2. Two Simple Models

We consider a simple prediction market in which traders buy and sell an all-or-nothing contract (a binary option) paying $1 if a specific event occurs, and nothing otherwise. There is heterogeneity in beliefs among the trading population, and following Manski's notation, we denote trader j's belief that the event will occur as q_j. These beliefs are orthogonal to wealth levels (y), and are drawn from a distribution, $F(q)$. Individuals are price-takers and trade so as to maximize their subjectively expected utility. Wealth is only affected by the event via the prediction market, so there is no hedging motive for trading the contract.

We first consider the case where traders have log utility, and we endogenously derive their trading activity, given the price of the contract is π. Thus, in deciding how many contracts, x, to buy, traders solve the following problem:

$$\begin{aligned} &\underset{\{x\}}{Max}\ EU_j = q_j Log[y + x_j(1-\pi)] + (1-q_j) Log[y - x_j \pi] \\ &yielding: \quad x_j^* = y\frac{q_j - \pi}{\pi(1-\pi)} \end{aligned} \tag{1}$$

Thus, individual demand is:

- *Zero when prices equal beliefs*: When traders believe that the event is more likely than the price ($q>\pi$) they have positive demand, and they are net suppliers if $q<\pi$.
- *Linear in beliefs*: For traders with a given wealth level (y), demand increases linearly with their beliefs.
- *Decreasing in risk*: Greater risk (π close to ½) yields smaller demand.
- *Homothetic*: Demand for these contracts rises proportionately with initial wealth, y.
- *Unique*: Only for prices between 0 or 1. (We will confine our attention to interior solutions.)

The prediction market is in equilibrium when supply equals demand:

$$\int_{-\infty}^{\pi} y\frac{q-\pi}{\pi(1-\pi)} f(q)dq = \int_{\pi}^{\infty} y\frac{\pi-q}{\pi(1-\pi)} f(q)dq \tag{2}$$

If beliefs (q) and wealth (y) are independent, then this implies:

$$\frac{y}{\pi(1-\pi)}\int_{-\infty}^{\pi}(q-\pi)f(q)dq=\frac{y}{\pi(1-\pi)}\int_{\pi}^{\infty}(\pi-q)f(q)dq \tag{3}$$

and hence: $\pi=\int_{-\infty}^{\infty}qf(q)dq=\overline{q}$ (4)

Thus, in this simple model, market prices are equal to the mean belief among traders. The source of the heterogeneity in beliefs merits some attention. An attractive interpretation may be that individual subjective beliefs reflect private, but noisy signals of the likelihood that the event will occur. If the noise term is normally distributed, then the mean of these private signals is an efficient estimate of the true likelihood of the event occurring, and hence these models yield conditions under which the prediction market price is a sufficient statistic for this private information.

Our second "model" is even simpler, and we sketch it not for its formal elegance, but because it may be descriptively accurate for the types of low-stakes entertainment-motivated trading in certain prediction markets. Suppose beliefs about probability are common and all trade is noise trading. That is, we consider traders as motivated simply by a desire for "action". Given this motivation (and a desire to minimize the costs of "action"), demand and supply are flat at a price corresponding to the common prior. Alternatively if external reasons lead more traders to be attracted to one side of the bet rather than the other (eg betting the Red Sox), then we also require a sufficient supply of either cost-minimizing action-motivated noise traders, or profit-motivated arbitrageurs to offset any such bias.

3. Generalizing the Model

We now turn to relaxing some of our assumptions. To preview, relaxing the assumption that budgets are orthogonal to beliefs yields the intuitively plausible result that prediction market prices are a wealth-weighted average of beliefs among market traders. And second, the result that the equilibrium price is exactly equal to the (weighted) mean of beliefs reflects the fact that demand for the prediction security is linear in beliefs, which is itself a byproduct of assuming log utility. Calibrating alternative utility functions, we find that prices can systematically diverge from mean beliefs, but that this divergence is typically small. (The advantage of our very simple model is that it is easily amenable to calibration exercises.)

What if Beliefs and Wealth are Correlated?

Consider traders drawn from a distribution *F(q,y)*, where *E[q,y]≠0*. Thus, equation (*1*) continues to describe the demand of individual traders, given their beliefs and wealth levels. However, our equilibrium condition now changes, as traders with specific beliefs will have greater weight in the market. As before, equilibrium requires that supply equals demand:

$$\int y\frac{q-\pi}{\pi(1-\pi)}dF(q\leq\pi,y)=\int y\frac{\pi-q}{\pi(1-\pi)}dF(q\geq\pi,y) \text{ and hence: } \pi=\int q\frac{y}{\overline{y}}F(q,y) \quad (5)$$

Thus, as before, the market equilibrium price is an average of beliefs in the trading population, although each belief is weighted according to the average wealth of traders holding that belief, relative to the average wealth level ($\overline{y}$).

If long-run market forces lead those with a history of accurate evaluation to become wealthier, then this wealth-weighted average may be a more accurate predictor than an unweighted average. Furthermore, if we are interested in using the prediction market price as a proxy for the beliefs of the marginal investor in other asset markets and if the wealthy trade more in these other markets, then the wealth-weighted average may again be closer to the object of interest.

Alternative Utility Functions

The key to the analytic simplicity of the above results is that individual demands in equation (1) are linear functions of each agent's beliefs. This, in turn, reflects the convenient assumption of log utility. As we relax this assumption, simple analytic results will be less easy to obtain.

Before analyzing the general case, it is worth analyzing an interesting special case, involving demand functions and distributions of beliefs that are symmetric. If:

i) Individual demand for the prediction security is a function of the difference between beliefs and market prices and symmetric around zero:
$x(q-\pi)=-x(\pi-q))$.