As $\bar{d}_i \in A$, clearly $A \cap P = (A \cup \bar{c}\bar{d}_i) \cap P$, hence $tp(\bar{c}\bar{d}_i/A) \in S_*(A)$. Hence $tp_{\psi_i}(\bar{c}\bar{d}_i/P^{\mathfrak{C}})$ is defined by $\Psi_{\psi_i}(\bar{y}, \bar{b}_i)$ with $\bar{b}_i \subseteq P \cap (A \cup \bar{c}\bar{d}_i) = A \cap P$, where Ψ_{ψ_i} is as in Fact 4.3.

So $\theta_i(\bar{x}) := (\forall \bar{y} \subseteq P)[\psi_i(\bar{x}, \bar{d}_i, \bar{y}) \equiv \Psi_{\psi_i}(\bar{y}, \bar{b}_i)]$ belongs to $tp(\bar{c}/A) = p$; hence

$R_A(q \cup \{\theta_i(\bar{x})\}, \Delta_1, \Delta_2, \lambda) \geq R_A(p, \Delta_1, \Delta_2, \lambda) = k$.

Now by (iv) of Definition 5.2, $R_A(p, \Delta_1, \Delta_2, \lambda) \geq k+1)$, and we are done.

■

Theorem 5.5. *The following are equivalent:*

(i) A is stable.

(ii) For every finite Δ_1 and finite n there are some finite Δ_2 and finite m such that $R^n_A(\bar{x} = \bar{x}, \Delta_1, \Delta_2, 2) \leq m$.

Proof: $(ii) \Rightarrow (i)$: Suppose (ii) holds. Since condition (ii) speaks only about $Th(\mathfrak{C}|_A)$, it suffices to prove $|S_*(A)| \leq |A|^{|T|}$ as the same proof works for every $A' \equiv A$.

Let $\lambda := |A|^{|T|}$ and assume that for $i < \lambda^+$ there are distinct types $p_i = tp(\bar{c}_i/A) \in S_*(A)$. By Fact 5.3 (ii) for every $i < \lambda^+$ and finite Δ_1, Δ_2 we can find a finite $p = p_{i,\Delta_1,\Delta_2} \subseteq p_i$ such that $R_A(p, \Delta_1, \Delta_2, 2) = R_A(p_i, \Delta_1, \Delta_2, 2)$. Let $q_i = \bigcup_{\Delta_1,\Delta_2} p_{i,\Delta_1,\Delta_2}$. So $q_i \subseteq p_i, |q_i| \leq |T|$ and by Fact 5.3 (i) $R_A(p_i, \Delta_1, \Delta_2, 2) = R_A(q_i, \Delta_1, \Delta_2, 2)$ for every finite Δ_1, Δ_2.

The function F with $domF = \lambda^+, F(i) = \langle p_i|_\psi : \psi \in L(T)\rangle$ is one-to-one where $p|_\psi = \{\pm\psi(\bar{x}, \bar{y}) : \pm\psi(\bar{x}, \bar{z}) \in p\}$. Hence, $\lambda^+ \leq \prod_{\psi \in L(T)} |\{p_i|\psi : i < \lambda^+\}|$. If for every ψ, $|\{p_i|\psi : i > \lambda^+\}| \leq \lambda$, we get a contradiction as $\lambda^{|T|} = \lambda$. Choose ψ^* such that $|\{p_i|\psi^* : i < \lambda^+\}| = \lambda^+$. By renaming we can assume that $\{p_i|\psi^* : i < \lambda^+\}$ are pairwise distinct.

There cannot be more than $(|A| + |T|)^{|T|} = |A|^{|T|} = \lambda$ different q_i, so without loss of generality, $q_i = q^*$ for all i. Also we can assume that all p_i's are n-types for some fixed n. Applying condition (ii) to n and $\Delta_1 := \{\psi^*\}$ there is a finite set Δ_2 and $m < \omega$ such that $R^n_A(\bar{x} = \bar{x}, \Delta_1, \Delta_2, 2) \leq m$.

Let $k := R^n_A(q^*, \Delta_1, \Delta_2, 2)$. Since q^* is consistent and by monotonicity (Fact 5.3 (i)) and the fact that $\{\bar{x} = \bar{x}\} \subseteq q^*$, it follows that $0 \leq k \leq m < \omega$.

Recall also that by the construction of q^*, for all i

$$R^n_A(p_i, \Delta_1, \Delta_2, 2) = R^n_A(q^*, \Delta_1, \Delta_2, 2) = k$$

We are now going to show that $R^n_A(q^*, \Delta_1, \Delta_2, 2) \geq k+1$, hence obtaining a contradiction. Note that by 5.4, k is even.

From now on, for simplicity of notation, let R denote R^n_A.

As $p_0|\psi^* \neq p_1|\psi^*$, there is $\psi^*(\bar{x}, \bar{b}), \bar{b} \in A$ such that $\psi^*(\bar{x}, \bar{b}) \in p_0$ and $\neg\psi^*(\bar{x}, \bar{b}) \in p_1$ (or conversely). Now $R(q^* \cup \{\pm\psi^*(\bar{x}, \bar{b})\}, \Delta_1, \Delta_2, 2) \geq k$ as this type is contained in p_0 or p_1, hence by monotonicity $R(q^*, \Delta_1, \Delta_2, 2) = R(q^* \cup \{\pm\psi^*(\bar{x}, \bar{b})\}, \Delta_1, \Delta_2, 2)$. So $R(q^*, \Delta_1, \Delta_2, 2) > k$, a contradiction.

This finishes the proof of one direction.

In order to prove the other direction, assume that condition (ii) fails. We will prove a strong version of $\neg(i)$. Let (ii) fail through Δ_1. So for all finite Δ_2, $R(\bar{x}=\bar{x},\Delta_1,\Delta_2,2)\geq\omega$.

Let $\lambda=\lambda^{<\lambda}>|T|$ (which exists by our set theoretic assumption). Let $B\equiv A$ be saturated, $|B|=\lambda$.

An m-type $p(\bar{x})$ over B (in $L(T)$) is called *large* (for Δ_1) if for all finite Δ_2, $R(p(\bar{x}),\Delta_1,\Delta_2,2)\geq\omega$.

So $\neg(ii)$ says that $\bar{x}=\bar{x}$ is large for Δ_1. It will be enough to prove the following claims:

Assume $p(\bar{x})$ is over B and is large, $|p|<\lambda$. Then the following holds:

(a) For any $\bar{b}\in B,\psi=\psi(\bar{x},\bar{y},\bar{z})$ there exists $\bar{d}\subseteq P\cap B$ such that $p(\bar{x})\cup\{\forall\bar{z}\subseteq P[\psi(\bar{x},\bar{b},\bar{z})\equiv\Psi_\psi(\bar{z},\bar{d})]\}$ is large.

(b) For $\bar{b}\subseteq B$ at least one of $p(\bar{x})\cup\{\pm\psi(\bar{x},\bar{b})\}$ is large.

(c) For some $\psi(\bar{x},\bar{y})\in\Delta_1$ and $\bar{b}\in B$ we have $p(\bar{x})\cup\{\pm\psi(\bar{x},\bar{b})\}$ are large.

Note that from $(a)-(c)$ (and Fact 4.4(ii)) it follows that $|S_*(B)|=2^\lambda$; in fact, even $|\{p|_{\Delta_1}:p\in S_*(B)\}|=2^\lambda$.

This already contradicts stability. But we can say more. For at least one $p\in S_*(B)$, $p|_{\Delta_1}$ is not definable. Hence (by [Sh8]) there exists some B, $B\equiv A$, $|B|=\lambda$ such that $|S_*(B)|\geq Ded(\lambda)$, assuming for simplicity that $Ded(\lambda)$ is obtained. This is a strong negation of (i).

It is left to show $(a)-(c)$:

(a): Without loss of generality assume that $p(\bar{x})$ is closed under conjunction. So we have to find $\bar{d}$ such that for all $\rho=\langle\Delta_2,n,\theta(\bar{x},\bar{e})\rangle$ where $n<\omega,\theta(\bar{x},\bar{e})\in p(\bar{x})$, and $\Delta_2\subseteq L(T)$ finite,

$$(*_\rho)\quad R(\theta(\bar{x},\bar{e}))\wedge(\forall\bar{z}\subseteq P)(\psi(\bar{x},\bar{b},\bar{z})\equiv\Psi_\psi(\bar{z},\bar{d})),\Delta_1,\Delta_2,2)\geq n.$$

For every such ρ there are $\bar{e}^*_\rho\subseteq B,\chi_\rho\in L(T$ such that for $\bar{d}\subseteq P\cap B, B\models\chi_\rho(\bar{d},\bar{e}^*_\rho)$ iff $*_\rho$ holds for $\bar{d}$.

As B is λ-saturated, $|p(\bar{x})|<\lambda$ it suffices to show that for every relevant $\rho_1,\dots,\rho_n$ there is $\bar{d}\subseteq P\cap B$ satisfying $*_{\rho_1}\wedge\cdots\wedge*_{\rho_n}$.

By monotonicity properties of rank and the fact that p is closed under conjunction, it is enough to consider one ρ. But $R(\theta(\bar{x},\bar{e}),\Delta_1,\Delta_2,2)=\omega>n+2$. So by the definition of rank there is suitable $\bar{d}\subseteq P\cap B$ satisfying $*_\rho$.

(b) follows from (a) with $\bar{z}$ empty.

For (c) assume first that $\Delta_1=\{\psi\}$. Repeat the proof of (a) conjuncting over $\pm\psi$, using the other clause in the definition of rank.

If $|\Delta_1|>1$, assume (c) doesn't hold. So for every $\psi\in\Delta_1$ there is some finite $q_\psi\subseteq p(\bar{x})$ such that stops $p(\bar{x})\cup\{\pm\psi(\bar{x},\bar{b}\}$ from being large. Now use $R(\bigcup_{\psi\in\Delta_1}q_\psi,\Delta_1,\Delta_2,2)\geq n+2$ to get a contradiction. ∎

The following two corollaries follow from the proof of (the second direction of) Theorem 5.5.

Corollary 5.6. *In Definition 4.1(iv), it is not necessary to consider all $A' \equiv A$. More specifically, a complete set A is stable if and only if $|S_*(A')| \leq |A'|^{|T|}$ for some $A' \equiv A$ saturated, $|A'| > |T|$.*

Corollary 5.7. *Let A be complete unstable and saturated of cardinality $\lambda > |T|$. Then $|S_*(A)| = 2^\lambda$. In fact, $|S_{*,\Delta}(A)| = 2^\lambda$, where Δ is some finite set of formulas, and*

$$S_{*,\Delta}(A) = \{p{\restriction}\Delta \colon p \in S_*(A)\}$$

From now on, we will often omit the superscript and the subscript in the rank R^n_A, and write simply R (at least when n and A are easily deduced from the context).

In conclusion, we observe that every type $p \in S_*(A)$ over a stable set A is internally definable (see Definition 5.1).

Corollary 5.8. (i) *If A is stable, then for every $\psi(\bar{x}, \bar{y}) \in L(T)$ there is Ψ_ψ in $L(A)$ such that if $p \in S_*(A)$, then for some $\bar{b} \subseteq A, \Psi_\psi(\bar{y}, \bar{b})$ defines $p|\psi$ in $\mathfrak{C}_A$.*

Specifically, for every $\bar{c} \in A$, $\psi(\bar{x}, \bar{c}) \in p$ if and only if $A \models \Psi_\psi(\bar{c}, \bar{b})$.

(ii) *Moreover, if $|A| \geq 2$, then for every $\psi(\bar{x}, \bar{y})$, there is a definition $\Psi_\psi(\bar{x}, \bar{y})$ as above which works uniformly for all $B \equiv A$ and $p \in S_*(B)$.*

Proof:

(i) Let $\Delta_1 = \{\psi\}$. Then there is some finite Δ_2 such that $R_A(\bar{x} = \bar{x}, \Delta_1, \Delta_2, 2) = n^* < \omega$. Let $\theta(\bar{x}) \in p(\bar{x})$ such that $n = R_A(\theta(\bar{x}), \Delta_1, \Delta_2, 2) = R_A(p(\bar{x}), \Delta_1, \Delta_2, 2) \leq n^*$.

Recall that n is even (Fact 5.4). Since $\Delta_1 = \{\psi\}$, there is no $\bar{b} \in A$ such that $R(\theta(\bar{x}) \wedge \pm\psi(\bar{x}, \bar{b}), \Delta_1, \Delta_2, 2) = n$.

But $\psi \in p(\bar{x}) \Rightarrow R(\theta(\bar{x}) \wedge \psi, \Delta_1, \Delta_2, 2) = n$.

So for $\bar{b} \subseteq A$, we have

$$\psi(\bar{x}, \bar{b}) \in p(\bar{x}) \Leftrightarrow R(\theta(\bar{x}) \wedge \psi(\bar{x}, \bar{b}), \Delta_1, \Delta_2, 2) \geq n.$$

By Fact 5.3(*iii*) the right hand side is definable in A. More precisely, the predicate

$$\Psi(\bar{y}) = [R_A(\theta(\bar{x}) \wedge \psi(\bar{x}, \bar{y}), \Delta_1, \Delta_2, 2) \geq n]$$

is definable in A; note that it may have parameters in A.

This proves that for each type $p \in S_*(A)$, there is some Ψ_ψ.

(ii) Now use a compactness argument to find a uniform definition Ψ_ψ for all $B \equiv A$ and $p \in S_*(B)$. The argument is pretty standard, but we have chosen to include it due to the (unusual) additional requirement of uniformity for all B.

Let $\psi(\bar{x}, \bar{y}) \in L(T)$. Expand $L(T)$ with a new monadic predicate $Q(y)$, which is interpreted in $\mathfrak{C}$ as the set A. Let T^* be the theory of the