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Developing Functional Thinking: from Concrete to Abstract Through an Embodied Design

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Abstract
In addressing the challenge of fostering functional thinking (FT) in secondary school students, our research centered on the question of how an embodied design can enhance FT's different aspects, including input–output, covariation, and correspondence views. Drawing from embodied cognition theory and focusing on an action- and perception-based task design that uses light ray contexts and different function representations, we developed a digital-embodied learning environment, using the nomogram as a central representation. Our pilot study involving four eighth-grade students provided insights into their physical interactions with these modules through a multi-touch digital interface. Analysis of video and audio recordings from the pilots, including students' hand gestures and verbal expressions, was guided by comparing hypothetical learning activities with the actual learning activities. The results show that (1) a concrete light ray context enables students to ground the abstract mathematical function concept; (2) the bimanual coordinating motion tasks, incorporating the covariation aspect of FT, allow students to connect their bodily experience with function properties; and (3) our embodied and dragging tasks support insight in the conversion between nomograms and graphs of functions, encouraging students' correspondence thinking by providing multiple perspectives to understand, reason about, and manipulate the function. In conclusion, our findings suggest the potential of digital-embodied tasks in fostering FT, evident in students' diverse strategies and reasoning.

Keywords Educational technology · Embodied design · Functional thinking · Mathematics education · Nomogram

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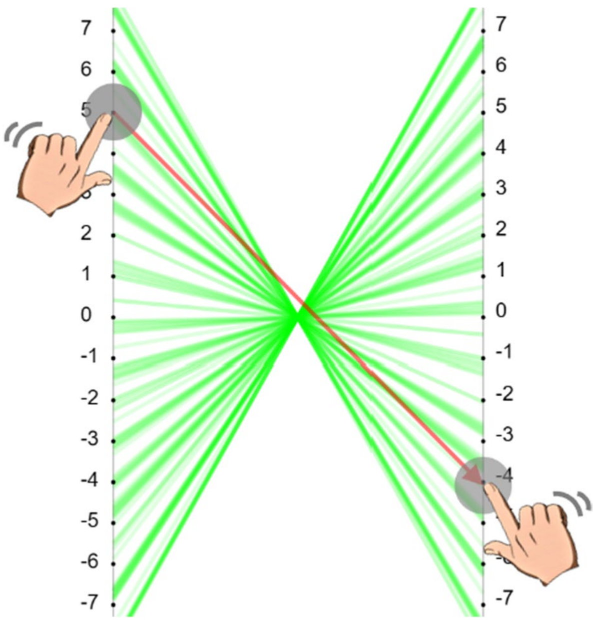
Introduction

A mathematical function is an abstraction—it is an abstract concept representing a relationship between input and output, irrespective of the concrete meanings of those inputs and outputs. Still, concrete contexts can make abstract functions more meaningful to students, providing opportunities to attach meaning to the mathematical constructs the students develop (Van den Heuvel-Panhuizen & Drijvers, 2020). Students need FT—thinking in terms of relationships, interdependencies, and change—for both later professional and daily life (FunThink team, 2021). One of the main challenges teachers face is how to foster the different aspects of FT, especially the abstract ideas of variation and covariation that lay the epistemological foundation for students to develop robust conceptions of functions (Thompson & Carlson, 2017).

A specific representation called a nomogram may play an important role in the development of FT. Nomograms support FT by incorporating various representations and contexts and have been emphasized by Thompson and Carlson (2017) for their importance in incorporating number lines and uniting two quantities' values in future study. The initial use of nomograms within a digital learning environment can be traced back to Nachmias and Arcavi (1990), who termed it the parallel axes representation (PAR). Various forms of this function representation exist, such as the horizontally oriented dynagraph (Sinclair et al., 2009). In our current research, and increasingly prevalent in recent studies, we use the term "nomogram" to describe two number lines linked by a bundle of arrows, which shows how each input number on the left number line corresponds to an output number on the right (see Fig. 1). It is a useful tool for developing FT due to its visual nature and ability to represent functional relationships using arrows.

Fig. 1 Movement on a nomogram



The integration of embodied actions within abstract mathematical thinking enriches meaning-making processes by incorporating various sensory channels such as perceptual, auditory, tactile, and kinesthetic (Radford, 2009). Bimanual movement, referring to the coordinated use of both hands, has been incorporated into mathematical education as a means to foster an understanding of mathematical concepts, such as proportion (Abrahamson et al., 2016). The importance of the bimanual movement in mathematics education originates from Piaget's work, which proposed that children's understanding of their world and the concepts within it is deeply rooted in their physical interactions with the environment. More recent work in the field of embodied cognition expands upon Piaget's theories, arguing that our understanding of abstract mathematical concepts is grounded in the physical and sensorimotor experiences of our bodies (Abrahamson & Lindgren, 2014; Tall, 2004).

In the context of our digital learning environment that features nomograms, the multi-touch and real-time feedback capabilities of digital technology (DT) enable students to investigate and construct nomograms using bodily movements, specifically a bimanual dragging motion (Fig. 1). Correctly positioning the two points on respective number lines causes these arrows to change to green, with their trajectory remaining visible, allowing for real-time feedback akin to the principles seen in the Mathematical Imagery Trainer for Proportion (MIT-p) (Abrahamson & Trninic, 2011). By traversing the entire nomogram, the students can visualize the complete function, which may include intersecting arrows, parallel arrows, and other distinctive features. However, if the placement of the two points does not accurately represent an input–output pair, the arrow between them will turn red. Inspired by the MIT-p, our evolved digital-embodied nomograms, potentially named MIT-f(x), extend to various functional relationships, making it a more versatile tool for developing FT. It achieves a range of instructional strategies, from using number lines and fostering corresponding quantities' relationship to reinforcing the facilitation of smooth continuous covariational and correspondence reasoning by providing continuous movements on the two number lines. In this regard, the main research question guiding our study is as follows: *How can an embodied design using nomograms foster functional thinking?* In order to explore the multifaceted nature of FT, we have formulated the following specific research questions through the lens of three aspects of FT: input–output, covariation, and correspondence:

- RQ1: How does a light ray context foster the students' meaning-making of nomograms?
- RQ2: How do bimanual movement tasks foster covariational thinking?
- RQ3: How do different function representations and their conversions support a correspondence view on functions?

Theoretical Framework

Two theoretical lenses—FT and embodied learning—serve as the foundations for our investigation in this study. By intertwining these theoretical perspectives, we aim to examine how the interactive, dynamic, and physically engaging nature of embodied learning can bolster FT.

Functional Thinking

FT, as a process of describing, building, and reasoning about/with functions (Pittalis et al., 2020; Stephens et al., 2017; Thompson & Carlson, 2017), consists of four main aspects: input–output thinking, covariation thinking, correspondence thinking, and mathematical object thinking (Confrey & Smith, 1995; Doorman et al., 2012; Vollrath, 1986; Wei et al., 2023). These four aspects indicate how to understand the concept of mathematical function through different characteristics of functions:

1. Input–output thinking: A function is regarded as an input–output assignment that helps organize and carry out a calculation process (Doorman et al., 2012). It is considered as the initial stage of understanding function, especially with the help of a special representation, an arrow chain (Freudenthal, 1983). Moreover, recognizing patterns and structures is linked to this aspect. For example, the recursive pattern is seen as how to find a number in a sequence when the previous number or numbers are given (Frey et al., 2022; Stephens et al., 2017).
2. Covariation thinking: This aspect emphasizes the relationship between two variables, primarily focusing on how changes in the independent variable cause corresponding shifts in the dependent variable. The emphasis is on the simultaneous change or movement of both variables (Confrey & Smith, 1995; Doorman et al., 2012; Thompson & Carlson, 2017).
3. Correspondence thinking: It is more about the pairing relationship between the two variables and being able to represent them with multiple representations, such as arrow chains, tables, graphs, formulas, and phrases (Doorman et al., 2012). Correspondence thinking highlights that each value of the independent variable aligns with a unique value of the dependent variable. Instead of emphasizing the simultaneous change, as in covariation, correspondence thinking underscores the direct association or pairing of values between the two variables (Pittalis et al., 2020; Smith, 2008). This aspect incorporates the mapping view, facilitating a holistic understanding of functional relationships (FunThink team, 2021).
4. Mathematical object thinking: A function, in this aspect, is seen as a mathematical object with its own representations and properties. Within this aspect, a function is recognized as part of a family of functions (Sfard, 1991), subject to higher-order operations such as composition, transposition, and differentiation. Within the scope of our study, this aspect receives minimal emphasis.

Our focus in this paper is specifically on the first three aspects of FT. Various aspects of FT are embedded in our task based on function representations, like tables, graphs, and formulas. Some studies argue that dynamic visualizations could be significantly more beneficial for learning functions than static representations (Brown, 2015; Falcade et al., 2007; Lindenbauer, 2019, Ten Voorde et al., 2023). For example, some tasks with interactive dynamic visualizations support students in observing and exploring the influence of parameters on function graphs. In addition, there are other advantages of dynamic visualizations, such as providing the possibility to create more interesting learning environments and facilitating understanding of the relationship and transitions between different representations (Günster & Weigand, 2020; Lindenbauer, 2019; Rolfes et al., 2020; Roux et al., 2015). Therefore, the aforementioned interactive, dynamic digital-embodied nomogram serves as a central feature.

In addition, we considered the design heuristic of the emergent model (Gravemeijer, 1999). The emergent model includes four levels of activity: task setting, referential, general, and formal. To support the design hierarchy of our tasks, we work with emergent modeling activity in three levels: situational, referential, and general, following our previous literature study (Wei et al., 2023). The use of these three levels will be further discussed in the Design section, with a combination of hypothetical learning trajectories.

Embodied Learning

Embodied learning is an educational approach that integrates bodily movements and physical experiences into the learning process. It operates on the premise that cognition not only is confined to the mind but also involves the entire body (Barsalou, 1999; Lakoff & Núñez, 2000). The theory of embodied cognition is foundational to understanding the mechanisms and efficacy of embodied learning, as it provides the theoretical underpinning for how bodily engagement can enhance cognitive processes. According to this perspective, cognitive activities such as problem-solving, memory, and learning are not just abstract mental tasks but are connected to sensory-motor systems. This connection implies that physical actions and sensory-motor experiences can shape and facilitate cognitive processes (Barsalou, 2008; Glenberg, 1997). Highlighting the role of the body in cognition and learning, embodied learning has gained quite some attention in mathematics education research (Bos et al., 2022; Drijvers, 2019; Lakoff & Núñez, 2000; Shvarts et al., 2021). The theoretical foundations concerning embodied learning in our study include the following two aspects: Embodied design and embodied instrumentation. Embodied design leverages embodied cognition to create learning environments and materials that prompt students to engage physically and perceptually with mathematical concepts. Similarly, embodied instrumentation combines embodied cognition with an instrumental approach to learning, emphasizing the coupling between the learner, the physical tools or artifacts, and the tasks at hand (Shvarts et al., 2021).

Embodied Design

Regarding the design and use of embodied cognition in the mathematics classroom, Abrahamson (2009) introduced a well-defined notion of embodied design (a term first coined by Van Rompay and Hekkert (2001)) as a systematic and procedural design method. It consists of two types: action-based design and perception-based design (Abrahamson, 2009; Abrahamson & Lindgren, 2014).

Action-based designs aim to ground mathematical concepts in students' natural capacity to adaptively solve sensorimotor problems. In action-based design, the sense of meaning comes from being able to achieve a target outcome using both a naive and an instrumented strategy with a technological system. For example, to teach the concept of a parabola, learners can be encouraged to manually plot a series of green isosceles triangles, which collectively form a U-shaped trace (Palatnik et al., 2023; Shvarts & Abrahamson, 2019). In this case, the sensorimotor coordination pattern manifesting the parabola concept necessitated preserving the triangle by equating the distances from a point to the directrix (CB) and focus (CA) (see Fig. 2). This task requires students to engage in a physical exploration of the parabola's geometric properties, specifically its reflective symmetry and the definition involving distances to the focus and directrix. This method allows them to intuitively grasp the parabola shape by acting on, combining naive (manually tracing) and instrumented (keeping isosceles triangles equal to learn about the parabola) strategies within a technological system (the graph and its representation of geometric figures).

Perception-based designs aim to ground mathematical concepts in students' natural perceptual ability in their naive perceptions of a situation. Like the action-based genre, it is followed by a phase of reflection in which these views are developed. This approach involves the manipulation of learners' perceptual fields or having them engage in activities where they discern patterns, identify relationships, or perceive variations. For instance, in a study on teaching the gradient using an augmented reality sandbox (Bos et al., 2022), students were invited to roll a marble down a plane while adjusting the plane's orientation and steepness (Fig. 3). The sandbox projected real-time height lines onto the plane, with the marble's trajectory perpendicular to these lines, indicating the steepest direction. In perception-based design, the sense of meaning arises when someone can make the same inferences from both direct and indirect observations of a given phenomenon. From the rolling marble experience, students are invited to consider the fact that the height lines and the gradient are perpendicular. Action-based and perception-based designs help create a richer, multisensory learning environment where learners can make sense of abstract concepts by enacting and perceiving them physically.

Building upon the existing literature on embodied design, our study also draws on the embodied-design procedure to inform our task design. Three fundamental steps are emphasized by Abrahamson (2014), namely, *phenomenalization*, *concretization*, and *dialog*. *Phenomenalization* involves creating an intuitive situation

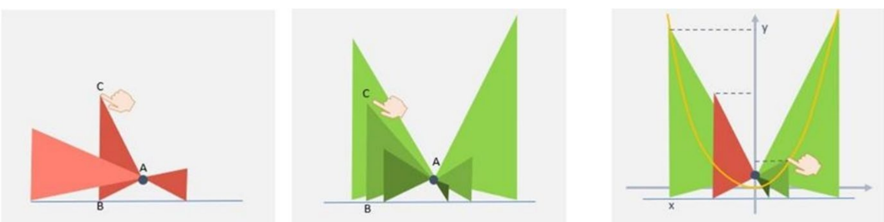


Fig. 2 Sensorimotor coordination patterns of a parabola $y = x^2$ (Palatnik et al., 2023, p.170)

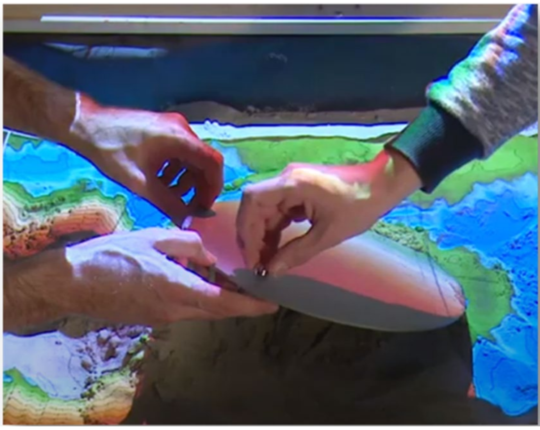


Fig. 3 Rolling a marble down a plane in the augmented reality sandbox (Bos et al., 2022)

related to the topic being learned. It starts by identifying a generic schema or pattern underlying the topic and then developing a scenario where applying this schema provides a solution. *Concretization* involves creating a visual model of the situation. The goal here is to decide on a formal disciplinary model related to the problem, devise a visual version of it, identify symbols that can represent the learner's solution strategy, and create incentives for the learner to use these symbols to understand the problem. In the *dialog* stage, the learner is guided through the process of using informal actions to solve the problem situation, constructing a formal visual solution, and reflecting on the relationship between their intuitive understanding and the visualization of the situation. The application of embodied design in mathematics education can profoundly reshape teaching and learning experiences. By weaving these stages into our task design (see details in the Design section), we aim to offer a more accessible and engaging learning experience, promoting a deeper comprehension of FT.

Embodied Instrumentation

In line with the design views of this study, we considered the embodied instrumentation theory (Drijvers, 2019) for the elaboration of task designs. As a combination of embodied cognition and an instrumental approach, embodied instrumentation underscores the amalgam between the body, artifact, and the cognitive scheme involved when DT is used in mathematics education. The term "instrumentation" here refers to the process through which an artifact (a tool, technology, etc.) becomes a part of the learner's conceptual scheme. A scheme is an invariant organization of activity for a certain type of situation (Vergnaud, 1998). Expanding on this idea, Shvarts et al. (2021) emphasize the nuanced and complex nature of action regulation, which occurs through dynamic functional systems involving both the body and the artifact in perception–action loops (Fig. 4). Significantly, perception–action loops are the lynchpin of this body-artifact functional system. These loops are central to a complex dynamic system

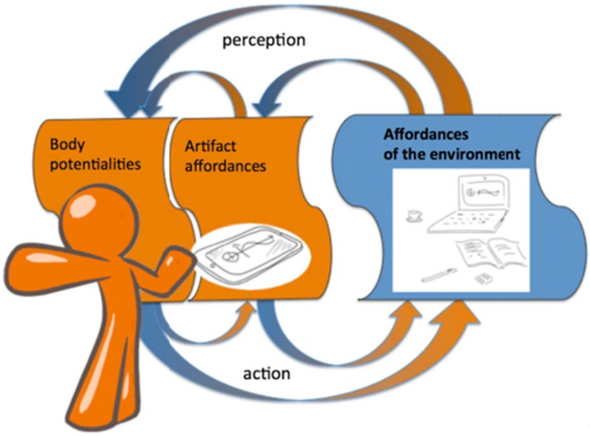


Fig. 4 Body-artifact functional system in interaction with the environment (Shvarts et al., 2021, p.451)

of behavior, with perception and action existing as intertwined processes within the interaction and coupling with the learning environment. For example, initial perception emanates from the interaction with and/or observation of the artifact, guiding the learners' actions at the same time. Concurrently, the actions reciprocally offer feedback and verification, thereby generating a new perception or preserving the existing one. Unlike conventional mental schemes, these functional systems are decentralized and can be expanded through the inclusion of artifacts. In this context, an artifact not only is an external tool but also becomes a part of the system, contributing to the way learners interact with and comprehend mathematical concepts.

The aforementioned theories offer a framework to develop FT, particularly in the context of using digital technologies. The embodied design theory, with its bifocal approach of action-based and perception-based designs, promotes the grounding of mathematical concepts within students' sensory-motor coordination. The embodied instrumentation theory stresses the relationship between the body, artifact, and cognitive scheme when employing DT in mathematics education. These two theoretical perspectives both center on the principle of embodied cognition. They emphasize the need to consider students' physical interactions and perceptual experiences and underscore the role that artifacts play in shaping these experiences. The fusion of these theories directs the way of designing a natural and engaging learning environment for the development of FT.

Methods

To address the research questions, we conducted a design-based study. This study was structured into two main phases: the initial design phase, where we developed and refined the digital-embodied learning materials and the subsequent case study phase. In the case study phase, we observed and analyzed the experiences of two pairs of 14-year-old students.

Design

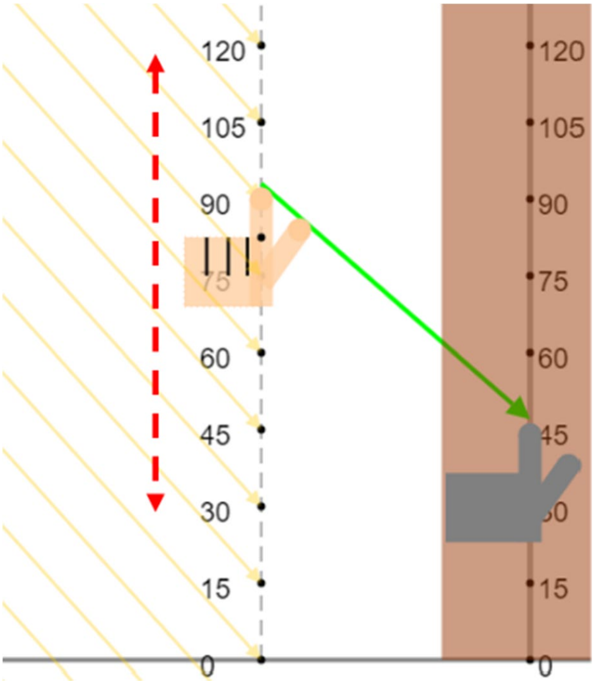
Our aim is to design a digital-embodied learning environment for fostering FT. As a first step, we drew on the principles of the embodied design framework (Abrahamson, 2014) to construct an overarching architecture for FT. This architecture, which is grounded in our theoretical foundations, includes three critical stages: *phenomenalization* — light ray context, *concretization* — embodied nomogram model, and *dialog* — a series of questions/tasks. The core concept is FT, and the implementation of a light ray context in our embodied design builds upon the historical use of nomograms, PAR, and dynagraphs in mathematics education (Nachmias & Arcavi, 1990; Sinclair et al., 2009).

This series of designs introduces the concept of function through a real-life context that is familiar to students, namely light rays. The innate connection between the light ray context and the nomogram model serves as the cornerstone of our design. Imagine a scenario with a sun or a light bulb illuminating an object, casting shadows in varying positions (Fig. 5).[1] We hypothesize that the students' existing familiarity with light rays and their effects, such as shadows, will help them understand the linear patterns that form the basis of nomograms. Specifically, light travels in straight lines, and objects can obstruct light, creating shadows. Therefore, students' perceptual experience that their shadow follows their bodily movement affects predictable object-shadow positions and can help them grasp the concept of predictable relationships between variables in nomograms. Students mathematize their intuitive perception of the situation using nomograms. The illustrative diagrams, including Fig. 5, are intended to support this foundational experience rather than to provide a comprehensive exploration of light and shadow geometry (Gravemeijer & Doorman, 1999).

We use the light ray context as the model for *phenomenalization*. In particular, two aspects of the context implicitly convey fundamental properties of the nomogram. First, the directedness of the light, from hand to shadow, carries over to the nomogram where arrows go from input element to image element. Second, in a nomogram, one only draws a finite number of arrows, even though arrows are virtually sprouting from every point on the input axis. The situation is the same for light rays: we draw only a finite number, even though we know there are light rays through every point. As described in the Introduction section, digital-embodied nomograms can offer a tangible, hands-on experience that enables learners to comprehend the relationship between two variables. This understanding is facilitated through both visual (perceptual) and tangible (action) modes, and these nomograms can effectively communicate a learner's solution strategy using specific visual cues,

[1] The geometric representations in Figs. 5 and 6 are intentionally simplified for clarity, despite deviations from shadow shapes in reality. Empirical evidence from our later study indicates that these simplifications did not detract from student understanding or engagement, supporting their use for educational purposes.

Fig. 5 A hand and its shadow under the sun



such as the trace of an arrow and algebraic symbols. Hence, digital-embodied nomograms serve as an ideal model for the *concretization* phase of the learning process. The *dialog* stage is where the mathematical concepts are consolidated and where the learners can see the relationships between their intuitive actions or strategies and the formal mathematical concepts. It encourages learners to navigate through problem scenarios using their informal approaches, subsequently guiding them to craft formal mathematical solutions. Our learning environment is structured to enhance this experience: students are presented with questions, urging them to consolidate and articulate their insights. Additionally, the collaborative design of some tasks stimulates interactive dialogue between peers. And there is another opportunity for students to seek clarifications from tutors and ensures that misconceptions are addressed, thereby ensuring a holistic, dialogic learning journey.

Beyond the general design principles discussed above, the specific design ideas for each task are outlined in an HLT (Bakker, 2018; Simon & Tzur, 2012). We elaborated the HLT based on the embodied instrumentation theory (Drijvers, 2019) and emergent modeling (Gravemeijer, 1999). A full description of the sequence for each learning module, referred to as the HLT, can be accessed at the provided link: https://bit.ly/FTnomogram. The table describing the HLT comprises multiple horizontally arranged components, which include task numbers, task descriptions, mathematical objectives, students' activities (incorporating practices/techniques for utilizing artifacts and levels of the adapted emergent mode), and the conceptualization of various aspects of FT. The arrangement of the three HLTs is sequentially organized in a vertical manner, reflecting both the aspects of FT and the levels of the emergent model.

In Module 1, tasks begin with situational activities that incorporate input–output thinking within a light ray context. Following several varied light ray tasks, the real-life context fades, and the required movement shifts from unimanual to bimanual. This gradual shift introduces covariation thinking through referential activities. To illustrate, Fig. 6a presents a task built on the context of a cardboard tree and its shadow on a screen. Here, students have the opportunity to delve into the geometric meaning of both additive and multiplicative terms of the linear relation $y = 3x - 2$ by manipulating the input, either moving the tree's apex/base or the entire tree. When students modify the tree's apex or base position, a commensurate change in the shadow's size is observable, governed by a specific multiplicative factor, in this case 3, which is geometrically an enlargement factor. The additive factor, in this case -2, is the image of the zero on the input axis.

In Module 2, the tasks initiate with referential activities that adopt semi-nomograms (nomograms without numbers), eventually leading to general activities. This module emphasizes mathematical contexts, highlighting covariation thinking and initial correspondence thinking. The presentation of various representations, such as nomograms and formulas, lays the foundation for introducing correspondence thinking. An example can be discerned in Fig. 1, where a task is based on the function $y = -x$. In this activity, students are tasked with acting in a specific bimanual motion—moving both hands in opposite directions at the same speed. This motion mirrors the mathematical relationship encapsulated in the function $y = -x$: a positive increment in x induces an equal decrement in y and vice versa. Such a coordinated movement not only embodies the interrelation between the two variables but also holds the promise of assisting students in transitioning from concrete actions to abstract mathematical conceptualization of functions. The act of moving both hands in opposite directions at the same speed embodies covariation thinking. It enables students to directly perceive that during the act of moving, every position of the left hand on the input number line corresponds to a right-hand position on the output

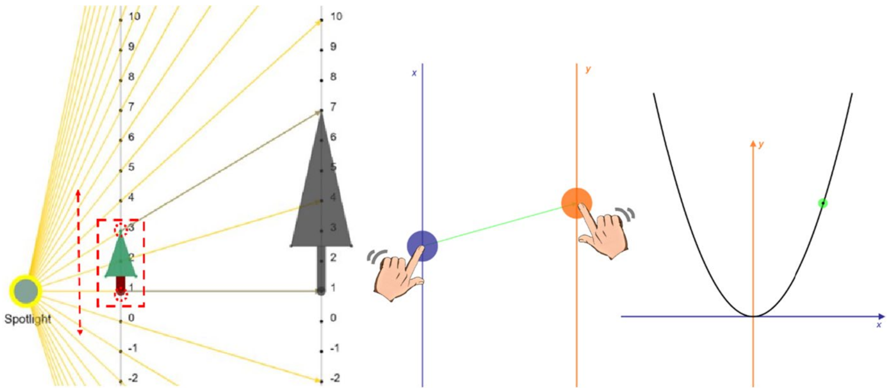


Fig. 6 **a** Cardboard tree and shadow task. **b** Maintaining a point on the function graph

line, as a result of an inverse movement. This kinesthetic experience is expected to reinforce the understanding of functions as covariational relationships between variables where the change of one variable directly influences the change of another in a specific manner.

Module 3 comprises general activities centered on covariational thinking, integrating various representations of functions. The core idea of this learning module is the conversion between these representations, with special emphasis on the transition between nomograms and function graphs. Using both unimanual and bimanual motions, students delve into the correspondence aspect of FT. Through an action-based design, depicted in Fig. 6b, students can adjust arrows on the nomogram, observing the resulting point on the corresponding function graph: $y = x^2$. The semi-coordinate system offers a visible and tangible representation of these shifts: the left hand's movement directly influences the point's horizontal position, while the right hand dictates its vertical movement. Notably, students are exposed to the subtleties of their hand movements' acceleration, which mirrors the change in the derivative of the function $y = x^2$. As the students approach $x = 1$, the left hand's movement needs to decelerate relative to the right hand. In contrast, distancing from $x = 1$ necessitates the left hand to progressively outpace the right. This hands-on experience emphasizes the covariational relationship between the two variables, demanding both speed and coordination to align the point accurately on the function graph. Consequently, students are expected to get insights into the conversions between nomogram and function graph, all while grounding their understanding in an intuitive, embodied experience.

Case Study Participants

For our case study, we selected two pairs of students who were in the pre-university stream of secondary education in the Netherlands, each pair comprising students aged 14 years, to participate in the intervention. These students were chosen by their teacher for their collaborative and communicative abilities and their willingness to take part.

The starting points and preliminary knowledge of the students were as follows: experience with number lines, basic algebra to describe relations between quantities/variables, using algebra for modeling situations, and basic skill with graphs in coordinate systems. This foundational knowledge is essential to grasp the concepts introduced in the intervention and effectively engage in the learning modules.

Intervention

The intervention was carried out with the two pairs of students, first in November 2022 and then in January 2023. Each intervention session, covering three learning modules, took 90 min, followed by a 15-min interview to gather the students' reflections and insights. Students were given digital-embodied nomogram tasks within a learning environment that featured a multi-touch screen (Fig. 7). This setup provided an interactive platform for the students to explore and engage with the tasks.

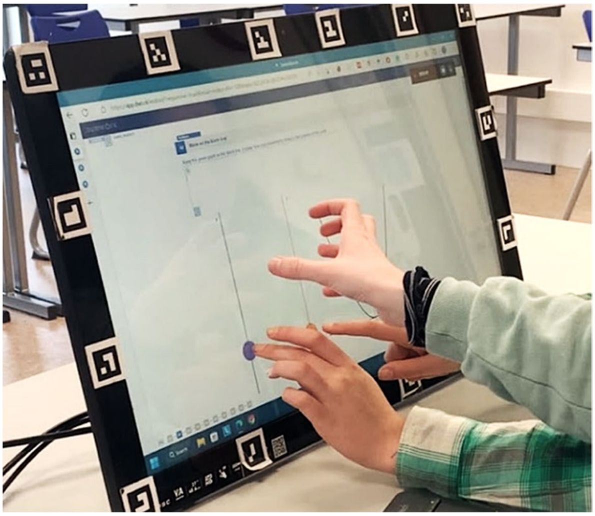


Fig. 7 Students worked on a multi-touch screen

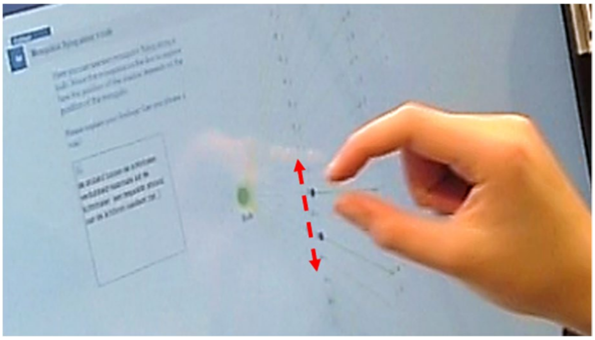
Throughout the intervention, the students were encouraged to collaborate and ask questions to the teacher during their work.

Data Collection and Analysis

The two primary data collection methods were video and audio recordings. The video recordings served a dual purpose: first, they tracked the physical activities of the students, including their hand motions and gestures, which are of prime interest in our study. Second, they captured the students' on-screen activities and solutions in the answer boxes for each task. This gave a clear picture of how students were engaging with the digital-embodied nomogram tasks and how they manipulated the tools provided within the digital learning environment. In addition, audio recordings of the post-intervention interviews were collected. This dataset was then fully transcribed to facilitate the subsequent analysis.

Central to our data analysis was understanding students' usage of digital-embodied nomogram affordances in their conceptualization of FT. We selected key segments where these affordances were distinctly used for further exploration. We qualitatively analyzed participants' actions, explanations, and discussions, comparing our anticipated outcomes (hypothetical learning activities, HLA) with the actual events (actual learning activities, ALA). This comparative lens provided insights into several key areas: practices and techniques with artifacts, conceptualizations of FT, action-perception loops in action-based tasks, and attentional anchors for action-based tasks. This comparative analysis is important for addressing the research questions. For instance, comparing conjectured and actual practices of and techniques for using artifacts helped us to assess the degree to which the light ray context promotes students' interpretative capabilities with respect to nomograms. Likewise, we could determine how the bimanual tasks have been a catalyst for fostering covariational thinking by contrasting hypothetical and actual action-perception loops. Through this comparison process, enriched

Fig. 8 Students' hand gestures while exploring the bulb-mosquito task



with participant quotes, we identified patterns in the students' interactions and gained insights in our design's efficacy and suggested areas for improvement.

Results

The Results section provides empirical data on how FT can be fostered through the three digital-embodied learning modules, each corresponding to one aspect of FT. For each learning module, we present one or two exemplary tasks and describe participants' activity. We compare the HLA with the ALA, highlighting the similarities and discrepancies between them, accompanied by examples and quotes from participants. The redesign ideas are also presented.

Learning Module 1: Light Ray Context and Nomogram

Example 1: In the bulb and mosquitos' shadows task (see Fig. 8), students can move the positions of two mosquitos and observe how the positions of their shadows change correspondingly. The relationship between the position of the mosquito and its shadow is as follows: height_shadow = height_mosquito • 2.

HLA	Hypothetical practices/techniques for using artifacts: By moving one input (mosquito) once on the number line, students can recognize and distinguish various patterns of relationship (light rays). The relationship established between the mosquito and its corresponding shadow in this instance is a proportional relationship. Hypothetical conceptualizations of FT: During the exploration of the bulb-mosquito context, students are able to manipulate the mosquito's position and subsequently observe the resulting shadow location, thus facilitating an understanding of the input (mosquito) and output (mosquito's shadow) relationship (coordination of input values and output values), and covariation thinking (output covaries when students change the input).

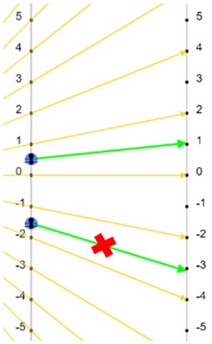
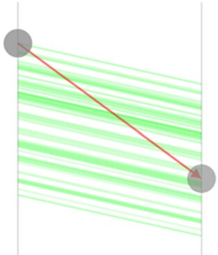
ALA	Actual practices/techniques for using artifacts: Student Pair 1: The students adjusted the position of each mosquito individually and noted the gradient of the light rays (arrows). After the tutor explained the question, their attention shifted toward the input-output (mosquito-shadow) relationship. Student Pair 2: The students manipulated the positions of the two mosquitos simultaneously, assessing the inter-mosquito distance and the distance between their shadows. Rather than a point-to-point relationship, they focused more on the interval-to-interval relationship (Chunky continuous covariation). They prioritized the distance between the two mosquitos in relation to the corresponding distance between their shadows, instead of observing how the output depends on the input. One of the students said: ‘The distance between all the lines is getting bigger and bigger... Well, I don’t know if I’m saying it right, But the distance between that line is getting bigger’. Actual conceptualizations of FT: Student Pair 1: The students phrased their findings, saying, “The shadow is at double the height of the mosquito’s height.” This description shows their comprehension of the relationship between input (height of the mosquito) and output (height of the shadow). Student Pair 2: The students gave a more descriptive explanation, focusing on the distance between the two mosquitoes. They described the rule as “the distance between the light rays doubles, as the light rays are further away from the light source”. They showed an understanding of chunky covariation (interval-to-interval), an advanced level of FT, compared to the input-output (point-to-point) level.
Comparison	While the actual learning trajectory of Pair 1 closely followed the hypothetical trajectory, Pair 2 showed an unexpected but advanced level of FT. In other words, concerning the potential given by the task, students could prioritize the inter-mosquito distance and the distance between their shadows, instead of observing how the output depends on the input (mosquito position). The students may have found the interval-to-interval relationship more intuitive or engaging to explore (because there are two mosquitos available), which diverted their attention from the point-to-point relationship.
Ideas for Redesign	 To prevent the unnecessary distractions on the screen, we plan to reduce the complexity by removing one of the mosquitos. Notably, these students are relatively high-achievers, and pair 2 invested six times more time to solve the task.

Fig. 9 Students explored nomograms **a** collaboratively and **b** individually

Learning Module 2: Bimanual Nomogram Tasks

Example 2.1: In the “Keep the arrow green with your neighbor” task (see Fig. 9a), students can construct the arrows of the semi-nomogram (without numbers) with their peers by controlling one point per person. The relationship between the input and output is as follows: output = input – 2.

HLA	Hypothetical action-perception loops: During the exploration, students adjust the arrow’s endpoints; one student moves one end, while the other student adjusts the opposite end. They aim to maintain the arrow’s green color, indicating a correct relationship between the heights of the two ends. If the points misrepresent the function’s nomogram, the arrow turns red, prompting students to correct their positioning to maintain the green indication. Initially, students employ subtle or slight movements in segmented (“chunk”) motion strategies, which help them get familiar with this new movement. As students traverse the input axis, they visualize the function’s complete pattern, observing intersecting and parallel arrows among other features. Hypothetical practices/techniques for using artifacts: Students move two points on two lines vertically with height differences on both sides and adjust the moving speed (same speed for both sides) and direction (same direction on both sides) based on the feedback of the arrow (green = positive, red = negative). Hypothetical conceptualizations of FT: Students’ simultaneous and smooth manipulation of the two points can foster an understanding of covariation between two variables based on bimanual movements. This can be achieved by drawing an analogy between their physical experiences (heights of hands) and mathematical meaning (dependent and independent values). In addition, the whole set of arrows, as shown by the trace of them, can contribute to the development of correspondence thinking.

ALA	Actual action-perception loops: Student Pair 1: The students were observed to adjust the position of one end of the arrow (student 1) and then move the other end (student 2) until the arrow turned green. After finding several green arrows, students started to move the two points together and keep the arrow green all the time. In the end, they adjusted the arrow smoothly and perceived the relation between the heights of the two ends of the arrow to meet the positive feedback, which is a green arrow. Student Pair 2: The students adopted a strategy similar to pair 1. Actual practices/techniques for using artifacts: Student Pair 1: The students became adept at moving vertically along two straight lines with both hands, which laid the foundation for the later tasks. Student Pair 2: The students followed a practice similar to pair 1. In addition, it was observed that the students focused on the speed of their movements and came to the insight that the speed of the two points' movements must be consistent to maintain parallelism between the arrows in the nomogram of output = input − 2. Actual conceptualizations of FT: Student Pair 1: The students connected their observations to the previous learning module. They referred to the light-shadow context, which was not actually present; "The sun's rays come from one side again, so the green arrows run in one direction." This task saw the emergence of situational reasoning, since the students used the light-shadow context to explain what they observed from this new task, but limited covariational thinking was observed. Student Pair 2: The students made a connection between their physical movements and the geometric attributes of the nomogram, stating, "By trying to both go down/up at the same speed…make sure you both move your fingers down at the same speed so that the lines stay parallel all the time". This enabled them to comprehend how one variable changed according to another based on their bodily experience.
Comparison	The HLA and ALA are primarily aligned, as students followed the anticipated process of adjusting the arrow and maintaining the green color; moving two points on two lines vertically and adjusting their movement based on the arrow's feedback. Their reflection suggests a strong bond between their bodily experience and the mathematical meaning. The HLA was an accurate description of the learning process, and the instructional strategies and materials were successful in guiding students along the desired learning trajectory.
Ideas for Redesign	 The alignment between the HLA and ALA indicates that this series of tasks was successful in facilitating the desired learning outcomes. There will be no adaptation.

Example 2.2: In the task "Describe the rule with x and y" (see Fig. 9b), students need to complete the nomogram (with numbers) by moving first and then describe the rule with x and y. The relationship between the input and output is as follows: $y = -2 \bullet x$.

HLA	Hypothetical action-perception loops: During the exploration, students are expected to move the two points together while keeping the arrow green to find the moving patterns based on the numbers on the two lines Hypothetical practices/techniques for using artifacts: Students are expected to move two points on the two number lines simultaneously and maintain the moving speed (different speeds for both sides) and direction (opposite direction on both sides) to hold positive feedback Hypothetical attentional anchor: An intersection point of the traces of the arrow Hypothetical conceptualizations of FT: With the appearance of numbers on both lines, students are expected to strengthen their understanding of input–output pairs and covariational quantities with referential reasoning. Students can develop a connection between a rule that determines their movement to a more abstract rule/relationship between two variables, which entails a form of correspondence thinking
ALA	Actual action-perception loops: Student Pair 1: The students were found to first move one unit on the left number line and then adjust the right point until they observed the arrow turning green. After noticing a pattern in the traces of the arrows, they adjusted their movement based on the trace Student Pair 2: These students quickly found some green arrows and subsequently shifted their attention to the point where the existing arrows intersected. They then moved their hands and completed the nomogram by aligning the arrows with the intersection point Actual practices/techniques for using artifacts: Student Pair 1: Students practiced a strategy of moving one unit by one unit, conjecturing the possible moving pattern, and then following the traces of the arrow to move two points smoothly Student Pair 2: The students noticed that a shortcut to creating green arrows easily on the nomogram involved "rotating" the arrow around the intersection point Actual attentional anchor: An intersection point of the traces of the arrow Actual conceptualizations of FT: Student Pair 1: The students adopted a top-down strategy, first identifying several input–output pairs, such as 0 to 0, 2 to -4, and -1 to 2, and then performing calculations to give the rule with x and y. They primarily focused on the mathematical aspects rather than the motion aspects of this task, which led them to covariation thinking Student Pair 2: The students connected their movement pattern and the geometric attributes of the nomogram. Their statement is as follows: "we have to make sure the line stays on this center point (intersection point) … and from there, one point (moves) down and the other (point moves) up." They quickly recognized that the opposite motion of their hands corresponds to a "-" symbol in the formula. Subsequently, by analyzing number pairs on the nomogram, they deduced the additive factor. This indicates their realization of the initial covariational relationship between the coordinated hand movements
Comparison	In the action-perception loops, the discrepancy lies in the students' (Pair 1) initial approach to moving one unit on the left number line and adjusting the right point until the arrow got green, instead of moving the two points together while keeping the arrow green. In the attentional anchor, the students focused on the intersection point of the traces of the arrow, rather than the green arrow itself. This shows that this task might not have been perceived as action based. The students paid more attention to the mathematical aspects, viewing the nomogram as a representation rather than a tool that provides physical experiences to facilitate mathematization

Ideas for redesign	The step-by-step approach could become an alternative pathway to understanding the moving patterns. As for students’ focus on the intersection point to complete the nomogram, future study will consider if it is necessary to redirect students’ focus to the green arrow, for example, make the traces invisible while students move the two points and reveal the traces when the movement covers most of the target traces

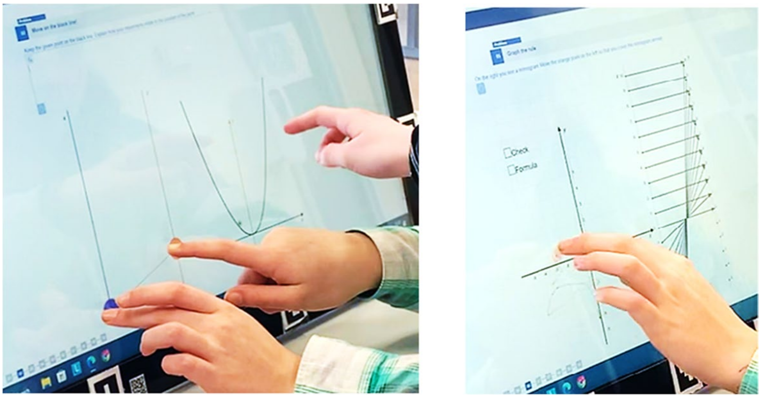


Fig. 10 Students explored the transition between **a** a nomogram and corresponding function graph and **b** a function graph and corresponding nomogram

Learning Module 3: Transition Tasks of Multiple Representations for Functions

Example 3.1: In the task “Move on the line” (see Fig. 10a), students can adjust the arrow in the nomogram and try to keep the corresponding point moving on the function graph. The relationship between x and y is as follows: $y = x^2$.

HLA	Hypothetical action-perception loops: Students are expected to explore the relationship between their hand movements and the corresponding point in the coordinate system, observing how the left- and right-hand movements affect the point’s horizontal and vertical positions, respectively Hypothetical practices/techniques for using artifacts: Students should become aware of the relationship between their hand movements and the corresponding point’s movements to accurately adjust their actions; left-hand movement controls the horizontal movement of the point (left/right), while right-hand movement controls the vertical movement (up/down) Hypothetical conceptualizations of FT: Speed and coordination are crucial to keep the point on the function graph. Students are expected to comprehend how their hand movements simulate the covariational relationship between the two variables. They can develop an understanding of the conversion between representations, such as from an arrow in a nomogram to a point on the function graph, based on their physical experience. Additionally, students are expected to recognize that the vertical axes in both representations remain consistent. Another crucial observation is the orientation of the horizontal axes; they are rotated at a right angle, and a geometric detail can be mirrored in the student’s movement

ALA	Actual action-perception loops: Student Pair 1: Initially, students moved the right point on the nomogram, observing its impact on the vertical movement of the green point in the Cartesian coordinate system. They then moved the left point and recognized its influence on the point's horizontal movement. After figuring out the effect of each hand's movement on the point's position in the Cartesian coordinate system, they started to move the two hands simultaneously on the nomogram to ensure the point moved along the corresponding function graph Student Pair 2: These students adopted a reverse strategy. They were observed initially moving both hands together to see the point's movement in the Cartesian coordinate system. They then experimented with moving one point at a time to understand the influence of the left number line on the point's horizontal movement and the right number line on the vertical movement Actual practices/techniques for using artifacts: Student Pair 1: The students demonstrated the same understanding as hypothesized, using their left hand to control the point's horizontal movement (left/right) and their right hand to control the vertical movement (up/down) Student Pair 2: Similar to Pair 1 Actual conceptualizations of FT: Student Pair 1: The students built the full connection between the two representations, nomograms and function graphs. They described the findings of the function $y = x^2$ as "you have to move the x circle up slowly and the y circle you have to move up faster," indicating that when x is greater than 1, y changes at a faster rate than x. This shows their understanding of the correspondence relationship of x and y for the function $y = x^2$. They incorporated their bodily experiences to explain the conversion between different function representations Student Pair 2: The students gave a similar statement on the conversion between nomograms and function graphs
Comparison	Both HLA and ALA demonstrated similar practices/techniques for using artifacts and shared similar action-perception loops, indicating that the hypotheses were effective and accurate in guiding students' engagement and focus. However, the difference lies in the initial strategy used by the students in Pair 1 and Pair 2. Pair 1 started by moving the right point and then the left point, while Pair 2 initially moved both hands together. Concerning the conceptualization of FT, both pairs were able to build the full connection between the two representations and used their physical experiences to explain the conversion between different function representations. These findings suggest that different students may adopt different strategies through exploration, but either way can lead them to the final learning goal
Ideas for redesign	The HLA and ALA have no conflicts on the learning goal of this task, which is "Conversion between different representations of functions, nomogram and function graph." There will be no adaptation for this task

Example 3.2: In the task "Graph the rule" (see Fig. 10b), students can use a digital pen on the screen to plot the function graph according to a given nomogram. There is a colorful arrow in the nomogram showing the position of the pencil in real time. The functional rule in this task is as follows: $y = |x|$.

HLA	Hypothetical action-perception loops: When completing the function graph, students are expected to move the digital pen (orange point) in the Cartesian coordinate system and observe the corresponding color-changing arrow on the nomogram. Students can use a top-down approach, first having an overview impression of the nomogram and guessing the function, and then moving the digital pen to draw the graph, with the color-change arrow signifying the location of the point in the Cartesian coordinate system. When the point aligns with the target function graph, the corresponding arrow in the nomogram turns green, and the trace of the pen remains visible. Or students can adopt a trial-and-error method, working in small steps to keep the green arrow on the given nomogram by loops Hypothetical practices/techniques for using artifacts: Students are expected to first move the pen horizontally or vertically to determine the effect of the movement and then focus on plotting several separate points that fit within the nomogram. Eventually, they should be able to plot the function graph. It is plausible they may need to erase and restart the entire canvas several times before they can smoothly plot the function graph Hypothetical conceptualizations of FT: It is a reverse task of conversion between nomogram and function graph. By "matching" the color-changing arrows on the nomogram and the pen's position in the Cartesian coordinate system, students' comprehension of input–output pairs and their corresponding locations in the Cartesian coordinate system is reinforced. The integration of nomogram, function graph, and formula in this task allows students to experience how different representations work together to represent one rule/function, which leads to correspondence thinking
ALA	Actual action-perception loops: Student Pair 1: The students first found a point (1, 1) that turned the arrow green. They then began to freely move the pen to observe its effect on the color-changing arrow on the nomogram. After discerning the connection between the pen and the color-changing arrow, they tried to plot the left part (second quadrant) of the function graph in small steps by loops Student Pair 2: The students initially moved the pen in an unstructured manner, attempting to find number pairs and to turn the color-changing arrow on the nomogram green. They found that even with free pen movement, when it reaches a certain area, the trace of the pen can remain visible. After being redirected by the tutor, they refocused on the task goal, which was to plot the function graph based on the given nomogram, rather than to move the pen freely Actual practices/techniques for using artifacts: Student Pair 1: Initially, students plotted a few points in the coordinate system while observing the corresponding nomogram and then lined up these points to complete the function graph Student Pair 2: The students first moved the pen randomly and got some pen traces, which formed part of the accurate function graph. Then they used a strategy similar to Pair 1 to plot the graph Actual conceptualizations of FT: Student Pair 1: The students identified the connection between the movement of the pen and the color-changing arrow on the nomogram and then further got an understanding of the conversion between the function graph and the nomogram. Their use of the number pairs indicates a deep development of the input–output aspect of FT. When giving the formula of the function, they described it as "x times -1 (second quadrant)…Always the same number (first quadrant)." They have not learned absolute function, so provided this kind of stepwise formula, when $x < 0, y = -x$ and when $x > 0, y = x$. This shows a strong ability to transfer between different function representations, which exemplifies the correspondence aspect of FT Student Pair 2: The students also grasped the conversion between the function graph and the nomogram. They figured out the patterns of the number pairs from the given nomogram. When explaining the findings, they described as "(in the first quadrant) the graph should be a 45-degree straight line,…, it is also a 45-degree straight line but towards the opposite direction (in the second quadrant)." Their explanation suggests that they have also got an understanding of correspondence thinking

Comparison	The difference in the action-perception loops is the initial strategy used by both pairs of students. In terms of the use of artifacts, both pairs demonstrated a grasp of the connection between pen movement and the color-changing arrows on the nomogram and applied similar strategies to plot the function graph. In conceptualizing FT, both pairs of students recognized the conversion between the function graph and the nomogram, and identified patterns from the given nomogram, indicating developed correspondence thinking
Ideas for redesign	The confusion about the task goal implies a need for clearer task instructions. To avoid this confusion, the task goal of graphing the function in the coordinate system will be explicitly stated at the beginning to guide student focus, for example, highlighting the goal in the instruction and the title of the tasks. In addition, the pen's traces could potentially cause a misinterpretation of the task. We will modify this aspect to preserve all traces rather than exclusively maintaining the correct one

The Results section elucidates how FT can be fostered through three digital-embodied learning modules, each targeting a different aspect of FT. Through detailed examples and participant activities, we compared the differences HLA and ALA. Moreover, the comparison and redesign ideas presented suggest ways for refining the design to better align with students' perceptual and physical experiences and learning needs. These results set the stage for a broader research of the implications of digital-embodied learning.

Conclusion and Discussion

The overarching question addressed in this paper is how an embodied design can foster abstract FT. To provide more comprehensive insights, we divided it into three sub-questions, each focusing on a specific aspect of FT. In the following, we reflect on how the results obtained from each learning module contribute to answering these sub-questions.

RQ1: How Does a Light Ray Context Foster the Students' Meaning-Making of Nomograms?

The light ray context, an integral part of our embodied design, served as an important instrument in facilitating students' understanding of nomograms and the input–output aspect of FT. As advocated by Abrahamson and Lindgren (2014), students need guidance to take action and move their bodies in specific ways, simulating key mechanisms and spatial relations. This approach helps them understand and apply functional analogies in the targeted knowledge domain. In our scenario, we have embedded numerous elements within the light ray context that render the nomogram as a function representation. First, the orientation of the light rays (arrows) underscores the principle that, with a function, it is the inputs that consistently map onto the outputs, and not the reverse. Second, a single light ray can only correspond to a unique point in the shadow, similar to the function rule where input can only determine one specific output. Third, in nomograms, only a finite number of arrows is drawn, despite every point on the input axis theoretically having an arrow, similar to how we depict only a limited number of light rays even though they pass through every point. Lastly, the visualization of two hands in the hand-shadow tasks offers an analogy for the subsequent dual-hand motion task on a nomogram. The object placed in front of the light source symbolizes the input, while the resulting shadow represents the output. The direction of the arrows offers students clear guidance regarding the mapping from input to output values.

Though still partial and vague in this learning module, students' grasping of the mathematical meaning of the nomogram—as a function representation—was accomplished through and manifested in their bodily actions, gestures, artifacts (the learning environment), and mathematical symbols (Radford, 2009). As shown in our observations, students were able to draw connections between their actions—such as changing the light source's type or position—and the resulting changes in the light ray patterns on the nomogram. While experimenting with two light sources, the sun or a bulb, the learning environment supports the students in the meaning-making of nomograms for two types of functions. The parallel nomograms are interpreted as the result of sunlight, which represents adding to or subtracting from the input values. The divergent nomograms, with a focal point left of the input lines, are ascribed to a bulb or spotlight, which enlarges the input values to some extent (see Example 1). In addition, the perceptual experience provided by the light ray contexts allows students to construct the mathematical meaning of using nomograms. For instance, the geometric patterns resulting from different light ray contexts left a deep impression on the students, enabling them to refer back to these contexts even in subsequent learning modules (as seen in Example 2.1).

In conclusion, the light ray context has been a productive situational tool in our embodied design, fostering a deeper understanding of the function representation of nomograms through a tactile and sensory-engaging approach.

RQ2: How Do Bimanual Movement Tasks Foster Covariational Thinking?

In our embodied nomogram tasks, the bimanual movements offer an opportunity for students to physically explore the relationship between two variables in a function, with the left hand representing x and the right hand y, respectively. This enables students to physically experience the covariation between the two variables. As prompted by Alberto et al. (2022), our embodied nomogram tasks also cover two learning phases, a qualitative stage (nomograms without numbers) and a quantitative stage (nomograms with numbers). In the qualitative stage, by moving both hands to maintain the green status of the arrow, students understood that these variables were related and their hands' movements should be coordinated (see Example 2.1). In the quantitative stage, students adopted a more systematic, quantitative approach to their movement patterns. For example, they used a strategy of moving one unit at a time on the left number line and then adjusting the right point accordingly (see Example 2.2). This shows they made connections between their movement and mathematical reasoning of the discrete numerical values associated with each point.

Concerning the bimanual movement in the nomogram-function graph tasks, students used both hands to manipulate the points that represented the value of the variables in the nomogram and observe the corresponding changes of the variables in the Cartesian coordinate system. A clear example of how this facilitated understanding of the covariational relationship can be seen in a task that involved keeping a point moving along a function graph (see Example 3.1). To keep a point moving along the function graph $y = x^2$, students quickly realized they had to move their right hand (controlling y) faster than their left hand (controlling x) when x was bigger than 1. This active engagement provided a tactile foundation for their comprehension of the different function representations. This grasp was evident in their explanations such as "the (y) point goes faster than the (x) point" or "when I move my left hand, the point goes left or right (on the nomogram), and when I move my right hand the point runs vertically (in the Cartesian coordinate system)."

Moreover, in line with continuous feedback used in previous studies in mathematics education, a notable feature embedded in our tasks was the continuous real-time feedback provided by the color-changing arrow on nomograms (Abrahamson, 2014; Alberto et al., 2019; Shvarts et al., 2021). Continuous feedback has proven to be a promising tool, promoting new sensorimotor coordination through students' exploration and interaction with the learning environment. This was apparent in our tasks as students adapted their hand movements in real time to maintain the "green" of the color-changing arrow. Students could immediately see the impact of their hand movements on the function and adjust their strategies accordingly. This loop of action, observation, and reaction has the potential to reinforce understanding continuous covariation in a dynamic, iterative way. The continuous feedback provided in these tasks was not only real time but also visually intuitive, using color changes (i.e., the arrow turning green or red) as an indication of correctness (Alberto et al., 2022). Such feedback, coupled with the simultaneous manipulation of the variables, enabled the students to experience the complex interrelationship between the two variables, thereby constituting a body-artifact functional system for covariational reasoning using nomograms.

In this manner, the integration of bimanual movement into the learning process not only aligns with the properties of functions itself but also leverages recent advancements in DT to provide a novel, hands-on approach to the foster of covariation thinking.

RQ3: How Do Different Function Representations and Their Conversions Support a Correspondence View on Functions?

Various function representations, such as arrow chains, tables, graphs, formulas, and nomograms, allow for different types of functional reasoning, fostering a holistic understanding of functions, which is the core of correspondence thinking. In our tasks, the conversions between different representations—nomogram, formula, and function graph—were intentionally designed to help students transfer them smoothly based on concrete experience. The formula and function graph, a more conventional representation, helped students further consolidate the functional relationship in a symbolic and graphical way.

According to previous research (e.g., Ainsworth, 1999; Duval, 2006), using multiple representations, especially the transitions between them, can deepen students' conceptual understanding and encourage more flexible thinking. When students are trained to transition smoothly between different function representations, they are better positioned to anticipate and operate on the function. This anticipatory ability allows students to deduce implications in one representation based on insights obtained from another. For instance, during the plotting function graph task (see Example 3.2), students were given the opportunity to conduct the unimanual movement on one representation (function graph) while observing the corresponding changes on another (nomogram). The changes induced by the unimanual movement were related to the overall shape of the nomogram or slope of the function graph, encouraging students to anticipate outcomes before initiating the plotting process.

In summary of the findings on RQ3, the use of different function representations and conversions between them in our study encouraged correspondence thinking by providing multiple perspectives to observe, understand, reason about, and manipulate the function. These tasks boosted an understanding of the function as a correspondence relationship, promoting a general understanding towards an object view of function.

In addressing the limitations of this case study, several elements deserve further consideration. First, the subjects of our study are students from the pre-university stream, suggesting that these students may have a solid foundational knowledge, potentially enabling them to better grasp abstract concepts than their peers. This skews the findings, as the approaches used might have different effects on students from various learning backgrounds. Other uncontrolled factors could have influenced the outcomes as well, like learning perceptual preferences and familiarity with digital tools. Second, although the students' performance was closely observed and analyzed, their strategies and thought processes were inferred from their behaviors and verbal expressions, possibly introducing some degree of bias. For instance, although some students employed a top-down strategy yielding correct responses, their subsequent explanations connected these answers to our questions, including embodied elements we expected. This could lead to the misinterpretation that tasks were addressed using a merely embodied approach. Third, the design intricacies, despite their novelty, might have been overly complicated for some participants. The confusion evident in initial tasks emphasizes the need for more explicit instructions in future designs. Adjustments, such as explicitly stating the task goal and refining the movement traces to maintain clarity, could help avoid misinterpretations. Last, the embodied nature of our tasks relies heavily on the nomograms. We could question if students are truly grasping the mathematical concepts or if they are mastering the manipulation of this specific tool.

Adding to our conclusions, we want to emphasize the importance of design considerations for effective embodied tasks. First, it is important to provide students with concrete experience, as exemplified by the light ray contexts in our design. This concrete, situational context served to anchor abstract mathematical concepts, allowing students to build on their intuitive understanding of the physical world—such as noticing how the presence of light sources affects the environment around them—how sunlight creates shadows that change during movement, or how a flashlight casts a shadow when its beam is obstructed. We acknowledge that while students might not have a detailed understanding of the geometric properties of light diffusion, their general awareness of how light and shadows interact due to movement is sufficient for the learning goals. And students frequently referred to this context in subsequent tasks and modules, illustrating the lasting value of such metaphors in function learning with nomograms. Another crucial design consideration is the integration of an interactive, dynamic learning environment that offers real-time feedback (Abrahamson, 2014; Alberto et al., 2022; Shvarts et al., 2021). The students responded positively to the interactive nature of our tasks, with both pairs indicating that they enjoyed engaging with the tasks and observing the changes in a real-time manner. Notable comments included the rewarding experience of seeing the arrows turn green, the ease of understanding how the graphs work through the dynamic lines, and the preference for this interactive learning environment over traditional textbooks. This immediate, sensory feedback provided by the tasks fosters a more engaging, intuitive, and satisfying learning experience, highlighting the potential benefits of integrating such elements into mathematics learning. At the heart of our embodied design is the intentional and tight coupling of learning goals with target tasks. We aimed to impart an understanding of functions as relationships between two variables. To this end, the two-hand coordinating motion served as an ideal task: it is an accessible action that affords both stability and dynamism. The stability of this action—consistent action type across all tasks, regardless of function type—could facilitate the emergence of body-artifact functional systems (Shvarts et al., 2021), while the dynamics allows for a wide range of movement patterns, mirroring the various properties of functions. This close alignment between task and learning goal was crucial in ensuring students develop new body potentialities and create new affordances under the embodied learning environment.

In conclusion, this study highlights the advantages of integrating digital-embodied nomogram tasks to foster FT. Such an approach appears to deepen students' grasp of abstract mathematical concepts by providing concrete experience. Furthermore, these findings offer a robust foundation for future research in FT, embodied learning, and the role of DT in mathematics education.

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Author Contribution All authors set up the outline of the paper and took part in the data collection. Hang Wei carried out the data analysis. Hang Wei wrote the main manuscript text. Rogier Bos and Paul Drijvers edited the draft manuscript. Hang Wei prepared Figs. 1, 2, 4–10; Rogier Bos prepared Fig. 3. All authors reviewed the manuscript.

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Data Availability The video and audio data cannot be openly shared due to concerns regarding the protection of study participant privacy, as the participants are younger than 15 years old. Additionally, during the ethics review process approved by Utrecht University, we committed to ensuring that the data availability would be restricted only to our research team. Therefore, the raw data supporting the conclusions of this article will not be made publicly available.

Declarations

Competing Interests The authors declare no competing interests.

References

Abrahamson, D. (2009). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics, 70*(1), 27–47. https://doi.org/10.1007/s10649-008-9137-1

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction, 2*(1), 1–16. https://doi.org/10.1016/j.ijcci.2014.07.002

Abrahamson, D., Shayan, S., Bakker, A., & Van Der Schaaf, M. (2016). Eye-tracking Piaget: Capturing the emergence of attentional anchors in the coordination of proportional motor action. *Human Development, 58*(4–5), 218–244. https://doi.org/10.1159/000443153

Abrahamson, D., & Trninic, D. (2011). Toward an embodied-interaction design framework for mathematical concepts. In P. Blikstein & P. Marshall (Eds.), *Proceedings of the 10th Annual Interaction Design and Children Conference (IDC 2011)* (Vol. "Full papers," pp. 1–10). IDC. https://doi.org/10.1145/1999030.1999031

Abrahamson, D., & Lindgren, R. (2014). Embodiment and embodied design. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (2nd ed., pp. 358–376). Cambridge University Press. https://doi.org/10.1017/cbo9781139519526.022

Ainsworth, S. (1999). The functions of multiple representations. *Computers & Education, 33*(2–3), 131–152. https://doi.org/10.1016/S0360-1315(99)00029-9

Alberto, R., Bakker, A., Walker-van Aalst, O., Boon, P., & Drijvers, P. (2019). Networking theories in design research: An embodied instrumentation case study in trigonometry. In U. T. Jankvist, M. H. A. M. v. d. Heuvel-Panhuizen, & Veldhuis M. (Eds.), *Proc. of the 11th Congress of the European Society for Research in Mathematics Education* (pp. 3088–3095). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME. https://hal.science/hal-02418076/document

Alberto, R., Shvarts, A., Drijvers, P., & Bakker, A. (2022). Action-based embodied design for mathematics learning: A decade of variations on a theme. *International Journal of Child-Computer Interaction, 32*, 1–23. https://doi.org/10.1016/j.ijcci.2021.100419

Bakker, A. (2018). *Design research in education*. Routledge. https://doi.org/10.4324/9780203701010

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences, 22*(4), 577–660. https://doi.org/10.1017/S0140525X99002149

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617–645. https://doi.org/10.1146/annurev.psych.59.103006.093639

Bos, R., Doorman, M., Drijvers, P., & Shvarts, A. (2022). Embodied design using augmented reality: The case of the gradient. *Teaching Mathematics and Its Applications: An International Journal of the IMA, 41*(2), 125–141. https://doi.org/10.1093/teamat/hrab011

Brown, J. P. (2015). Complexities of digital technology use and the teaching and learning of function. *Computers & Education, 87*, 112–122. https://doi.org/10.1016/j.compedu.2015.03.022

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86. https://doi.org/10.5951/jresematheduc.26.1.0066

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: From repeated calculations to functional thinking. *International Journal of Science and Mathematics Education, 10*(6), 1243–1267. https://doi.org/10.1007/s10763-012-9329-0

Drijvers, P. (2019). Embodied instrumentation: Combining different views on using digital technology in mathematics education. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 8–28). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME. https://hal.science/hal-02436279v1

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics, 66*(3), 317–333. https://doi.org/10.1007/S10649-006-9072-Y

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Reidel.

Frey, K., Sproesser, U., & Veldhuis, M. (2022). What is functional thinking? Theoretical considerations and first results of an international interview study. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education* (pp. 497–504). Free University of Bozen-Bolzano and ERME. https://hal.science/hal-03744607/document

FunThink team (2021). *Vision document on functional thinking*. Retrieved September 17, 2023, from https://www.funthink.eu/fileadmin/user_upload/io1_vision_document_version_2.0.pdf

Glenberg, A. M. (1997). Mental models, space, and embodied cognition. In T. B. Ward, S. M. Smith, & J. Vaid (Eds.), *Creative thought: An investigation of conceptual structures and processes* (pp. 495–522). American Psychological Association. https://doi.org/10.1037/10227-018

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*, 111–129. https://doi.org/10.1023/A:1003749919816

Günster, S. M., & Weigand, H. G. (2020). Designing digital technology tasks for the development of functional thinking. *ZDM - Mathematics Education, 52*(7), 1259–1274. https://doi.org/10.1007/s11858-020-01179-1

Lakoff, G., & Núñez, R. (2000). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. Basic books.

Lindenbauer, E. (2019). Investigating students' use of dynamic materials addressing conceptions related to functional thinking. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 2876–2883). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME. https://hal.science/hal-02428705/

Nachmias, R., & Arcavi, A. (1990). A parallel representation of linear functions using a microcomputer-based environment. *Journal of Computers in Mathematics and Science Teaching, 9*(4), 79–88.

Palatnik, A., Abrahamson, D., Baccaglini-Frank, A., Ng, O. L., Shvarts, A., & Swidan, O. (2023). Theory and practice of designing embodied mathematics learning. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 159–189). University of Haifa, Israel: PME.

Pittalis, M., Pitta-Pantazi, D., & Christou, C. (2020). Young students' functional thinking modes: The relation between recursive patterning, covariational thinking, and correspondence relations. *Journal for Research in Mathematics Education, 51*(5), 631–674. https://doi.org/10.5951/jresematheduc-2020-0164

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*(2), 111–126. https://doi.org/10.1007/s10649-008-9127-3

Rolfes, T., Roth, J., & Schnotz, W. (2020). Learning the concept of function with dynamic visualisations. *Frontiers in Psychology, 11*, 1–16. https://doi.org/10.3389/fpsyg.2020.00693

Roux, A., Nieuwoudt, H., & Nieuwoudt, S. (2015). Conceptual learning of functions in a technologically enhanced environment. *African Journal of Research in Mathematics Science and Technology Education, 19*(3), 289–305. https://doi.org/10.1080/10288457.2015.1089674

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36. https://doi.org/10.1007/BF00302715

Shvarts, A., & Abrahamson, D. (2019). Dual-eye-tracking Vygotsky: A microgenetic account of a teaching/learning collaboration in an embodied-interaction technological tutorial for mathematics. *Learning, Culture and Social Interaction, 22*, 100316. https://doi.org/10.1016/j.lcsi.2019.05.003

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics, 107*(3), 447–469. https://doi.org/10.1007/s10649-021-10053-0

Simon, M. A., & Tzur, R. (2012). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91–104. https://doi.org/10.1207/s15327833mtl0602_2

Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: Narrative thinking with dynamic geometry. *ZDM – Mathematics Education, 41*, 441–452. https://doi.org/10.1007/s11858-009-0180-x

Smith, E. (2008). 5 Representational thinking as a framework for introducing functions in the elementary curriculum. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Studies in mathematical thinking and learning. Algebra in the early grades* (pp. 133–160). LEA.

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning, 19*(3), 143–166. https://doi.org/10.1080/10986065.2017.1328636

Tall, D. (2004). Thinking through three worlds of mathematics. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, (Vol 4, pp. 281–288). Bergen University College.

Ten Voorde, A., Piroi, M., & Bos, R. (2023). Dynamic visualization in animated mathematics videos I: A classification of didactic roles. *Teaching Mathematics and Applications: An International Journal of the IMA*. [Manuscript accepted]

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. In S. Lerman (Ed), *Encyclopedia of Mathematics Education* (pp.713–717). Springer. https://doi.org/10.1007/978-3-030-15789-0_170

Van Rompay, TJL., & Hekkert, PPM. (2001). Embodied design: The role of bodily experiences in product design. In M. G. Helander, H. M. Khalid, & M. P. Tham (Eds.), *Proceedings of the international conference on affective human factors design, June 27–29* (pp. 39–46). Asean Academic Press.

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *Journal of Mathematical Behavior, 17*(2), 167–181. https://doi.org/10.1016/s0364-0213(99)80057-3

Vollrath, H. J. (1986). Search strategies as indicators of functional thinking. *Educational Studies in Mathematics, 17*(4), 387–400. https://doi.org/10.1007/BF00311326

Wei, H., Bos, R., & Drijvers, P. (2023). An embodied approach to abstract functional thinking using digital technology: A systematic literature review. *The International Journal for Technology in Mathematics Education, 20*(2), 75–92. https://doi.org/10.1564/tme_v30.2.2

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