MARKOV MODELS FOR STUDYING SOCIAL PROCESSES

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1. INTRODUCTION AND BACKGROUND

Markov models have been used to analyze structural mechanisms that underlie social processes. Broadly, Markov models can be used to project future change in the state distribution of some population. Myriad analyses have been performed using Markov Models. Their popularity may be due to their applicability to dynamical systems and relative simplicity when compared to more complex methods (i.e. systems of differential equations).

Markov Models are used to describe the probabilities of a sequence of random variables, or chained events, where a prediction about a future state can be accurately provided using only the information given by the current state. In other words, any state prior to the current state doesn't impact a future state except by way of the current state. Consider a sequence of random variables,

$$x_1, x_2, ..., x_{t-1}, x_t, x_{t+1}, ..., x_n.$$

The **Markov assumption** is that the state at time t is dependent only on the state at time $t-1$. Mathematically, this assumption is expressed as

$$P(x_t = a|x_1, x_2, ..., x_{t-1}) = P(x_t = a|x_{t-1}).$$

This paper will first survey current applications of Markov modelling techniques in the social sciences and economics. We then present a practice exercises to better understand the properties of transition matrices and their applications.

1.1. Social Mobility and the Markov Chain

The simplest type of Markov model is the Markov chain. The Markov chain may be visualized as a graph, or expressed as a **transition matrix**. Rows of the transition matrix describe the probability of moving from the state represented by that row to the other states. For example, row 1 of a 3×3 transition matrix has as its first entry the probability of *staying* in state 1, its second entry the probability of moving to state 2, and its third entry the probability of moving to state 3.

The transition matrix, T, is said to be *historyless*, meaning that future states are dependent only on the current state – not any prior state. For a **regular Markov chain**, powers of T approach a matrix V where all rows are the same. In other words, a regular Markov matrix T converges to a **stationary point** V. That is,

$$\lim_{n \longrightarrow \infty} T^n = V$$

This does not imply an absence of movement between classes. Individuals still may transfer between classes; however, individual movements into one class are exactly counterbalanced by individual movements out of the class.

The properties of the regular Markov transition matrix will be explained in the context of Spaseki & Hasanociv's 2016 paper on *Inter-generational Social Mobility as a Markov Process*. The authors of the paper claim that social mobility in the UK can be entirely represented by a regular Markov matrix. Using data from the *Study of Intergenerational Change in Status* by Glass & Hall, they propose the following transition matrix:

$$P = \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}.$$

This matrix represents the probabilities of moving between classes between generations. Let class 1 be lower class, 2 be middle, and 3 be upper. Then p_{11} may be thought of as the probability that a person in the lower class has offspring in the lower class. Similarly, p_{13} would the probability that a person in the upper class has offspring in the lower class. Squaring P, we find

$$P^2 = \begin{bmatrix} 0.473 & 0.395 & 0.132 \\ 0.219 & 0.556 & 0.225 \\ 0.194 & 0.462 & 0.344 \end{bmatrix},$$

whose ij^{th} entry represents the probability that an individual in the i^{th} class has *grandchildren* the j^{th} class. After approximately 13 generations, the transition matrix reaches its stationary point:

$$P^{12+n} \approx P^{12} = \begin{bmatrix} 0.286 & 0.488 & 0.225 \\ 0.286 & 0.488 & 0.225 \\ 0.286 & 0.488 & 0.225 \end{bmatrix}.$$

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The implication here is that without any policy alteration, after about 12 generations the distribution of individuals across social classes will be approximately constant over time.

1.2. Markov Models and the Growth of Non-religion

Religious trends have been predicted using time-series forecasts and other statistical modelling tools. In 2015, Stinespring and Cragun developed a **Markov switching model** using General Social Survey data from 1973 to 2012. Markov switching model is the name given by economists to models describing regular Markov chains. The authors calculated a range of Markov models and compared them with real-world data to assess validity.

Using data from the past 20 years yielded a transition matrix that projected the most rapid growth in nonreligion:

$$S_{20} = \begin{bmatrix} 0.92 & 0.07 \\ 0.08 & 0.93 \end{bmatrix},$$

where $s_{11} = 0.07$ is the conversion rate (non-religious individuals becoming religious) and s_{22} is the leaving rate (religious individuals becoming nonreligious). This matrix converges to V_{20} with $v_{11} = v_{21} = 0.47$, the proportion of nonreligious. For context, an estimated $20 - 29\%$ of individuals in the United States currently identify as nonreligious.

The authors also built a more conservative model using the 40-year mean:

$$S_{40} = \begin{bmatrix} 0.825 & 0.175 \\ 0.06 & 0.94 \end{bmatrix},$$

where V_{40}, the stationary matrix corresponding to S_{40} has $v_{11} = v_{21} = 0.26$. There is a significant disparity between the two projections, highlighting a limitation of Markov models. The assumption of the model is that the state of an individual at time $n + 1$ is *only* dependent on the state of that individual at time n. This ignores other factors that may be essential to understanding the state distribution of a population.

1.3. Graphical Markov Models and Attitudes Towards Environmental Protection

Graphical Markov models can be used to understand some statistical models. A paper by Nanny Wermuth at University of Mainz explored their explanatory power using General Social Survey data to understand how political perceptions evolve, specifically concerning environmental issues. Respondents were asked a series of binary questions which admit the following variables:

Primary Response:

- Individual has no concern about protecting the environment

Intermediate:

- No own political impact expected
- At risk of social exclusion
- Lower education level

The study exhaustively fit logistic regression models and built a graph indicating the direction and strength of dependencies. Using this technique, the authors were able to show that women are at higher risk for social exclusion, that those at risk for social exclusion are less likely to believe in having personal political impact, and that those who perceive no individual political impact are more likely to be unconcerned about the environment.

1.4. Hidden Markov Models

Regular Markov models work well when all states in a sequence are observable. There are also phenomena where events of interest are *hidden*. In other words, you are not directly observing the event you're interested in; rather, you observe some outcome you suspect is caused by the event of interest. To illustrate the process, consider the following example:

You're an analytically-minded ski bum and you want to model the amount of new snow at Beaver Mountain across the season. Unfortunately, all lengthwise measuring devices have been confiscated by the government due to a recent study linking their usage to chronic insecurity. Instead of measuring the snow directly, you collect a daily count of the number of attendees, assuming a direct link between the number of skiers and the amount of new snow on a given day.

A **Hidden Markov Model** is derived from augmenting the Markov chain. The models discussed in previous sections are *observed markov models*, meaning the states can be measured and probabilities assigned. By contrast, *hidden* markov models rely on making inference about a probabilistic relationship without explicitly measuring values of a response variable. Hidden Markov models are most commonly used in biosciences and behavioral social sciences. Their use in sociology and demography could be profound, but they are often ignored likely due to mathematical complexity.

2. WORKED PROBLEMS

This section contains a few worked problems from the textbook.

1. Determine if the following statements are true or false.

   (a) **Claim.** If $A \in M_{n\times n}(C)$ and $\lim_{m\to\infty} A^m = L$, then for any invertable matrix $Q \in M_{n\times n}(C)$ we have $\lim_{m\to\infty} QA^mQ^{-1} = QLQ^{-1}$.

   Proof. Take

$$\lim_{m\to\infty} A^m = L.$$

   Left and right multiply by Q and Q^{-1} to get

$$Q\left(\lim_{m\to\infty} A^m\right)Q^{-1} = QLQ^{-1}.$$

   The constant multiple rule for limits applies to constant matrices, so we can pass these matrices into the limit to get

$$\lim_{m\to\infty} QA^mQ^{-1} = QLQ^{-1}.$$

   (c) **Claim.** Any vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$$

   such that

$$x_1 + x_2 + ... + x_n = 1$$

   is a probability vector.

   Proof. The main criteria for probability vectors is to contain only non-negative entries whose sum is zero. Suppose $x_1 = -50$ and $x_n = 50$ with $x_2, ..., x_{n-1}$ non-negative entries whose sum is zero. This vector satisfies the criteria in the claim but fails the criteria of a probability vector, so the claim is **false**.

   (i) **Claim.** If A is a transition matrix then $\lim_{n\to\infty} A^n$ exists.

   Proof. Suppose

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   A is a transition matrix since it is **doubly stochastic** (contains non-negative real entries with each row and column summing to one). Now take

$$A^2 = A' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and}$$

$$A^3 = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   The sequence then becomes

$$\{A, A', A, A', ...\}$$

   which alternates between two distinct values and does not converge. Therefore, the claim is **false**.

2. Determine whether $lim_{m\to} A^m$ exists for each of the following matrices A, and compute the limit if it exists.

   (a)

$$A = \begin{bmatrix} 0.1 & 0.7 \\ 0.7 & 0.1 \end{bmatrix}.$$

   If a matrix has a limit, that is

$$\lim_{m\to\infty} A^m = L,$$

   Then we can use a property of diagonalizable matrices, which is

$$\lim_{m\to\infty} (QAQ^{-1})^m = \lim_{m\to\infty} QL^mQ^{-1}$$

$$= Q\left(\lim_{m\to\infty} L^m\right)Q^{-1}.$$

   Using python,

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.8 & 0 \\ 0 & -0.6 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}.$$

   $Q^m = Q$ and $\left(Q^{-1}\right)^m = Q^{-1}$ since they are idempotent. All we need to do now is compute

$$\lim_{m\to\infty} L^m = \begin{bmatrix} (-0.8)^m & 0 \\ 0 & (0.6)^m \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(g)

$$A = \begin{bmatrix} -1.8 & 0 & -1.4 \\ -5.6 & 1 & -2.8 \\ 2.8 & 0 & 2.4 \end{bmatrix}.$$

With python,

$$A = \begin{bmatrix} 0 & -0.5 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.4 \end{bmatrix} \begin{bmatrix} -4 & 1 & -2 \\ 2 & 0 & 2 \\ -2 & 0 & -1 \end{bmatrix}.$$

The limit of the second matrix shall be

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now computing QDQ^{-1}, we get

$$\lim_{m\to\infty} A^m = QDQ^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -4 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}.$$

3. **Claim.** If $A_1, A_2, ...$ is a sequence of $n \times p$ matrices with complex entries such that $\lim_{m\to\infty} A_m = L$, then $\lim_{m\to\infty} (A_m)^t = L^t$.

   Proof. Consider the ij^{th} entries of A and L respectively, a_{ij} and l_{ij}. Now,

   $$\lim_{m\to\infty} (A_m)^t_{ij} = \lim_{m\to\infty} (A_m)_{ji} = l_{ji}.$$

   Now l_{ji} is just l^t_{ij}, so we have shown that for an arbitrary entry in A and L

   $$\lim_{m\to\infty} (A_m)^t = L^t.$$

4. **Claim.** If $A \in M_{n\times n}(C)$ is diagonalizable and $L = \lim_{m\to\infty} A^m$ exists, then either $L = I_n$ or $rank(L) < n$.

   Proof. Given the definition of diagonalizability,

   $$Q^{-1}AQ = D.$$

   It has been discussed already that

   $$\lim_{m\to\infty} \left(Q^{-1}AQ\right)^m = \lim_{m\to\infty} D^m = QLQ^{-1}.$$

   Now, $\lim_{m\to\infty} D^m$ only exists when the absolute value of eigenvalues in A is less then 1, or when they equal one. If eigenvalues in A are all equal to one then $L = I_n$, the identity matrix. If they are less then one, as m approaches ∞ the entry in D approaches 0 which means rank$(L) < n$.

6. A hospital trauma unit has determined that 30% of its patients are ambulatory and 70% are bedridden at the time of arrival at the hospital. A month after arrival, 60% of the ambulatory patients have recovered, 20% remain ambulatory, and 20% have become bedridden. After the same amount of time, 10% of the bedridden patients have recovered,20% have become ambulatory, 50% remain bedridden, and 20% have died. Determine the percentages of patients who have recovered, are ambulatory, are bedridden, and have died 1 month after arrival. Also determine the eventual percentages of patients of each type.

   Answer. Let's begin by building our transition matrix A. The first column will represents recovered patients and assumes that once recovered a patient will not get sick again. The second column represents ambulatory patients, who recover with probability 0.6, remain ambulatory with probability 0.2, and become bedridden with probability 0.2. The third column represents bedridden patients and is constructed similarly. The final column represents patients who have died, and we assume that a dead patient cannot un-die. Therefore,

   $$A = \begin{bmatrix} 1 & 0.6 & 0.1 & 0 \\ 0 & 0.2 & 0.2 & 0 \\ 0 & 0.2 & 0.5 & 0 \\ 0 & 0 & 0.2 & 1 \end{bmatrix}.$$

   We are asked to determine the distribution of patients after 1 month in the hospital given initial probability vector

   $$\vec{p} = \begin{bmatrix} 0 \\ 0.3 \\ 0.7 \\ 0 \end{bmatrix}.$$

   We can find the distribution with some simple multiplication,

   $$A\vec{p} = \begin{bmatrix} 1 & 0.6 & 0.1 & 0 \\ 0 & 0.2 & 0.2 & 0 \\ 0 & 0.2 & 0.5 & 0 \\ 0 & 0 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.3 \\ 0.7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.2 \\ 0.41 \\ 0.14 \end{bmatrix}.$$

   This suggests that after one month, given initial probability vector $\vec{p}$, 25% of patients will have recovered, 20% of patients will be ambulatory, 41% of patients will be bedridden, and 14% of patients will have died. Now we need to find the eventual percentages of each patient category which we will do by finding

$$\left(\lim_{m\to\infty} A^m\right)\vec{p} = L\vec{p}.$$

Once again, we diagonalize A to find

$$A = Q\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} Q^{-1}.$$

Taking the limit results in

$$\lim_{m\to\infty} A^m = Q\left(\lim_{m\to\infty}\begin{bmatrix} 1^m & 0 & 0 & 0 \\ 0 & 0.1^m & 0 & 0 \\ 0 & 0 & 0.6^m & 0 \\ 0 & 0 & 0 & 1^m \end{bmatrix}\right) Q^{-1}$$

$$= Q\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} 1 & 0.889 & 0.556 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.11 & 0.44 & 1 \end{bmatrix}.$$

Finally, we compute

$$L\vec{p} = \begin{bmatrix} 1 & 0.889 & 0.556 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.11 & 0.44 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.3 \\ 0.7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.66 \\ 0 \\ 0 \\ 0.34 \end{bmatrix}.$$

This result tells us that in our system, all patients either fully recover (with probability 0.66) or die (with probability 0.34).

8. Determine if the following transition matrices are regular.

(a)

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$

A matrix is called **regular** if some power of the matrix contains only positive entries.

$$A^2 = \begin{bmatrix} 0.38 & 0.37 & 0.25 \\ 0.37 & 0.38 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

has only positive entries, therefore A is a **regular** matrix.

(c)

$$C = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Each time C is multiplied with itself, the second and third columns simply interchange. Each of these columns have 0 entries (not strictly positive). Since they do not change with powers of C, C is **not regular**.

f.

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0.2 \\ 0 & 0.3 & 0.8 \end{bmatrix}.$$

The first column of F never changes (same as first column of identity). With 0 entries, the matrix is **not regular**.

10. Each of the matrices that follow is a regular transition matrix for athree-state Markov chain. In all cases, the initial probability vector is

$$\vec{p} = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$$

For each transition matrix, compute the proportions of objects in each state after two stages and the eventual proportions of objects in each state by determining the fixed probability vector.

(a)

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.9 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{bmatrix}.$$

After two stages, we have

$$A^2\vec{p} = \left(\begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.9 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{bmatrix}\right)^2 \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.225 \\ 0.441 \\ 0.334 \end{bmatrix}.$$

The fixed probability vector can be found by finding $(A - I)\vec{v} = 0$. This gives

$$(A-I)\vec{v} = \begin{bmatrix} -0.4 & 0.1 & 0.1 \\ 0.1 & -0.1 & 0.2 \\ 0.5 & 0.1 & -0.4 \end{bmatrix} \vec{v} = 0,$$

so a basis for the nullspace is

$$\vec{v} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

which can be normalized to find the fixed probability vector,

$$\vec{p} = \begin{bmatrix} 0.2 \\ 0.6 \\ 0.2 \end{bmatrix}.$$

11. In 1940, a county land-use survey showed that 10% of the county land was urban, 50% was unused, and 40% was agricultural. Five years later, a follow-up survey revealed that 70% of the urban land had remained urban, 10% had become unused, and 20% had become agricultural. Likewise, 20% of the unused land had become urban, 60% had remained unused, and 20% had become agricultural. Finally, the 1945 survey showed that 20% of the agricultural land had become unused while 80% remained agricultural. Assuming that the trends indicated by the 1945 survey continue, compute the percentages of urban, unused, and agricultural land in the county in 1950 and the corresponding eventual percentages.

    First we want to build a Markov transition matrix to show the probability that a plot of land in one state moves to the next state after a period of 5 years. Let the first column be the distribution of urban land after 5 years, which is 70% urban, 10% unused, and 20% agricultural. The second column corresponds to the distribution of unused land after 5 years, and the final column corresponds to the distribution of agricultural land after 5 years. Thus, the Markov transition matrix should be

$$A = \begin{bmatrix} 0.7 & 0.2 & 0 \\ 0.1 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.8 \end{bmatrix}.$$

The initial distribution of land can be expressed by the vector

$$\vec{v} = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.4 \end{bmatrix}.$$

So to find the distribution of land in 1950 (two "generations" after the initial survey), we take

$$A^2\vec{v} = \left(\begin{bmatrix} 0.7 & 0.2 & 0 \\ 0.1 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} \right)^2 \begin{bmatrix} 0.1 \\ 0.5 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.20 \\ 0.34 \\ 0.46 \end{bmatrix}.$$

Now to find the eventual percentage we can just find some $\vec{u}$ such that $(A-I)\vec{u} = 0$. This gives

$$= (A-I)\vec{u} = \left(\begin{bmatrix} -0.3 & 0.2 & 0 \\ 0.1 & -0.4 & 0.2 \\ 0.2 & 0.2 & -0.2 \end{bmatrix} \right)^2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is solved by

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix},$$

and can be re-expressed with entries adding to 1 by dividing by the sum of the entries. Thus,

$$\vec{u} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix},$$

suggesting that the eventual distribution of land is 20% urban, 30% unused, and 50%agricultural.

15. **Claim.** If a 1-dimensional subspace W of R^n contains a nonzero vector with all nonnegative entries, then W contains a unique probability vector.

    Proof. Let $\vec{x} \in W$, the nonzero vector with all nonnegative entries. Suppose the sum of all entries in $\vec{x}$ is equal to k. Then a probability vector in W is easily found as

$$\vec{p} = \frac{1}{k} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}.$$

Now we show uniqueness. Suppose there is another probability vector, $\vec{y}$. These vectors should be parallel, which can be expressed mathematically as

$$\vec{y} = m\vec{x}.$$

Now in order for these both to be probability vectors, m must equal 1 else the entries in one vector or the other will not sum to 1. Thus,

$$\vec{y} = \vec{x}$$

and the probability vector is unique.

18. **Claim.** Suppose A is a egular transition matrix that is diagonalizable. Then $\lim_{m\to\infty} A^m$ exists.

    Proof. There are two conditions that must be satisfied for this limit to exist.

    i. Every eigenvalue of A $|\lambda_i| < 1$ or $\lambda_i = 1$. This is because the diagonal matrix in the diagonalization of A must have only have nonzero values $|a_{ii}| < 1$ or $a_{ii} = 1$. For other values, $\lim_{n\to\infty} a_{ii}^n$ is unbounded. Since A is a regular transition matrix, its largest eigenvalue is 1 and its other eigenvalues are $-1 < \lambda_i < 1$, so the condition is satisfied.

    ii. The second condition is that A must be diagonalizable. A is diagonalizable because it is a regular transition matrix.

19. Suppose that M and M are $n \times n$ transition matrices.

    (a) **Claim.** If M is regular and N is any $n \times n$ transition matrix, and c is a real number such that $0 < c << 1$, then $cM + (1-c)N$ is a regular transition matrix.

    Proof. Let $\vec{v}$ be a probability vector. Then consider the expression,

    $$(cM + (1-c)N)\vec{v}.$$

    Transition matrices do not lose their defining properties under transposition, so take the transpose to get

    $$= (cM + (1-c)N)\vec{v} = cM^T\vec{v} + (1-c)N^T\vec{v}$$

    Now the product of a transition matrix and a probability vector is just another probability vector, so this expression reduces to

    $$cM^T\vec{v} + (1-c)N^T\vec{v} = \vec{v},$$

    which is a probability vector thus showing that $(cM + (1-c)N)$ must be a transition matrix. Now to show it is regular we need to show that for some n,

    $$(cM + (1-c)N)^N$$

    is positive. Note that M is regular so M must be positive. Thus, some M^k is also positive. This matrix reduces to the sum of c^kM^k terms which are by definition positive. Also included in the sum are nonnegative lower order terms. Therefore,

    $$(cM + (1-c)N)$$

    is a regular transition matrix.

3. REFERENCES

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