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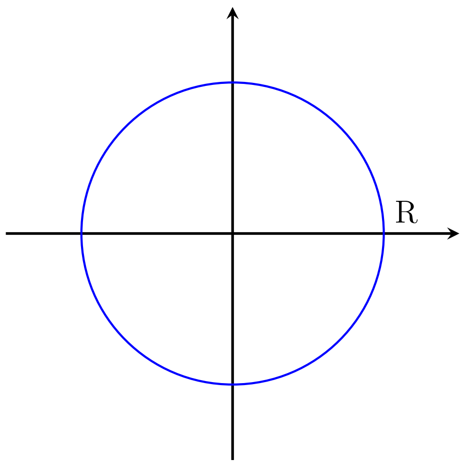
Find the curvature of the following curves.

Problem 1. A circle defined by the function $r(t) = (R\cos t, R\sin t)$.



Solution:

$$r(t) = (R\cos t, R\sin t)$$
$$r'(t) = (-R\sin t, R\cos t)$$

$$\begin{aligned} s(t) &= \int_0^t \|r'(t)\|_2 \, dx \\ &= \int_0^t R dx = Rt \end{aligned} \tag{1}$$

From equation 1, let $t = \frac{s}{R}$.

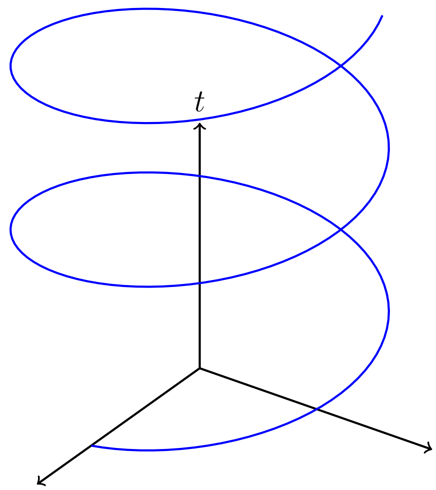
$$\begin{aligned} \tilde{r}(t) &= r\left(\frac{s}{R}\right) = \left(R, R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right) \\ \tilde{r}'(t) &= \left(0, \frac{R}{R}\cdot\left(-\sin\frac{s}{R}\right), \frac{R}{R}\cdot\left(\cos\frac{s}{R}\right)\right) \\ &= \left(0, -\sin\frac{s}{R}, \cos\frac{s}{R}\right) \\ \tilde{r}''(t) &= \left(0, -\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right) \end{aligned} \tag{2}$$

From equation 2, curvature κ is

$$\begin{aligned} \kappa = \|\tilde{r}''(t)\|_2 &= \sqrt{0^2 + \left(-\frac{1}{R}\cos\frac{s}{R}\right)^2 + \left(-\frac{1}{R}\sin\frac{s}{R}\right)^2} \\ &= \sqrt{\frac{1}{R^2}\cos^2\frac{s}{R} + \frac{1}{R^2}\sin^2\frac{s}{R}} = \sqrt{\frac{1}{R^2}} = \frac{1}{R} \end{aligned}$$

□

Problem 2. A helix defined by the function $(t, \cos t, \sin t)$.



Solution:

$$r(t) = (t, \cos t, \sin t)$$
$$r'(t) = (1, -\sin t, \cos t)$$

$$\begin{aligned} s(t) = \int_0^t \|r'(t)\|_2 \, dx &= \int_0^t \sqrt{1^2 + (\sin t)^2 + (\cos t)^2} dx \\ &= \int_0^t \sqrt{1 + \sin^2 t + \cos^2 t} dx = \sqrt{2} t \end{aligned} \tag{3}$$

From equation 3, let $t = \frac{s}{\sqrt{2}}$.

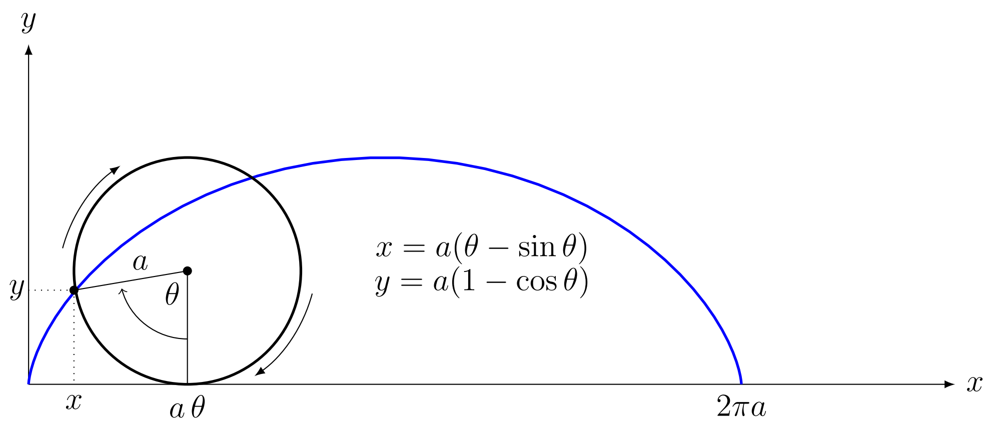
$$\begin{aligned} \tilde{r}(t) = \tilde{r}\left(\frac{s}{\sqrt{2}}\right) &= \left(t, \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right)\right) \\ \tilde{r}'(t) &= \left(0, \frac{s}{\sqrt{2}} \cdot \left(-\sin\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \cdot \left(\cos\frac{s}{\sqrt{2}}\right)\right) \\ &= \left(0, -\frac{1}{\sqrt{2}} \sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos\frac{s}{\sqrt{2}}\right) \\ \tilde{r}''(t) &= \left(0, -\frac{1}{2} \cos\frac{s}{\sqrt{2}}, -\frac{1}{2} \sin\frac{s}{\sqrt{2}}\right) \end{aligned} \tag{4}$$

From equation 6, curvature κ is

$$\begin{aligned} \kappa = \|\tilde{r}''(t)\|_2 &= \sqrt{0^2 + \left(-\frac{1}{2}\cos\frac{s}{\sqrt{2}}\right)^2 + \left(-\frac{1}{2}\sin\frac{s}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{1}{4}\cos^2\frac{s}{\sqrt{2}} + \frac{1}{4}\sin^2\frac{s}{\sqrt{2}}} = \sqrt{\frac{1}{4}} = \frac{1}{2} \end{aligned}$$

$\square$

Problem 3. A cycloid defined by the function $(a\theta - a\sin\theta, a - a\cos\theta)$.



Solution:

$$\begin{aligned} r(\theta) &= (a\theta - a\sin\theta, a - a\cos\theta) \\ r'(\theta) &= (a - a\cos\theta, a\sin\theta) \end{aligned}$$

$$\begin{aligned} s(t) = \int_0^{2\pi a} \|r'(\theta)\|_2 dx &= \int_0^{2\pi a} \sqrt{(a - a\cos\theta)^2 + (a\sin\theta)^2} dx \\ &= a \end{aligned} \tag{5}$$

From equation 5, let $\theta = 4\sin^{-1}\sqrt{\frac{s}{8a}} = 2\cos^{-1}(1 - \frac{s}{4a})$

$$\begin{aligned} \tilde{r}(t) = \tilde{r}\left(4\sin^{-1}\sqrt{\frac{s}{8a}}\right) &= \left(4a\sin^{-1}\sqrt{\frac{s}{8a}} - 4a\sqrt{\frac{s}{8a}}, a - a\cos\left(4\sin^{-1}\sqrt{\frac{s}{8a}}\right)\right) \\ &= \left(4a\sin^{-1}\sqrt{\frac{s}{8a}} - 4a\sqrt{\frac{s}{8a}}, a - a\left(1 - 8\left(\sqrt{\frac{s}{8a}}\right)^2\left(1 - \left(\sqrt{\frac{s}{8a}}\right)^2\right)\right)\right) \\ &= \left(4a\sin^{-1}\sqrt{\frac{s}{8a}} - 4a\sqrt{\frac{s}{8a}}, \frac{s(8a+s)}{8a^2}\right) \\ \tilde{r}'(t) &= \left(0, \frac{s}{\sqrt{2}} \cdot \left(-\sin\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \cdot \left(\cos\frac{s}{\sqrt{2}}\right)\right) \\ &= \left(0, -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}\right) \\ \tilde{r}''(t) &= \left(0, -\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}}\right) \end{aligned} \tag{6}$$

□