**Problem 1.** Show that $u_{\alpha\alpha} + u_{\beta\beta} = 0$. s.t. $u_{xx} - 6u_{xy} + 12u_{yy} = 0$.

*Solution:*

$$\begin{aligned} & u_{xx} - 6u_{xy} + 12u_{yy} = 0 && (1)\\ & \Leftrightarrow (\partial_{xx} - 6\partial_{xy} + 12\partial_{yy})u = 0 && (2)\\ & \Leftrightarrow ((\partial_x - 3\partial_y)^2 + 3\partial_{yy})u = 0 && (3)\end{aligned}$$

From equation (3), we can get

$$x = 1 \cdot \alpha + 0 \cdot \beta = \alpha \tag{4}$$
$$y = -3 \cdot \alpha + \sqrt{3} \cdot \beta = -3\alpha + \sqrt{3}\beta \tag{5}$$

From equation (4) and (5), we can get

$$u_\alpha = 1 \cdot u_x + (-3) \cdot u_y = u_x - 3u_y \tag{6}$$
$$u_\beta = 0 \cdot u_x + \sqrt{3} \cdot u_y = \sqrt{3}u_y \tag{7}$$

From equation (6) and (7), we can get

$$\partial_\alpha = \partial_x - 3\partial_y \tag{8}$$
$$\partial_\beta = \sqrt{3}\partial_y \tag{9}$$

And, we can get

$$\begin{aligned}\partial_{\alpha\alpha} &= \partial_\alpha\partial_\alpha = \partial_\alpha^2\\ &= (\partial_x - 3\partial_y)^2 (\because \text{Eq.}(8))\\ &= \partial_{xx} - 6\partial_{xy} + 9\partial_{yy} && (10)\end{aligned}$$

$$\begin{aligned}\partial_{\beta\beta} &= \partial_\beta\partial_\beta = (\partial_\beta)^2\\ &= (\sqrt{3}\partial_y)^2 (\because \text{Eq.}(9))\\ &= 3\partial_{yy} && (11)\end{aligned}$$

Thus,

$$\begin{aligned}u_{\alpha\alpha} + u_{\beta\beta} &= \underbrace{(\partial_{xx} - 6\partial_{xy} + 9\partial_{yy})}_{\text{Eq.}(10)} + \underbrace{(3\partial_{yy})}_{\text{Eq.}(11)}\\ &= \partial_{xx} - 6\partial_{xy} + 12\partial_{yy}\\ &= 0 (\because \text{Eq.}(2))\end{aligned}$$

□