Bayesian Decision Theory

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- Bayesian Decision Theory–Continuous Features
- Minimum-Error-Rate Classification
- Classifiers, Discriminant Functions and Decision Surfaces
- The Normal Density
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- Bayes Decision Theory – Discrete Features

Introduction

- The sea bass/salmon example
  - State of nature, prior
    - State of nature is a random variable
    - The catch of salmon and sea bass is equiprobable
      - $P(\omega_1) = P(\omega_2)$ (uniform priors)
      - $P(\omega_1) + P(\omega_2) = 1$ (exclusivity and exhaustivity)

- Decision rule with only the prior information
  - Decide ω_1 if $P(\omega_1) > P(\omega_2)$ otherwise decide ω_2
- Use of the class –conditional information
- $P(x \mid \omega_1)$ and $P(x \mid \omega_2)$ describe the difference in lightness between populations of sea and salmon

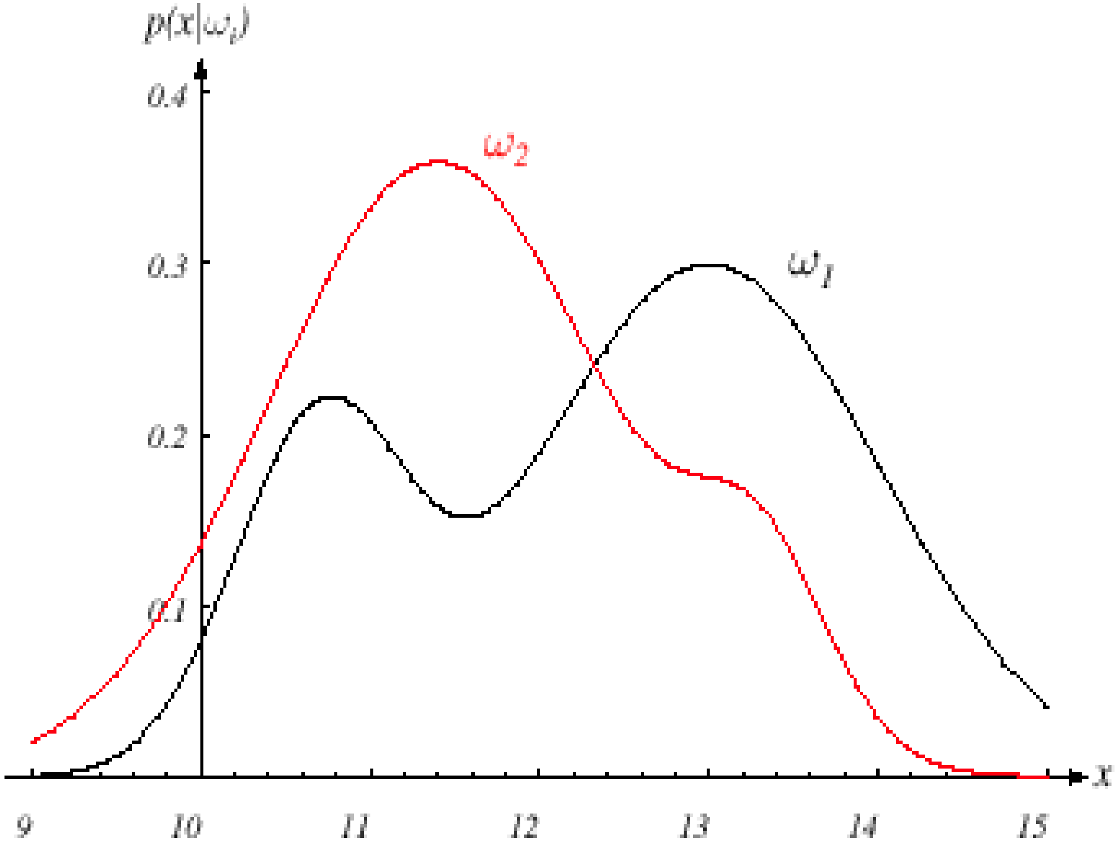

FIGURE 2.1. Hypothetical class-conditional probability density functions show the probability density of measuring a particular feature value x given the pattern is in category ω_i. If x represents the lightness of a fish, the two curves might describe the difference in lightness of populations of two types of fish. Density functions are normalized, and thus the area under each curve is 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

- Posterior, likelihood, evidence
  - $P(\omega_j \mid x) = P(x \mid \omega_j) \,.\, P(\omega_j) / P(x)$
  - Where in case of two categories

$$P(x) = \sum_{j=1}^{j=2} P(x \mid \omega_j) P(\omega_j)$$

  - Posterior = (Likelihood. Prior) / Evidence

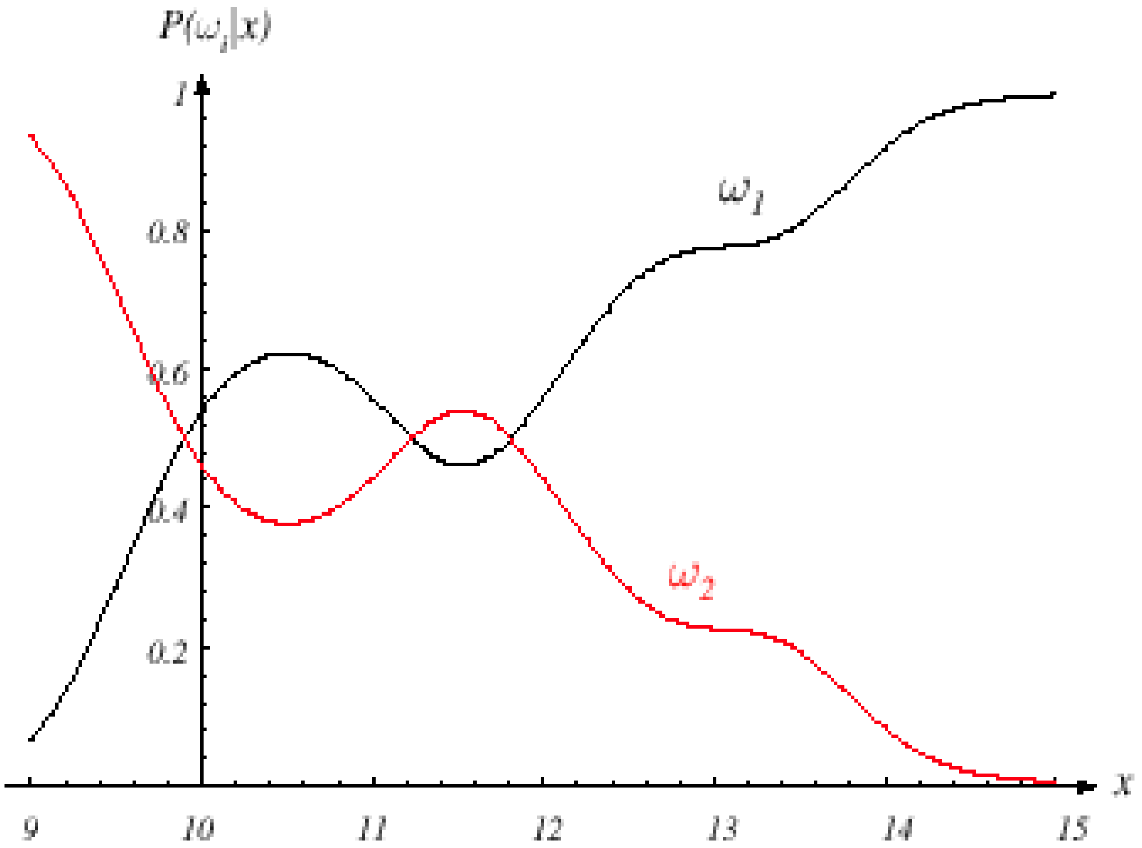

FIGURE 2.2. Posterior probabilities for the particular priors $P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$ for the class-conditional probability densities shown in Fig. 2.1. Thus in this case, given that a pattern is measured to have feature value $x = 14$, the probability it is in category ω_2 is roughly 0.08, and that it is in ω_1 is 0.92. At every x, the posteriors sum to 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

- Decision given the posterior probabilities

X is an observation for which:

if $P(\omega_1 \mid x) > P(\omega_2 \mid x)$ ⇒ True state of nature = ω_1
if $P(\omega_1 \mid x) < P(\omega_2 \mid x)$ ⇒ True state of nature = ω_2

Therefore:
whenever we observe a particular x, the probability of error is :

$$P(error \mid x) = P(\omega_1 \mid x) \text{ if we decide } \omega_2$$
$$P(error \mid x) = P(\omega_2 \mid x) \text{ if we decide } \omega_1$$

- Minimizing the probability of error

- Decide ω_1 *if* $P(\omega_1 \mid x) > P(\omega_2 \mid x)$; otherwise decide ω_2

  Therefore:

$$P(error \mid x) = \min\,[P(\omega_1 \mid x),\, P(\omega_2 \mid x)]$$

(Bayes decision)

- Two-category classification

α_1 : *deciding* ω_1

α_2 : *deciding* ω_2

$\lambda_{ij} = \lambda(\alpha_i \mid \omega_j)$

loss incurred for deciding ω_i when the true state of nature is ω_j

Conditional risk:

$$R(\alpha_1 \mid x) = \lambda_{11}P(\omega_1 \mid x) + \lambda_{12}P(\omega_2 \mid x)$$
$$R(\alpha_2 \mid x) = \lambda_{21}P(\omega_1 \mid x) + \lambda_{22}P(\omega_2 \mid x)$$

Our rule is the following:

if $R(\alpha_1 \mid x) < R(\alpha_2 \mid x)$

action α_1: "decide ω_1" is taken

This results in the equivalent rule :

decide ω_1 if:

$$(\lambda_{21} - \lambda_{11})\, P(x \mid \omega_1)\, P(\omega_1) > (\lambda_{12} - \lambda_{22})\, P(x \mid \omega_2)\, P(\omega_2)$$

and decide ω_2 otherwise

Likelihood ratio:

The preceding rule is equivalent to the following rule:

$$\text{if } \frac{P(x\,|\,\omega_1)}{P(x\,|\,\omega_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)}$$

Then take action α_1 (decide ω_1)

Otherwise take action α_2 (decide ω_2)

Exercise

Select the optimal decision where:

$\Omega = \{\omega_1, \omega_2\}$

$P(x \mid \omega_1)$ → N(2, 0.5) (Normal distribution)

$P(x \mid \omega_2)$ → N(1.5, 0.2)

$P(\omega_1) = 2/3$

$P(\omega_2) = 1/3$

$$\lambda = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Minimum-Error-Rate Classification

- Actions are decisions on classes

  If action α_i is taken and the true state of nature is ω_j then: the decision is correct if $i = j$ and in error if $i \neq j$

- Seek a decision rule that minimizes the *probability of error* which is the *error rate*

- Introduction of the zero-one loss function:

$$\lambda(\alpha_i, \omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases} \qquad i, j = 1, \dots, c$$

Therefore, the conditional risk is:

$$R(\alpha_i \mid x) = \sum_{j=1}^{j=c} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid x)$$

$$= \sum_{j \neq i} P(\omega_j \mid x) = 1 - P(\omega_i \mid x)$$

- Minimize the risk requires maximize $P(\omega_i \mid x)$ (since $R(\alpha_i \mid x) = 1 - P(\omega_i \mid x)$)

- For Minimum error rate
  - Decide ω_i *if* $P(\omega_i \mid x) > P(\omega_j \mid x) \quad \forall j \neq i$

- Regions of decision and zero-one loss function, therefore:

$$Let\ \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}}\cdot\frac{P(\omega_2)}{P(\omega_1)}=\theta_\lambda\ then\ decide\ \omega_1\ \ if:\ \frac{P(x\,|\,\omega_1)}{P(x\,|\,\omega_2)}>\theta_\lambda$$

- If λ is the zero-one loss function which means:

$$\lambda=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$then\ \ \theta_\lambda=\frac{P(\omega_2)}{P(\omega_1)}=\theta_a$$

$$if\ \ \lambda=\begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}then\ \ \theta_\lambda=\frac{2P(\omega_2)}{P(\omega_1)}=\theta_b$$

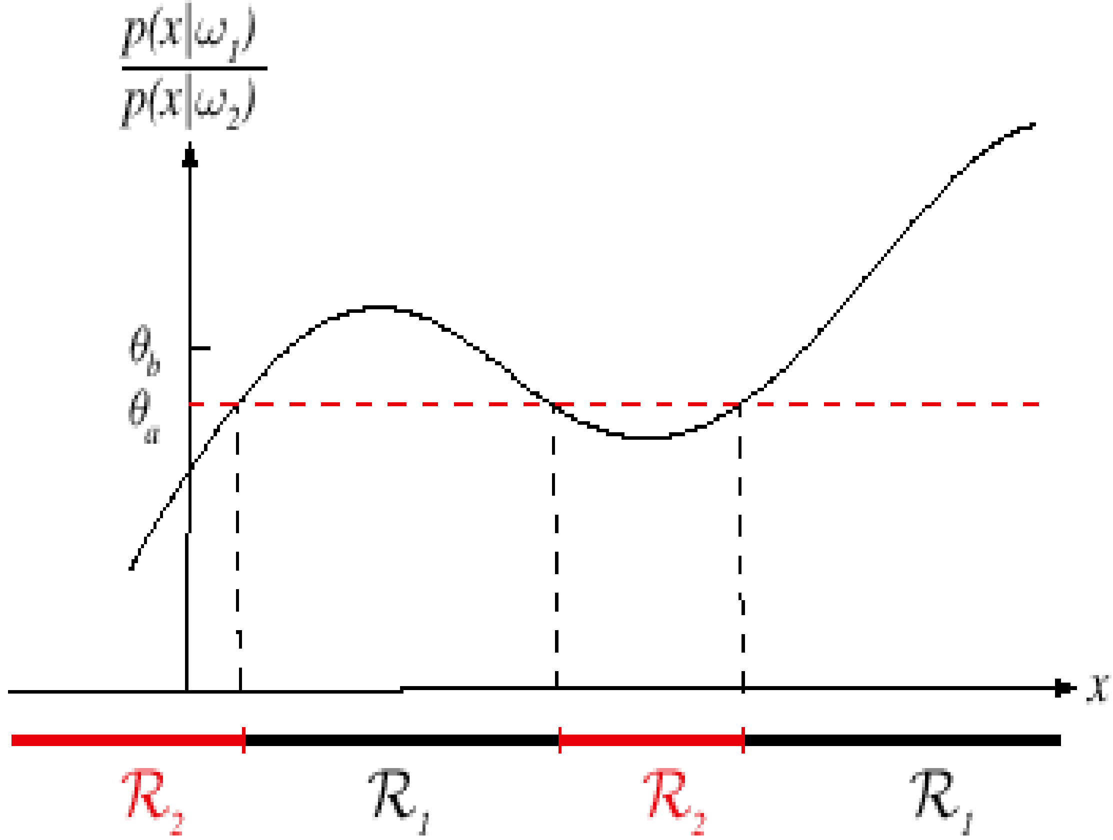

FIGURE 2.3. The likelihood ratio $p(x|\omega_1)/p(x|\omega_2)$ for the distributions shown in Fig. 2.1. If we employ a zero-one or classification loss, our decision boundaries are determined by the threshold θ_a. If our loss function penalizes miscategorizing ω_2 as ω_1 patterns more than the converse, we get the larger threshold θ_b, and hence $\mathcal{R}_1$ becomes smaller. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.