Classification over a predicate - the general case Part I - structure theory

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Abstract

We begin a systematic development of structure theory for a first order theory, which is stable over a monadic predicate. We show that stability over a predicate implies quantifier free definability of types over stable sets, introduce an independence notion and explore its properties, prove stable amalgamation results, and show that every type over a model, orthogonal to the predicate, is generically stable.

1 Introduction

The classical Classification Theory [She90] deals with a first order theory T, how complicated its models can be, and to which extent they can be characterized by cardinal invariants.

For algebraically closed fields, or for divisible abelian groups, there is such a structure theory. In general, there is a division into theories that have a structure and such for which we have a non-structure theorem.

Consider now vector spaces over a field. Do we have a structure theory? It depends on how you ask the question: We have a structure theory for the vector space over the field, but not necessarily for the field.

Problem: Classify pairs (T, P) where T is a first order theory and P is a monadic predicate. We want to know how much $M \models T$ is determined by $M|P^M$. So in case T is the two-sorted theory of a vector space over a field, and P is the predicate for the field, we ask how much we can know about a vector space over a field F once F is flixed (obviously, we know quite a bit, especially if the cardinality of the vector space is fixed).

Although at the first glance the problem above may appear close to classical (first order) model theory, this context actually exhibits behavior which

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is more similar to that of some non-elementary classes (classes of models of a sentence in an infinitary logic, or abstract elementary classes). See e.g. Hart and Shelah [HS90]. An intuitive "explanation" for this is that fixing P is similar to insisting on omitting a certain type (the type enlarging P), which immediately puts one in a non-elementary context.

Much work has already been done on classification theory over P. Relative categoricity for particular theories was investigated by e.g. Hodges et al [Hod99, HY09, Hod02]; countable categoricity over P was studied by Pillay [Pil83]. Pillay and the first author laid the foundations for the study of stability and related properties in this context in [PS85]. Here we are going to continue their investigation and build upon their results.

In [She86] the first author proved an analogue of Morley's Theorem over P under the assumption of "no two-cardinal models", which means that for all $M \models T$, $|P^M| = |M|$. However, this assumption is very strong. Even for the the case of uncountable categoricity (a notion that we discuss below in Definition 1.3), it would be nice to be able to prove an analogue of Morley's Theorem without making this assumotion it *a priori*. Furthermore, the "no two cardinal model" assumption does not hold in many natural examples that should "morally" be quite tame. For example, the general theory developed in [She86] does not even cover the example of a vector space over a field, already mentioned briefly above (Example 1.2). Apparently, from the point of view of model theory, this example is not that easy (but is quite instructive).

A much more interesting example that we hope will eventually be included in our treatment is the theory of so-called "Zilber's field", more precisely, the theory of exponentially closed fields of characteristic 0 [Zil05, KZ14, Hen14]. In our context, one considers the theory of an exponentially closed field (F, e^z) over P, which is the kernel of the exponential function e^z. For example, it follows from the results in [KZ14] that this theory is stable over P.

Another natural example to consider in this setting is the theory of algebraically closed field of characteristic 0 with a generic automorphism σ, where P is the fixed field of σ. This theory has been studied by Chatzidakis in a recent preprint [Cha20]. This example is different than the ones mentioned above, since in this case P is model theoretically "tame" (it is pseudo-finite), and the full theory T is also very well understood (it is simple, has QE, etc). Nevertheless, considering it in this new framework, offers further insight.

Some results in [Cha20] can be derived directly from the more general analysis obtained in our paper. For example, Chatzidakis proves (Theorem 3.14 in [Cha20]) the existence of a κ-prime κ-atomic model over an algebraically closed difference field of characteristic 0 with a κ-saturated and pseudo-finite P-part (for κ uncountable or $\aleph_\varepsilon$). We prove an analogous result in our general context (see Theorem 6.5 and Corollary 6.6 thereafter) for the case that the P-part is saturated (of cardinality κ). So in particular, the case $\kappa = \aleph_\varepsilon$ is not covered. A more nuanced analysis, which would imply stronger results closer to Theorem 3.14 in [Cha20] goes beyond the scope of this article. This will be investigated in a future work.

Let us make the above discussion a bit more precise.

One rough measure of complexity of a theory is (as usual) the number of its non-isomorphic models in different cardinalities:

Definition 1.1. (i) $I(\lambda, N)$ = cardinality of $\{M/_{\cong_N} : M \models T, M|_P = N, |M| = \lambda\}$, where $\cong_N$ means isomorphic over N and $M/_{\cong_N}$ denotes the isomorphism class.

(ii) $I(\lambda, \mu) = \sup\{I(\lambda, N) : |N| = \mu\}$.

(iii) $I(\lambda) =: I(\lambda, \lambda)$.

Example 1.2. (Back to vector spaces over a field). Let T be the theory of two-sorted models (V, F) where V is a vector space over the field F, and let P be a predicate for the field F. If the cardinality of F is $\aleph_\alpha$, then $I(\aleph_\alpha, F) = |\omega + \alpha|$ and for $\beta > \alpha$, $I(\aleph_\beta, F) = 1$.

So we want to divide the pairs (T, P) according to how much freedom we have to determine M knowing $M|P$.

Definition 1.3. (i) (T, P) is *categorical in* (λ_1, λ_2) when the following holds: If $M_1|P = M_2|P, M_1$ and M_2 are models of T and $|M_l| = \lambda_1$, $|P^{M_l}| = \lambda_2$ for $l = 1, 2$, then M_1 and M_2 are isomorphic over P^{M_l}.

(ii) We write *categorical in* λ instead of (λ, λ).

(iii) We say *totally categorical* if (ii) holds for all λ.

For our purpose, categorical pairs are the simplest. However, we will deal here with a more general context of *stability* over P. For example, the class of vector spaces over a field is not categorical in λ the sense of the definition above; still, it is *almost* categorical, and it would obviously be desirable to develop a general theory that covers this example.

Recall that much of the work in classification theory follows the following general recipe. First, we assume that the theory T (or, more generally, the class of models under investigation) has a particular "bad" model theoretic property: e.g., is unstable. Under this assumption, we prove a non-structure theorem: e.g., T has many non-isomorphic models of some big enough cardinality λ (normally, in many, if not all, such cardinalities). Since we are ultimately interested in the "good case" – for example, if we are trying to prove an analogue of Morley's Categoricity Theorem, we only care about theories (or classes) with few models – we may assume from now on that T falls on the "right" side of the dividing line: e.g., is stable. This way we can use good properties of stability in order to investigate properties of our class further.

As perhaps should be clear from the title, in this paper we focus on the development of *structure theory*. In particular, we do not prove any new non-structure results here. However, we do follow "recipe" described above. That is, we recall (mostly from [PS85] and [She86]) that certain "bad" properties (e.g. instability) imply non-structure, by which we normally mean many non-isomorphic models over P – sometimes only in a forcing extension of the