Proof: By Fact 5.3 (i), $\langle R_{A_\mu}(tp(\bar{c}/A_\alpha), \{\theta\}, \Delta_2, 2) : \alpha < \mu\rangle$ is a non-increasing sequence of natural numbers $\leq n_\theta$.

So there are $n \leq n_\theta, 0 = i_0 < \cdots < i_n = \mu$ such that

$$i_l \leq \alpha \leq \beta < i_{l+1} \Rightarrow R_{A_\mu}(tp_\theta(\bar{c}/M_\alpha), \{\theta\}, \Delta_2, 2) = R_{A_\mu}(tp_\theta(\bar{c}/M_\beta), \{\theta\}, \Delta_2, 2).$$

Again by Fact 5.3 (more specifically, by the proof of Corollary 5.8 – the nature of the defining scheme), for each l there is $\bar{d}_l \in A_{i_l}$ and $\Psi_\theta(\bar{x}, \bar{w}, \bar{d}_l)$ which defines $tp_\theta(\bar{c}/A_{i_l})$ so that $\Psi_\theta(\bar{x}, \bar{w}, \bar{d}_l)$ actually defines $tp_\theta(\bar{c}/A_\alpha)$ for $i_l \leq \alpha < i_{l+1}$. But if $i_l \leq \alpha < \beta < i_{l+1}$, then $tp(\bar{a}_\alpha/A_{i_l}) = tp(\bar{a}_\beta/A_{i_l})$, hence for every $\bar{m} \subseteq A$, $\models \Psi_\theta(\bar{a}_\alpha, \bar{m}, \bar{d}_l) \equiv \Psi_\theta(\bar{a}_\beta, \bar{m}, \bar{d}_l)$, hence $\models \theta(\bar{a}_\alpha, \bar{m}, \bar{c}) \equiv \theta(\bar{a}_\beta, \bar{m}, \bar{c})$, as required.

∎

Theorem 9.9. *<u>(The symmetry theorem for models)</u>. If $tp(\bar{a}\bar{b}/M) \in S_*(M)$ where $tp(\bar{b}/M \cup \bar{a})$ is the stationarization of $tp(\bar{b}/M)$, then $tp(\bar{a}/M \cup \bar{b})$ is the stationarization of $tp(\bar{a}/M)$.*

Proof: Assume we have a counterexample. Let $\lambda = \lambda^{<\lambda} \geq |M| + |T|^+$. Without loss of generality M is saturated of cardinality λ. We define $\bar{a}_i, \bar{b}_i, M_i$ by induction on $i < \lambda$ such that $P^{M_i} = P^{M_0}$, M_{i+1} is λ-saturated of power λ, M_i increasing continuously, $\bar{a}_i\bar{b}_i \subseteq M_{i+1}$ and $tp(\bar{a}_i\bar{b}_i/M_i)$ is the stationarization of $tp(\bar{a}\bar{b}/M)$. This is straightforward. Let $M_\lambda = \bigcup_{i<\lambda} M_i$.

Since all models are stable, we are clearly in the situation of Lemma 9.8. Note that $P^{M_\lambda} = P^M \subseteq M$, hence by conclusion of the Lemma we get:

For every $\bar{c} \in M_\lambda$ and $\psi(\bar{x}, \bar{y}, \bar{z})$ there are $n \leq n_\theta$ and $0 = i_0 < i_1 < \cdots < i_n = \lambda$ and $p_0, \ldots, p_{n-1}$ such that for all $m < n, i_m < i < i_{m+1}$ implies $tp_\psi(\bar{a}_i\bar{b}_i/\bar{c} \cup P^M) = p_m$.

From the uniqueness and definability of stationarizations it follows that $\langle \bar{a}_i\bar{b}_i : i < \lambda\rangle$ is indiscernible over M_0. That is, if $i_0 < i_1 < \cdots < i_n < \lambda$ and $j_0 < \cdots < j_n < \lambda$ then $tp(\bar{a}_{i_0}\bar{b}_{i_0} \ldots \bar{a}_{i_n}\bar{b}_{i_n}/M_0) = tp(\bar{a}_{j_0}\bar{b}_{j_0} \ldots \bar{a}_{j_n}\bar{b}_{j_n}/M_0)$.

Now let $R = \{\bar{a}_i\bar{b}_i : i < \lambda\}$ and let $<$ be the order on R defined so that $\bar{a}'\bar{b}' < \bar{a}"\bar{b}"$ iff there are $i < j$ with $\bar{a}'\bar{b}' = \bar{a}_i\bar{b}_i$ and $\bar{a}"\bar{b}" = \bar{a}_j\bar{b}_j$. The model $(M_\lambda, R, <)$ has a saturated extension, $(M^*, R^*, <^*)$ of power λ. So for some linear order I, $R^* = \{\bar{a}_t\bar{b}_t : t \in I\}$ where $\bar{a}_s\bar{b}_s < \bar{a}_t\bar{b}_t$ whenever $I \models s < t$.

Note that n depends on ψ and $\bar{c}$, but the bound n_θ depends on ψ only. Therefore the following is true:

Fact 9.10. For every $\bar{c} \in M^*$ and $\psi(\bar{x}, \bar{y}, \bar{z})$ there are $n < n_\theta < \omega$ and $t_0 < \cdots < t_n$ where $t_0 \in I$ is the first element, and $p_0, \ldots p_n$ so that $t_l < t < t_{l+1}$ or $t_l < t, l = n$ implies $tp_\psi(\bar{a}_t\bar{b}_t/P^{M^*} \cup \bar{c}) = p_l$.

Now we shall finish proving the Symmetry Lemma: Since I is a λ-saturated linear order of power λ, it has 2^λ Dedekind cuts $\{(I_\alpha, J_\alpha) : \alpha < 2^\lambda\}$. Let $p_\alpha = \{\psi(\bar{x}, \bar{y}, \bar{d}); \bar{d} \in M^*$ and for some $s_\alpha \in I_\alpha, t_\alpha \in J_\alpha$, if $v \in I$ and $s_\alpha < v < t_\alpha$ then $\models \psi(\bar{a}_v, \bar{b}_v, \bar{d})\}$.

By Fact 9.10, p_α is a complete type over M^*. In fact, $p_\alpha \in S_*(M^*)$: indeed,if $[\exists \bar{z} \in P\psi(\bar{z}, \bar{m}, \bar{x}, \bar{y})] \in p_\alpha$ (where $\bar{m} \subseteq M^*$), then (by definition of p_α) there exist s_α, t_α such that for all $v \in (s_\alpha, t_\alpha)$, we have $\models \exists \bar{z} \in P\psi(\bar{z}, \bar{m}, \bar{a}_v, \bar{b}_v)$; hence M^* satisfies this sentence as well, so there exists such $\bar{d} \subseteq P^{M^*}$. A priori perhaps d depends on v; however, recall (Fact 9.10) that for some s'_α, t'_α all a_v, b_v (for $v \in (s'_\alpha, t'_\alpha)$) have the same ψ-type over $\bar{m} \cup P^M*$, so choosing d for any such v, we get that for all $v \in (s'_\alpha, t'_\alpha)$ the formula $\psi(\bar{d}, \bar{m}, \bar{a}_v, \bar{b}_v)$ holds, hence by the definition $\psi(\bar{d}, \bar{m}, \bar{x}, \bar{y}) \in p_\alpha$. Now by Observation 4.11, $p_\alpha \in S_*(M^*)$, as required.

If $i < j, tp(\bar{b}_j/M \cup \bar{a}_i) \subseteq tp(\bar{b}_j/M_j)$ is a stationarization of $tp(\bar{b}/M)$. By the uniqueness of stationarizations and the assumption that $tp(\bar{b}/M \cup \bar{a})$ is the stationarization of $tp(\bar{b}/M)$ it follows that $tp(\bar{b}_j\bar{a}_i/M) = tp(\bar{b}\bar{a}/M)$.

Similarly $tp(\bar{a}_1/M \cup \bar{b}_0)$ is the stationarization of $tp(\bar{a}_1/M)$ and hence $\neq tp(\bar{a}_0/M \cup \bar{b}_0)$ as we assumed that $\bar{a}\bar{b}$ form a counter-example to symmetry. So for some $\bar{e} \in M$ and θ we have $\models \theta(\bar{a}, \bar{b}, \bar{e}) \wedge \neg\theta(\bar{a}_1, \bar{b}_0, \bar{e})$. Therefore we get:

$$j \geq i \Rightarrow \models \theta(\bar{a}_i b_j); \qquad j < i \Rightarrow \models \neg\theta(\bar{a}_i b_j)$$

So $\theta(\bar{x}, \bar{b}_t) \in p_\alpha(\bar{x}, \bar{y})$ if and only if $t \in J_\alpha$. Hence if $\alpha \neq \beta$, then $p_\alpha \neq p_\beta$. So we have too many types in $S_*(M^*)$, contradicting stability of M^*. ■

Definition 9.11. (i) We say that $I = \{a_\alpha : \alpha < \alpha^*\}$ is *convergent* over a set A if it is an infinite indiscernible set such that for every ψ and $\bar{d} \in A$ there is some n_ψ such that the type $tp_\psi(\bar{a}_\alpha/P^{\mathfrak{C}} \cup \bar{d})$ is the same for all but $\leq n_\psi$ many α's.

(ii) For any such I let the *average* of I over A be $Av(I/A) = \{\phi(\bar{x}, \bar{b}) : \bar{b} \in A, (\exists^\infty \alpha)\phi(\bar{a}_\alpha, \bar{b})\}$.

Corollary 9.12. *If $\alpha \geq \omega$ and for $i \leq \alpha, tp(\bar{a}_i/M \cup \bigcup_{j<i} \bar{a}_j)$ is a stationarization of $tp(\bar{a}/M)$, then $\{\bar{a}_i : i < \alpha\}$ is an indiscernible set over M. That is, if $i_1, \ldots, i_n < \alpha$ are distinct, then $tp(\bar{a}_{i_1} \ldots \bar{a}_{i_n}/M) = tp(\bar{a}_1 \ldots \bar{a}_n/M)$. In addition, if $I = \{\bar{a}'_\alpha : \alpha < \alpha_0\}$ is an indiscernible set with the same type over M, it is convergent over any $M' \succ M$, and for $M' \succ M, Av(I/M')$ is the stationarization of $tp(\bar{a}_0/M)$ over M'.*

Proof: It follows from Symmetry (Theorem 9.9) that any such sequence is an indiscernible set over M. Now a standard argument shows that for indiscernible sets weak convergence (Lemma 9.8) implies convergence. ■

We can therefore conclude:

Corollary 9.13. *The global stationarization of a type orthogonal to P over a model $p \in S_*(M)$ is a generically stable type, as defined in [PT11, GOU13].*

Proof: First note that since M is a model, p has a (unique) global stationarization p^*, which is definable (hence invariant) over M. By the previous Corollary, every Morley sequence in p^* is convergent over $\mathfrak{C}$, in particular, p^* is generically stable (as defined in Definition 1 of [PT11]). ■

One can now use some basic machinery of generic stability to throw more light on *-types over models and the concept of stationarization, for example:

Corollary 9.14. *Let M be a model and $p \in S_*(M)$. Then the unique global stationarization of p is also the unique global non-forking extension of p.*

Proof: By Proposition 1 in [PT11]. ∎

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