Problem 1. (b) Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

$$
\begin{aligned}
A \cup (B \cap C) &\Leftrightarrow x \in A \,\text{or}(x \in B \,\text{and}\, x \in C) && (1)\\
&\Leftrightarrow (x \in A \,\text{or}\, x \in B)\,\text{and}(x \in A \,\text{or}\, x \in C) && (2)\\
&\Leftrightarrow (A \cup B) \cap (A \cup C) && (3)
\end{aligned}
$$

$$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

□

Problem 2. (c) Show $(A \cup B)^c = A^c \cap B^c$.

Proof.

$$
\begin{aligned}
(A \cup B)^c &\Leftrightarrow x \notin (A \cup B) && (4)\\
&\Leftrightarrow x \notin A \,\text{and}\, x \notin B && (5)\\
&\Leftrightarrow x \in A^c \,\text{and}\, x \in B^c && (6)\\
&\Leftrightarrow A^c \cap B^c && (7)
\end{aligned}
$$

$$\therefore (A \cup B)^c = A^c \cap B^c$$

□

Problem 3. (d) Show $(A \cap B)^c = A^c \cup B^c$.

Proof.

$$
\begin{aligned}
(A \cap B)^c &\Leftrightarrow x \notin (A \cap B) && (8)\\
&\Leftrightarrow x \notin A \,\text{or}\, x \notin B && (9)\\
&\Leftrightarrow x \in A^c \,\text{or}\, x \in B^c && (10)\\
&\Leftrightarrow A^c \cup B^c && (11)
\end{aligned}
$$

$$\therefore (A \cap B)^c = A^c \cup B^c$$

□

Problem 4. If set A has n elements, then show that $\mathcal{P}(A)$ has 2^n elements.

Proof. A combination of the presence or absence of each element in a set A.

$$\overbrace{2 \times 2 \times \cdots \times 2}^{n\,\text{times}} = 2^n$$

□

Problem 5. Show $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

Proof. ① $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$

$$
\begin{aligned}
x \in A_1 \cup A_2 &\Leftrightarrow x \in A_1 \,\text{or}\, x \in A_2 && (12)\\
&\Leftrightarrow f(x) \in f(A_1) \,\text{or}\, f(x) \in f(A_2) && (13)\\
&\Leftrightarrow f(x) \in f(A_1) \cup f(A_2) && (14)
\end{aligned}
$$

$$\therefore f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$$

② $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$

$$
\begin{aligned}
y \in f(A_1) \cup f(A_2) &\Leftrightarrow y \in f(A_1) \text{ or } y \in f(A_2) && (15)\\
&\Leftrightarrow \exists x \in A_1 \mid f(x) = y && (16)\\
&\quad\ \text{or } \exists x \in A_2 \mid f(x) = y \\
&\Leftrightarrow \exists x \in A_1 \cup A_2 \mid f(x) = y && (17)
\end{aligned}
$$

$$\therefore f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$$

From ① and ②, $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. □

Problem 6. Show $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

Proof.

$$
\begin{aligned}
x \in A_1 \cap A_2 &\Leftrightarrow x \in A_1 \,\text{and}\, x \in A_2 && (18)\\
&\Leftrightarrow f(x) \in f(A_1) \,\text{and}\, f(x) \in f(A_2) && (19)\\
&\Leftrightarrow f(x) \in f(A_1) \cap f(A_2) && (20)
\end{aligned}
$$

$$\therefore f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

□