is a model) has a precise technical meaning, as the notion of independence defined below coincides with the usual non-forking independence; see Corollary 9.14.

Definition 8.2. Suppose A is stable, $p \in S_*(A)$ and $A \subseteq_t B$. Then $q \in S(B)$ is a *stationarization* of p over B if for every $\psi \in L$ there is some definition $\Psi_\psi(\bar{y}, \bar{a}_\psi)$ with $\bar{a}_\psi \subseteq A$ that defines both p_ψ and q_ψ.

Notation 8.3. (i) *We write $\bar{a} \perp_A B$ if A is stable and $q = \mathrm{tp}(\bar{a}/B)$ is a stationarization of $p = \mathrm{tp}(a/A)$ (so in particular $p \in S_*(A)$ and $A \subseteq_t B$). In this case, will also write $q = p|B$.*

(ii) *We write $C \perp_A B$ if for every $\bar{a} \in C$ we have $\bar{a} \perp_A B$.*

If $\bar{a} \perp_A B$ or $C \perp_A B$, we say that $\bar{a}$ (or C) is *independent* from B over A.

Let us point out some basic properties of the notions defined above.

Lemma 8.4. (i) *$A \subseteq_t B$ if and only for every quantifier free formula $\varphi(\bar{x})$ over A, if there exists $\bar{b} \in B$ such that $\models \varphi(\bar{b})$, then $A \models \exists \bar{x} \varphi(\bar{x})$.*

(ii) *If A is λ-saturated then $A \subseteq_t B$ if and only for every (partial) type $p(\bar{x})$ over a subset of A of size $< \lambda$, if p is realised by some $\bar{b} \in B$, then it is realised by some $\bar{a} \in A$.*

Proof:

(i) *By quantifier elimination and the assumption that there are no function symbols (so every subset is a substructure).*

(ii) *By quantifier elimination, $p(x)$ is equivalent (in $\mathfrak{C}$) to a quantifier free type $\Delta(\bar{x})$. By part (i), Δ is finitely satisfiable in A, and by saturation there exists $\bar{a} \in A$ such that $A \models \theta(\bar{a})$ for all $\theta(\bar{x}) \in \Delta$. Since Δ is quantifier free, the truth value of $\theta(\bar{a})$ is preserved between A and $\mathfrak{C}$; so $\Delta(\bar{x})$ is realised by $\bar{a}$, hence so is $p(\bar{x})$.*

$\blacksquare$

Lemma 8.5. *Assume A is stable, $A \subseteq_t B$ and $p \in S_*(A)$. Then:*

(i) p has a stationarization q over B.

(ii) It is unique: We can replace "some $\Psi_\psi(\bar{y}, \bar{a}_\psi)$" by "every...", so q does not depend on its choice.

(iii) If B is complete, $q \in S_*(B)$.

Proof: (i) By Theorem 7.2 there are quantifier free formulas $\Psi_\psi(\bar{y}, \bar{a}_\psi)$ with $\bar{a}_\psi \in A$ defining $p|_\psi$. Let $q = \{\psi(\bar{x}, \bar{b}); \bar{b} \subseteq B \text{ and } \models \Psi_\psi(\bar{b}, \bar{a}_\psi)\}$.

q is consistent: If not, there are $n < \omega, \psi_l(\bar{x}, \bar{b}_l) \in q (l = 1, \ldots, n)$ such that $\models \neg \exists \bar{x} (\bigwedge_{l=1}^n \psi_l(\bar{x}, \bar{b}_l))$. So $\models \bigwedge_{l=1}^n \Psi_{\psi_l}(\bar{b}_l, \bar{a}_{\psi_l}) \wedge \neg \exists \bar{x} (\bigwedge_{l=1}^n \psi_l(\bar{x}, \bar{b}_l))$. This is a formula in $L(T)$. So there are $\bar{b}_l \in A, (l = 1, \ldots, n)$ such that $\models \bigwedge_{l=1}^n \Psi_{\psi_l}(\bar{b}'_l, \bar{a}_{\psi_l}) \wedge \neg \exists \bar{x} (\bigwedge_{l=1}^n \psi_l(\bar{x}, \bar{b}'_l))$. But $\bigwedge_l \psi_l(\bar{x}, \bar{b}'_l) \in p$, contradicting the consistency of p.

q is complete: Assume $\bar{b} \subseteq B, \psi \in L$ and $\psi(\bar{x}, \bar{b}), \neg\psi(\bar{x}, \bar{b}) \notin q$. So $\models \neg\Psi_\psi(\bar{b}, \bar{a}_\psi) \wedge \neg\Psi_{\neg\psi}(\bar{b}, \bar{a}_{\neg\psi})$. By the definition of q and $A \subseteq_t B$ there is some $\bar{b}' \subseteq A$ so that $\models \neg\Psi_\psi(\bar{b}', \bar{a}_\psi) \wedge \neg\Psi_{\neg\psi}(\bar{b}', \bar{a}_{\neg\psi})$. But then $\psi(\bar{x}, \bar{b}'), \neg\psi(\bar{x}, \bar{b}') \notin p$ and p is complete, a contradiction.

(ii) Same proof: Let Ψ, q be as in (i), and suppose that $\Psi'_\psi(\bar{y}, \bar{a}'_\psi)$ (not necessarily quantifier free) with $\bar{a}'_\psi \in A$ also defines $p|_\psi$, and assume that it defines a ψ-type q'_ψ over B. If $q' \neq q$, that is, for example, $\psi(\bar{x}, \bar{b}) \in q$, $\psi(\bar{x}, \bar{b}) \notin q'$, then $B \models \Psi_\psi(\bar{b}) \wedge \neg\Psi'_\psi(\bar{b})$. Since $A \subseteq_t B$, there exists $\bar{a} \in A$ such that $\Psi_\psi(\bar{a}) \wedge \neg\Psi'_\psi(\bar{a})$, which is clearly absurd, since both schemata Ψ and Ψ' define the same type $p \in S(A)$.

(iii) Let B be complete. We consider $\phi(\bar{x}, \bar{y}, \bar{z})$ with $\phi'(\bar{x}, \bar{b}) := (\exists \bar{z} \in P)\phi(\bar{x}, \bar{b}, \bar{z}) \in q, \bar{b} \in^{\omega>} B$. We have to show that for some $\bar{c} \in^{\omega>} (B \cap P), \phi(\bar{x}, \bar{b}\bar{c}) \in q$. Without loss of generality $\phi(\bar{x}, \bar{y}, \bar{z}) \vdash \bar{z} \subseteq P$. So in the problematic case $\bar{b} \in^{\omega>} B, \models \Psi_{\phi'}(\bar{b})$, but for no $\bar{c} \subseteq {}^{\omega>}(B \cap P)$ is $\models \Psi_\phi(\bar{b}, \bar{c}, \bar{a}_\phi)$. But since B is complete this implies $\models \Psi_{\phi'}(\bar{b}) \wedge (\neg\exists \bar{z} \subseteq P)\Psi'_\phi(\bar{b}, \bar{z})$, so this is satisfied by some $\bar{b}' \in A$ (since the definitions are over A and $A \subseteq_t B$), and we get a contradiction to $p \in S_*(A)$. ■

Corollary 8.6. *(of the proof): If $A \subseteq_t B, A$ stable, $\bar{c} \subseteq \bar{b}, tp(\bar{b}/A) \in S_*(A)$, then $tp(\bar{c}/A) \in S_*(A)$ and the stationarization of $tp(\bar{b}/A)$ over B includes the stationarization of $tp(\bar{c}/A)$ over B.*

Corollary 8.7. *Let $q \in S_*(B)$ definable over $A \subseteq_t B$, A a stable set. Then q is the stationarization of $q{\restriction}A$.*

The following example illustrates the importance of the condition $A \subseteq_t B$ in the definition of the stationarization (and in Lemma 8.5(i)).

Example 8.8. *In [Cha20] Chatzidakis explores the theory $T = ACFA_0$ over $P = Fix(\sigma)$ and proves that for all uncountable λ, if A is a substructure with P^M λ-saturated, there exists a λ-primary model N of T over A. This result immediately implies (by e.g. Theorem 7.3) that $ACFA_0$ is stable over P. In this example, in the construction of the primary model over A, one has to address the following situation: $A \subseteq B$, $B^P = M^P$, B is complete and stable (this is not specifically stated in [Cha20], but is a posteriori clear, since the construction in particular yields a λ-primary model N over B), $d \in B$, and the type (extending) the difference equation $\sigma(x) = d \cdot x$ is not realized in B. Clearly (by e.g. Lemma 4.12), this types extends to $p \in S_*(B)$, and one has to realize p in order to complete the construction of N.*

Now consider $a \models p$. One may ask: can we have $b \in N$ such that $b \models p|Ba$ (so $a, b \models p$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_B b$)? The answer is clearly no, since in this case $\frac{a}{b}$ would be a new element in P. So why does N not realize the stationarization of p to Ba? The issue is that if B does not realize p, then $B \not\subseteq_t Ba$. And indeed in this case such a stationarization does not exist.

Let us make a few further remarks.

Remark 8.9. (i) *If $A \subseteq B \subseteq C$ and $A \subseteq_t C$, then $A \subseteq_t B$. In particular, if M is a model, then $M \subseteq_t C$ for any $C \supseteq M$. So $p \in S_*(M)$ has a (unique) stationarization over any superset.*

(ii) *If, under the assumptions of* (i), *q is the stationarization of $p \in S_*(A)$ over C (so in particular A is stable), and B is stable, then $q \restriction B$ is the stationarization of p over B.*

(iii) *If $A \subseteq_t B$, $tp(ab/B)$ is the stationarization of $tp(ab/A) \in S_*(A)$, and Ab is stable, then $tp(a/Bb)$ is the stationarization of $tp(a/Ab)$*

(iv) *In the previous clause, if b is finite, then the assumption on Ab is redundant, that is, it follows from the other assumptions.*

Proof: Easy. For clause (iv), note that since $tp(ab/A) \in S_*(A)$, the set Ab is complete; and since A is stable and b is finite, Ab is stable as well. ■

Lemma 8.10. *Let A, B, C be sets such that $A \subseteq_t B$, $A \subseteq C$, $C \mathop{\smile\!\!\!\!\!\!|}_A B$ (see Notation 8.3). Let F be an elementary map from B onto B', G be an elementary map from C onto C' such that $F \restriction A = G \restriction A$. Then $F \cup G$ is elementary.*

Proof: Let $A' = F(A) = G(A)$. Let C'' be such that $tp(C''/B)$ is the stationarization of $tp(C'/A)$. Then clearly there is an elementary map G' such that $G'(C') = C''$, and $G'' \restriction A'$ is the identity. First we claim that $F \cup (G' \circ G)$ is elementary.

Indeed, let $\bar{c} \in C$. Then for every $\varphi(\bar{x}, \bar{y})$ there exists $\bar{d} \in A$ such that $\Psi_\varphi(\bar{y}, \bar{d})$ defines the φ-type of $\bar{c}$ over A. In other words, $\varphi(\bar{c}, \bar{a})$ if and only if $\Psi_\varphi(\bar{a}, \bar{d})$ for all $\bar{a} \in A$. Then (since G is elementary) we have $\varphi(\bar{c}', \bar{a}')$ if and only if $\Psi_\varphi(\bar{a}', \bar{d}')$ for all $\bar{a}' \in A'$, where $\bar{c}'\bar{d}' = G(\bar{c}\bar{d})$. Hence (since G' is elementary) we have $\varphi(\bar{c}'', \bar{a}')$ if and only if $\Psi_\varphi(\bar{a}', \bar{d}')$ for all $\bar{a}' \in A'$, where $\bar{c}'' = G'(c)$. Recalling that $G'(\bar{a}'\bar{d}) = \bar{a}'\bar{d}$, we get:

$$\varphi(\bar{c}'', \bar{a}'') \Longleftrightarrow \Psi_\varphi(\bar{a}'', \bar{d}'')$$

for all $\bar{a}'' \in A''$, where $\bar{c}''\bar{d}'' = G' \circ G(\bar{c}\bar{d})$.

In other words, $tp(\bar{c}''/A')$ is definable. Since $tp(c''/B')$ is the stationarization of $tp(\bar{c}''/A')$, by Lemma 8.5, the same definition works for since $tp(c''/B')$. Hence for all $\bar{b}'' \in B'$, letting (as before) $\bar{c}''\bar{d}'' = G' \circ G(\bar{c}\bar{d})$

$$\varphi(\bar{c}'', \bar{b}'') \Longleftrightarrow \Psi_\varphi(\bar{b}'', \bar{d}'')$$

In particular, the equivalence above holds for $\bar{b}'' = F(\bar{b})$. Recall that since $\bar{d} \in A$, we have $F(\bar{d}) = G(d) = G' \circ G(\bar{d})$. Hence

$$\varphi(\bar{c}'', \bar{b}'') \Longleftrightarrow \Psi_\varphi(\bar{b}'', \bar{d}'') \Longleftrightarrow \Psi_\varphi(F(\bar{b}), F(\bar{d}))$$

(the rightmost equivalence holds since F is elementary). Now, by the choice of Ψ_φ, we also have

$$\varphi(G' \circ G(c), F(b)) = \varphi(\bar{c}'', \bar{b}'') \Longleftrightarrow \Psi_\varphi(F(\bar{b}), F(\bar{d})) \Longleftrightarrow \varphi(b, d)$$