(ii) A is *stable over* P, or simply *stable*, if (it is complete and) for all A' with $A' \equiv A$, $|S_*(A')| \leq |A'|^{|T|}$.

Remark 4.9. (i) *Even though "stability over P" is a more appropriate and accurate name for our notion of stability of a set (and the term "stable set" exists in literature, and has a different meaning), since we have only one notion of stability in this article (stability over P), we will mostly omit "over P" and simply write "stable".*

(ii) *Sometimes we refer to types in $S_*(A)$ as* complete types over A which are orthogonal to P.

Remark 4.10. *Note that by Observation 4.6, if A is complete and $\bar{c}$ a tuple such that $tp(\bar{c}/A) \in S_*(A)$, then* $\mathrm{acl}(A \cup \bar{c}) \cap P^{\mathfrak{C}} = P^A$.

The last remark also follows from the following more general criterion for a complete type being a *-type:

Observation 4.11. *Let A be a complete set and $\bar{c}$ a tuple. Then $tp(\bar{c}/A) \in S_*(A)$ if and only if for every formula $\psi(\bar{x}, \bar{a}, \bar{y})$ over A we have*

$$\models \exists \bar{x} \in P\, \psi(\bar{x}, \bar{a}, \bar{c}) \Longrightarrow \exists \bar{b} \in P \cap A \textit{ such that } \models \psi(\bar{b}, \bar{a}, \bar{c})$$

Proof: Let $tp(\bar{c}/A) \in S_*(A)$, and let $\psi(\bar{x}, \bar{a}, \bar{y})$ be a formula as above satisfying $\models \exists \bar{x} \in P\, \psi(\bar{x}, \bar{a}, \bar{c})$. Since the set $A \cup \bar{c}$ is complete, the type $\mathrm{tp}(\bar{a}\bar{c}/P^{\mathfrak{C}})$ is definable over $P \cap (A \cup \bar{c}) = P \cap A$. Hence the set $D = \{\bar{x} \in P\colon \models \psi(\bar{x}, \bar{a}, \bar{c})\}$ is definable in $\mathfrak{C}|_P$ over P^A. This set is not empty by the assumption. Since A is complete, $P^A \prec P^{\mathfrak{C}}$, hence the set $P^A \cap D$ is nonempty as well, as required for the "only if" direction.

Now assume that the right hand of the equivalence holds. First we note that $P \cap (A \cup \bar{c}) = P \cup A$. Indeed, if $d \in P \cup (A \cup \bar{c})$, then the formula $x = d$ has a solution in P, hence in P^A, so $d \in P^A$. Now obviously (since A is complete) $P^{A \cup \bar{c}} \prec P^{\mathfrak{C}}$. By Observation 4.2, $A \cup \bar{c}$ is complete. ∎

The following lemma shows that, although the definition of types orthogonal to P may seem quite strong, such types are not very hard to come by.

Lemma 4.12. *(The Small Type Extension Lemma) If $A \prec \mathfrak{C}$ is saturated (or just $A \cap P$ is $|A|$-compact) and $p(\bar{x})$ is an $L(T)$-type over A of cardinality $< |A|$, then there is some $p^*(\bar{x}) \in S_*(A)$ extending p.*

Proof: To prove this, we have to show that p is realized by some $\bar{c} \in \mathfrak{C}$ such that $P^{\mathfrak{C}} \cap (A \cup \bar{c}) = P^{\mathfrak{C}} \cap A$ and $A \cup \bar{c}$ is complete. So we have to extend p in such a way that whatever part of p that can be realized inside P, has to be realized in $P \cap A$.

Let $\langle \psi_i(\bar{z}, \bar{b}, \bar{x})\colon i < |A| \rangle$ list all the formulas over A. Now define inductively for $i < |A|$ and $\psi_i(\bar{z}, \bar{b}, \bar{x})$ (so $\bar{b} \in A$), a consistent T-type $p_i(\bar{x}), |p_i| < |p|^+ + |i|^+ + \aleph_0$ and p_i increasing continuously, making sure that the requirements are met.

Let $p_0 = p$. If $p_i(\bar{x}) \cup \{(\not\exists \bar{z} \in P)\psi(\bar{z}, \bar{b}, \bar{x})\}$ is consistent, let it be p_{i+1}. If not, consider the type $q(\bar{z}) = \{\exists \bar{x}\phi(\bar{x}) \wedge \psi(\bar{z}, \bar{b}, \bar{x})\colon \phi(\bar{x}) \in p_i\} \cup \{\bar{z} \in P\}$. By compactness and saturation of A there is some $\bar{c} \subseteq P \cap A$ such that $p_{i+1} := p_i(\bar{x}) \cup \{\psi(\bar{c}, \bar{b}, \bar{x})\}$ is consistent.

In fact, since A is complete, by Corollary 4.7, q is equivalent to a T^P-type over $P \cap A$. Hence (since $|q| < |A|$), it is enough to assume that $P \cap A$ is $|A|$-compact.

By compactness $p^* = \bigcup_i p_i$ is realized by some $\bar{a} \in \mathfrak{C}$. By Observation 4.11, $\operatorname{tp}(\bar{a}/A) \in S_*(A)$, as required.

■

This Lemma yields another characterization of completeness, justifying, in some sense, the original motivation behind this definition (see the discussion in the very beginning of this section).

Proposition 4.13. *Suppose that $A \cap P$ is $|A|$-compact. Then A is complete if and only if there exists an $M \prec \mathfrak{C}$ with $P^M \subseteq A \subseteq M$. If $|A| = |A|^{<|A|} > |T|$, we can add "M saturated".*

Proof: The direction $\Leftarrow$ is trivial. For the other direction we construct a model inductively, using Fact 4.4 for the limit stages and the previous lemma for the successor stages. ■

As we want to reconstruct M from P^M using some complete A as an approximation, it makes sense to look at $S_*(A)$ as candidates for types to be realized. So with $S_*(A)$ small (A stable), this task appears to be easier, as there is less choice.

The rest of the paper is devoted to the study of stable (in particular complete) sets.

In light of Lemma 4.5, since, in the definition of a stable set, instead of looking at a specific set A, we need to consider the class of all subsets A' of $\mathfrak{C}$ with $A' \equiv A$, it would be very useful to ensure that T has quantifier elimination. Fortunately, by the discussion in the previous section (specifically, by Observation 3.6), we can Morleyize T with not much cost. Therefore for the rest of the paper we strengthen Hypothesis 1 and add

Hypothesis 4.14. *(Hypothesis 1')*

(iii) T has quantifier elimination, even to the level of predicates.

5 Stability and rank

In this section we begin the investigation of stable sets, and introduce a notion of rank that "captures" stability.

Definition 5.1. We say that a type $p \in S(A)$ is *internally definable* if for every formula $\phi(x, y)$, the set $\{a \in A\colon \varphi(x, a) \in p\}$ is internally definable in $\mathfrak{C}|_A$. We say that p is internally definable over $B \subseteq A$ if the set above is internally definable in $\mathfrak{C}|_A$ with parameters in B.

Definition 5.2. For a complete set A, a (partial) n-type $p(\bar{x})$ (with parameters in $\mathfrak{C}$), sets Δ_1, Δ_2 of formulas $\psi(\bar{x}, \bar{y})$, and a cardinal λ, we define when $R^n_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha$. We usually omit n.

(i) $R_A(p, \Delta_1, \Delta_2, \lambda) \geq 0$ if $p(\bar{x})$ is consistent.

(ii) For α a limit ordinal: $R_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha$ if $R_A(p, \Delta_1, \Delta_2, \lambda) \geq \beta$ for every $\beta < \alpha$.

(iii) For $\alpha = \beta + 1$ and β even: For $\mu < \lambda$ and finite $q(\bar{x}) \subseteq p(\bar{x})$ we can find $r_i(\bar{x})$ for $i \leq \mu$ such that;

1. Each r_i is a Δ_1-type over A,
2. For $i \neq j, r_i$ and r_j are explicitly contradictory (i.e. for some ψ and $\bar{c}$, $\psi(\bar{x}, \bar{c}) \in r_i, \neg\psi(\bar{x}, \bar{c}) \in r_j$).
3. $R_A(q(\bar{x}) \cup r_i(\bar{x}), \Delta_1, \Delta_2, \lambda) \geq \beta$ for all i.

(iv) For $\alpha = \beta + 1 : \beta$ odd: For $\mu < \lambda$ and finite $q(\bar{x}) \subseteq p(\bar{x})$ and $\psi_i \in \Delta_2, \bar{d}_i \in A$ $(i \leq \mu)$, there are $\bar{b}_i \in A \cap P$ such that $R(r_i, \Delta_1, \Delta_2, \lambda) \geq \beta$ where $r_i = q(\bar{x}) \cup \{(\forall \bar{z} \subseteq P) \left[\psi_i(\bar{x}, \bar{d}_i, \bar{z}) \equiv \Psi_{\psi_i}(\bar{z}, \bar{b}_i)\right] : i < \mu\}$ where Ψ_{ψ_i} is as in Fact 4.3.

$R^n_A(p, \Delta_1, \Delta_2, \lambda) = \alpha$ if $R^n_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha$ but not $R^n_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha + 1$. $R^n_A(p, \Delta_1, \Delta_2, \lambda) = \infty$ iff $R^n_A(p, \Delta_1, \Delta_2, \lambda) \geq \alpha$ for all α.

The main case for applications will be $\lambda = 2$. Note that the larger $R^n_A(p, \Delta_1, \Delta_2, \lambda)$, the more evidence there is for the existence of many types $q(\bar{x}) \in S_*(A)$ consistent with $p(\bar{x})$.

Fact 5.3. (i) The rank $R^n_A(p, \Delta_1, \Delta_2, \lambda)$ is increasing in A, Δ_1 and decreasing in p, Δ_2, λ.

(ii) For every p there is a finite $q \subseteq p$, such that $R_A(p, \Delta_1, \Delta_2, \lambda) = R_A(q, \Delta_1, \Delta_2, \lambda)$.

(iii) For any A and any finite $\Delta_1, \Delta_2, \lambda, m$ and $\psi(\bar{x}, \bar{y})$ there is a formula $\theta(\bar{y}) \in L(\mathfrak{C}|A)$ such that for $\bar{b} \subseteq A$ $R_A(\psi(\bar{x}, \bar{b}), \Delta_1, \Delta_2, \lambda) \geq m$ iff $A \models \theta(\bar{b})$. The formula θ doesn't really depend on A but has quantifiers ranging on A.

Fact 5.4. Let A be complete, $p \in S_*(A)$, $q^* \subseteq p$, and assume

$$R^n_A(q^*, \Delta_1, \Delta_2, \lambda) = R^n_A(p, \Delta_1, \Delta_2, \lambda) = k < \infty$$

Then k is even.

Proof: Assume k is odd; we shall show that $R^n_A(q^*, \Delta_1, \Delta_2, \lambda) \geq k$ implies $R^n_A(q^*, \Delta_1, \Delta_2, \lambda) \geq k + 1$.

As in Definition 5.2(iv), let $\mu < \lambda$, $q(\bar{x}) \subseteq q^*(\bar{x})$ finite, $\psi_i \in \Delta_2, \bar{d}_i \in A$ $(i \leq \mu)$.

Let $\bar{c} \models p$; so $A \cup \{\bar{c}\}$ is complete by the assumption $p \in S_*(A)$.