expanded monster model. Consider the following set of formulae in the expanded language:

$$T^* \bigwedge \{\forall \bar{z} \in Q \exists \bar{y} \in Q\ \psi(\bar{x}, \bar{y}) \not\leftrightarrow [Q \models \Psi(\bar{y}, \bar{z})] \colon \Psi(\bar{y}, \bar{z}) \in L(T)\}$$

Assume that this set is finitely satisfiable. Let M^* be a model of T^* in which the tuple $\bar{a}$ realize the above set, and let $M \models T$ be reduct of M^* to $L(T)$. Denote $B = Q^{M^*}$. We now have (all formulas and types below are in the original language $L(T)$):

- $B \equiv A$
- For all $\Psi(\bar{y}, \bar{z}) \in L(T)$ and $\bar{c} \in B$, $\Psi(\bar{y}, \bar{c})$ does not internally (in B) define $\mathrm{tp}_\psi(\bar{a}/B)$.

So for every $\psi(\bar{x}, \bar{y})$ there are finitely many $\Psi_i(\bar{y}, \bar{z})$ such that for any $B \equiv A$ and any $\bar{a} \in \mathfrak{C}$, the type $\mathrm{tp}_\psi(\bar{a}/B)$ is defined by $\Psi_i(\bar{x}, \bar{b})$ for some i and $\bar{b} \in B$. Now (since $|A| \geq 2$, hence so is any $B \equiv A$), we can combine these Ψ_i's into one formula $\Psi(\bar{x}, \bar{y})$ that works for all $B \equiv A$ and $\bar{a} \in \mathfrak{C}$, as required.

∎

So we have obtained a uniform (for all $B \equiv A$) notion of definability of types in $S_*(A)$ for a stable set A. Note, however, that what we got is not the "usual" notion of definability of types; this is different than saying that p is definable (in $\mathfrak{C}$). Specifically, unless $A \prec \mathfrak{C}$, Ψ_ψ might have quantifiers. In order to obtain quantifier free definitions, we will need to make yet another "structure" assumption; see Hypothesis 7.1.

6 Stability and primary models

In this section our goal is to obtain a characterization of stable sets that strengthens the characterization of complete sets with a saturated P-part (Proposition 4.13). Specifically, we will show that if A is (also) stable, then, in addition, one can have the model M in Proposition 4.13 be "constructible" over A in a nice way.

First, we strengthen the Small Type Extension Lemma (Lemma 4.12) in this context.

Lemma 6.1. (i) *Assume B is stable, $|B| = |P^B| = \lambda$, P^B is saturated. Let $p(\bar{x})$ an m-type over B, $|p(\bar{x})| < \lambda$, then there is $q(\bar{x})$ such that $|q(\bar{x})| \leq |T|$, $p(\bar{x}) \cup q(\bar{x})$ consistent and there is $r \in S_*(B)$ such that $p(\bar{x}) \cup q(\bar{x}) \equiv r(\bar{x})$.*

In particular, $r(\bar{x})$ is λ-isolated.

(ii) *The previous clause is also true if $\bar{x}$ is an infinite tuple with $< \lambda$ variables, but in this case we can only require that $|q| < \lambda$.*

Specifically, if $|\bar{x}| = \kappa < \lambda$, then there exists $|q| \leq |T| \cdot \kappa$ as above.

Proof:

(i) Let $\{\psi_i(\bar{x}, \bar{y}_i) : i < |T|\}$ list all formulas of $L(T)$. Let Δ_i be finite such that $R(\bar{x} = \bar{x}, \{\psi_i\}, \Delta_i, 2) < \omega$ (where $R = R^m_B$). Define $q_i(\bar{x})$ by induction on $i < |T|$ such that

  (a) q_i is finite and is over B,
  (b) $p(\bar{x}) \cup \bigcup_{j \leq i} q_j(\bar{x})$ is consistent, and
  (c) $R(p \cup \bigcup_{j \leq i} q_j, \{\psi_i\}, \Delta_i, 2)$ is minimal with respect to (a) and (b).

By Lemma 4.12, there is $p^* \in S_*(B)$ extending $p \cup \bigcup_{j \leq i} q_j$. Clearly, $R(p^*, \{\psi_i\}, \Delta_i, 2) \leq R(p \cup \bigcup_{j \leq i} q_j, \{\psi_i\}, \Delta_i, 2)$, and for some finite $q' \subseteq p^*$ we have $R(p^*, \{\psi_i\}, \Delta_i, 2) = R(q', \{\psi_i\}, \Delta_i, 2)$. If $R(p^*, \{\psi_i\}, \Delta_i, 2) < R(p \cup \bigcup_{j \leq i} q_j, \{\psi_i\}, \Delta_i, 2)$, setting $q'_i = q_i \cup q'$ would contradict the "minimality" of q_i (clause (c) above). Hence $R(p^*, \{\psi_i\}, \Delta_i, 2) = R(p \cup \bigcup_{j \leq i} q_j, \{\psi_i\}, \Delta_i, 2)$.
By Fact 5.4, $R(p^*, \{\psi_i\}, \Delta_i, 2)$ is even, hence so is $R(p \cup \bigcup_{j \leq i} q_j, \{\psi_i\}, \Delta_i, 2)$. In particular, we have:

  (d) For no $\bar{b} \subseteq B$ do we have $R(p \cup \bigcup_{j \leq i} q_j \cup \{\pm\psi_i(\bar{x}, \bar{b})\}, \{\psi_i\}, \Delta_i, 2) \geq R(p \cup \bigcup_{j \leq i} q_j, \{\psi_i\}, \Delta_i, 2)$

Clearly $q = |\bigcup_{j \leq |T|} q_j| \leq |T|$. By Lemma 4.12 there is some $r \in S_*(B)$ such that $p \cup \bigcup_{j < |T|} q_j \subseteq r$. By (c) and (d) above it follows that $p \cup \bigcup_{j < |T|} q_j \vdash r$.

(ii) Let $\langle \psi_i(\bar{x}_i, \bar{y}_i) \colon i < \kappa \rangle$ list all the formulas where $\bar{x}_i$ is a finite tuple from $\bar{x}$ (so $|T| \leq \kappa < \lambda$), and define q_i on induction on κ just as in the proof of the previous clause. Since each q_i is finite, $q = |\bigcup_{j \leq |\kappa|} q_j| \leq \kappa < \lambda$, and we can again use Lemma 4.12 in order to obtain $r \in S_*(B)$ as required.

■

Proposition 6.2. *Assume that B is stable, $|B| = |P^B| = \lambda$, P^B is saturated. Let $C \supset B$ such that:*

- *C is complete*
- $P^C = P^B$
- $|C \setminus B| < \lambda$
- $\mathrm{tp}(C/B)$ *is λ-isolated*

Let $p(\bar{x})$ an m-type over C, $|p(\bar{x})| < \lambda$. Then there is a λ-isolated $r \in S_*(C)$ extending p.

Proof: Let $\bar{c} = \langle c_i \colon i < \kappa \rangle$ list $C \setminus B$ (so $\kappa < \lambda$), and let $B_0 \subseteq B$ be such that $|B_0| < \lambda$ and $\mathrm{tp}(\bar{c}/B_0) \equiv \mathrm{tp}(\bar{c}/B)$.

Separating the parameters of p, we can think of $p(\bar{x})$ as the type $p(\bar{x}, \bar{c})$ over $B\bar{c}$. By replacing all occurrences of c_i by a new variable y_i, we therefore obtain a type $p(\bar{x}, \bar{y})$ over B.

Denote $\hat{p}(\bar{x}, \bar{y}) = p(\bar{x}, \bar{y}) \cup \mathrm{tp}(\bar{c}/B_0)$. By Lemma 6.1(ii), there is $\hat{r} \in S^*(B)$ extending $\hat{p}$ which is λ-isolated. Let $\hat{\bar{a}}\hat{\bar{c}}$ realize $\hat{r}$ in $\mathfrak{C}$. Note that $\hat{\bar{c}} \equiv_B \bar{c}$. Let

$\bar{a} \in \mathfrak{C}$ such that $\bar{a}\bar{c} \equiv_B \hat{\bar{a}}\hat{\bar{c}}$. Note that it is still the case that $\bar{a}C \cap P = P^B$ and $\bar{a}C$ is complete, so in particular $\text{tp}(\bar{a}/C) \in S_*(C)$.

Finally, clearly $\text{tp}(\bar{a}/C)$ is λ-isolated: indeed, if $B_1 \subseteq B$ is such that $\text{tp}(\bar{a}\bar{c}/B_1) \vdash \text{tp}(\bar{a}\bar{c}/B)$, then $\text{tp}(\bar{a}/\bar{c}B_1) \vdash \text{tp}(\bar{a}/\bar{c}B) = \text{tp}(\bar{a}/C)$, so we are done. ∎

Definition 6.3. Let N be a model, $P^N \subseteq B \subseteq N$.

(i) We say that a model N is λ-*prime* over a B if N is λ-saturated, and it can be elementarily embedded over B into any λ-saturated model containing B.

(ii) We say that N is λ-*atomic* over B if for every $\bar{d} \subseteq N$, $tp(\bar{d}, B)$ is λ-isolated over some $B_{\bar{d}} \subseteq B, |B_{\bar{d}}| < \lambda$.

(iii) We say that the sequence $\bar{d} = \{d_i : i < \alpha\} \subseteq N$ is a λ-construction over B in N if for all $i < \alpha$, the type $tp\{d_i/B \cup \{d_j : j < i\})$ is λ-isolated.

(iv) We say that a set $C \subseteq N$ is N is λ-*constructible* over B in N if there is a λ-construction $\bar{d}$ over B in N.

In particular, we say that N is λ-*constructible* over B if there is a construction $N = B \cup \{d_i : i < \lambda\}$ such that for all $i < \lambda$ the type $tp\{d_i/B \cup \{d_j : j < i\})$ is λ-isolated.

(v) We say that a model N is λ-*primary* over B if it is λ-constructible and λ-saturated.

Remark 6.4. (i) *If N is λ-primary over B, then it is λ-prime over B.*

(ii) *(λ regular) If N is λ-constructible over B witnessed by a construction $N = B \cup \{d_i : i < \lambda\}$, then for every $\alpha < \lambda$,* $\text{tp}(\{d_i : i < \alpha\}/B)$ *is λ-isolated. Hence N is λ-atomic over B.*

Recall that given a complete set B satisfying $|B| = \lambda = \lambda^{<\lambda}$ with a saturated P-part, Proposition 4.13 implies that B can be extended to a saturated model of cardinality λ with the same P-part. We now show that in case B is stable, this model can be chosen to be λ-primary (hence λ-prime) over B.

Theorem 6.5. *Assume that B is a stable set, $|P^B| = |B| = \lambda = \lambda^{<\lambda}$, P^B is saturated. Then there is $N \supseteq B$ which is λ-primary over B.*

Proof: By using Proposition 6.2 repeatedly, one constructs $N = B \cup \{d_i \colon i < \lambda\}$ by induction on $i < \lambda$ such that $\text{tp}(d_i/B \cup \{d_j \colon j < i\})$ is λ-isolated, while making sure that for all $i < \lambda$, all types over subsets of $B_i = B \cup \{d_j \colon j < i\}$ of cardinality smaller that λ are realized by some d_j for $j < i$.

Specifically, we construct by induction on i a sequence $\langle \bar{d}_i : i < \lambda \rangle$ of sequences such that:

- $\bar{d}_i = \langle d_\alpha : \alpha < \alpha_i \rangle$ (where d_α is a singleton) with $\alpha_i < \lambda$ (so in particular $|\bar{d}_i| < \lambda$)
- $\bar{d}_i$ is increasing and continuous with i (so in particular $\alpha_i \leq \alpha_j$ for $i < j$)