From now on, to simplify the notation, when no confusion should arise, we will write P for $P^{\mathfrak{C}}$. Also, for a set A, we will often denote by A both the set and the substructure of $\mathfrak{C}$ with universe A. So for example, when we write that $A \cap P^{\mathfrak{C}}$ is λ-saturated, or just that $A \cap P$ is λ-saturated, we mean that the substructure $\mathfrak{C}|_{A \cap P^{\mathfrak{C}}}$ is a λ-saturated model of the appropriate theory.

4 Completeness and relevant types

In trying to reconstruct M from $M|P^M$ one needs to work with sets A satisfying $P^M \subseteq A \subseteq M$. Such A have the following property (which under a certain assumption on saturation of P^M characterizes such sets; see Proposition 4.13 below), that can be viewed as an analogue to Tarski-Vaught Criterion for being an elementary submodel:

Definition 4.1. $A \subseteq \mathfrak{C}$ is *complete* if for every formula $\psi(\bar{x}, \bar{y})$ and $\bar{b} \subseteq A, \models (\exists \bar{x} \in P)\psi(\bar{x}, \bar{b})$ implies $(\exists \bar{a} \subseteq P \cap A) \models \psi(\bar{a}, \bar{b})$.

The following useful characterization offers a different understanding of the notion of completeness:

Observation 4.2. *A set A is complete if and only if for every $\bar{a} \subseteq A$ and $\phi(\bar{x}, \bar{y})$ the ϕ-type $tp_\phi(\bar{a}/P^{\mathfrak{C}})$ is definable over $A \cap P^{\mathfrak{C}}$ and $A \cap P^{\mathfrak{C}} \prec P^{\mathfrak{C}}$.*

Proof: Assume that A is complete. First we use the Tarski-Vaught criterion in order to show that $A \cap P^{\mathfrak{C}} \prec P^{\mathfrak{C}}$. Indeed, if $P^{\mathfrak{C}} \models \exists x \theta(x, \bar{b})$ with $\bar{b} \in A$, then (recall Remark 2.2) $\mathfrak{C} \models \exists x P(x) \wedge \hat{\theta}(x, \bar{b})$. By completeness of A, there exists $c \in A$ such that $\mathfrak{C} \models P(c) \wedge \hat{\theta}(c, \bar{b})$, hence $P^{\mathfrak{C}} \models \theta(c, \bar{b})$, as required.

Now let $\bar{a} \subseteq A$. We know that $tp_\phi(\bar{a}/P^{\mathfrak{C}})$ is definable over $P^{\mathfrak{C}}$; so suppose $\mathfrak{C} \models \forall \bar{y} \theta(\bar{y}, \bar{c}) \iff \phi(\bar{a}, \bar{y})$ for some $\bar{c} \in P^{\mathfrak{C}}$. By completeness, such a $\bar{c}$ exists already in $A \cap P^{\mathfrak{C}}$.

For the other direction, assume that $\mathfrak{C} \models (\exists \bar{x} \in P)\phi(\bar{x}, \bar{a})$ with $\bar{a} \in A$. Since $tp_\phi(\bar{a}/P^{\mathfrak{C}})$ is definable over $A \cap P^{\mathfrak{C}}$, the formula $\phi(\bar{x}, \bar{a})$ is equivalent to a formula $\theta(\bar{x}, \bar{d})$ for some $\bar{d} \in A \cap P^{\mathfrak{C}}$. So $\mathfrak{C} \models (\exists \bar{x} \in P)\theta(\bar{x}, \bar{a})$, hence $P^{\mathfrak{C}} \models \hat{\theta}(\bar{c}, \bar{d})$ for some $\bar{c}$. Since $A \cap P^{\mathfrak{C}} \prec P^{\mathfrak{C}}$, such $\bar{c}$ exists also in $A \cap P^{\mathfrak{C}}$; and we have $\models \theta(\bar{c}, \bar{d})$, as required.

■

Fact 4.3. For any complete A there are $\langle \Psi_\psi : \psi(\bar{x}, \bar{y}) \in L(T) \rangle$ (depending on A) such that for all $\bar{a} \subseteq A$, $tp_\psi(\bar{a}/P \cap A)$ is definable by $\Psi_\psi(\bar{y}, \bar{c})$ for some $\bar{c} \subseteq A \cap P$.

Proof: By compactness, go to a $|T|^+$-saturated model. So, for each ψ we have but finitely many candidates $\Psi^1_\psi, \dots, \Psi^n_\psi$ (or else, by compactness, there is an undefinable type). As without loss of generality $|P^{\mathfrak{C}}| \geq 2$, we can manipulate these as in [She90] II§2 to an Ψ_ψ. ■

The following properties of complete sets are clear:

Fact 4.4. (i) If $M \prec \mathfrak{C}$ and $P^M \subseteq A \subseteq M$, then A is complete.

(ii) If $\langle B_i : i < \delta \rangle$ is an increasing sequence of complete sets, then $\bigcup_{i<\delta} B_i$ is complete.

Furthermore, if T has QE, then the property of completeness for a set A depends only on its first order theory (as a substructure of $\mathfrak{C}$):

Lemma 4.5. *(T has QE)*

(i) If $A_1 \equiv A_2$, then A_1 is complete iff A_2 is complete.

(ii) A is complete iff whenever the sentence

$$\theta =: (\forall \bar{y})[S(\bar{y}) \leftrightarrow (\exists x \in P) R(x, \bar{y})]$$

for quantifier free R, S is satisfied in $\mathfrak{C}$, then A satisfies θ.

Furthermore, if T has QE down to the level of predicates (e.g., T has been Morleyized), then it is enough to consider all the formulas above with R, S predicates.

Proof:

(i) Follows easily from (ii), but we also give a direct proof. Assume that $A_1 \equiv A_2$ and A_1 is complete. Let us prove this for A_2.

Let $\varphi(\bar{x}, \bar{y})$ be a formula. Since T has QE, there are quantifier free formulae $\theta(\bar{y})$ and $\theta'(\bar{x}, \bar{y})$ such that:

(1)
$$\mathfrak{C} \models \forall \bar{y} \left[(\exists \bar{x} \in P \varphi(\bar{x}, \bar{y})) \longleftrightarrow \theta(\bar{y}) \right]$$

(2)
$$\mathfrak{C} \models \forall \bar{x} \bar{y} \left[\varphi(\bar{x}, \bar{y}) \leftrightarrow \theta'(\bar{x}, \bar{y}) \right]$$

Since A_1 is complete (combining the definition with (1) and (2) above), we have that whenever $\mathfrak{C} \models \theta(\bar{a}_1)$ for $\bar{a}_1 \in A_1$, there is some $\bar{c} \in A_1 \cap P$ such that $\mathfrak{C} \models \theta'(\bar{c}, \bar{a}_1)$. But since these formulae are quantifier free, clearly A_1 (as a substructure) satisfies

$$A_1 \models \forall \bar{y} \theta(\bar{y}) \longrightarrow \exists \bar{x} \in P \theta'(\bar{x}, \bar{y})$$

Since $A_1 \equiv A_2$, so does A_2, which implies completeness.

(ii) Similar [and easier]: If A is complete and $\mathfrak{C} \models \theta$, assume that $A \models S(\bar{b})$ for some $\bar{b}$; then (since S is quantifier free), so does $\mathfrak{C}$, hence $\mathfrak{C} \models (\exists \bar{x} \in P) R(\bar{x}, \bar{b})$, and by completeness of A, there is such $\bar{x}$ already in $P \cap A$. So $A \models \theta$. Conversely, assume that whenever $\mathfrak{C} \models \theta$, so does A. Let $\varphi(\bar{x}, \bar{y})$ be a formula, and assume that $\mathfrak{C} \models (\exists \bar{x} \in P) \varphi(\bar{x}, \bar{b})$.

As in the proof of (iv), let $R(\bar{x}, \bar{y})$ and $S(\bar{y})$ be quantifier free such that

(1)
$$\mathfrak{C} \models \forall \bar{y} \left[(\exists \bar{x} \in P \varphi(\bar{x}, \bar{y})) \longleftrightarrow S(\bar{y}) \right]$$

(2)

$$\mathfrak{C} \models \forall \bar{x}\bar{y}\,[\varphi(\bar{x},\bar{y}) \leftrightarrow R(\bar{x},\bar{y})]$$

Clearly, $\mathfrak{C} \models \theta$ (with these specific R and S), hence so that A. But $\mathfrak{C} \models S(\bar{b})$, hence so does A; so $A \models (\exists \bar{x} \in P)R(\bar{x},\bar{b})$, hence $A \models R(\bar{a},\bar{b})$ for some $a \in A \cap P$, hence so does $\mathfrak{C}$, and we are clearly done.

The "futhermore" part is trivial.

■

We make a few remarks on the relation between P and the algebraic closure in complete sets.

Observation 4.6. *Let A be a complete set. Then:*

(i) $A \cap \mathrm{acl}(P^{\mathfrak{C}}) \subseteq \mathrm{acl}(P^A)$

(ii) $\mathrm{acl}(A) \cap P^{\mathfrak{C}} = P^A$

Proof:

(i) Let $d \in A \cap \mathrm{acl}(P^{\mathfrak{C}})$ (in this proof we will distinguish between tuples and singletons). Then there is a formula $\varphi(x,\bar{a})$ with $\bar{a} \in P^{\mathfrak{C}}$ which has $k < \omega$ solutions in $\mathfrak{C}$ such that $\models \varphi(d,\bar{a})$. Since A is complete (see Observation 4.2), the type $\mathrm{tp}(d/P^{\mathfrak{C}})$ is definable over P^A, so the set

$$\{\bar{y} \in P \colon \varphi(d,\bar{y}) \wedge \exists^k x \varphi(x,\bar{y})\}$$

is definable in $\mathfrak{C}|_P$ by a formula $\theta(\bar{y})$ over P^A. Again, since A is complete, $P^A \prec P^{\mathfrak{C}}$, so $\theta(\bar{y})$ has a solution in P^A, which finishes the proof.

(ii) Let $d \in \mathrm{acl}(A) \cap P$. So there is $\varphi(x,\bar{a})$ with $\bar{a} \in A$ with $k < \omega$ solutions in $\mathfrak{C}$ such that $\varphi(d,\bar{a})$ holds. Just like in the proof of clause (i), the set $D = \{x \in P \colon \models \varphi(x,\bar{a})\}$ is definable in $\mathfrak{C}|_P$ over P^A and is non-empty. Clearly, D is finite. Since $P^A \prec P^{\mathfrak{C}}$, $D \subseteq P^A$, as required.

■

The characterization of completeness (Observation 4.2), in combination with Corollaries 3.2 and 3.4, yields the following important property of complete sets:

Corollary 4.7. *Let A be complete and let p be a (partial) type over A with "$\bar{x} \subseteq P$" $\in p$. Then p is equivalent both to a T-type and a T^P-type over P^A. If, in addition, P^A is λ-compact, and $|p| < \lambda$, then p is realized in P^A.*

We are now ready to define one of the most basic objects of our study: the relevant notion of type for this context.

Definition 4.8. (i) For a complete set A, let

$$S_*(A) = \{tp(\bar{c}/A) : P \cap (A \cup \bar{c}) = P \cap A \text{ and } A \cup \bar{c} \text{ is complete}\}$$