\frac{\lambda q}{f};q\Big)_\infty}{\Big(\lambda q,\frac{\lambda q}{ef},\frac{aq}{e},\frac{aq}{f};q\Big)_\infty}{}_8W_7\Big(\lambda;\tilde b,\tilde c,\tilde d,e,f;q,\frac{aq}{ef}\Big), \tag{C.5}$$

where

$$\lambda=\frac{qa^2}{bcd},\quad \tilde b=\frac{\lambda b}{a},\quad \tilde c=\frac{\lambda c}{a},\quad \tilde d=\frac{\lambda d}{a}. \tag{C.6}$$

Note that the dual version of the first relation of (C.6) is given by

$$a=\frac{q\lambda^2}{\tilde b\tilde c\tilde d}. \tag{C.7}$$

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[4] The basic hypergeometric series

$$\sum_{n=0}^{\infty}\frac{(a,\alpha_1,\cdots,\alpha_s;q)_n}{(q,\beta_1,\cdots,\beta_s;q)_n}z^n \tag{C.3}$$

is called well-poised when $aq=\alpha_1\beta_1=\cdots=\alpha_s\beta_s$. ${}_8W_7(a;b,c,d,e,f;q,z)$ is called very well-poised when $z=\lambda q/ef$.

Thus the transformation of the parameters $(a,b,c,d) \to (\lambda,\tilde{b},\tilde{c},\tilde{d})$ in (C.5) is a $\mathbb{Z}_2$ involution.[5] We can apply the Bailey transformation (C.5) to our case (C.2) by setting

$$\begin{aligned} a &= q^{-1+\Delta_1+2\Delta_2} e^{\mathrm{i}(\theta_2+\theta_3)}, \\ b &= q^{\Delta_1} e^{\mathrm{i}(\theta_2+\theta_3)}, \\ c &= q^{\Delta_2} e^{\mathrm{i}(\theta_2-\theta_1)}, \\ d &= q^{\Delta_2} e^{\mathrm{i}(\theta_3-\theta_4)}, \\ e &= q^{\Delta_2} e^{\mathrm{i}(\theta_1+\theta_2)}, \\ f &= q^{\Delta_2} e^{\mathrm{i}(\theta_3+\theta_4)}. \end{aligned} \tag{C.8}$$

Then the dual parameters are given by

$$\begin{aligned} \lambda &= q^{-1+\Delta_1+2\Delta_2} e^{\mathrm{i}(\theta_1+\theta_4)}, \\ \tilde{b} &= q^{\Delta_1} e^{\mathrm{i}(\theta_1+\theta_4)}, \\ \tilde{c} &= q^{\Delta_2} e^{\mathrm{i}(\theta_4-\theta_3)}, \\ \tilde{d} &= q^{\Delta_2} e^{\mathrm{i}(\theta_1-\theta_2)}. \end{aligned} \tag{C.9}$$

We can see that the mapping from (a,b,c,d) in (C.8) to $(\lambda,\tilde{b},\tilde{c},\tilde{d})$ in (C.9) corresponds to the crossing symmetry $(12) \leftrightarrow (34)$. We can also check that the prefactor of ${}_8W_7$ in (C.2) is correctly transformed under the Bailey transformation (C.5). Finally we find that $R(\theta_1,\theta_2,\theta_3,\theta_4)$ in (C.2) is invariant under the crossing symmetry (4.15).

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