which proves that $F \cup (G' \circ G)$ is elementary.

Now clearly
$$F \cup G = (G')^{-1} \circ \left[F \cup (G' \circ G)\right]$$
is also elementary, and we are done.

■

Lemma 8.11. *Let A be λ-saturated and stable, $A \subseteq_t B$, N a λ-saturated model λ-atomic over A such that $N \mathop{\smile\!\!\!\!\!|}_A B$.*

Then $tp(N/A) \vdash tp(N/B)$; so the types $tp(N/A)$ and $tp(B/A)$ are weakly orthogonal.

Proof: Let $\bar{c} \in N$ and $\bar{c}' \in \mathfrak{C}$ such that $tp(\bar{c}/A) = tp(\bar{c}'/A)$. Assume towards contradiction that for some formula $\varphi(\bar{z}, \bar{b})$ (with $\bar{b} \subseteq B$) so that $\models \varphi(\bar{c}, \bar{b}) \wedge \neg\varphi(\bar{c}', \bar{b})$.

The type $tp(\bar{c}/A)$ is isolated by $\Theta(\bar{z})$, a partial type over a subset of A of cardinality less than λ.

Consider the following partial type:

$$\pi(\bar{y}) = \left\{\exists \bar{z}\bar{z}' \left[\theta(\bar{z}) \wedge \theta(\bar{z}') \wedge \varphi(\bar{z}, \bar{y}) \wedge \neg\varphi(\bar{z}', \bar{y})\right] : \ \theta \in \Theta\right\}$$

It is realized by $\bar{b} \in B$, hence, by 8.4(ii), it is also realized by some $\bar{a} \in A$.

Now consider the following type:

$$\left\{\theta(\bar{z}) \wedge \theta(\bar{z}') \wedge \varphi(\bar{z}, \bar{a}) \wedge \neg\varphi(\bar{z}', \bar{a}) : \ \theta \in \Theta\right\}$$

It is finitely satisfiable in N (precisely because $\bar{a} \models \pi(\bar{y})$), and since N is λ-saturated, it is realized by some $\bar{c}_1, \bar{c}_2$ in N (recall that Θ is over a "small" set). But $\Theta(\bar{z})$ implies a complete type over A; a contradiction.

■

9 Main consequences

Theorem 9.1. *(Stable Amalgamation for models). If $M_l, l = 1, 2$ is saturated of power λ (or just P^{M_0} is saturated), $P^{M_l} \subseteq M_0 \prec M_l$, then we can find $M \supseteq M_0 \supseteq P^M$ and elementary embeddings f_l of M_l into M over M_0 such that $tp(\bar{c}/f_2(M_2)) \in S_*(f_2(M_2))$ for all $\bar{c} \in f_1(M_1)$, and, moreover it is the stationarization of $tp(\bar{c}/M_0)$ over $f_2(M_2)$; that is, $f(M_1) \mathop{\smile\!\!\!\!\!|}_{M_0} f(M_2)$.*

If $\lambda = \lambda^{<\lambda}$, then M can be chosen to be saturated.

Proof: We can find an elementary mapping f_1 from M_1 to $\mathfrak{C}$ such that $f_1|_{M_0} = id$ and for all $\bar{c} \subseteq M_1$, $tp(f_1(\bar{c})/M_2)$ is the stationarization of $tp(\bar{c}/M_0)$: Since for $\bar{c} \in M_1, P^{M_1} \subseteq M_0 \cup \bar{c} \subseteq M_1$, $M \cup \bar{c}$ is complete, hence $tp(\bar{c}/M_0)$ is in $S_*(M_0)$, and by Lemma 8.5 has a stationarization $q_{\bar{c}}$ over M_2 ($M_0 \subseteq_t M_2$, of course). By the previous corollary all these types $q_{\bar{c}}$ are compatible (being a directed system), so we can define f_1 as an elementary map so that $f_1|_{M_0} = id, dom f_1 = M_1$ and $f_1(\bar{c})$ realizes $q_{\bar{c}}$: so for $c_1, \ldots, c_n \in M_1$,

$M_2 \cup \{f_1(c_1), \dots, f_1(c_n)\}$ is complete, $P \cap (M_2 \cup \{f_1(c_1), \dots, f_1(c_n)\}) = P \cap M_2$ by Lemma 8.5 (iii). Hence by Fact 4.4(i) $M_2 \cup f_1(M_1)$ is complete, $P \cap (M_2 \cup \{f_1(c) : c \in M_1\}) = P \cap M_2 = P \cap M_0$. As $P \cap (M_2 \cup f_1(M_1))$ is saturated of power λ, by Proposition 4.13 there is some M with $M_2 \cup f_1(M_1) \subseteq M$ and $P^M = P^M \cap (M_2 \cup f_1(M_1)) = P^{M_0}$ as required.

If $\lambda = \lambda^{<\lambda}$, then in Proposition 4.13, M can be chosen to be saturated. ■

We can now deduce amalgamation over stable sets with a saturated P-part.

Theorem 9.2. *(Stable Amalgamation for types over stable sets). Let A be stable and saturated (or just $A \cap P$ is $|A|$-compact). If $|T| < \lambda^{<\lambda} = \lambda = |A|$, then we have amalgamation in $S_*(A)$. That is, if $tp(\bar{a}\bar{b}/A) \in S_*(A), tp(\bar{a}\bar{c}/A) \in S_*(A), \bar{a}, \bar{b}, \bar{c}$ of length $< \lambda$, then for some $\bar{a}', \bar{b}', \bar{c}', tp(\bar{a}'\bar{b}'/A) = tp(\bar{a}\bar{b}/A), tp(\bar{a}'\bar{c}'/A) = tp(\bar{a}\bar{c}/A)$ and $tp(\bar{a}'\bar{b}'\bar{c}'/A) \in S_*(A)$.*

Proof: Note that $P \cap A\bar{a}\bar{b} = P \cap A$ is saturated, $A\bar{a}\bar{b}$ is complete. Hence by Proposition 4.13 we can find a model $M_{\bar{b}}$, λ-saturated of cardinality λ such that $A \cup \bar{a}\bar{b} \subseteq M_{\bar{b}}$, and $P^{M_{\bar{b}}} \subseteq A$. Similarly, we can choose $M_{\bar{c}}$, λ-saturated of cardinality λ with $A \cup \bar{a}\bar{c} \subseteq M_{\bar{c}}$, and $P^{M_{\bar{c}}} \subseteq A$. By Theorem 7.3 (ii) there is a model $M_{\bar{a}}$ of cardinality λ, $A \cup \bar{a} \subseteq M_{\bar{a}}$, and $P^{M_{\bar{a}}} \subseteq A$ such that $M_{\bar{a}}$ is λ-primary over $A \cup \bar{a}$. Hence there is an elementary embedding $f_{\bar{b}} : M_{\bar{a}} \to M_{\bar{b}}, f_{\bar{b}}|_{(A \cup \bar{a})} = id$. Similarly, there is an elementary embedding $f_{\bar{c}} : M_{\bar{a}} \to M_{\bar{c}}, f_{\bar{c}}|_{(A \cup \bar{a})} = id$. By the previous theorem, there are elementary mappings $g_{\bar{b}}, g_{\bar{c}}$ and a model M with $g_{\bar{b}} : M_{\bar{b}} \to M, g_{\bar{c}} : M_{\bar{c}} \to M$ and $g_{\bar{b}} \circ f_{\bar{b}} = g_{\bar{c}} \circ f_{\bar{c}}$. So in particular $g_{\bar{b}}|(A \cup \bar{a}) = g_{\bar{c}}|(A \cup \bar{a}) = id$ and $P^M \subseteq A$. Now $a^\frown g_{\bar{b}}(\bar{b})^\frown g_{\bar{c}}(\bar{c})$ is as required. ■

Definition 9.3. (i) We say that a model N is λ-full over a set A if:
N is saturated, $P^N \subseteq A$, and for every $B \subseteq N$, $|B| < \lambda$, every $p \in S_*(A \cup B)$ is realized in N.
If $\lambda = |A|$, we omit it.

(ii) We say that a model N is λ-homogenous for sequences if whenever $\langle a_i : i < \alpha \rangle, \langle b_i : i < \alpha \rangle, \alpha < \lambda$, realize the same type in N, then for every $a_\alpha \in N$ there is $b_\alpha \in N$ such that $\langle a_i : i \leq \alpha \rangle, \langle b_i : i \leq \alpha \rangle$ realize the same type in N.

Remark 9.4. *If N is λ-full over A, then $(N, a)_{a \in A}$ is λ-homogenous for sequences.*

Proof: Note that $P^N \subseteq A$, so for every sequence $\langle b_i : i \leq \alpha \rangle$, the type $tp(b_\alpha / A \cup \{b_i : i < \alpha\})$ is in $S_*(A \cup \{b_i : i < \alpha\})$. ■

Lemma 9.5. *Let A is stable, P^A λ-saturated with $\lambda = |A| = \lambda^{<\lambda} > |T|$, then there is M such that:*

(i) *$P^M \subseteq A$; and moreover*

(ii) *Every $p \in S_*(A)$ is realized.*

(iii) *M is λ-saturated of cardinality λ.*

Proof: Let $\langle p_i : i < \lambda \rangle$ list $S_*(A)$ (note: A is stable, $\lambda = \lambda^{|T|}$).

By induction on $i \leq \lambda$ choose A_i increasing continuously with $A_0 = A$ and $|A_{i+1} \setminus A_i| < \lambda$, such that A_i is complete, and A_{i+1} realizes p_i. Use Amalgamation over A (Theorem 9.2) to amalgamate A_i and $a_i \models p_i$ at successor stages (recall that $\lambda = \lambda^{<\lambda}$) and Fact 4.4(ii) for limit stages.

Since A_λ is complete, $P \cap A_\lambda = P \cap A$ saturated, by Proposition 4.13 there is M as required (since $\lambda = \lambda^{<\lambda}$, M is also saturated).

∎

Corollary 9.6. *If A is stable and λ-saturated with $\lambda = |A| = \lambda^{<\lambda} > |T|$, there is M of cardinality λ which is full over A.*

Proof: Let M_0 be as in the previous Lemma. Now construct M_i increasing (for $i < \lambda$) such that M_i satisfies the requirements (i) – (iii) of the Lemma with A there replaced with $\bigcup_{j<i} M_j$ (note that all models are stable, and $P^{M_i} = P^{M_0} = P \cap A$).

Clearly M_λ is as required (note that λ is regular). ∎

Corollary 9.7. *If A is stable and λ-saturated with $\lambda = |A| = \lambda^{<\lambda} > |T|$, there is M such that:*

(i) *M is λ-saturated of cardinality λ with $P^M \subseteq A$; and moreover*

(ii) *$(M, a)_{a \in A}$ is λ-homogenous for sequences and every $p \in S_*(A)$ is realized.*

Our next goal is a Symmetry Lemma for stationarizations over a model.

We begin by showing that every "Morley sequence" (that is, a sequence of stationarizations of a given type) has a certain weak convergence property (which may remind the reader of the behaviour of indiscernible sequences in dependent theories). After having proved symmetry, we will conclude true convergence (since we will know that every such sequence is in fact an indiscernible set). However, we need the weak convergence property for the proof of symmetry, hence we deal with it first.

Lemma 9.8. *(Weak Convergence over stable sets). Let $\langle A_i \colon i \leq \mu \rangle$ be a sequence of stable sets increasing continuously, $A \subseteq_t A_i$, $\bar{a}_i \subseteq A_{i+1}$ and $tp(\bar{a}_i/A_i)$ is the stationarization of $tp(\bar{a}_0/A)$. Let $\bar{c} \in A_\mu$ and $\psi(\bar{x}, \bar{z}, \bar{w})$ a formula, $\theta(\bar{z}, \bar{x}, \bar{w}) := \psi(\bar{x}, \bar{z}, \bar{w})$. Let Δ_2 be finite such that $n_\theta := R_{A_\mu}(\bar{x} = \bar{x}, \{\theta\}, \Delta_2, 2) < \omega$ (A_μ is stable, so such Δ_2, n_θ exist).*

Then there are $n \leq n_\theta$, $0 = i_0 < i_1 < \cdots < i_n = \lambda$, and $p_0(\bar{x}), \ldots, p_{n-1}(\bar{x})$ such that for all $m < n, i_m < i < i_{m+1}$ implies $tp_\psi(\bar{a}_i/\bar{c} \cup A) = p_m$.