A dimension-independent strict submultiplicativity for the transposition map in diamond norm

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Abstract

We prove that there exists an absolute constant $\alpha < 1$ such that for every finite dimension d and every quantum channel T on $\mathsf{L}(\mathbb{C}^d)$, $\|\Theta \circ (\mathrm{id} - T)\|_\diamond \le \alpha \|\Theta\|_\diamond \|\mathrm{id} - T\|_\diamond$, where Θ is the transposition map. In fact we show the explicit choice $\alpha = 1/\sqrt{2}$ works.

1 Setup and definitions

Let $\mathcal{H} \cong \mathbb{C}^d$ be a d-dimensional Hilbert space. We write $\mathsf{L}(\mathcal{H})$ for the space of linear operators on $\mathcal{H}$, and

$$\mathsf{D}(\mathcal{H}) = \{\rho \in \mathsf{L}(\mathcal{H}) : \rho \succeq 0,\ \mathrm{tr}(\rho) = 1\}$$

for the set of density operators.

Transposition map. Fix an orthonormal basis $\{|1\rangle, \dots, |d\rangle\}$ of $\mathcal{H}$. The transposition map $\Theta : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H})$ is defined by

$$\Theta(X) = X^{\mathsf{T}},$$

which depends on the chosen basis.

Quantum channels. A *quantum channel* $T : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H})$ is a completely positive trace-preserving (CPTP) linear map.

Norms. The trace norm and Hilbert–Schmidt norm of an operator X are

$$\|X\|_1 = \mathrm{tr}\sqrt{X^\dagger X}, \qquad \|X\|_2 = \sqrt{\mathrm{tr}(X^\dagger X)}.$$

The *diamond norm* of a linear map $\Phi : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H})$ is

$$\|\Phi\|_\diamond := \sup_{k \ge 1} \sup_{\rho \in \mathsf{D}(\mathcal{H} \otimes \mathbb{C}^k)} \|(\Phi \otimes \mathrm{id}_k)(\rho)\|_1 .$$

It is standard that for maps acting on $\mathsf{L}(\mathbb{C}^d)$, the supremum over k can be restricted to $k = d$. We will use this without further comment.

2 Main result

Theorem 1. *Let $\mathcal{H} \cong \mathbb{C}^d$ and let $T : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H})$ be a quantum channel. Then*

$$\|\Theta \circ (\mathrm{id} - T)\|_\diamond \ \le\ \alpha \|\Theta\|_\diamond \|\mathrm{id} - T\|_\diamond$$

with $\alpha = 1/\sqrt{2}$*. In particular, since* $\|\Theta\|_\diamond = d$ *[1],*

$$\|\Theta \circ (\mathrm{id} - T)\|_\diamond \;\leq\; \frac{d}{\sqrt{2}}\, \|\mathrm{id} - T\|_\diamond .$$

The proof proceeds by reducing to a matrix inequality for the partial transpose applied to traceless Hermitian matrices. The specific form $\Theta \circ (\mathrm{id} - T)$ appears in [2], but the analysis therein relied on the standard diamond-norm submultiplicativity corresponding to the case $\alpha = 1$. The possibility of obtaining a *dimension-independent* constant $\alpha < 1$ for a stronger submultiplicativity result remained open.

2.1 Auxiliary lemmas

Lemma 1. *If* $X = X^\dagger$ *and* $\mathrm{tr}(X) = 0$*, then*

$$\|X\|_2 \leq \frac{1}{\sqrt{2}} \|X\|_1 .$$

Proof. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of X (real because X is Hermitian). Write the positive eigenvalues as $p_1, \dots, p_r \geq 0$ and the negative eigenvalues as $-q_1, \dots, -q_s$ with $q_1, \dots, q_s > 0$. Since $\mathrm{tr}(X) = \sum_i \lambda_i = 0$, we have

$$\sum_{i=1}^r p_i \;=\; \sum_{j=1}^s q_j \;=:\; t.$$

Hence

$$\|X\|_1 = \sum_i |\lambda_i| = \sum_{i=1}^r p_i + \sum_{j=1}^s q_j = 2t.$$

Also,

$$\|X\|_2^2 = \sum_i \lambda_i^2 = \sum_{i=1}^r p_i^2 + \sum_{j=1}^s q_j^2 \leq \Big(\sum_{i=1}^r p_i\Big)^2 + \Big(\sum_{j=1}^s q_j\Big)^2 = t^2 + t^2 = 2t^2, \tag{1}$$

where we used the elementary inequality $\sum a_i^2 \leq (\sum a_i)^2$ for nonnegative a_i. Therefore $\|X\|_2 \leq \sqrt{2}\,t = \|X\|_1 / \sqrt{2}$. □

Lemma 2. *Let* Y *be an operator on an* N*-dimensional Hilbert space. Then*

$$\|Y\|_1 \leq \sqrt{N}\, \|Y\|_2 .$$

Proof. Let $s_1, \dots, s_N$ be the singular values of Y (padding with zeros if necessary). Then $\|Y\|_1 = \sum_i s_i$ and $\|Y\|_2 = \sqrt{\sum_i s_i^2}$. By Cauchy–Schwarz,

$$\Big(\sum_{i=1}^N s_i\Big)^2 \leq N \sum_{i=1}^N s_i^2, \tag{2}$$

which gives the claim. □

Lemma 3. *Let* $\mathcal{H}, \mathcal{K}$ *be finite-dimensional and let* Θ *denote transposition on* $\mathcal{H}$*. Then for all* $X \in \mathsf{L}(\mathcal{H} \otimes \mathcal{K})$*,*

$$\|(\Theta \otimes \mathrm{id})(X)\|_2 = \|X\|_2 .$$

Proof. In the computational basis on $\mathcal{H}$, the transposition map is an isometry for the Hilbert–Schmidt inner product:

$$\langle A, B\rangle_{HS} := \mathrm{tr}(A^\dagger B), \qquad \mathrm{tr}\left((A^\mathsf{T})^\dagger B^\mathsf{T}\right) = \mathrm{tr}(A^\dagger B).$$

Applying this entrywise on $\mathcal{H}$ and trivially on $\mathcal{K}$ yields

$$\mathrm{tr}\Big(\big((\Theta \otimes \mathrm{id})(X)\big)^\dagger (\Theta \otimes \mathrm{id})(X)\Big) = \mathrm{tr}(X^\dagger X),$$

i.e. $\|(\Theta \otimes \mathrm{id})(X)\|_2 = \|X\|_2$. □

2.2 Proof of Theorem 1

Proof of Theorem 1. Let $\Phi := \mathrm{id} - T$. Since T is CPTP, the map Φ is Hermiticity-preserving and trace-annihilating:

$$\mathrm{tr}(\Phi(Z)) = \mathrm{tr}(Z) - \mathrm{tr}(T(Z)) = 0 \qquad \forall\, Z \in \mathsf{L}(\mathcal{H}).$$

Fix an ancilla $\mathcal{K} \cong \mathbb{C}^d$. Let $\rho \in \mathsf{D}(\mathcal{H} \otimes \mathcal{K})$ and define

$$X := (\Phi \otimes \mathrm{id})(\rho) \in \mathsf{L}(\mathcal{H} \otimes \mathcal{K}). \tag{3}$$

Then:

- X is Hermitian, since Φ is Hermiticity-preserving and ρ is Hermitian.
- X is traceless:
$$\mathrm{tr}(X) = \mathrm{tr}\big((\Phi \otimes \mathrm{id})(\rho)\big) = \mathrm{tr}\big(\Phi(\mathrm{tr}_{\mathcal{K}}\, \rho)\big) = 0,$$
because Φ is trace-annihilating.
- $(\Theta \circ \Phi \otimes \mathrm{id})(\rho) = (\Theta \otimes \mathrm{id})(X)$.

We now bound $\|(\Theta \otimes \mathrm{id})(X)\|_1$ in terms of $\|X\|_1$. The operator $(\Theta \otimes \mathrm{id})(X)$ acts on $\mathcal{H} \otimes \mathcal{K}$, which has dimension $N = d^2$. By Lemma 2,

$$\|(\Theta \otimes \mathrm{id})(X)\|_1 \leq \sqrt{d^2}\, \|(\Theta \otimes \mathrm{id})(X)\|_2 = d\, \|(\Theta \otimes \mathrm{id})(X)\|_2 .$$

By Lemma 3, $\|(\Theta \otimes \mathrm{id})(X)\|_2 = \|X\|_2$, hence

$$\|(\Theta \otimes \mathrm{id})(X)\|_1 \leq d\, \|X\|_2 .$$

Finally, since X is traceless Hermitian, Lemma 1 gives

$$\|X\|_2 \leq \frac{1}{\sqrt{2}} \|X\|_1 .$$

Combining these inequalities yields

$$\|(\Theta \otimes \mathrm{id})(X)\|_1 \leq \frac{d}{\sqrt{2}} \|X\|_1 .$$

Substituting back $X = (\Phi \otimes \mathrm{id})(\rho)$ gives, for every $\rho \in \mathsf{D}(\mathcal{H} \otimes \mathcal{K})$,

$$\|(\Theta \circ \Phi \otimes \mathrm{id})(\rho)\|_1 \leq \frac{d}{\sqrt{2}} \|(\Phi \otimes \mathrm{id})(\rho)\|_1 .$$

Taking the supremum over ρ proves

$$\|\Theta \circ \Phi\|_\diamond \leq \frac{d}{\sqrt{2}} \|\Phi\|_\diamond .$$

Thus

$$\|\Theta \circ (\mathrm{id} - T)\|_\diamond \leq \frac{d}{\sqrt{2}} \|\mathrm{id} - T\|_\diamond .$$

Since $\|\Theta\|_\diamond = d$, this is equivalent to

$$\|\Theta \circ (\mathrm{id} - T)\|_\diamond \leq \frac{1}{\sqrt{2}} \|\Theta\|_\diamond \|\mathrm{id} - T\|_\diamond ,$$

completing the proof. $\square$

Remark 1. *The constant $1/\sqrt{2}$ is universal (independent of d and T) and arises from the fact that $X = \big((\mathrm{id} - T) \otimes \mathrm{id}\big)(\rho)$ is always traceless because $\mathrm{id} - T$ is trace-annihilating.*

2.3 Positive gap in finite dimension

Regrettably, the inequality in Theorem 1 is still not tight for any finite d, except in the trivial case $T = \mathrm{id}$, where both LHS and RHS become zero. Fix any nonzero channel difference $\Phi = \mathrm{id} - T \neq 0$ (equivalently $T \neq \mathrm{id}$). Rewrite the pointwise inequality (for every ρ):

$$\underbrace{\|(\Theta \circ \Phi \otimes \mathrm{id})(\rho)\|_1}_{=:L(\rho)} \leq \frac{d}{\sqrt{2}} \underbrace{\|(\Phi \otimes \mathrm{id})(\rho)\|_1}_{=:R(\rho)}. \tag{4}$$

Taking suprema gives $\sup_\rho L(\rho) \leq \frac{d}{\sqrt{2}} \sup_\rho R(\rho)$. In finite dimension, the suprema are maxima, so equality

$$\sup_\rho L(\rho) = \frac{d}{\sqrt{2}} \sup_\rho R(\rho)$$

can only happen if there exists some $\rho_\star$ such that $R(\rho_\star) = \sup_\rho R(\rho)$ and the pointwise inequality Eq. (4) is tight at $\rho_\star$, i.e., $L(\rho_\star) = \frac{d}{\sqrt{2}} R(\rho_\star)$. Let $X_\star := (\Phi \otimes \mathrm{id})(\rho_\star)$. Since $\Phi \neq 0$ and $\rho_\star$ maximizes R, we have $X_\star \neq 0$. Now, the chain of inequalities shows

$$\|(\Theta \otimes \mathrm{id})(X_\star)\|_1 \leq d \|X_\star\|_2 \leq \frac{d}{\sqrt{2}} \|X_\star\|_1 .$$

So equality in Theorem 1 forces equality in Lemmas 1 and 2 at some nonzero $X_\star$. But the equality conditions of Lemmas 1 and 2 are incompatible (unless $X = 0$).

Equality condition for Lemma 1. For a nonzero traceless Hermitian X, equality in Lemma 1 holds *iff* X has exactly two nonzero eigenvalues, $+t$ and $-t$. This follows from Eq. (1), where equality requires all cross terms to vanish. Equivalently,

$$\mathrm{rank}(X_\star) = 2 \qquad \text{(for nonzero equality cases)}. \tag{5}$$

Equality condition for Lemma 2. Equality in Lemma 2 holds *iff* s in Eq. (2) is proportional to $(1, \ldots, 1)$, i.e., all singular values are equal. If $Y \neq 0$, that forces all N singular values to be the same positive number, hence

$$\mathrm{rank}\big((\Theta \otimes \mathrm{id})(X_\star)\big) = \mathrm{rank}(Y_\star) = N = d^2. \tag{6}$$

To proceed, we need the following definitions and identities.

Definition 2.1 (Column-major vectorization)**.** For a matrix $A \in \mathbb{C}^{d \times d}$ with entries $A_{ij} = \langle i|A|j\rangle$, define

$$|\mathrm{vec}(A)\rangle := \sum_{i=1}^{d} \sum_{j=1}^{d} A_{ij} |i\rangle_{\mathcal{H}} \otimes |j\rangle_{\mathcal{K}}.$$

Definition 2.2. The swap operator $F \in \mathsf{L}(\mathcal{H} \otimes \mathcal{K})$ is defined by:

$$F(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle.$$

Lemma 4. *For X in Eq.* (3), $\mathrm{tr}_{\mathcal{H}}(X) = 0$.

Proof. There exists a set of Kraus operators $\{E_k\}$ acting on $\mathcal{H}$ such that

$$T(X) = \sum_k E_k X E_k^\dagger, \qquad \sum_k E_k^\dagger E_k = \mathrm{id}.$$

Let Z be an operator on $\mathcal{H} \otimes \mathcal{K}$. We can write Z as

$$Z = \sum_j A_j \otimes B_j,$$

where $A_j \in \mathsf{L}(\mathcal{H})$ and $B_j \in \mathsf{L}(\mathcal{K})$. By linearity we have

$$(T \otimes \mathrm{id})Z = \sum_j T(A_j) \otimes B_j = \sum_j \left(\sum_k E_k A_j E_k^\dagger \right) \otimes B_j.$$

Now we take the partial trace $\mathrm{tr}_\mathcal{H}$:

$$\begin{aligned}\mathrm{tr}_\mathcal{H}\big((T \otimes \mathrm{id})Z\big) &= \sum_j \mathrm{tr}_\mathcal{H}\left(\sum_k E_k A_j E_k^\dagger \otimes B_j \right) = \sum_j \left(\sum_k \mathrm{tr}(E_k A_j E_k^\dagger) \right) B_j \\ &= \sum_j \mathrm{tr}\left(\left(\sum_k E_k^\dagger E_k \right) A_j \right) B_j = \sum_j \mathrm{tr}(A_j) B_j.\end{aligned}$$

Meanwhile, by the definition of the partial trace on the original operator $Z = \sum_j A_j \otimes B_j$,

$$\mathrm{tr}_\mathcal{H}(Z) = \sum_j \mathrm{tr}(A_j) B_j.$$

Thus,

$$\mathrm{tr}_\mathcal{H}\big((T \otimes \mathrm{id})(Z)\big) = \mathrm{tr}_\mathcal{H}(Z)$$

and finally

$$\mathrm{tr}_\mathcal{H}(X) = \mathrm{tr}_\mathcal{H}\big((\Phi \otimes \mathrm{id})(\rho)\big) = \mathrm{tr}_\mathcal{H}\big(\rho - (T \otimes \mathrm{id})(\rho)\big) = \mathrm{tr}_\mathcal{H}(\rho) - \mathrm{tr}_\mathcal{H}(\rho) = 0.$$

□

Lemma 5. *For any $M \in \mathbb{C}^{d \times d}$ on $\mathcal{H}$ and $N \in \mathbb{C}^{d \times d}$ on $\mathcal{K}$,*

$$(M \otimes N)|\mathrm{vec}(A)\rangle = |\mathrm{vec}(MAN^\top)\rangle.$$

Lemma 6. *For any $X, Y \in \mathbb{C}^{d \times d}$,*

$$F(X \otimes Y) = (Y \otimes X)F.$$

Lemma 7. $(\Theta \otimes \mathrm{id})\big(|\mathrm{vec}(A)\rangle\langle \mathrm{vec}(B)|\big) = (\mathrm{id} \otimes A^\top)F(\mathrm{id} \otimes B^*)$.

Since $X_\star$ is Hermitian of rank 2 and traceless, we can write its spectral decomposition as

$$X_\star = t\big(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|\big),$$

for some orthogonal $|\psi\rangle, |\phi\rangle \in \mathcal{H} \otimes \mathcal{K}$ and some $t > 0$. From Lemma 4 we have $\mathrm{tr}_\mathcal{H}(X_\star) = 0$, which implies

$$\mathrm{tr}_\mathcal{H} |\psi\rangle\langle\psi| = \mathrm{tr}_\mathcal{H} |\phi\rangle\langle\phi|.$$

By Uhlmann's theorem [3], there exists a unitary U on $\mathcal{H}$ such that

$$|\phi\rangle = (U \otimes \mathrm{id})|\psi\rangle.$$

Now choose A such that

$$|\psi\rangle = |\mathrm{vec}(A)\rangle.$$

Then, using Lemma 5,

$$|\phi\rangle = (U \otimes \mathrm{id})|\mathrm{vec}(A)\rangle = |\mathrm{vec}(UA)\rangle.$$

By linearity,

$$Y_\star = t\big((\Theta \otimes \mathrm{id})(|\psi\rangle\langle\psi|) - (\Theta \otimes \mathrm{id})(|\phi\rangle\langle\phi|)\big).$$

Using Lemma 7 we get

$$(\Theta \otimes \mathrm{id})(|\psi\rangle\langle\psi|) = (\mathrm{id} \otimes A^\top)F(\mathrm{id} \otimes A^*)$$

and

$$
\begin{aligned}
(\Theta \otimes \mathrm{id})(|\phi\rangle\langle\phi|) &= \big(\mathrm{id} \otimes (UA)^\top\big) F \big(\mathrm{id} \otimes (UA)^*\big) \\
&= (\mathrm{id} \otimes A^\top U^\top) F (\mathrm{id} \otimes U^* A^*) \\
&= (\mathrm{id} \otimes A^\top)(\mathrm{id} \otimes U^\top) F (\mathrm{id} \otimes U^*)(\mathrm{id} \otimes A^*).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Y_\star &= t(\mathrm{id} \otimes A^\top) \left[F - (\mathrm{id} \otimes U^\top) F (\mathrm{id} \otimes U^*)\right] (\mathrm{id} \otimes A^*) \\
&= t(\mathrm{id} \otimes A^\top) F (\mathrm{id} - U^\top \otimes U^*)(\mathrm{id} \otimes A^*),
\end{aligned}
$$

where the second equality follows from Lemma 6. From submultiplicativity of rank under multiplication,

$$
\mathrm{rank}(Y_\star) \leq \mathrm{rank}(\mathrm{id} - U^\top \otimes U^*).
$$

Now diagonalize U as $U = VDV^\dagger$ with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$, $|\lambda_i| = 1$. Then

$$
U^\top = V^* D V^\top, \qquad U^* = V^* D^* V^\top,
$$

so

$$
U^\top \otimes U^* = (V^* \otimes V^*)(D \otimes D^*)(V^\top \otimes V^\top).
$$

Thus $U^\top \otimes U^*$ is similar to $D \otimes D^*$, whose eigenvalues are $\lambda_i \lambda_j^*$. In particular, for every i,

$$
\lambda_i \lambda_i^* = 1,
$$

so

$$
\dim\ker(\mathrm{id} - U^\top \otimes U^*) \geq d \quad \Rightarrow \quad \mathrm{rank}(\mathrm{id} - U^\top \otimes U^*) \leq d^2 - d.
$$

Hence

$$
\mathrm{rank}(Y_\star) \leq d^2 - d < d^2,
$$

which contradicts Eq. (6).

3 Discussion

The result in Section 2.3 leaves us with the following stronger open problem:

Problem 1. *Does there exist an absolute constant $\alpha < 1/\sqrt{2}$ for Theorem 1?*

References

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