ME EN 2450 Design Project

Name: ________________________________

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Objective

This will be a team-based design project, where you will work with your teammates and submit a combined report. The final report must clearly indicate the team members and their contributions to the code writing. The final design project involves extending your `moving_train` program from Lab 08 to include the remaining aspects of the design project. Note, you will not be building any physical object for the project, but designing the physical dimensions of the train and locomotion system.

In Lab 08, the mathematical model was extended to include the deceleration stage. Also in lab 08, the numerical methods aspect was extended to solve the resulting system of ODEs using 4^{th}-order Runge-Kutta. There are now two remaining aspects. First, the mathematical model will be extended to include the decrease in pressure inside the tank as the train moves forward. Second, an optimization code will be included to determine the design parameters (variables for which we have control and flexibility over the train characteristics) that result in the fastest possible train, while abiding by the rules of the race.

Much of what follows in this handout is a repeat of the previous labs and AIDs and included here to have all the details in a single document. All of the details regarding grading of the various aspects of this project are included in the `Gradesheet.pdf` and `GradesheetRubric.pdf` files.

Train Design Competition

The race track consists of a 10 m section of straight track with a single tunnel that starts 6 m after the start line, see Figure 1. There is a 1.5 m length of set-up track before the start line, and a 2.5 m length of run-out track after the finish line. The tunnel has a minimum internal width of 0.2 m, a length of 1 m, and a maximum internal height of 0.23 m above the track. The track used in the competition is standard model G-scale railroad track (rails spaced 45 mm apart). The train should use steel axles and wheels with radius of 20 mm. The objective of the competition is to design a train that completes the race from start to finish in the least amount of time without damaging the tunnel and without running off the end of the track.

The train is to be powered by compressed air stored on-board in the pressurized cylindrical tank. The tank will be filled prior to the start of the race using a manual bicycle pump with an inline dial-type pressure gauge. According to the rules of the competition, the initial tank pressure must not exceed 30 psig (gauge pressure in psi). The train is propelled forward through a pneumatic piston that is actuated by the stored pressure in the tank. The piston is connected to a rack and pinion gear that drives the rear axle of the train. Lastly, we restrict the actuation of the piston to a single stroke.

The locomotion system will provide an initial push to accelerate the train up to its maximum velocity; after which time, the train simply coasts (decelerates) to the finish line. In this manner, the propulsion system is only active during a portion of the race, as illustrated in Figure 1. We will denote the acceleration stage from $0 \leq x \leq L_a$ and the deceleration stage from $L_a < x \leq L$, where L is the total length traveled. Here, $x(t)$ represents the distance traveled by the model train as a function of time, where $x = 0$ represents the starting line.

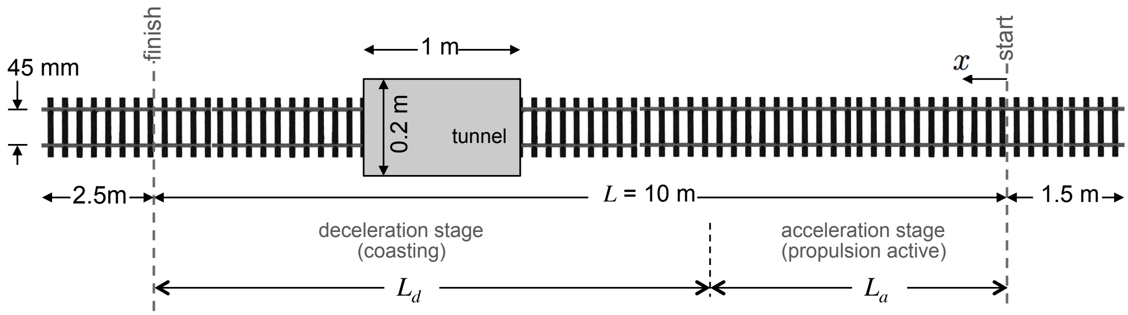

Figure 1: Schematic highlighting the locomotion strategy whereby the train is initially accelerated over a distance $0 \leq x \leq L_a$, and then allowed to coast (decelerate) to the finish line over the distance $L_a < x \leq L$.

Physical Model

First, as illustrated in Figure 1, acceleration and deceleration stages are necessary such that the train can reach the finish line, but come to a stop from frictional forces only (no braking).

We start with a free-body diagram of the train (Figure 2), and Newton's 2^{nd} law. The forces acting on the train still (as with the previous labs and AIDs) include the rolling friction, F_r, the aerodynamic drag force, F_d, and the traction force, F_t. The traction force is only applicable during the *acceleration stage* and, consequently, two 2^{nd}-order ODEs must now be defined:

$$\text{acceleration:} \qquad m\frac{d^2x}{dt^2} = F_t - F_d - F_r, \tag{1}$$

$$\text{deceleration:} \qquad m\frac{d^2x}{dt^2} = \quad - F_d - F_r. \tag{2}$$

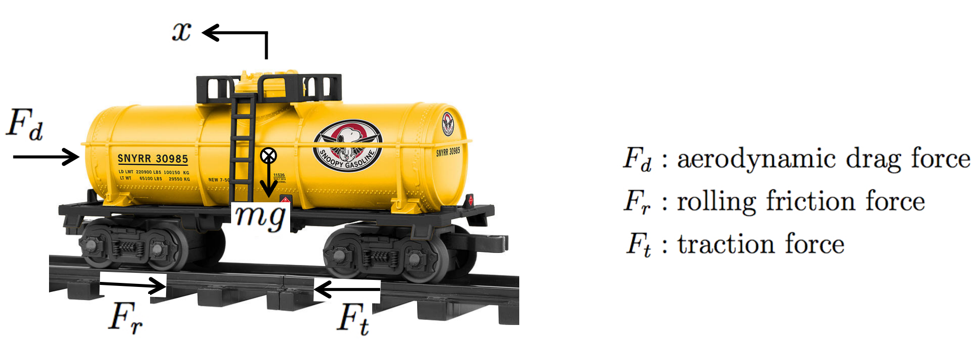

Figure 2: Free body diagram of the forces acting on the train during the race.

where m denotes the total mass of the train and $a = d^2x/dt^2$ denotes the acceleration if positive and deceleration if negative.

This train is powered by a single-stroke piston. Consequently, once the piston stroke has been fully extended the train will lose power and the deceleration stage begins. The length of track that will be covered during the acceleration stage, L_a, is therefore a function of the piston stroke length, L_s, wheel radius, r_w, and gear radius, r_g.:

$$L_a = \frac{L_s r_w}{r_g} \tag{3}$$

The aerodynamic drag force can be written in terms of a drag coefficient as:

$$F_d = \frac{1}{2} C_d \rho A v^2,$$

where ρ denotes the density of the air, A is the frontal area of the train, and v is the velocity of the train ($v = dx/dt$). The drag coefficient C_d depends on the shape of the object and the surface roughness. The model train to be used in the race will consist of a pressurized cylindrical tank on wheels. A good estimate for the drag coefficient of a circular cylinder oriented axially to the flow is $C_d \approx 0.8$. The rolling friction force between the wheel of the train and the rail of the train track is parameterized by the rolling resistance coefficient C_r according to the expression:

$$F_r = C_r m g.$$

One can empirically (using experiments) determine a value for the rolling resistance. However, we will assume a value of $C_r \approx 0.03$. We will assume that C_d and C_r are constant, which is a very good assumption under steady state conditions (i.e., the train is traveling at a constant speed). During the race, the model train never reaches a steady state condition; therefore, some error is expected by assuming a constant C_d. Substituting the expressions for F_d and F_r into Equations (1) and (2) yields:

$$\text{acceleration:} \qquad m\frac{d^2x}{dt^2} = F_t - \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g, \tag{4}$$

$$\text{deceleration:} \qquad m\frac{d^2x}{dt^2} = \quad - \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g. \tag{5}$$

Traction Force

The traction force F_t is the frictional force that the track rail exerts on the rotating train wheel. The traction force is applied at the point of contact between the wheel and the rail (see Figure 3). The traction force can be related to the applied torque on the wheel through conservation of angular momentum, which states that the sum of the torque is equal to the moment of inertia of the wheel (I) times the angular acceleration (α) of the wheel. Applying conservation of momentum about the axis of the wheel gives

$$T - r_w F_t = I\alpha,$$

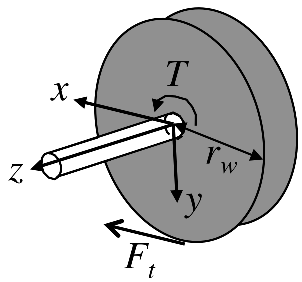

Figure 3: Schematic of the train wheel illustrating the traction force, F_t, between the wheel and rail (not shown).

where r_w = 20 mm is the radius of the train wheel. Approximating the train wheel as a solid disc, we can write the moment of inertia of the wheel as $I = \frac{1}{2} m_w r_w^2$, where m_w denotes the mass of the wheel. Note, since there are two wheels per axle, we need to multiply I by two. Therefore, the traction force is

$$F_t = \frac{T}{r_w} - m_w r_w \alpha. \tag{6}$$

Since the pinion gear and train wheel are connected to the same axle, the wheel/gear rotation is determined by the distance traveled,

$$x = r_w \theta. \tag{7}$$

This means we can write both the angular rotation of the wheel, θ, and the angular acceleration of the wheel, α, in terms of x as

$$\theta = \frac{x}{r_w} \qquad \text{and} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{1}{r_w}\frac{d^2x}{dt^2}. \tag{8}$$

Note, wheel slip will occur if the traction force is greater than the static friction force,

$$\text{wheel-slip criterion:} \quad F_t > \mu_s \tfrac{m}{2} g, \tag{9}$$

where μ_s is the coefficient of static friction between the wheel and the rail. The factor of $\frac{1}{2}$ is used because we assume the propulsion force only drives one of the axles, and that the total mass of the train is distributed equally between the front and rear axles. The value of μ_s depends on the type of materials in contact, but 0.7 is a reasonable value to use for this project. Your code will need to check for wheel-slip criterion in Equation 9 and print an error message if it is violated.

Applied Torque

The applied torque driving the wheel, T in Eqn. (6), is produced by the locomotion system, which in this case includes a rack and pinion gear as shown in Figure 4. Therefore, the applied torque is equal to the driving force of the rack, F_p, multiplied by the radius of the pinion gear, r_g, i.e.,

$$T = r_g \cdot F_p. \tag{10}$$

The rack is to be connected to a pneumatic piston as shown in Figure 5. An example pneumatic piston that can be purchased off-the-shelf is shown in Figure 6.

The force, F_p, driving the rack and pinion is equal to the *gauge pressure* acting on the piston times the area of the piston head A_p,

$$F_p = P_{\text{gauge}} \cdot A_p. \tag{11}$$

To model the reduction in pressure as the piston is actuated (discussed next), we must decompose the gauge pressure into its internal and atmospheric pressure components. The gauge pressure is equal to the pressure inside the piston chamber, P, minus the pressure acting on the outside of the piston which is assumed to be atmospheric pressure, i.e., $P_{\text{gauge}} = P - P_{\text{atm}}$. Therefore, we can write:

$$F_p = (P - P_{\text{atm}})A_p. \tag{12}$$

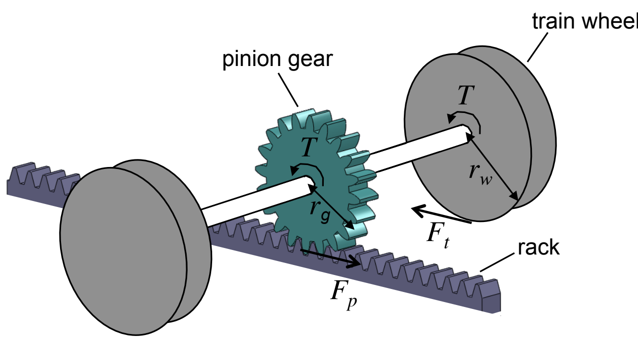

Figure 4: Schematic of the train axle showing one possible configuration of the rack/pinion.

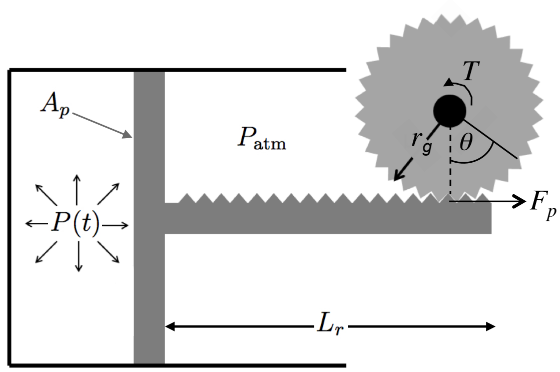

Figure 5: Schematic of the piston pushing against the rack thereby causing an applied torque T on the pinion gear.

Piston Pressure

The pressure inside the piston chamber will be decreasing with time as the piston is actuated. This decrease is governed by the ideal gas law:

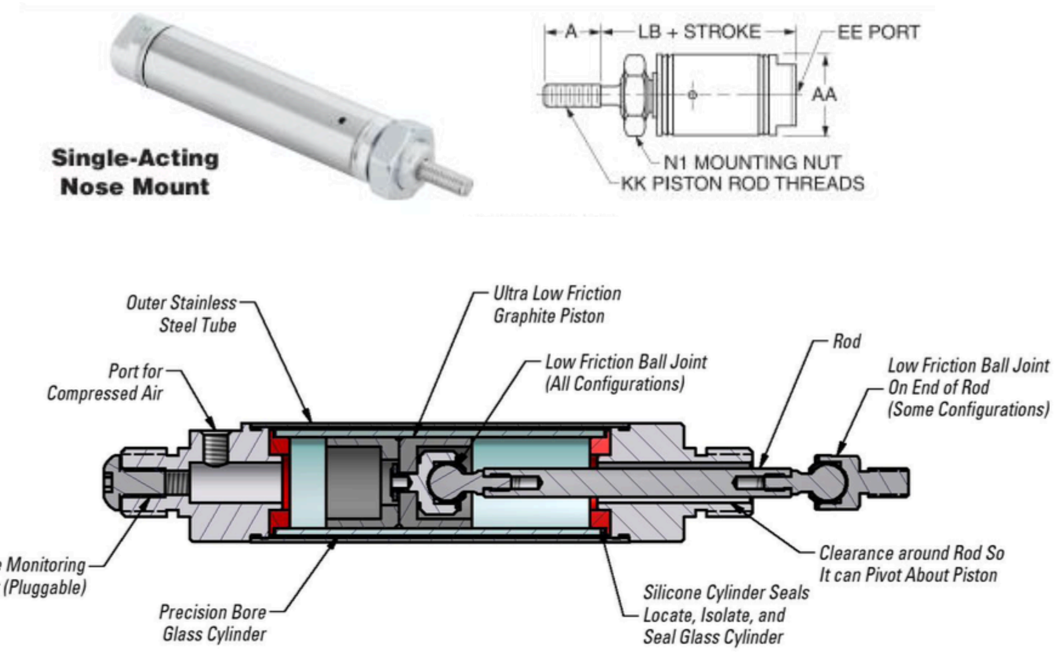

Figure 6: Example of a pneumatic piston that can be purchased off-the-shelf.

$$PV = nRT, \tag{13}$$

where V is the volume of the gas in the tank, n is the amount of gas in moles, R is the universal gas constant, and T is the absolute temperature of the gas. Note, P in (13) must be expressed in terms of an absolute pressure and NOT a gauge pressure. Since no gas is exhausted to the surroundings during locomotion, the mass of gas, and hence n, remains constant. If we assume that the expansion of the piston occurs isothermally, i.e., temperature remains constant, then the left hand side of (13) must be constant and equal to its initial value. This means (13) can be rewritten as

$$PV = P_0 V_0, \tag{14}$$

where P_0 and V_0 denote the initial tank pressure and volume, respectively. At a given time t, the total volume of the gas is equal to the initial tank volume plus the volume of the piston chamber, $V = V_0 + \Delta V$. Solving (14) for P gives

$$P = \frac{P_0 V_0}{V_0 + \Delta V}. \tag{15}$$

As the piston is actuated, the volume increases by an amount $\Delta V = A_p \ell$ as shown in Figure 5. Since the piston is connected to the rack and pinion gear, ℓ is determined by the angular rotation of the pinion gear, $\ell = r_g \theta$, as illustrated in Figure 5. Therefore, (15) becomes

$$P = \frac{P_0 V_0}{V_0 + A_p r_g \theta}. \tag{16}$$

Final System of ODEs

Finally, we can substitute (16), (12), and (10) into (6) to obtain an expression for the traction force in terms of wheel rotation θ,

$$F_t = \frac{r_g A_p}{r_w}\left[\frac{P_0 V_0}{V_0 + A_p r_g \theta} - P_{\text{atm}}\right] - m_w r_w \alpha. \tag{17}$$

Substituting Eqn. (8) and (9) into (17) yields

$$F_t = \frac{r_g A_p}{r_w}\left[\frac{P_0 V_0}{V_0 + A_p \frac{r_g}{r_w} x} - P_{\text{atm}}\right] - m_w \frac{d^2 x}{dt^2}. \tag{18}$$

Note, the traction force is inversely proportional to x.

Finally, we can substitute (18) back into the original equations of motion

$$\text{acceleration:} \quad m\frac{d^2x}{dt^2} = \frac{r_g A_p}{r_w}\left[\frac{P_0 V_0}{V_0 + A_p \frac{r_g}{r_w} x} - P_{\text{atm}}\right] - m_w \frac{d^2 x}{dt^2} - \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g, \tag{19}$$

$$\text{deceleration:} \quad m\frac{d^2x}{dt^2} = - \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g. \tag{20}$$

Dividing through by m and rearranging the top equation to get the derivative term on the left hand side produces the final set of governing equations

$$\text{Acceleration:} \quad \frac{d^2x}{dt^2} = \frac{1}{m + m_w}\left[A_p \frac{r_g}{r_w}\left(\frac{P_0 V_0}{V_0 + A_p (r_g/r_w) x} - P_{\text{atm}}\right) - \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g\right], \tag{21}$$

$$\text{Deceleration:} \quad \frac{d^2x}{dt^2} = \frac{1}{m}\left[- \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g\right]. \tag{22}$$

Recall, your code will need to test when the acceleration stage ends and the deceleration stage begins. This is set by the length of the rack L_r as shown in Figure 5. Accordingly, the propulsion system will continue to apply a traction force as long as $r_g \theta < L$, or substituting (8) for θ, gives the following conditions

$$\text{acceleration stage:} \quad x \le L_r \frac{r_w}{r_g} \tag{23}$$

$$\text{deceleration stage:} \quad x > L_r \frac{r_w}{r_g} \tag{24}$$

Solution of Equations (21) and (22) for a competitive train should look like the plot in Figure 7, which shows the distance traveled (left vertical axis) and velocity (right vertical axis) versus time. The train crosses the finish line at a time $t_f \approx 5.6$ s, and comes to a full stop before reaching the end of the run-out track.

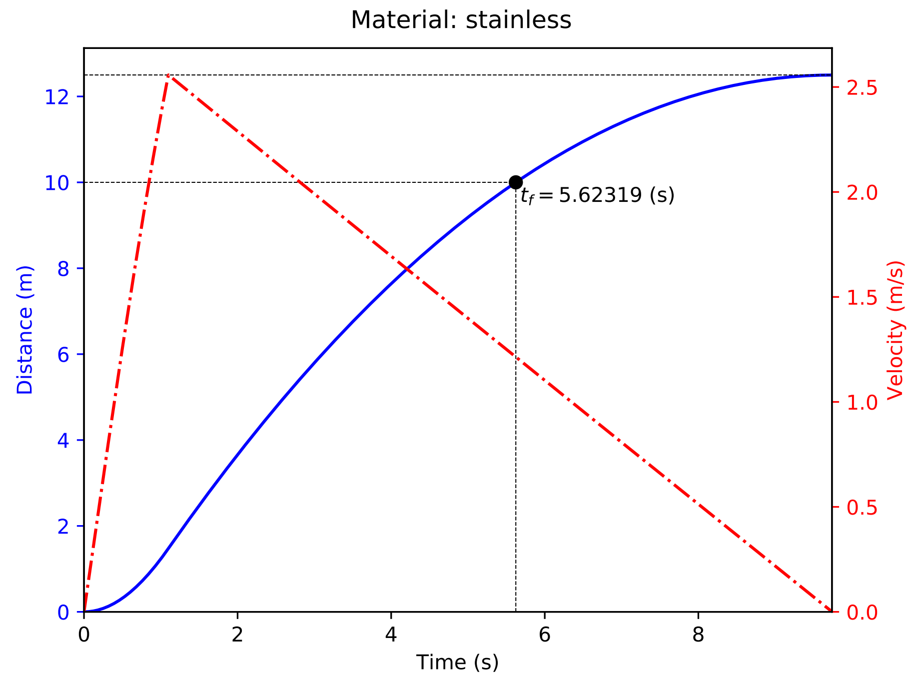

Figure 7: Behavior of a competitive train, where $x(t)$ represents the distance traveled as a function of time t.

Multidimensional Optimization Problem

Tables 1 and 4 list relevant physical parameters that appear in equations (21) and (22) and, therefore, can affect the performance of the train. The parameters in Table 1 are fixed at the values shown; whereas, the parameters in Table 4 will be optimized as part of the design solution.

Ranges of realistic values for each of the design parameters are given in Table 2, with the exception of density which is explained in the next paragraph. The objective of the optimization aspect of the final project is then to determine the set of parameters, in the defined ranges, that minimizes the time it takes the train to cross the finish line (without going off the end).

The model train is assumed to be made in the shape of a cylinder, consisting of a pipe/tube with PVC end caps. The cylinder body can be made out of PVC, steel, aluminum, or any other material that is readily purchased in tube/pipe form. You should consider at least three different materials in the design process. The densities of various tube/pipe materials is given in Table 3. A reasonable way to consider various materials is simply to run your optimization code separately for each material (density) and choose the fastest result among them.

Table 1: Relevant Physical Parameters

Parameter	**Symbol**	**Value**	**Units**
air density	ρ_a	1	kg/m^3
atmospheric pressure	P_{atm}	101325	Pa
drag coefficient	C_d	0.8	
rolling friction coefficient	C_r	0.03	
coefficient of static friction	μ_s	0.7	
wheel radius	r_w	0.02	m
mass of wheels and axles	m_w	0.1	kg

Table 2: Design Parameters

Parameter	**Symbol**	**Range of Values**	**Units**
length of train	L_t	(0.2, 0.3)	m
radius of tank	r_o	(0.05, 0.2)	m
density of train material	ρ_t	choose material	kg/m^3
initial tank gauge pressure	$P_{0_{\text{gauge}}}$	(70000, 200000)	Pa
pinion gear radius	r_g	(0.002, 0.01)	m
length of piston stroke	L_r	(0.1, 0.5)	m
radius of piston	r_p	(0.02, 0.04)	m

Table 3: Density of Various Materials used for Pipes/Tubes

Material	**Density** (kg/m^3)
PVC	1400
acrylic	1200
galvanized steel	7700
stainless steel	8000
titanium	4500
copper	8940
aluminum – 6061	2700

Your program should include the following features:

1. Define ranges for physical parameters

   Use the specified values for the *fixed* parameters in Table 1. Your optimization procedure should eliminate any combination of parameters that violates the design constraints (see Item 2).

   For simplicity, assume the following:

   - The inside radius r_i of the pipe tube is proportional to the outer radius ($r_o = 1.15r_i$).
   - The mass of the pneumatic piston is proportional to its volume, i.e., $M_p = \rho_p(\pi r_p^2 L_p)$, with a proportionality constant of $\rho_p = 1250$ kg/m^3.
   - The total length of the piston is proportional to its stroke ($L_p = 1.5L_r$).
   - The length of the rack is equal to the length of the stroke of the pneumatic piston.

   Note, you should consider where the piston will be placed. The options are underneath the train or behind the train like a caboose. If the piston is placed underneath the train, then enough clearance must be ensured, which will affect the total height of the train. If the piston is placed behind the train, then the total length of the train is the length of the tank plus the length of the piston. There are design constraints on both the total length of the train and the height of the train as discussed below.

2. Check design constraints

   Your code should eliminate any set of parameters that violates one or more of the design constraints:

   train height/width the train must be capable of passing through the tunnel. Therefore, the height cannot exceed 0.23 m and the width cannot exceed 0.2 m.

   train length the entire train must fit on the track at the starting line. Since the length of "set-up" track is only 1.5 m, the total length of the train plus propulsion system cannot exceed 1.5 m.

   gear radius the radius of the pinion gear must be less than the radius of the train wheel (i.e., $r_g/r_w < 1$).

   wheel slippage the wheels will slip if the maximum traction force exceeds the force due to static friction, as described by (9). The maximum traction force will be experienced at start up when the pressure inside the tank is a maximum. Using the relation for the traction force in equation (17), one can derive a condition on the maximum allowable initial tank pressure to avoid wheel slippage. If the initial tank pressure exceeds the maximum allowable pressure, it is inadmissible and an appropriate error should be raised.

   travel distance The train must cross the finish line and come to a full stop before traveling 12.5 m (running beyond the end of the track).

3. Perform optimization

   Set up and run an optimization problem to determine the optimal train parameters. For this, use a built-in optimization function in Matlab or Python. In Matlab, `fmincon` is an example of a constrained optimization code where bounds on each design variable can be passed. This method will guarantee that the design parameters stay within their specified bounds as specified in Table 2.

There are 2 things to keep in mind when using `fmincon`. First, it can not handle `inf` or `nan` values. Consequently, in your objec-tive function (where you're computing and returning the time taken to cross the finish line) you should not return a value of infinity if the train wheels slip or the train runs off the track. In-stead, you should just return an arbitrary, high value (on the order of 100 seconds will work). Alternatively, there are two unconstrained optimization algorithms in Matlab that will likely re-turn a more optimal result, `fminsearch` and `fminunc`. Similarly, Python has several optimization methods in `scipy.optimize.minimize`. Here, a constrained optimization algorithm is specified by providing the optional `bounds` argument. Similar to Matlab, however, there are unconstrained optimization algorithms in Python that will probably provide a more optimal set of design parameters. This can be specified by not provid-ing the optional `bounds` argument and adding the argument `method='Nelder-Mead'`, which will be similar to the unconstrained Matlab algorithms.

Tips:

- If unconstrained optimization algorithms are utilized, keep in mind that it is possible that some of the returned design parameters could violate the bounds. These unconstrained methods can be used as long as the returned design parameters are checked for being within the desired bounds.
- This design problem is very sensitive to having a good initial guess from which these optimization functions will start. You should run a brute force search to determine good initial guesses for the design parameters for input to those built-in functions. If you have any trouble getting the brute force step to work, you may resort to using the following initial guesses which work well for a stainless steel train:

Table 4: Example Set of Design Parameter Initial Guesses

Parameter	**Symbol**	**Range of Values**	**Units**
length of train	L_t	0.25	m
radius of tank	r_o	0.115	m
density of train material	ρ_t	8000	kg/m^3
initial tank gauge pressure	$P_{0_{\text{gauge}}}$	96000	Pa
pinion gear radius	r_g	0.005	m
length of piston stroke	L_r	0.3	m
radius of piston	r_p	0.032	m

Note that you will need to calculate quantities such as the frontal area of the train A, the initial tank volume V_0, the total mass of the train m, etc., based on the values input for the design parameters.

4. Plot distance traveled and train velocity versus elapsed time (optimum design solution)

   For the optimum set of parameters obtained, plot x versus t and v versus t based on your numerical simulation. You can use the provided plotting function, or write your own. Be sure to include the plots in your memo.

5. Display optimum design solution

   Once you have found the optimum set of design parameters, your code should calculate the following quantities, which are to be included in your memo in tabular format as shown below. Be sure to include the table in your memo.

Parameter	**Symbol**	**Optimum Value**	**Units**
length of train	L_t		m
outer radius of train	r_o		m
height of train	H_t		m
material of train	–		
total mass of train	m		kg
train frontal area	A		m^2
initial tank gage pressure	$P_{0_{\text{gage}}}$		Pa
tank volume	V_0		m^3
pinion gear radius	r_g		m
length of piston stroke	L_r		m
total length of piston	L_p		m
radius of piston	D_p		m
mass of piston	M_p		kg

6. Select *realistic* components

   You may not be able to build an actual model train having the exact specifications given by your optimum solution because materials and components will typically only be available in standard sizes. You are now tasked with selecting realistic components for your train that match as close as possible to the parameter values given by your optimum design solution. In order to do this, you will need to search the internet to identify manufacturers that make pipe/tubing, pneumatic pistons, and rack/pinion gears. Based on your research put together a parts list for your train including the size, model number, vendor, and estimated price for each item. Include the parts lists in your memo.

   If your *realistic* train has different parameter values than your optimum numerical solution, then you will need to rerun your simulation on the *realistic* train in order to verify that the *realistic* train still completes the race without running off the track. In your memo, you should state the time to cross the finish line for your *realistic* train.

Provided Codes

Two functions are provided that may be used to help complete the design project. First, you are encouraged to use the provided `rk4` code (for Matlab or Python). The `rk4` function performs the 4th-Order Runge-Kutta method to solve a system of ODEs with an arbitrary number of equations. Second, you are encouraged to use the provided `plot_results` function (for Matlab or Python) to generate the required plots for the project report. In both cases, you are free to use your own code for these two functions instead: these two functions are just being provided to provide additional assistance, if needed.