Elementary Math
Elementary Proof
Semester 1
Hyukdong Kim Assignment 3

Lemma 1. Let $x, y \in \mathbb{Q}$. Then $x + y, xy \in \mathbb{Q}$.

Proof. Let $x = \frac{a}{b}, y = \frac{c}{d}$ such that $(a, b, c, d) \in \mathbb{Z}, (b, d) \neq 0$. Then,

$$\begin{aligned} x + y &= \frac{a}{b} + \frac{c}{d} && (1) \\ &= \frac{ad + bc}{bd} \in \mathbb{Q} && (2) \\ & (\because ad + bc, bd \in \mathbb{Z}) && (3) \end{aligned}$$

□

Lemma 2. Let $x \in \mathbb{Q}$, $y \in \mathbb{Q}^c$. Then (1) $x + y \in \mathbb{Q}^c$, (2) $xy \in \mathbb{Q}^c$.

Proof. (1) Suppose $y \in \mathbb{Q}$. By **Lemma 1**, $y = \underbrace{x + y}_{\in \mathbb{Q}} - \underbrace{x}_{\in \mathbb{Q}} \in \mathbb{Q}$.

(2) Suppose $y \in \mathbb{Q}$. By **Lemma 1**,

$$y = y\frac{x}{x} = \frac{\overbrace{xy}^{\in \mathbb{Q}}}{\underbrace{x}_{\in \mathbb{Q}}} \in \mathbb{Q}$$

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Problem 1. Show given any two number district real numbers, there is at least one retional number and one irrational number between them.

Proof.

Case 1. Let $x, y \in \mathbb{Q}$. Then, there exists $\frac{1}{2}(x + y) \in \mathbb{Q}(\because$ **Lemma 1**).

Case 2. Let $x \in \mathbb{Q}^c$, $y \in \mathbb{R}$ such that $x < y$. Since $x < y$, $y - x > 0$ then there exists $n \in \mathbb{N}$ such that $n(y - x) > \sqrt{2}$.

$$\begin{aligned} & n(y - x) > \sqrt{2} && (4) \\ & \Leftrightarrow y - x > \frac{\sqrt{2}}{n} && (5) \\ & \Leftrightarrow x + \frac{\sqrt{2}}{n} < y && (6) \end{aligned}$$

Then, $x < x + \underbrace{\frac{\sqrt{2}}{n}}_{\in \mathbb{Q}^c} < y$.

□

Problem 2. Show $\sqrt{2} \in \mathbb{Q}^c = \mathrm{I}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$. Then, There exists number a, b satisfying $\sqrt{2} = \frac{a}{b}$ such that $(a, b) \in \mathbb{Z}, \gcd\{a, b\} = 1$. Thus, $2b^2 = a^2$ is even number. Then, a is also even number. Let $a = 2k$, $2b^2 = a^2 = (2k)^2 = 4k^2$. Then, b is even number($\because b^2$ is even number). a and b are even number that has common divisor 2. □