universe, or under mild set-theoretic assumptions. Non-structure theorems become quite hard in this context, so at the moment we are happy with just consistency results. Since we are ultimately interested in absolute properties (e.g., stability, categoricity), there does not seem to be much loss in this approach (although non-structure results in ZFC are definitely on our to-do list). Then we restrict our attention to theories on the "good side" of all the dividing lines, and focus on proving structure results for this case. There is, of course, also a need for better and stronger non-structure theorems, that would "justify" restricting onelsef to even nicer contexts; this is a topic for future work.

This paper is organized as follows.

In Section 2 we recall non-structure results from [PS85] and make our most basic working assumptions, e..g., every type over P is definable.

Section 3 is devoted to some of the less obvious consequences of the assumptions mentioned above (already discussed to a large extent in [PS85]). In particular, we explain why we may assume that T has quantifier elimination.

In Section 4 we revisit some basic stability theory over P originally developed in [PS85] and [She86] and take it further. We introduce some of the major players necessary for analyzing models over their P-part: complete sets and "good" types, which we call *-types; these are types "orthogonal to P", that is, types, realizing which do not increase the P-part. Based on these notions, we define the relevant notion of stability: a set A is called stable (over P) if there are "few" *-types over all sets elementarily equivalent to A. This concept and many of its basic properties already appear in the previous works mentioned above. However, it is our goal to make this paper reasonably self-contained, so we include all the definitions as well as details of proofs.

Section 5 revisits a notion of rank (already discussed in [PS85, She86]) for *-types that captures stability over P.

Section 6 is devoted to the existence of "small" (atomic, primary) saturated models over stable sets with a saturated P-part.

In Section 7 we make an additional working assumption: every model of T is stable over P (this hypothesis is justified by a non-structure result proved in [She86]). We then use stability of models together with the existence of primary models (the results of section 6) in order to obtain quantifier-free internal definitions for *-types. This is, in a sense, the main technical result of this paper, on which the rest of our analysis is built. In particular, we immediately conclude that *-types over stable sets are definable (in $\mathfrak{C}$).

In section 8 we define stationarization – a version of non-forking independence appropriate for our context – and examine its basic properties.

Finally, in Section 9 we prove the main structure results of this article. In particular, we establish stable amalgamation for models followed by a similar result for types over stable sets, show symmetry of independence over models, and draw several conclusions.

One basic corollary of our analysis is that if every model is stable over P (as mentioned earlier, a non-structure result from [She86] implies that this is a natural assumption in our context; in addition, it holds in many interesting examples), then every $*$-type over a model is generically stable. In particular, we conclude that over models, independence derived from stationarization coincides with non-forking.

Acknowledgements

The authors are grateful to Mariana Vicaria for a careful reading, and for numerous comments, questions, suggestions, and corrections. We also thank Zoé Chatzidakis for helpful comments, clarifications, and instructive questions, which led to improving the readability and clarity of this paper.

The second author thanks Jakob Kellner for several conversations, motivating the authors to expand the background and the introductory sections, hopefully making them more clear and accessible to non-experts, and to eventually finalize the paper. Usvyatsov is also thankful to Alf Onshuus for inspiring conversations at early stages of the writing of this paper, and to Jonathan Kirby for bringing the paper [KZ14] to the authors' attention.

2 The Gross Non-structure Cases

Convention 2.1. *Let T be a complete first order theory, P a monadic predicate in its vocabulary.*

Let $\mathfrak{C}$ be the monster model of T. From now on, we assume that all models of T are elementary submodels of $\mathfrak{C}$, and all sets are subsets of $\mathfrak{C}$.

For $M \models T$, we denote by $M|_P$ the set P^M viewed as a substructure of M. Similarly, for a subset $A \subseteq M$, we denote by $M|_A$ the substructure of M with universe A. We write $A \equiv B$ if $Th(\mathfrak{C}|_A) = Th(\mathfrak{C}|_B)$.

We also denote $T^P = Th(\mathfrak{C}|_P)$. For a set A, we denote $P^A = A \cap P^{\mathfrak{C}}$.

Our first dividing line concerns the connection between subsets of $P^{\mathfrak{C}}$ that are 0-definable externally (in $\mathfrak{C}$) and internally (in $\mathfrak{C}|_P$, that is, in $P^{\mathfrak{C}}$ viewed as a substructure of $\mathfrak{C}$). One direction of this correspondence is straightforward:

Remark 2.2. *For every formula $\theta(\bar{x})$ there exists a formula $\theta^P(\bar{x})$ such that for every $\bar{b} \in P$ we have $\mathfrak{C}|_P \models \theta(\bar{b})$ if and only $\mathfrak{C} \models \theta^P(\bar{b})$. Moreover, for every $M \models T$ we have $M|_P \models \theta(\bar{b})$ if and only $M \models \theta^P(\bar{b})$*

Proof: By induction on θ, by replacing quantifiers in θ by quantifiers restricted to P (e.g., $\exists y \in P$). ∎

Corollary 2.3. *Let A be a set.*

(i) *If A is a model (so $A \prec \mathfrak{C}$), then $(A|_P =)\mathfrak{C}|_{PA} \prec \mathfrak{C}|_P$.*

(ii) *Every (partial) T^P-type $q(\bar{x})$ over P^A is equivalent to a T-type $q^P(\bar{x})$ (over P^A) with $[\bar{x} \subseteq P] \in q^P$.*

In particular, if A is a λ-saturated model, then $A|_P = \mathfrak{C}|_{PA}$ is a λ-saturated model of T^P.

Proof:

(i) Assume $\mathcal{C}|_P \models \theta(\bar{b})$ with $\bar{b} \in P^A$. Then (see Remark 2.2) $\mathcal{C} \models \theta^P(\bar{b})$, hence $A \models \theta^P(\bar{b})$. Again, by Remark 2.2, $A|_P \models \theta(\bar{b})$.

(ii) Let $\theta(\bar{x}, \bar{b})$ be a formula such that $A|_P \models \exists \bar{x} \theta(\bar{x}, \bar{b})$. Then

$$A \models \exists \bar{x} \in P \; \theta^P(\bar{x}, \bar{b})$$

Hence given a T^P-type $q(\bar{x})$ over P^A, the collection $q^P = \{\theta^P(\bar{x}) \colon \theta(\bar{x}) \in q\}$ is indeed a T-type, and the rest should be clear.

■

On the other hand, it is not clear (and, in general, not true) that any externally definable subset of $P^{\mathcal{C}}$ is definable internally in $\mathcal{C}|_P$.

Question 2.4. *If $M \models T$, and $\psi(\bar{x})$ is a relation on P^M which is first order definable in M, is ψ definable in $M|_P$, possibly with parameters?*

If the answer to this question is "no", then, as shown in [PS85], for every $\lambda \geq |T|$, $I(\lambda) \geq Ded(\lambda)$, where $Ded(\lambda) = \sup\{|I| : I$ is a linear order with dense subset of power $\lambda\}$. The proof relies on earlier papers by Chang, Makkai, Reyes and Shelah.

Following the "general recipe of Classification Theory" outlined in the introduction, we therefore assume that the answer to the Question 2.4 is "yes".

Furthermore we expand T by the necessary individual constants (maybe working in $\mathcal{C}^{eq}$) in order to assume that all such relations are parameter-free definable in $\mathcal{C}|_P$. Note that we only have to add elements in $dcl(P^{\mathcal{C}})$ inside $\mathcal{C}^{eq}$.

Specifically, if $\mathcal{C} \models (\exists \bar{y} \in P)(\forall \bar{x} \in P)(\psi(\bar{x}) = \theta^P(\bar{x}, \bar{y}))$, we define an equivalence relation on tuples of sort $\bar{y}$ as follows:

$$\bar{y}_1 \equiv \bar{y}_2 \iff P(\bar{y}_1) \wedge P(\bar{y}_2) \wedge \forall \bar{x} \left[\theta^P(\bar{x}, y_1) \leftrightarrow \theta^P(\bar{x}, y_2)\right]$$

and expand $\mathcal{C}^{eq}$ with names for the relevant equivalence classes; that is, for each $\psi(\bar{x})$ as above, expand the language by a constant $[\bar{y}]_\psi$ for the class of a tuple $\bar{b}$ for which $\theta^P(\bar{x}, \bar{b})$ is equivalent to $\psi(\bar{x})$. Clearly, this does not increase the cardinality of the language.

As a result, given a formula $\psi(\bar{x})$ defining a relation on P and a formula $\theta(\bar{x}, \bar{b})$ defining this relation in $\mathcal{C}|_P$, we have that e.g. the formula $\exists \bar{y}\, ([\bar{y}] = [\bar{y}]_\psi) \wedge \theta(\bar{x}, \bar{y})$ defines the same relation without parameters. Note that we add the new constants to P and to the language of T^P.

Remark 2.5. See [PS85] for the number of such expansions for models of T. If the new theory is T^+, note that $I_T(\lambda, N) = \sup\{I_{T^+}(\lambda, N^+) : N^+$ an expansion of N as described above$\}$. In particular $I_T(\lambda, \mu)^{|T|} \geq I_T(\lambda, \mu) \geq I_{T^+}(\lambda, \mu)$, so our non-structure results are not affected.