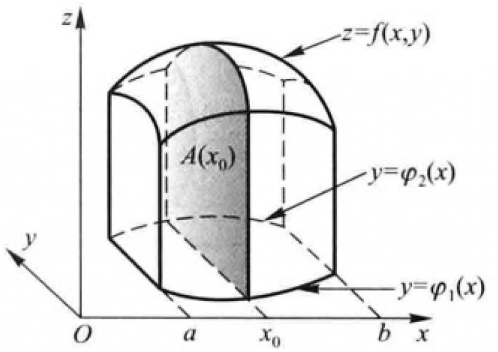
6

多元函数与重积分

6.3 重积分的求解

直角坐标系

计算二重积分 $\iint_D f(x, y)\,\mathrm{d}\delta$



积分区域D: $a < x < b,\ \varphi_1(x) < y < \varphi_2(x)$

截面面积：$A(x_0) = \int_{\varphi_1(x_0)}^{\varphi_2(x_0)} f(x_0, y_0)\,\mathrm{dy}$→任意截面面积 $A(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,\mathrm{dy}$

得体积： $V = \int_a^b A(x)\,\mathrm{dx} = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,\mathrm{dy}\right]\mathrm{dx}$

即二重积分 $\iint_D f(x, y)\,\mathrm{d}\delta = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,\mathrm{dy}\right]\mathrm{dx} = \int_a^b \mathrm{dx} \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,\mathrm{dy}$

计算 $\iint_D x\,y\,\mathrm{d}\delta$,其中D是直线y=1，x=2，及y=x所围成的闭区域.

```
p1 = ParametricPlot[{{x, x}, {2, y}, {x, 1}},
     └绘制参数图
  {x, 1, 2}, {y, 1, 2}, PlotRange → {{0, 2.1}, {0, 2.1}}]
                        └绘制范围
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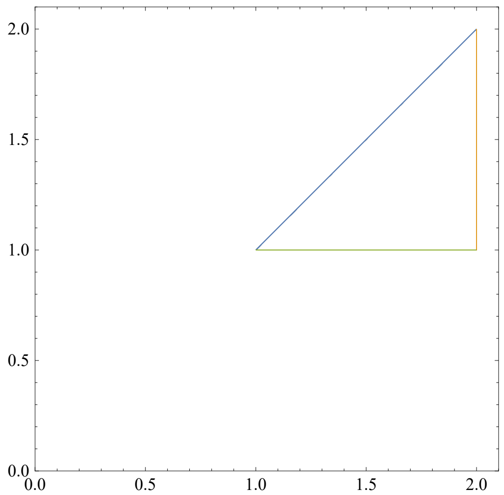
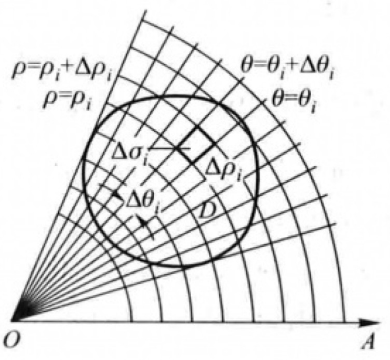


$$\iint_D x\,y d\delta= \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy = \int_1^2 dx \int_1^x x\, y dy = \int_1^2 dx\left[x\,\frac{y^2}{2}\right]_1^x = \int_1^2 dx\left(\frac{x^3}{2} - \frac{x}{2}\right) = \left[\frac{x^4}{8} - \frac{x^2}{4}\right]_1^2 = \frac{9}{8}$$

极坐标系

极坐标与直角坐标的转化关系：$\begin{cases} x = \rho\,\mathrm{Cos}\,\theta & \rho = \sqrt{x^2 + y^2} \\ y = \rho\,\mathrm{Sin}\,\theta & \mathrm{Tan}\,\theta = \dfrac{y}{x} \end{cases}$

二重积分定义：$\iint_D f(x, y)\, d\,\delta = \lim_{\lambda\to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\, \Delta\delta_i$



$$\Delta\delta_i = \frac{1}{2}(\rho_i + \Delta\rho_i)^2\, \Delta\theta_i - \frac{1}{2}\rho_i^2\, \Delta\theta_i = \frac{1}{2}(2\,\rho_i + \Delta\rho_i)\, \Delta\rho_i\, \Delta\theta_i = \frac{\rho_i + (\rho_i + \Delta\rho_i)}{2}\, \Delta\rho_i\, \Delta\theta_i = \overline{\rho}_i\, \Delta\rho_i\, \Delta\theta_i$$

$$\iint_D f(x, y)\, d\,\delta = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i, \eta_i)\, \Delta\delta_i \quad = \quad \lim_{\lambda\to 0}\sum_{i=1}^{n} f\left(\overline{\rho}_i\,\mathrm{Cos}\,\overline{\theta}_i, \overline{\rho}_i\,\mathrm{Sin}\,\overline{\theta}_i\right)\overline{\rho}_i\, \Delta\rho_i\, \delta\theta_i \quad = \quad \iint_D f(\rho\,\mathrm{Cos}\,\theta, \rho\,\mathrm{Sin}\,\theta)\, \rho\, d\rho\, d\theta \quad = \int_\alpha^\beta d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\,\mathrm{Cos}\,\theta, \rho\,\mathrm{Sin}\,\theta)\, \rho\, d\rho$$

计算 $\iint_D e^{-x^2-y^2}\, dx\, dy$,其中D是由圆心在原点、半径为a的圆周所围成的闭区域。

积分区域D在极坐标可以方便表示为$0\le\rho\le a, 0\le\theta\le 2\pi$.

$$\iint_D f(x, y)\, d\delta= \iint_D f(\rho\,\mathrm{Cos}\,\theta, \rho\,\mathrm{Sin}\,\theta)\, \rho\, d\rho\, d\theta= \iint_D e^{-\rho^2}\, \rho\, d\rho\, d\theta= \qquad \int_\alpha^\beta d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\,\mathrm{Cos}\,\theta, \rho\,\mathrm{Sin}\,\theta)\, \rho\, d\rho=$$

$\int_0^{2\pi} d\theta \int_0^{a} E^{-\rho^2} \rho \, d\rho =. .. = \pi\left(1 - e^{-a^2}\right)$