Find the torsion of the followings.

Problem 1. $r = r(t) = (a\cos t, a\sin t, bt)$ s.t. $a > 0, b \neq 0, t \in \mathbb{R}$.

Solution:

$$\begin{aligned} r(t) &= (a\cos t, a\sin t, bt) \\ r'(t) &= (-a\sin t, a\cos t, b) \end{aligned}$$

$$\begin{aligned} s(t) = \int_0^t \|r'(t)\|_2 \, dx &= \int_0^t \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} dx \\ &= \int_0^t \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} dx = \sqrt{a^2+b^2}t \end{aligned} \tag{1}$$

From equation 1, let $t = \frac{s}{\sqrt{a^2+b^2}}$.

$$\begin{aligned} \tilde{r}(t) = \tilde{r}\left(\frac{s}{\sqrt{a^2+b^2}}\right) &= \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, b\frac{s}{\sqrt{a^2+b^2}}\right) \\ \tilde{r}'(t) &= \left(-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) \\ \tilde{r}''(t) &= \left(-\frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}, -\frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right) \end{aligned} \tag{2}$$

From equation 2, curvature κ is

$$\begin{aligned} \kappa(s) = \|\tilde{r}''(t)\|_2 &= \sqrt{\left(-\frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}\right)^2 + \left(-\frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}\right)^2 + 0^2} \\ &= \sqrt{\left(-\frac{a}{a^2+b^2}\right)^2\left(\cos^2\frac{s}{\sqrt{a^2+b^2}} + \sin^2\frac{s}{\sqrt{a^2+b^2}}\right)} = \frac{a}{a^2+b^2} \end{aligned}$$

On the other hand, from $n(s) := \frac{1}{\kappa(s)}\tilde{r}''(t)$,

$$\begin{aligned} n(s) &= \frac{1}{\kappa(s)}\tilde{r}''(t) \\ &= \frac{a^2+b^2}{a}\cdot\left(-\frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}, -\frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right) \\ &= \left(-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right) \end{aligned}$$

On the other hand, from $b'(s) = t(s) \times n'(s)$,

$$\begin{aligned}
b'(s) = \tilde{r}'(t) \times n'(s) &= \begin{bmatrix} \tilde{r}'(t)_2 \cdot n'(s)_3 - \tilde{r}'(t)_3 \cdot n'(s)_2 \\ \tilde{r}'(t)_3 \cdot n'(s)_1 - \tilde{r}'(t)_1 \cdot n'(s)_3 \\ \tilde{r}'(t)_1 \cdot n'(s)_2 - \tilde{r}'(t)_2 \cdot n'(s)_1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} \cos \frac{s}{\sqrt{a^2+b^2}} \cdot 0 \\ \frac{b}{\sqrt{a^2+b^2}} \cdot \left(\frac{1}{\sqrt{a^2+b^2}} \sin \frac{s}{\sqrt{a^2+b^2}} \right) \\ \left(-\frac{a}{\sqrt{a^2+b^2}} \sin \frac{s}{\sqrt{a^2+b^2}} \right) \cdot \left(-\frac{1}{\sqrt{a^2+b^2}} \cos \frac{s}{\sqrt{a^2+b^2}} \right) \end{bmatrix} \\
&\quad - \begin{bmatrix} \frac{b}{\sqrt{a^2+b^2}} \cdot \left(-\frac{1}{\sqrt{a^2+b^2}} \cos \frac{s}{\sqrt{a^2+b^2}} \right) \\ \left(-\frac{a}{\sqrt{a^2+b^2}} \sin \frac{s}{\sqrt{a^2+b^2}} \right) \cdot 0 \\ \frac{a}{\sqrt{a^2+b^2}} \cos \frac{s}{\sqrt{a^2+b^2}} \cdot \left(\frac{1}{\sqrt{a^2+b^2}} \sin \frac{s}{\sqrt{a^2+b^2}} \right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{b}{a^2+b^2} \cos \frac{s}{\sqrt{a^2+b^2}} \\ \frac{b}{a^2+b^2} \sin \frac{s}{\sqrt{a^2+b^2}} \\ 0 \end{bmatrix}
\end{aligned}$$

From $b'(s) = \tau(s)(-n(s))$,

$$\begin{aligned}
b'(s) &= \tau(s)(-n(s)) \\
\Leftrightarrow \tau(s) &= -b'(s) \cdot n(s) \\
&= -\left[\frac{b}{a^2+b^2} \cos \frac{s}{\sqrt{a^2+b^2}}, \frac{b}{a^2+b^2} \sin \frac{s}{\sqrt{a^2+b^2}}, 0 \right] \begin{bmatrix} -\cos \frac{s}{\sqrt{a^2+b^2}} \\ -\sin \frac{s}{\sqrt{a^2+b^2}} \\ 0 \end{bmatrix} \\
&= \frac{b}{a^2+b^2} \cos^2 \frac{s}{\sqrt{a^2+b^2}} + \frac{b}{a^2+b^2} \sin^2 \frac{s}{\sqrt{a^2+b^2}} + 0 \\
&= \frac{b}{a^2+b^2} \left(\cos^2 \frac{s}{\sqrt{a^2+b^2}} + \sin^2 \frac{s}{\sqrt{a^2+b^2}} \right) = \frac{b}{a^2+b^2}
\end{aligned}$$

$$\therefore \tau(s) = \frac{b}{a^2+b^2}$$

$\square$

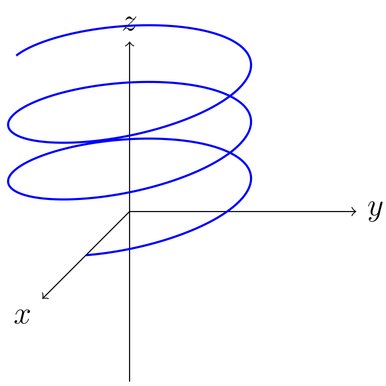

Figure 1: Helix with $b > 0$

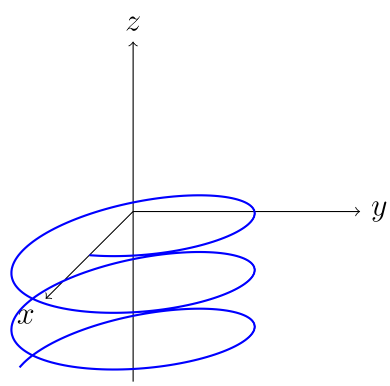

Figure 2: Helix with $b < 0$