when both $a(Lk+1-r) \geq d(Lt+1-r)$ and $(t-a)(Lk+1-r) \geq (Lt+1)(k-d)$, or equivalently,

$$\frac{d(Lt+1-r)}{Lk+1-r} \;\leq\; a \;\leq\; \frac{t(Ld+1-r)-k+d}{Lk+1-r}. \tag{11}$$

To achieve average row weight at most $Lt+2$, a must be such that the total weights in the upper and lower blocks are at most $d(Lt+2-r)$ and $(k-d)(Lt+2)$, respectively. This is guaranteed when both $a(Lk+1-r) \leq d(Lt+2-r)$ and $(t-a)(Lk+1-r) \leq (k-d)(Lt+2)$, or equivalently,

$$\frac{t(Ld+1-r)-2(k+d)}{Lk+1-r} \;\leq\; a \;\leq\; \frac{d(Lt+2-r)}{Lk+1-r}. \tag{12}$$

Finally, to ensure that each block of rows is tall enough to accommodate the 1's specified in the construction, it suffices that $0 \leq a \leq d$ and $0 \leq t-a \leq k-d$, which together are equivalent to $\max(0, d+t-k) \leq a \leq \min(d,t)$. By our assumptions, this reduces to $0 \leq a \leq t$.

We now show that these inequalities can all be simultaneously satisfied. In 11, the upper bound is at least the lower bound if and only if $r \geq \frac{k-t}{d-t}$, which holds for our choice of r, and in 12, the upper bound is at least the lower bound if and only if $r \leq \frac{k-t}{d-t} + \frac{k}{d-t}$, which holds since $\frac{k}{d-t} \geq 1$. Exchanging the upper/lower bound pairs in 11 and 12 results in two pairs of inequalities on a that are always satisfied. Finally, for $0 \leq a \leq t$, it suffices that at least one of the lower bounds is nonnegative, and at least one of the upper bounds is at most t. The lower bound in 11 is nonnegative if and only if $L \geq \frac{r-1}{t}$, and the upper bound is at most t if and only if $L \geq -\frac{1}{t}$, so it suffices to take $L \geq \frac{r-1}{t}$. The rest of the proof follows the same reasoning as in Proposition 21. ☐

Proposition 24. *Let $k \geq 3$, $1 < t < k$, and $1 \leq d < t$ be such that $t+d \leq k$ and $(k,t,d) \equiv (1,0,0) \pmod 2$. Let $\mathbf{x}$ be a tuple of weight d. Then $\mathrm{Pol}(\textbf{t-}\textbf{\textit{in}}\textbf{-k} \cup \{\mathbf{x}\}, \mathbf{NAE})$ does not have 2-block-symmetric functions of all odd arities.*

Proof. We place $L \geq 1$ copies of C_k^t in the odd block of our tableau, leaving $Lk-1$ columns to be filled in the even block for a total arity of $2Lk-1$. The three weights used are Lt, $Lt-1$, and $Lt-2$, with L determined later.
Case 1: weights Lt and $Lt-1$.
We take $L-1$ copies of C_k^t and one copy of C_k^{t-}, so that each row has weight Lt or $Lt-1$.
Case 2: weights Lt and $Lt-2$.
We take $L-1$ copies of C_k^t and one copy of C_k^{t-}, leaving t rows of weight $Lt-1$, and since t is even, these rows can be paired and values swapped so that each row has weight Lt or $Lt-2$. In particular, in columns $2,4,\ldots,t-2$, we swap the values in the pairs of rows $(1,2),(3,4),\ldots,(t-3,t-2)$, respectively. Finally, in column $k-1$, we swap the values in rows k and $t-1$.
Case 3: weights $Lt-1$ and $Lt-2$.

Let $r = \left\lceil \frac{k-t}{t-d} \right\rceil$ and let a be the average column weight in the upper d rows. To achieve average row weight at most $Lt-1$, a must be such that the total weights in the upper and lower blocks are at most $d(Lt-1-r)$ and $(Lt-1)(k-d)$, respectively. This is guaranteed when both $a(Lk-1-r) \leq d(Lt-1-r)$ and $(t-a)(Lk-1-r) \leq (Lt-1)(k-d)$, or equivalently,

$$\frac{t(Ld-1-r)+k-d}{Lk-1-r} \;\le\; a \;\le\; \frac{d(Lt-1-r)}{Lk-1-r}. \tag{13}$$

To achieve average row weight at least $Lt-2$, a must be such that the total weights in the upper and lower blocks are at least $d(Lt-2-r)$ and $(k-d)(Lt-2)$, respectively. This is guaranteed when both $a(Lk-1-r) \ge d(Lt-2-r)$ and $(t-a)(Lk-1-r) \ge (k-d)(Lt-2)$, or equivalently,

$$\frac{d(Lt-2-r)}{Lk-1-r} \;\le\; a \;\le\; \frac{t(Ld-1-r)+2(k-d)}{Lk-1-r}. \tag{14}$$

Finally, to ensure that each block of rows is tall enough to accommodate the 1's specified in the construction, it suffices that $0 \le a \le d$ and $0 \le t-a \le k-d$, which together are equivalent to $\max(0, d+t-k) \le a \le \min(d,t)$. By our assumptions, this reduces to $0 \le a \le d$.

We now show that these inequalities can all be simultaneously satisfied. In 13, the upper bound is at least the lower bound if and only if $r \ge \frac{k-t}{t-d}$, which holds for our choice of r, and in 14, the upper bound is at least the lower bound if and only if $r \le \frac{k-t}{t-d} + \frac{k}{t-d}$, which holds since $\frac{k}{t-d} \ge 1$. Exchanging the upper/lower bound pairs in 13 and 14 results in two pairs of inequalities on a that are always satisfied. Finally, for $0 \le a \le d$, it suffices that at least one of the lower bounds is nonnegative, and at least one of the upper bounds is at most d. The lower bound in 14 is nonnegative if and only if $L \ge \frac{r+2}{t}$, and the upper bound in 13 is at most d if and only if $t \le k$, so it suffices to take $L \ge \frac{r+2}{t}$. The rest of the proof follows the same reasoning as in Proposition 21. ☐

5 Removing tuples

In this section we prove Theorem 13 and show how it implies Theorem 14.

Schaefer's dichotomy theorem (Theorem 4) allows us to obtain a simple description of all $\mathbf{T}$ with $\mathrm{CSP}(\mathbf{T})$ tractable and $\mathbf{t\text{-}in\text{-}k} \to \mathbf{T}$.

Proposition 25. *Let $k \ge 3$, $1 \le t < k$, and suppose that $\mathbf{t\text{-}in\text{-}k} \to \mathbf{T}$. Then $\mathrm{CSP}(\mathbf{T})$ is tractable if and only if*

1. *$0^k \in \mathbf{T}$ or $1^k \in \mathbf{T}$, or*
2. *t is odd, k is even, and $\mathbf{T} = \mathbf{odd\text{-}in\text{-}k}$.*

Observe that Proposition 25 in particular implies Proposition 7, NP-hardness of $\mathrm{CSP}(\mathbf{t\text{-}in\text{-}k})$.

Proof. In Case 1, $\mathrm{Pol}(\mathbf{T})$ contains a constant function so $\mathrm{CSP}(\mathbf{T})$ is tractable by Theorem 4, and in Case 2, tractability is given by Proposition 17.

We now turn to hardness. For the rest of the proof, assume that neither (1) nor (2) of the proposition statement applies.

Suppose that $\mathbf{t\text{-}in\text{-}k} \to \mathbf{T}$ by the function $\phi : \{0,1\} \to \{0,1\}$. If ϕ is constant, then either $\mathbf{T} = \{0^k\}$ or $\mathbf{T} = \{1^k\}$ and we are in case (1), a contradiction. If $\phi(x) = 1-x$, note that $\phi(\mathbf{t\text{-}in\text{-}k})$ satisfies conditions (1) and (2) precisely when $\mathbf{t\text{-}in\text{-}k}$ does, so it suffices to consider only the case where ϕ is the identity and $\mathbf{t\text{-}in\text{-}k} \subseteq \mathbf{T}$.

We show that $\mathrm{Pol}(\mathbf{T})$ contains none of the functions AND_2, OR_2, MAJ_3, and XOR_3 from Theorem 4. To accomplish this, we assume that one of these functions f is present in $\mathrm{Pol}(\mathbf{T})$, and then show that repeated application of f to a certain set of tuples leads to (1) or (2) of the proposition, a contradiction. Recall that we denote by $f(R)$ the image of the relation R under f. We make seven claims below, which together exclude the four functions as polymorphisms: case (i) covers AND_2, case (ii) covers OR_2, (overlapping) cases (iii) and (iv) cover MAJ_3, and cases (v), (vi), and (vii) cover XOR_3. Case (vii) contradicts (2) of the proposition; all others contradict (1).[4]

We give more details for case (i) for illustration. Taking AND_2 of the two tuples, we get $\mathrm{AND}_2(1^t0^{k-t}, 01^t0^{k-t-1}) = 01^{t-1}0^{k-t}$, a tuple of weight $t-1$. By symmetry, we obtain all tuples of weight $t-1$, which is the statement of case (i). Continuing this way, we obtain all tuples of weight $t-2$, $t-3$, etc. until we eventually obtain 0^k, which gives a contradiction as we assume that (1) of the proposition does not apply, so $0^k \notin \mathbf{T}$.

i $(\mathbf{t-1})$**-in-k** $\subseteq \mathrm{AND}_2(\mathbf{t}$**-in-k**$)$

tuples: 1^t0^{k-t} and 01^t0^{k-t-1}

eventual output: 0^k

ii $(\mathbf{t+1})$**-in-k** $\subseteq \mathrm{OR}_2(\mathbf{t}$**-in-k**$)$

tuples: 1^t0^{k-t} and 01^t0^{k-t-1}

eventual output: 1^k

iii if $t \geq 2$ then $(\mathbf{t+1})$**-in-k** $\subseteq \mathrm{MAJ}_3(\mathbf{t}$**-in-k**$)$

tuples: $1^{t-2}0^{k-t-1}110, 1^{t-2}0^{k-t-1}101$, and $1^{t-2}0^{k-t-1}011$

eventual output: 1^k

iv if $t \leq k-2$ then $(\mathbf{t-1})$**-in-k** $\subseteq \mathrm{MAJ}_3(\mathbf{t}$**-in-k**$)$

tuples: $1^{t-1}0^{k-t-2}100, 1^{t-1}0^{k-t-2}010$, and $1^{t-1}0^{k-t-2}001$

eventual output: 0^k

v if t is even, then $(\mathbf{t-2})$**-in-k** $\subseteq \mathrm{XOR}_3(\mathbf{t}$**-in-k**$)$

tuples: $1^{t-2}0^{k-t-1}110, 1^{t-2}0^{k-t-1}101$, and $1^{t-2}0^{k-t-1}011$

eventual output: 0^k

vi if t and k are odd, then $(\mathbf{t+2})$**-in-k** $\subseteq \mathrm{XOR}_3(\mathbf{t}$**-in-k**$)$

tuples: $1^{t-1}0^{k-t-2}100, 1^{t-1}0^{k-t-2}010$, and $1^{t-1}0^{k-t-2}001$

eventual output: 1^k

vii if t is odd and k is even,
then $(\mathbf{t+2})$**-in-k** $\subseteq \mathrm{XOR}_3(\mathbf{t}$**-in-k**$)$ if $t < k-2$,
and $(\mathbf{t-2})$**-in-k** $\subseteq \mathrm{XOR}_3(\mathbf{t}$**-in-k**$)$ if $t > 2$.

tuples:

$1^{t-1}0^{k-t-2}100, 1^{t-1}0^{k-t-2}010$, and $1^{t-1}0^{k-t-2}001$

[4] Another way of establishing this result for XOR_3 is via linear algebra (being closed under XOR_3 is the same as being an affine subspace) and for MAJ_3 from the fact that relations closed under MAJ_3 are determined by their binary projections.