- For every $i < \lambda$, every $A \subseteq B$, $|A| < \lambda$, every type over $A \cup \bar{d}_i$ is realized by some d_α
- For every $i < \lambda$, the set $B_i = B \cup d_i$ is complete, and $P^{B_i} = P^B$
- For every $\alpha < \lambda$, the type $\operatorname{tp}(d_\alpha / B \cup \{d_\beta \colon \beta < \alpha\})$ is λ-isolated

If we succeed, then clearly $\bar{d}_\lambda = \bigcup_{i<\lambda} \bar{d}_i$ is a λ-construction of a λ-saturated model N over B.

Let $\bar{d}_0 = \langle\rangle$.

For i limit, take unions. For $i = j + 1$, let $B_j = B \cup \bar{d}_j$. Let the sequence $\langle p_{j,\gamma} \colon \gamma \in [j, \lambda)\rangle$ list all the types over subsets of B_j of cardinality $< \lambda$ (recall that $\lambda = \lambda^{<\lambda}$). Now consider the sequence $\langle p_{\ell,j} \colon \ell < i\rangle$.

Recall that by induction, $\bar{d}_j = \langle d_\alpha : \alpha < \alpha_j\rangle$. Now for $\ell < i$ and $\alpha = \alpha_j + \ell$, let d_α realize $p_{\ell,j}$ such that $\operatorname{tp}(d_\alpha / B \cup \{d_\beta \colon \beta < \alpha\}) \in S_*(B \cup \{d_\beta \colon \beta < \alpha\})$ is λ-isolated (this is possible by Proposition 6.2).

Clearly, setting $\alpha_i = \alpha_j + i$, the sequence $\bar{d}_i = \langle d_\alpha : \alpha < \alpha_i\rangle$ is as required.

■

In the proof of Theorem 6.5, the assumption that $\lambda = \lambda^{<\lambda}$ was only used in in order to ensure that at any stage $i < \lambda$ of the construction, the number of types over small subsets of B_i is bounded by λ. For a specific theory T, this assumption may hold for other cardinals λ (for instance, if T has a saturated model of cardinality λ).

So for example, the same proof as above gives the following stronger result:

Corollary 6.6. *Let A be a stable set such that there exists a saturated model N of cardinality λ containing A with $P^N = P^A$. Then there exists M, $A \subseteq M \prec N$, M is λ-primary over A.*

7 From Stability of Models: Quantifier Free Definitions

The goal of this section is to establish the major technical tool of this paper: quantifier free definability of types orthogonal to P over stable sets. However, as we have already pointed out at the end of section 5, for this we will need an additional hypothesis.

In [She86], the first author has shown (see Theorems 2.10 and 2.12 there) that if there is an unstable *model*, then there is a forcing extension in which there are many M_i pairwise non-isomorphic with $M_i|P = M_j|P$ (all of cardinality $|P| > \aleph_0$). We will, therefore, following the Classification Theory guidelines, add yet another hypothesis to Hypothesis 2.7. Specifically, from now on we assume the following:

Hypothesis 7.1. *(Hypothesis 2). Every $M \prec \mathfrak{C}$ is stable over P (as in Definition 4.8(ii))*

Now we are ready to prove that *-types over all stable sets (not just models) are quantifier free internally definable, and therefore are also definable in $\mathfrak{C}$ in the usual sense.

Theorem 7.2. *If A is stable, $|A| \geq 2$, then for every $\psi(\bar{x}, \bar{y})$ there is a quantifier free $\Psi_\psi(\bar{y}, \bar{z}) \in L(T)$ such that whenever $B \equiv A$ and $p \in S_*(B)$ then $p|\psi$ is defined by $\Psi_\psi(\bar{y}, \bar{d})$ for some $\bar{d} \subseteq B$, i.e. $p|\psi = \{\psi(\bar{x}, \bar{a}) : \bar{a} \in B, B \models \Psi_\psi(\bar{a}, \bar{d})\}$.*

Proof: Let $\lambda = \lambda^{<\lambda}, \lambda > |A| + |T|$ to make things simple.

Also note that if A is stable and $\bar{c}$ is finite with $tp(\bar{c}/A) \in S_*^n(A)$, then $A \cup \bar{c}$ is also stable (as every $p(x) \in S_*^1(A \cup \bar{c})$ gives rise to some type $q(\bar{x}) \in S_*^{n+1}(A)$.)

Let A, p be a counterexample, and $\bar{c}$ realize p. We can find B saturated of power λ such that $(\mathfrak{C}|(A \cup \bar{c}), A, \bar{c}) \prec (\mathfrak{C}|(B \cup \bar{c}), B, \bar{c})$. Clearly $B, \bar{c}$ form a counterexample too, and in particular $tp(\bar{c}/B) \in S_*(B)$. We will arrive at a contradiction by showing how to construct the required quantifier free definition. By Proposition 4.13 there is a model $M, P^M \subseteq B \cup \bar{c} \subseteq M$. $Th(M, B, \bar{c})$ has a saturated model of power λ preserving the relevant properties. So without loss of generality $(M, B, \bar{c})$ is saturated and $|M| = |B| = \lambda$.

By Theorem 6.5 and Remark 6.4, there is a λ-saturated model $N, P^N \subseteq B \subseteq N$ such that for every $\bar{d} \subseteq N$, $tp(\bar{d}, B)$ is λ-isolated, say over $B_{\bar{d}} \subseteq B, |B_{\bar{d}}| < \lambda$, and a construction $N = \{d_i : i < \lambda\}$ such that $tp\{d_i/B \cup \{d_j : j < i\})$ is λ-isolated. In particular, N is λ-prime over B.

Hence we can embed N into M over B. So without loss of generality $N \prec M$, and in particular $P^M = P \cap B = P \cap N = P^N$ and $tp(\bar{c}/N) \in S_*(N)$.

Hence there are formulas $\Psi_\psi(\bar{x}, \bar{e}_\psi) \in L(T), \bar{e}_\psi \subseteq N$ defining $tp_\psi(\bar{c}/N)$ for $\psi \in L$. Let $E = \bigcup_{\psi \in L(T)} \bar{e}_\psi \subseteq N$ and $B^* = \bigcup_{\bar{d} \subseteq E} B_{\bar{d}}$. So $|E| \leq |T|$ and $|B^*| < \lambda$.

Now, if $\bar{b}_1, \bar{b}_2 \in B$ realize the same type over B^* (in $\mathfrak{C}$), then they realize the same type over $B^* \cup E$ by choice of E. Hence, they realize the same type over $B^* \cup E \cup \bar{c}$.

For $\psi \in L(M, B)$ let

$$\Gamma_\psi = \{\psi(\bar{c}, \bar{y}_1) \equiv \neg\psi(\bar{c}, \bar{y}_2)\} \cup \{\chi(\bar{y}_1, \bar{d}) \equiv \chi(\bar{y}_2, \bar{d}) : \chi \in L(T), \bar{d} \subseteq B^*\} \cup \{\bar{y}_1 \bar{y}_2 \subseteq B\}$$

.

By the previous observation Γ_ψ is not realized in (M, B). By the fact that (M, B) is λ-saturated, and $|\Gamma_\psi| < \lambda$, it is inconsistent. By compactness there are $\chi_1, \ldots, \chi_n \in L(T)$ and $\bar{d}_1, \ldots, \bar{d}_n \in B^*$ such that

$$\Gamma_\psi^1 = \{\psi(\bar{c}, \bar{y}_1) \equiv \neg\psi(\bar{c}, \bar{y}_2)\} \cup \{\bar{y}_1 \bar{y}_2 \subseteq B\} \cup \{\chi_l(\bar{y}_1, \bar{d}_l) \equiv \chi_l(\bar{y}_2, \bar{d}_l) : l = 1, \ldots, n\}$$

is inconsistent.

So we can define $tp_\psi(\bar{c}/B)$ since

$$\models \psi(\bar{c}, \bar{b}) \Leftrightarrow [\{l : 1 \leq l \leq n, \models \chi_l(\bar{b}, \bar{d}_l)\} \text{ is in} P^*]$$

for some appropriate $P^* \subseteq \mathcal{P}\{1, \ldots, n\}$. Now apply compactness as in [Sh:c,II§2]. ∎

Note that we have used the assumption that models are stable in the proof.

Theorem 7.3. *Let A be complete and $\lambda = \lambda^{<\lambda}$. The following are equivalent:*

(i) *A is stable.*

(ii)$_\lambda$ *If $A' \equiv A$ is λ-saturated, $\lambda = |A'| > |T|$, then over A' there is a λ-primary model M.*

(iii)$_\lambda$ *If $A' \equiv A$ is λ-saturated, $\lambda > |T|$, then every m-type p over A, $|p| < \lambda$ can be extended to a λ-isolated $q \in S_*(A')$.*

(iv) *For every $A' \equiv A$ and $p \in S_*(A)$ and $\phi \in L(T)$, $p|\phi$ is definable by some $\Psi_\phi(\bar{y}, \bar{a}), \bar{a} \subseteq A, \Psi_\phi \in L(T)$.*

(v) *There is some collection $\langle \Psi_\phi; \phi \in L \rangle$ such that for every $A' \equiv A, p \in S_*(A')$ and $\psi \in L(T), p|\psi$ is definable by $\Psi_\psi(\bar{y}, \bar{a})$ for some $\bar{a} \in A'$.*

So $(ii)_\lambda, (iii)_\lambda$ do not depend on λ.

Proof: Included in the proofs of Theorem 7.2, Lemma 6.1, and Theorem 6.5. ∎

Theorem 7.4. *(T countable) If A is stable, $\bar{a} \in A$, and $\models \exists x \theta(\bar{x}, \bar{a})$, then there is $p \in S_*(A)$ such that $\theta(\bar{x}, \bar{a}) \in p$ and for every $\phi \in L(T)$ there is $\psi(\bar{x}, \bar{a}') \in p$ such that $\psi(\bar{x}, \bar{a}') \vdash p|\phi$ (i.e. p is locally isolated, i.e $\mathbf{F}^l_{\aleph_0}$-isolated. So the locally isolated types are dense in $S_*(A)$.)*

Proof: Again this is contained in the proofs of Theorem 7.2 and Lemma 6.1. ∎

8 Stationarization and Independence

The following definition mimics the Tarski-Vaught criterion, when one does not demand $A, B \prec \mathfrak{C}$.

Definition 8.1. $A \subseteq_t B$ if for every $\bar{a} \in A, \bar{b} \in B$ and $\psi \in L(T)$ such that $\models \psi(\bar{b}, \bar{a})$ there is some $\bar{b}' \subseteq A$ such that $\models \psi(\bar{b}', \bar{a})$

As a simple example, note that A is complete if and only if $A \cap P \subseteq_t P$.

We can now define "free" ("non-forking") extensions for *-types over stable sets. Such extensions will be defined only to supersets that are "elementary extensions" in the sense defined above.

The use of the term "non-forking" above is not just by analogy with classical stability theory, but (at least under certain circumstances, e.g., when A