Question 2.6. *For $\psi = \psi(\bar{x}, \bar{y})$, $\bar{c} \in \mathfrak{C}$, is $tp_\psi(\bar{c}/P^{\mathfrak{C}})$ definable?*

Recall that $tp_\psi(\bar{c}/A) = \{\psi(\bar{x}, \bar{a}) : \bar{a} \in A,\ \mathfrak{C} \models \psi(\bar{c}, \bar{a})\}$ and $p = tp_\psi(\bar{c}/A)$ is definable if it is definable by some $\theta(\bar{y}, \bar{d})$ with $\bar{d} \in A$, which means that for every $\bar{a} \in A$ we have $\psi(\bar{x}, \bar{a}) \in p \iff \mathfrak{C} \models \theta(\bar{a}, \bar{d})$. We restrict ourselves to ψ-types in order to be able to use compactness arguments.

Again, it is shown in [PS85] that if the answer to this question is "no", then $I(\lambda) \geq Ded(\lambda)$. The two questions are closely related (the second one is a version of the first one with external parameters), and the proofs are similar, and, in fact, both work for pseudo-elementary classes.

In conclusion, for the rest of the paper we make the following assumptions:

Hypothesis 2.7. *(Hypothesis 1)*
T is a complete first order theory , P a monadic predicate in the language of T, $\mathfrak{C}$ is the monster model of T such that

(i) *Subsets $P^{\mathfrak{C}}$ that are 0-definable (in $\mathfrak{C}$), are already 0-definable in $\mathfrak{C}|_P$.*

(ii) *Every type over $P^{\mathfrak{C}}$ is definable.*

Later we shall add an additional clause to this hypothesis regarding quantifier elimination of T; see Hypothesis 4.14.

For simplicity we also make the set theoretic assumption that for arbitrarily large λ, we have $\lambda^{<\lambda} = \lambda$ (in particular, T has arbitrarily large saturated models). Note that for any conclusion we draw which says something about every λ, this hypothesis can be eliminated.

3 Internal and external definability

In this section we observe several basic consequences of Hypothesis 1.

First we reformulate the assumption that every type over $P^{\mathfrak{C}}$ is definable in the following obviously equivalent way: every externally definable subset of $P^{\mathfrak{C}}$ is also internally definable in $\mathfrak{C}|_P$ (with parameters in $P^{\mathfrak{C}}$).

Observation 3.1. (i) *Let $\varphi(\bar{x}, \bar{a})$ define (in $\mathfrak{C}$) a subset of $P^{\mathfrak{C}}$. Then there exists a formula $\hat{\theta}(\bar{x}, \bar{b})$ with $\bar{b} \in P^{\mathfrak{C}}$ that defines the same set. Moreover, there exists a formula $\theta(\bar{x}, \bar{b})$ that defines the same set in $\mathfrak{C}|_P$.*

(ii) *Furthermore, for any $A \subseteq P^{\mathfrak{C}}$ such that $\mathrm{tp}(\bar{a}/P^{\mathfrak{C}})$ is definable over A, $\theta(\bar{x}, \bar{b})$ in the previous clause can be chosen so that $\bar{b} \in P^A$.*

Proof: By Hypothesis 2.7(ii), $\mathrm{tp}(a/P^{\mathfrak{C}})$ is definable. Hence there is $\hat{\theta}(\bar{x}, \bar{b})$ with $\bar{b} \in P^{\mathfrak{C}}$ such that $\mathfrak{C} \models \forall \bar{x} \left(\hat{\theta}(\bar{x}, \bar{b}) \longleftrightarrow \varphi(\bar{x}, \bar{a})\right)$. By Hypothesis 2.7(i), there exists $\theta(\bar{x}, \bar{y})$ such that for every $\bar{c}, \bar{d} \in P^{\mathfrak{C}}$ we have $\mathfrak{C} \models \hat{\theta}(\bar{c}, \bar{d})$ if and only if $\mathfrak{C}|_P \models \theta(\bar{c}, \bar{d})$. Clearly $\theta(\bar{x}, \bar{b})$ is as required (in both clauses of the Observation). ■

Corollary 3.2. (i) *Let $p(\bar{x})$ be a (partial) type over $\mathfrak{C}$ such that $P(\bar{x}) \in p$. Then p is equivalent to a T^P-type p' over $P^{\mathfrak{C}}$.*

(ii) *Let A be a set. Assume that for every $\bar{a} \in A$ the type $\operatorname{tp}(\bar{a}/P^{\mathfrak{C}})$ is definable over $P^A = P^{\mathfrak{C}} \cap A$. Let p be a (partial) type over A with $P(x) \in p$. Then p is equivalent to a (T-)type $\hat{p}$ over P^A, and to a T^P-type p' over P^A.*

The following corollary again follows immediately, but is very useful.

Observation 3.3. *Let A be a set such that for every $\bar{a} \in A$ the type $\operatorname{tp}(\bar{a}/P^{\mathfrak{C}})$ is definable over P^A. Let λ be a cardinal.*

(i) *Assume that $\mathfrak{C}|_{P^A}$ is a λ-saturated (or just λ-compact) model of T^P. Then every T-type of size $< \lambda$ over P^A which is finitely satisfiable in P^A is realized in P^A.*

In fact, it is enough to assume that any T^P-type of size $< \lambda$ over P^A which is finitely satisfiable in P^A is realized in P^A (so $\mathfrak{C}|_{P^A}$ does not need to be itself a model of T^P).

(ii) *If P^A satisfies the conclusion of the clause (i), then every type p of size $< \lambda$ over A with $P(x) \in p$ is realized in P^A.*

Proof: For clause (i), let p is a T-type of size $< \lambda$ over P^A finitely satisfiable in P^A. By Corollary 3.2(ii), it is equivalent to a T^P-type over P^A, and so is realized in P^A. For clause (ii), note that, again by Corollary 3.2(ii), such a type p is equivalent to a type of size $< \lambda$ over P^A (and clearly it is finitely satisfiable in P^A). ∎

Let us formulate a simple version of the last Observation, which will be particularly useful:

Corollary 3.4. *Let A be a set such that for every $\bar{a} \in A$ the type $\operatorname{tp}(\bar{a}/P^{\mathfrak{C}})$ is definable over P^A. Let λ be a cardinal, and assume that $\mathfrak{C}|_{P^A}$ is a λ-saturated model of T^P. Then every T-type p of size $< \lambda$ over A with $P(x) \in p$ is realized in P^A.*

Remark 3.5. In the rest of the paper, we will prove claims about sets and models under the assumption that their P-part is λ-saturated. As a matter of fact, all we'll need in these claims is the conclusion of Observation 3.3 (i). So in particular assuming that P^A is a "λ-compact subset" of $\mathfrak{C}$ (every T-type of size $< \lambda$ which is finitely satisfiable in P^A is realized in P^A) is enough. But we will not focus on this.

A perhaps less obvious consequence of Hypothesis 1 is that T can be Morleyrized without much additional cost.

In classical Classification Theory (classifying first order theories), assuming that T has quantifier elimination often comes without much cost (e.g., by Morleyzation). However, in our case, this assumption may appear less harmless. Indeed, in general, if $\mathfrak{C}$ is expanded with new predicates, then so is

$\mathfrak{C}|_P$, potentially enhancing the theory T^P, hence changing the original classification problem. Fortunately, since Morleyzation of T does not add new 0-definable sets to $\mathfrak{C}$, if Hypothesis 1 is true, no new 0-definable sets are added to $\mathfrak{C}|_P$ either. In other words:

Observation 3.6. *Let T' is the Morleyzation of T, and let $\mathfrak{C}'$ be the expanded monster model (of T'). Let T'^P be the theory of $\mathfrak{C}'|_P$. Then T'^P is a trivial expansion of the Morleyzation of T^P (that is, it is an expansion that adds no new 0-definable sets).*

Proof: On the one hand, if $\theta(\bar{x})$ be a T_P-formula, let $\theta^P(\bar{x})$ be the formula that defines the same subset of $P^{\mathfrak{C}}$ in $\mathfrak{C}$ (as in Remark 2.2). Let $R(\bar{x})$ be a T'-predicate equivalent (modulo T') to $\theta^P(\bar{x})$; then it also defines the same subset of $\mathfrak{C}'_P$, hence is equivalent to $\theta(\bar{x})$ modulo T'^P. In conclusion, T'_P *expands* the Morleyzation of T^P (this part is, of course, always true).

On the other hand, let $R(\bar{x})$ be a new predicate in T'^P. Then it is equivalent modulo T' to a T-formula

$$\bar{x} \subseteq P \bigwedge \varphi(\bar{x})$$

By Hypothesis 1, there is a T^P-formula $\theta(\bar{x})$ (without parameters) defining the same subset of $P^{\mathfrak{C}}$. Therefore any T'_P formula $\theta'(\bar{x})$ is equivalent to a T_P formula, as required. ■

We conclude this section with a few trivial consequences of quantifier elimination in T. Some more interesting ones will be discussed in the next section (e.g., Lemma 4.5).

First we note that some of the observations above become trivially true for any substructure of $\mathfrak{C}$ (which in our case just means a set containing all the individual constants), for example:

Remark 3.7. *Assume that T has QE and let A be a substructure of $\mathfrak{C}$.*

(i) *Any externally definable subset of A is definable internally in $\mathfrak{C}|_A$. A T-type over A is also a type in $\mathfrak{C}|_A$.*

(ii) *If $\mathfrak{C}|_A$ is λ-compact, that is, it is a λ-compact model of the theory $Th(\mathfrak{C}|_A)$. Then every T-type of size $< \lambda$ which is finitely satisfiable in A is realized in A.*

Quantifier elimination also adds to the understanding of T^P:

Remark 3.8. *Assume that T has QE and let A be a substructure of $\mathfrak{C}$.*

(i) *T^P has QE.*

(ii) *Every subset of $P^{\mathfrak{C}}$ definable in $\mathfrak{C}$ (with or without parameters) is definable by the same quantifier free formula (with or without parameters, respectively) both in $\mathfrak{C}$ and in $\mathfrak{C}|_P$.*

(iii) *Corollary 3.2(ii) can be strengthened to: both $\hat{p}$ and p' are equivalent to the same quantifier free type.*