22. Conditional Expectation given an R.V.

e.g. Let $X \sim N(0,1)$, $Y = X^2$. Then $E(Y|X) = E(X^2|X) = X^2 = Y$

$E(X|Y) = E(X|X^2) = 0$ since if observe $X^2 = a$, then $X = \pm\sqrt{a}$, equally likely.

e.g. $\underset{0 \;\; Y \;\; X \qquad\qquad 1}{\vdash\!\!+\!\!+\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\dashv}$ stick, break off random piece, break off another piece.

$X \sim \text{Unif}(0,1)$, $Y|X \sim \text{Unif}(0, X)$

$E(Y|X=x) = \frac{x}{2}$, so $E(Y|X) = \frac{X}{2}$, $E(E(Y|X)) = \frac{1}{4} = E(Y)$

· Useful properties

(1) $E(h(X)\,Y|X) = h(X)\,E(Y|X)$ [taking out what's known]

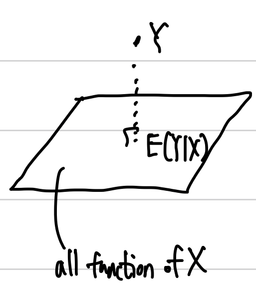
(2) $E(Y|X) = E(Y)$ if X, Y are independent

(3) $E(E(Y|X)) = E(Y)$ Iterated Expectation / Adam's Law

(4) $E((Y - E(Y|X))\,h(X)) = 0$, i.e., $Y - E(Y|X)$ (residual) is uncorrelated with $h(X)$

$Cov(Y - E(Y|X), h(X)) = E((Y - E(Y|X))\,h(X)) - \underbrace{E(Y - E(Y|X))}_{0}\,E(h(X))$

(5) $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ (Eve's Law)



$\langle X, Y \rangle = E(XY)$

· pf. of (4) $E(Y h(X)) - E(E(Y|X)h(X))$

$= E(Y h(X)) - E(E(h(X)\,Y|X))$

$= E(Y h(X)) - E(Y h(X)) = 0$

· pf. of (3) [discrete case]

Let $E(Y|X) = g(X)$

$E\,g(X) = \sum_x g(x) P(X=x) = \sum_x E(Y|X=x) P(X=x)$

$= \sum_x \left(\sum_y y P(Y=y|X=x) \right) P(X=x)$

$= \sum_y \sum_x y P(Y=y|X=x) = \sum_y y P(Y=y) = E(Y)$

Defn $Var(Y|X) = E(Y^2|X) - (E(Y|X))^2 = E((Y - E(Y|X))^2|X)$

e.g. pick random city, pick random sample of n people in that city,

X = (# with disease), Q = (proportion of people in the random city with disease)

Find $E(X)$, $Var(X)$, assuming $Q \sim \text{Beta}(a,b)$, $X|Q \sim \text{Bin}(n, Q)$.

$E(X) = E(E(X|Q)) = E(nQ) = \frac{na}{a+b}$

$Var(X) = E(Var(X|Q)) + Var(E(X|Q)) = E(nQ(1-Q)) + n^2 Var(Q)$

$E(Q(1-Q)) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 q^a (1-q)^b \, dq$

$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$ / $\Gamma(x+1) = x\Gamma(x)$

$= \frac{ab}{(a+b+1)(a+b)}$

$Var(Q) = \frac{\mu(1-\mu)}{a+b+1}$, $\mu = \frac{a}{a+b}$