5

定积分及其应用

5.2 定积分的求解

牛顿-莱布尼兹公式

微积分基本定理：如果函数F(x)是连续函数f(x)在区间[a,b]上的一个原函数，那么 $\int_a^b f(x)\,\mathrm{dx} = F(b) - F(a)$

计算定积分 $\int_0^1 x^2\,\mathrm{dx}$

$f(x) = x^2,\ F(x) = \frac{x^3}{3}.$

$\int_0^1 x^2\,\mathrm{dx} = F(1) - F(0) = \frac{1}{3}$

```
Integrate[x^2, {x, 0, 1}]
积分
```

$\frac{1}{3}$

换元法与分部积分法

- 换元法

定理：假设函数f(x)在区间[a,b]上连续，函数 x=φ(t)满足条件

▲ $\varphi(\alpha)$=a,$\varphi(\beta)$=b;

▲ φ(t)在[α,β]上具有连续导数，且其值域 $R_\varphi = [a, b]$

则有 $\int_a^b f(x)\,\mathrm{dx} = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,\mathrm{dt}.$

计算定积分 $\int_0^a \sqrt{a^2 - x^2}\ \mathrm{dx},\ (a > 0)$

设x=a sin t, 则 dx=a cos t dt, 当x = 0, 取t = 0, 当x = a, 取t = $\frac{\pi}{2}$

$\int_0^a \sqrt{a^2 - x^2}\ \mathrm{dx} = a^2 \int_0^{\frac{\pi}{2}} \mathrm{Cos}^2\, t\,\mathrm{dt} = \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 + \mathrm{Cos}\,2\,t)\,\mathrm{dt} = \frac{a^2}{2}\left[t + \frac{1}{2}\,\mathrm{Sin2t}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}$

- 分部积分法

$$\int_a^b u(x)\,v'(x)\,dx = \left[\int u(x)\,v'(x)\,\mathrm{dx}\right]_a^b = \left[u(x)\,v(x) - \int u'(x)\,v(x)\,\mathrm{dx}\right]_a^b = [u(x)\,v(x)]_a^b - \int_a^b v(x)\,u'(x)\,dx.\text{简记作}\int_a^b \mathrm{uv}'\,dx = [\mathrm{uv}]_a^b - \int_a^b \mathrm{vu}'\,dx$$

计算定积分$\int_0^1 e^{\sqrt{x}}\,dx$

先换元法，令 $\sqrt{x}=t$, 则 $x=t^2$, $dx=2\,t\,dt$, 且当 $x=0$ 时，$t=0$，当 $x=1$ 时，$t=1$.

$\int_0^1 e^{\sqrt{x}}\,dx = 2\int_0^1 \mathrm{t}e^t\,dt = 2\int_0^1 t\,d(e^t) = 2\left([t\,e^t]_0^1 - \int_0^1 e^t\,dt\right) = 2\left(e - [e^t]_0^1\right) = 2[e-(e-1)] = 2$