Definition 1. With $\frac{dy}{dx} + Py = Q$ as 1st differential equation, if $Pdx + Qdy = 0$ is not exact and $F(Pdx + Qdy) = 0$ is exact, the integrating factor F w.r.t. x defined as

$$F_x = \exp\left(\int \frac{\frac{\partial}{\partial y}P - \frac{\partial}{\partial x}Q}{Q}dx\right) = \exp\left(\int \frac{P_y - Q_x}{Q}dx\right). \tag{1}$$

And, the integrating factor F w.r.t. y defined as

$$F_y = \exp\left(\int \frac{\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P}{P}dx\right) = \exp\left(\int \frac{Q_x - P_y}{P}dy\right). \tag{2}$$

Problem 1. Determine a solution of $y' + ay = b$ $(a, b \in \mathbb{R})$.

Proof. From $y' + ay = \frac{dy}{dx} + ay = b \Leftrightarrow (ay - b)dx + dy = 0$, $P_y = \frac{\partial}{\partial y}(ay - b) = a$, $Q_x = \frac{\partial}{\partial x}1 = 0$. Because $P_y \neq Q_x$, $Pdx + Qdy = 0$ is not exact. Let $ay - b$ as P and 1 as Q. From equation (1),

$$F_x = \exp\left(\int \frac{P_y - Q_x}{Q}dx\right) = \exp\left(\int \frac{a - 0}{1}dx\right) = e^{ax}. \tag{3}$$

and from equation (2),

$$F_y = \exp\left(\int \frac{Q_x - P_y}{P}dy\right) = \exp\left(\int \frac{0 - a}{ay - b}dy\right) = e^{-\frac{1}{y^2}(ay - b + b\ln(|ay - b|))}. \tag{4}$$

Let $F = F_x$. Then,

$$\begin{aligned}
& y' + ay = b && (5)\\
& \Leftrightarrow e^{ax}y' + e^{ax}ay = e^{ax}b && (6)\\
& \Leftrightarrow \frac{d}{dx}e^{ax}y = e^{ax}b && (7)\\
& \Leftrightarrow \int \frac{d}{dx}e^{ax}ydx = \int e^{ax}bdx && (8)\\
& \Leftrightarrow e^{ax}y = \frac{b}{a}e^{ax}y + C && (9)\\
& \Leftrightarrow y = \frac{b}{a} + Ce^{ax}y && (10)
\end{aligned}$$

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