

THE GROEBNER BASIS AND SOLUTION SET OF A POLYNOMIAL SYSTEM RELATED TO THE JACOBIAN CONJECTURE

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ABSTRACT. We compute the Groebner basis of a system of polynomial equations related to the Jacobian conjecture, and describe completely the solution set.

1 Introduction

Let K be a characteristic zero field and let $K[y]((x^{-1}))$ be the algebra of Laurent series in x^{-1} with coefficients in $K[y]$. We will start from the following theorem, proved in [2, Theorem 1.9].

Theorem 1.1. *The Jacobian conjecture in dimension two is false if and only if there exist*

- $P, Q \in K[x, y]$ and $C, F \in K[y]((x^{-1}))$,
- $n, m \in \mathbb{N}$ such that $n \nmid m$ and $m \nmid n$,
- $\nu_i \in K$ ($i = 0, \dots, m+n-2$) with $\nu_0 = 1$,

such that

- C has the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \dots \quad \text{with each } C_{-i} \in K[y],$$

- $gr(C) = 1$ and $gr(F) = 2 - n$, where gr is the total degree,
- $F_+ = x^{1-n}y$, where F_+ is the term of maximal degree in x of F ,
- $C^n = P$ and $Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F$.

Furthermore, under these conditions (P, Q) is a counterexample to the Jacobian conjecture.

In [2], the authors consider the following slightly more general situation. Let D be a K -algebra (for example, in Theorem 1.1 we have $D = K[y]$), n, m positive integers such that $n \nmid m$ and $m \nmid n$, $(\nu_i)_{1 \leq i \leq n+m-2}$ a family of elements in K with $\nu_0 = 1$ and $F_{1-n} \in D$ (in Theorem 1.1 we take $F_{1-n} = y$). A Laurent series in x^{-1} of the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \dots \quad \text{with } C_{-i} \in D,$$

is a solution of the system $S(n, m, (\nu_i), F_{1-n})$, if there exist $P, Q \in D[x]$ and $F \in D[[x^{-1}]]$, such that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + F_{-1-n}x^{-1-n} + \dots, \quad \text{with } F_{1-n}, F_{-n}, \dots \text{ in } D$$

$$P = C^n \quad \text{and} \quad Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F.$$

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For example, if $n = 3$, then

$$\begin{aligned} P(\mathbf{x}) = C^3 &= \mathbf{x}^3 + 3C_{-1}\mathbf{x} + 3C_{-2} + (3C_{-1}^2 + 3C_{-3})\mathbf{x}^{-1} + (6C_{-1}C_{-2} + 4C_{-4})\mathbf{x}^{-2} \\ &\quad + (C_{-1}^3 + 3C_{-2}^2 + 6C_{-2}C_{-3} + 3C_{-5})\mathbf{x}^{-3} \\ &\quad + (3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6})\mathbf{x}^{-4} \\ &\quad + (3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 6C_{-7})\mathbf{x}^{-5} \\ &\quad + \dots \end{aligned}$$

and the condition $C^3 \in K[x]$ translates into the following conditions on C_{-k} :

$$\begin{aligned} 0 &= (C^3)_{-1} = 3C_{-1}^2 + 3C_{-3}, \\ 0 &= (C^3)_{-2} = 6C_{-1}C_{-2} + 4C_{-4}, \\ 0 &= (C^3)_{-3} = C_{-1}^3 + 3C_{-2}^2 + 6C_{-2}C_{-3} + 3C_{-5}, \\ 0 &= (C^3)_{-4} = 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6}, \\ 0 &= (C^3)_{-5} = 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 6C_{-7}, \\ &\vdots \end{aligned}$$

In general, the condition $P(x) = C^n \in K[x]$ yields equations $(C^n)_{-k} = 0$, whereas the condition $Q(x) = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F \in K[x]$ gives us the equations $\left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F \right)_{-k} = 0$, where we note that $F_{-k} = 0$ for $k = 1, \dots, n-2$.

It is easy to see (e.g. [2, Remark 1.13]) that the first $m+n-2$ coefficients determine the others, i.e., the coefficients $C_{-1}, \dots, C_{-m-n+2}$ determine univocally the coefficients C_{-k} for $k > m+n-2$. Moreover, the F_{-k} for $k > n-1$ depend only on F_{1-n} and C . Consequently, having a solution C to the system $S(n, m, (\nu_i), F_{1-n})$ is the same as having a solution $(C_{-1}, \dots, C_{-m-n+2})$ to the system

$$\begin{aligned} E_k &:= (C^n)_{-k} = 0, & \text{for } k = 1, \dots, m-1, \\ E_{m-1+k} &:= \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} \right)_{-k} = 0, & \text{for } k = 1, \dots, n-2, \\ E_{m+n-2} &:= \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} \right)_{1-n} + F_{1-n} = 0, \end{aligned} \tag{1.1}$$

with $m+n-2$ equations E_k and $m+n-2$ unknowns C_{-k} .

In order to understand the solution set of this system, it would be very helpful to find a Groebner basis for the ideal generated by the polynomials E_k in $D[C_{-1}, \dots, C_{m+n-2}]$. In this paper we compute such a Groebner basis of (1.1) in a very particular case: we assume $n = 3$, $m = 3r + 1$ or $m = 3r + 2$ for some integer $r > 0$, and $\nu_i = 0$ for $i > 0$. Moreover we consider $D = \mathbb{C}[y]$ and $F_{1-n} = y$, as in Theorem 1.1.

2 Computation of a Groebner basis for I_{m-1}

Assume $n = 3$, $3 \nmid m > 3$ and $\nu_i = 0$ for $i > 0$. Set also $D = \mathbb{C}[y]$ and $F_{1-n} = y$.

Then the system (1.1) reads

$$E_i = \begin{cases} (C^3)_{-i}, & i = 1, \dots, m-1 \\ (C^m)_{-1}, & i = m, \\ (C^m)_{-2} + y, & i = m+1, \end{cases} \quad (2.1)$$

where $(C^2)_{-i}$ denotes the coefficient of x^{-i} in the Laurent series C^3 . Explicitly, the polynomials E_i are given by

$$\begin{aligned} E_1 &= 3C_{-1}^2 + 3C_{-3}, \\ E_2 &= 6C_{-1}C_{-2} + 3C_{-4}, \\ E_3 &= C_{-1}^3 + 3C_{-2}^2 + 6C_{-1}C_{-3} + 3C_{-5}, \\ E_4 &= 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 3C_{-6}, \\ E_5 &= 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 3C_{-7}, \\ &\vdots \\ E_{m-1} &= (C^3)_{1-m}, \\ E_m &= (C^m)_{-1}, \\ E_{m+1} &= (C^m)_{-2} + y, \end{aligned} \quad (2.2)$$

Each E_i is a polynomial in the ring $\mathbb{C}[C_{-1}, C_{-2}, \dots, C_{m+1}, y]$, and the $m+1$ polynomials yield the ideal

$$I = \langle E_1, \dots, E_m, E_{m+1} \rangle.$$

Our goal is to find a Groebner basis for the ideal I , but we find it nearly explicit only for $I_{m-1} := \langle E_1, E_2, \dots, E_{m-2}, E_{m-1} \rangle$. For this we note that the equations are homogeneous, for the weight obtained by setting

$$w(C_{-i}) = i+1, \quad \text{and} \quad w(y) = m+n-1 = m+2.$$

We consider y as a variable, so the equations remain homogeneous. Then

$$w(E_k) = k+3, \quad \text{for } k = 1, \dots, m-1, \quad w(E_m) = m+1 \quad \text{and} \quad w(E_{m+1}) = m+2.$$

Note that for $k = 1, \dots, m-1$ we have

$$E_k := 3 \left(\sum_{\substack{i=-1 \\ 3i \neq k}}^{[\frac{k+1}{2}]} C_{-i}^2 C_{-(k-2i)} \right) + 6 \left(\sum_{\substack{0 < i < j \\ i+j=k+1}} C_{-i} C_{-j} \right) + 6 \left(\sum_{\substack{0 < i < j < l \\ i+j+l=k}} C_{-i} C_{-j} C_{-l} \right) + \varepsilon (C_{-\frac{k}{3}})^3, \quad (2.3)$$

where $\varepsilon = \begin{cases} 1, & 3|k \\ 0, & 3 \nmid k \end{cases}$. Note that $C_1 = 1$ and $C_0 = 0$, and so

$$3 \sum_{i=-1}^{[\frac{k+1}{2}]} C_{-i}^2 C_{-(k-2i)} = 3C_{k+2} + 3 \sum_{i=1}^{[\frac{k+1}{2}]} C_{-i}^2 C_{-(k-2i)} \quad (2.4)$$

In order to compute a Groebner basis we will consider the degree reverse lexicographic monomial order, but for the degree given by the above mentioned weight. This means that the monomial

order is given by the matrix

$$\text{wmat} = \begin{pmatrix} m+2 & m+1 & m & \dots & 4 & 3 & 2 & m+2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

on the variables $C_{-(m+1)}, C_{-m}, C_{-(m-1)}, \dots, C_{-3}, C_{-2}, C_{-1}, y$. We first compute the reduced Groebner basis $(\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{m-1})$ for the ideal $I_{m-1} := \langle E_1, E_2, \dots, E_{m-2}, E_{m-1} \rangle$.

Proposition 2.1. *The set $\{E_1, \dots, E_{m-1}\}$ is a Groebner basis of I_{m-1} . The reduced Groebner basis of I_{m-1} is given by polynomials \tilde{E}_k for $k = 1, \dots, m-1$, each of the form*

$$\tilde{E}_k = C_{-(k+2)} + R_k(C_{-1}, C_{-2}),$$

where $R_k(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ is an homogeneous polynomial in the variables C_{-1} and C_{-2} of weight $w(\tilde{E}_k) = w(E_k) = k+3$.

Proof. By (2.3) and (2.4) we know that E_k is of the form

$$E_k = 3C_{-k-2} + T(C_{-1}, \dots, C_{-k}), \quad \text{for } k = 1, \dots, m-1,$$

where T is a polynomial in the variables C_{-1}, \dots, C_{-k} . Then by Proposition 2.9.4 of [1], since

$$LCM(LT(E_i)/3, LT(E_j)/3) = LCM(C_{-i-2}, C_{-j-2}) = C_{-i-2}C_{-j-2} = (LT(E_i)/3)(LT(E_j)/3),$$

we have $S(E_i, E_j) \longrightarrow_G 0$, and so, by Theorem 2.9.3 of [1], the set $G = \{E_1/3, \dots, E_{m-1}/3\}$ is a Groebner basis of I_{m-1} . One verifies directly that it is a minimal Groebner basis, according to Definition 2.7.4 of [1]. If we apply the process described in the proof of [1, Proposition 2.7.6] to the Groebner basis $G = \{E_1/3, \dots, E_{m-1}/3\}$ we obtain that

$$\tilde{E}_1 = \overline{E_1/3}^{G \setminus \{\frac{E_1}{3}\}} = E_1/3 \quad \text{and} \quad \tilde{E}_1 = \overline{E_2/3}^{G \setminus \{\frac{E_2}{3}\}} = E_2/3.$$

Moreover, for $k = 3, \dots, m-1$, we define $G_k = \{\tilde{E}_1, \dots, \tilde{E}_{k-1}, E_k, \dots, E_{m-1}\}$ and then

$$\tilde{E}_k = \overline{E_k}^{G_k \setminus E_k},$$

Clearly the remainder can have only the variables C_{-1} and C_{-2} , hence \tilde{E}_k is of the form

$$\tilde{E}_k = C_{-(k+2)} + R_k(C_{-1}, C_{-2}),$$

as desired. □

Although we have no explicit formula for $R_k(C_{-1}, C_{-2})$, we can compute it for small k .

$$\begin{aligned}\tilde{E}_1 &= C_{-3} + C_{-1}^2, \\ \tilde{E}_2 &= C_{-4} + 2C_{-1}C_{-2}, \\ \tilde{E}_3 &= C_{-5} + C_{-2}^2 - \frac{5}{3}C_{-1}^3, \\ \tilde{E}_4 &= C_{-6} - 5C_{-1}^2C_{-2}, \\ \tilde{E}_5 &= C_{-7} + \frac{10}{3}C_{-1}^4 - 5C_{-1}C_{-2}^2.\end{aligned}$$

Dividing the polynomials E_m and E_{m+1} by the polynomials $\{\tilde{E}_{m-1}, \dots, \tilde{E}_2, \tilde{E}_1\}$ with respect to the given order, we obtain

$$\frac{\overline{E_m}^{G_m \setminus \{\frac{E_m}{3}\}}}{3} = \tilde{E}_m = R_m(C_{-1}, C_{-2}) \quad \text{and} \quad \frac{\overline{E_{m+1}}^{G_m \setminus \{\frac{E_{m+1}}{3}\}}}{3} = \tilde{E}_{m+1} = y + R_{m+1}(C_{-1}, C_{-2}),$$

where $R_m(C_{-1}, C_{-2}), R_{m+1}(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ are homogeneous polynomials such that $w(\tilde{E}_m) = w(E_m) = m+1$ and $w(\tilde{E}_{m+1}) = w(E_{m+1}) = m+2$.

Although we don't give an explicit description of the Groebner Basis of the whole system, in the next section we show how to determine the solution set of the polynomial system, using that

$$I = \langle E_1, E_2, \dots, E_m, E_{m+1} \rangle = \langle \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_m, \tilde{E}_{m+1} \rangle.$$

3 The solution set of the system of polynomial equations

In this section we analyze the solutions of the system of equations. Note that the partial system I_{m-1} shows that the values of C_{-1} and C_{-2} determine univocally the values of C_{-k} for $k > 2$. Moreover, C_{-1} and C_{-2} can be computed using the following two equations:

$$\tilde{E}_m = R_m(C_{-1}, C_{-2}) = 0 \tag{3.5}$$

and

$$\tilde{E}_{m+1} = y + R_{m+1}(C_{-1}, C_{-2}) = 0, \tag{3.6}$$

where $R_m(C_{-1}, C_{-2}), R_{m+1}(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ are homogeneous polynomials with respect to the weight considered before, i.e. $w(C_{-1}) = 2$, $w(C_{-2}) = 3$. Moreover $w(\tilde{E}_m) = m+n-2 = m+1$ and $w(\tilde{E}_{m+1}) = m+n-1 = m+2$. Then (3.5) and (3.6) read

$$\tilde{E}_m = \sum_{2i+3j=m+1} \lambda_m^{ij} C_{-1}^i C_{-2}^j \tag{3.7}$$

and

$$\tilde{E}_{m+1} = y + \sum_{2i+3j=m+2} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j, \tag{3.8}$$

for some constants $\lambda_m^{ij}, \lambda_{m+1}^{ij} \in K$. By (3.8) the two variables cannot be zero at the same time. We compute first the solutions in the cases where one of the variables is zero.

FIRST CASE: $C_{-1} = 0$ and $C_{-2} \neq 0$.

In this case the only term surviving in (3.7) is

$$0 = \tilde{E}_m = \lambda_m^{0j} C_{-2}^j,$$

with $3j = m + 1$. So necessarily

$$\lambda_m^{0,(m+1)/3} = 0 \quad \text{if } 3|m + 1. \quad (3.9)$$

Similarly, the only term surviving in the sum (3.8) has $i = 0$, and so we obtain

$$0 = \tilde{E}_{m+1} = y + \lambda_{m+1}^{0j} C_{-2}^j \quad \text{with } 3j = m + 2.$$

Since $y \neq 0$, necessarily $\lambda_{m+1}^{0j} \neq 0$ for $3j = m + 2$, and so $3|m + 2$, i.e. $m \equiv 1 \pmod{3}$. This shows that the condition (3.11) is trivially satisfied.

Lemma 3.1. *If $3|m + 2$, and $C_{-1} = 0$, then $\lambda_{m+1}^{0j} \neq 0$ for $3j = m + 2$.*

Proof. It is easy to check that $P = x^3 + 3C_{-2}$, and then, by Newtons binomial theorem we have

$$C^m = P^{m/3} = \sum_{k=0}^{\infty} \binom{m/3}{k} (3C_{-2})^k (x^3)^{\frac{m}{3}-k}. \quad (3.10)$$

Thus $\lambda_{m+1}^{0j} C_{-2}^j = (C^m)_{-2}$ is the coefficient of $x^{-2} = (x^3)^{\frac{m}{3}-j}$, since $m = 3j - 2$. Then

$$\lambda_{m+1}^{0j} = \binom{m/3}{j} 3^j \neq 0,$$

as desired. \square

Thus we have proved the following proposition.

Proposition 3.2. *If $(C_{-1}, C_{-2}, \dots, C_{-(m+1)})$ is a solution of the system (2.1), with $C_{-1} = 0$ and $C_{-2} \neq 0$, then*

- $m \equiv 1 \pmod{3}$,
- $\lambda_{m+1}^{0j} \neq 0$ for $j := \frac{m+2}{3}$,
- There are j solutions of the system (2.1) in $K[y^{1/j}]$, given by

$$C_{-1} = 0, \quad C_{-2} = \left(\frac{-y}{\lambda_{m+1}^{0j}} \right)^{\frac{1}{j}} \quad \text{and} \quad C_{-k} = -R_{k-2}(C_{-1}, C_{-2}) \quad \text{for } 3 \leq k \leq m+1.$$

SECOND CASE: $C_{-1} \neq 0$ and $C_{-2} = 0$.

In this case the only term surviving in (3.7) is

$$0 = \tilde{E}_m = \lambda_m^{i0} C_{-1}^i,$$

with $2i = m + 1$. So necessarily

$$\lambda_m^{(m+1)/2,0} = 0 \quad \text{if } 2|m + 1. \quad (3.11)$$

Similarly, the only term surviving in the sum (3.8) has $j = 0$, and so we obtain

$$0 = \tilde{E}_{m+1} = y + \lambda_{m+1}^{i0} C_{-1}^i \quad \text{with } 2i = m + 2.$$

Since $y \neq 0$, necessarily $\lambda_{m+1}^{i0} \neq 0$ for $2i = m + 2$, and so $2|m + 2$, i.e. m is even. This shows that the condition (3.11) is trivially satisfied.

Lemma 3.3. *If $2|m$ and $C_{-2} = 0$, then $\lambda_{m+1}^{i0} \neq 0$ for $2i = m + 2$.*

Proof. It is easy to check that $P = x^3 + 3xC_{-1}$, and then, by Newtons binomial theorem we have

$$C^m = P^{m/3} = \sum_{k=0}^{\infty} \binom{m/3}{k} (3xC_{-1})^k (x^3)^{\frac{m}{3}-k}. \quad (3.12)$$

Thus $\lambda_{m+1}^{i0} C_{-1}^i = (C^m)_{-2}$ is the coefficient of $x^{-2} = (x)^i (x^3)^{\frac{m}{3}-i}$, since $m = 2i - 2$. Then

$$\lambda_{m+1}^{i0} = \binom{m/3}{i} 3^i \neq 0,$$

as desired. \square

Thus we have proved the following proposition.

Proposition 3.4. *If $(C_{-1}, C_{-2}, \dots, C_{-(m+1)})$ is a solution of the system (2.1), with $C_{-1} \neq 0$ and $C_{-2} = 0$, then*

- $m \equiv 1 \pmod{3}$,
- $\lambda_{m+1}^{i0} \neq 0$ for $i := \frac{m+2}{2}$,
- There are i solutions of the system (2.1) in $K[y^{1/i}]$, given by

$$C_{-1} = \left(\frac{-y}{\lambda_{m+1}^{i0}} \right)^{\frac{1}{i}}, \quad C_{-2} = 0 \quad \text{and} \quad C_{-k} = -R_{k-2}(C_{-1}, C_{-2}) \quad \text{for } 3 \leq k \leq m+1.$$

THIRD CASE: $C_{-1} \neq 0$, $C_{-2} \neq 0$ and m even.

In this case we introduce a new auxiliary variable t satisfying $C_{-2}^2 = tC_{-1}^3$. The equality (3.7) now reads

$$\tilde{E}_m = \sum_{2i+3j=m+1} \lambda_m^{ij} C_{-1}^i C_{-2}^j = \sum_{2i+6r+3=m+1} \lambda_m^{i,2r+1} C_{-1}^i C_{-2}^{2r+1} = \sum_{2i+6r+2=m} \lambda_m^{i,2r+1} C_{-1}^{i+3r} C_{-2}^{2r},$$

since m even implies that the weight $2i + 3j = m + 1$ is odd, so j is odd and can be written as $2r + 1$. Moreover, for the terms in the sum we have $i + 3r = \frac{m-2}{2}$, and so we arrive at

$$0 = C_{-1}^{\frac{m-2}{2}} C_{-2} \sum_{\substack{2i+6r=m-2 \\ j=2r+1}} \lambda_m^{ij} t^r.$$

Thus t is a root of the polynomial

$$f(t) = \sum_{r=0}^{\lfloor \frac{m-2}{6} \rfloor} a_r t^r, \quad \text{where } a_r = \lambda_m^{\frac{m-2-6r}{2}, 2r+1}. \quad (3.13)$$

Let $\{t_1, \dots, t_s\}$ be the roots of the polynomial $f(t)$. Note that in the equality (3.8) the power j has to be even, since m is even and $2i + 3j = m + 2$. Hence, if we replace C_{-2}^2 by $t_l C_{-1}^3$ in (3.8), we obtain

$$\tilde{E}_{m+1} = y + \sum_{\substack{2i+3j=m+2 \\ j=2r}} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j = y + \sum_{2i+6r=m+2} \lambda_{m+1}^{i,2r} C_{-1}^{i+3r} t_l^r.$$

Note that for each of the terms in the last sum we have $i + 3r = \frac{m+2}{2}$, and so

$$0 = y + C_{-1}^{\frac{m+2}{2}} g(t_l), \quad \text{where } g(t) = \sum_{r=0}^{\lfloor \frac{m+2}{6} \rfloor} b_r t^r,$$

with $b_r = \lambda_{m+1}^{\frac{m+2-6r}{2}, 2r}$. It follows that

$$C_{-1} = \left(\frac{-y}{g(t_l)} \right)^{\frac{2}{m+2}}.$$

Thus we have arrived at the following result.

Proposition 3.5. *If $(C_{-1}, C_{-2}, \dots, C_{-(m+1)})$ is a solution of the system (2.1), with $C_{-1} \neq 0$, $C_{-2} \neq 0$ and m even, then the system has at most $s \cdot (m+2)$ solutions, where s is the number of roots of $f(t)$ defined in (3.13). Moreover, for every choice of a root t_l of f , the solutions are given by*

$$\begin{aligned} C_{-1} &= \left(\frac{-y}{g(t_l)} \right)^{\frac{2}{m+2}}, & \frac{m+2}{2} &\text{ choices,} \\ C_{-2} &= (t_l C_{-1}^3)^{\frac{1}{2}}, & 2 &\text{ choices,} \\ C_{-k} &= -R_{k-2}(C_{-1}, C_{-2}) \quad \text{for } 3 \leq k \leq m+1. \end{aligned}$$

FOURTH CASE: $C_{-1} \neq 0$, $C_{-2} \neq 0$ and m odd.

In this case we introduce a new auxiliary variable t satisfying $C_{-2}^2 = t C_{-1}^3$. The equality (3.7) now reads

$$\tilde{E}_m = \sum_{2i+3j=m+1} \lambda_m^{ij} C_{-1}^i C_{-2}^j = \sum_{2i+6r=m+1} \lambda_m^{i,2r} C_{-1}^i C_{-2}^{2r} = \sum_{2i+6r=m+1} \lambda_m^{i,2r} C_{-1}^{i+3r} t^r,$$

since m odd implies that the weight $2i+3j = m+1$ is even, so j is even and can be written as $2r$. Moreover, for the terms in the sum we have $i+3r = \frac{m+1}{2}$, and so we arrive at

$$0 = C_{-1}^{\frac{m+1}{2}} \sum_{\substack{2i+6r=m+1 \\ j=2r}} \lambda_m^{ij} t^r.$$

Thus t is a root of the polynomial

$$f(t) = \sum_{r=0}^{\lfloor \frac{m+1}{6} \rfloor} a_r t^r, \quad \text{where } a_r = \lambda_m^{\frac{m+1-6r}{2}, 2r}. \quad (3.14)$$

Let $\{t_1, \dots, t_s\}$ be the roots of the polynomial $f(t)$. Note that in the equality (3.8) the power j has to be odd, since m is odd and $2i+3j = m+2$. Hence, if we replace C_{-2}^2 by $t_l C_{-1}^3$ in (3.8), we obtain

$$\tilde{E}_{m+1} = y + \sum_{\substack{2i+3j=m+2 \\ j=2r+1}} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j = y + \sum_{2i+6r+3=m+2} \lambda_{m+1}^{i,2r+1} C_{-1}^{i+3r} C_{-2} t_l^r.$$

Note that for each of the terms in the last sum we have $i+3r = \frac{m-1}{2}$, and so

$$0 = y + C_{-1}^{\frac{m-1}{2}} C_{-2} g(t_l), \quad \text{where } g(t) = \sum_{r=0}^{\lfloor \frac{m-1}{6} \rfloor} b_r t^r,$$

with $b_r = \lambda_{m+1}^{\frac{m-1-6r}{2}, 2r+1}$. We also replace C_{-2} by $(t_l C_{-1}^3)^{\frac{1}{2}}$. It follows that

$$0 = y + C_{-1}^{\frac{m+2}{2}} (t_l)^{\frac{1}{2}} g(t_l),$$

and so

$$C_{-1} = \left(\frac{-y}{(t_l)^{\frac{1}{2}} g(t_l)} \right)^{\frac{2}{m+2}}.$$

Thus we have arrived at the following result.

Proposition 3.6. *If $(C_{-1}, C_{-2}, \dots, C_{-(m+1)})$ is a solution of the system (2.1), with $C_{-1} \neq 0$, $C_{-2} \neq 0$ and m odd, then the system has at most $2 \cdot s \cdot (m+2)$ solutions, where s is the number*

of roots of $f(t)$ defined in (3.14). Moreover, for every choice of a root t_l of f , we first choose a square root of t_l and then the solutions are given by

$$\begin{aligned} C_{-1} &= \left(\frac{-y}{(t_l)^{\frac{1}{2}} g(t_l)} \right)^{\frac{2}{m+2}}, & \frac{m+2}{2} & \text{ choices,} \\ C_{-2} &= (t_l C_{-1}^3)^{\frac{1}{2}}, & 2 & \text{ choices,} \\ C_{-k} &= -R_{k-2}(C_{-1}, C_{-2}) \quad \text{for } 3 \leq k \leq m+1. \end{aligned}$$

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