

# A dimension-independent strict submultiplicativity for the transposition map in diamond norm

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## Abstract

We prove that there exists an absolute constant  $\alpha < 1$  such that for every finite dimension  $d$  and every quantum channel  $T$  on  $\mathcal{L}(\mathbb{C}^d)$ ,  $\|\Theta \circ (\text{id} - T)\|_\diamond \leq \alpha \|\Theta\|_\diamond \|\text{id} - T\|_\diamond$ , where  $\Theta$  is the transposition map. In fact we show the explicit choice  $\alpha = 1/\sqrt{2}$  works.

## 1 Setup and definitions

Let  $\mathcal{H} \cong \mathbb{C}^d$  be a  $d$ -dimensional Hilbert space. We write  $\mathcal{L}(\mathcal{H})$  for the space of linear operators on  $\mathcal{H}$ , and

$$\mathcal{D}(\mathcal{H}) = \{\rho \in \mathcal{L}(\mathcal{H}) : \rho \succeq 0, \text{tr}(\rho) = 1\}$$

for the set of density operators.

**Transposition map.** Fix an orthonormal basis  $\{|1\rangle, \dots, |d\rangle\}$  of  $\mathcal{H}$ . The transposition map  $\Theta : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  is defined by

$$\Theta(X) = X^\top,$$

which depends on the chosen basis.

**Quantum channels.** A *quantum channel*  $T : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  is a completely positive trace-preserving (CPTP) linear map.

**Norms.** The trace norm and Hilbert–Schmidt norm of an operator  $X$  are

$$\|X\|_1 = \text{tr} \sqrt{X^\dagger X}, \quad \|X\|_2 = \sqrt{\text{tr}(X^\dagger X)}.$$

The *diamond norm* of a linear map  $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  is

$$\|\Phi\|_\diamond := \sup_{k \geq 1} \sup_{\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^k)} \|(\Phi \otimes \text{id}_k)(\rho)\|_1.$$

It is standard that for maps acting on  $\mathcal{L}(\mathbb{C}^d)$ , the supremum over  $k$  can be restricted to  $k = d$ . We will use this without further comment.

## 2 Main result

**Theorem 1.** *Let  $\mathcal{H} \cong \mathbb{C}^d$  and let  $T : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  be a quantum channel. Then*

$$\|\Theta \circ (\text{id} - T)\|_\diamond \leq \alpha \|\Theta\|_\diamond \|\text{id} - T\|_\diamond$$

with  $\alpha = 1/\sqrt{2}$ . In particular, since  $\|\Theta\|_\diamond = d$  [1],

$$\|\Theta \circ (\text{id} - T)\|_\diamond \leq \frac{d}{\sqrt{2}} \|\text{id} - T\|_\diamond.$$

The proof proceeds by reducing to a matrix inequality for the partial transpose applied to traceless Hermitian matrices. The specific form  $\Theta \circ (\text{id} - T)$  appears in [2], but the analysis therein relied on the standard diamond-norm submultiplicativity corresponding to the case  $\alpha = 1$ . The possibility of obtaining a *dimension-independent* constant  $\alpha < 1$  for a stronger submultiplicativity result remained open.

## 2.1 Auxiliary lemmas

**Lemma 1.** *If  $X = X^\dagger$  and  $\text{tr}(X) = 0$ , then*

$$\|X\|_2 \leq \frac{1}{\sqrt{2}} \|X\|_1.$$

*Proof.* Let  $\{\lambda_i\}_{i=1}^n$  be the eigenvalues of  $X$  (real because  $X$  is Hermitian). Write the positive eigenvalues as  $p_1, \dots, p_r \geq 0$  and the negative eigenvalues as  $-q_1, \dots, -q_s$  with  $q_1, \dots, q_s > 0$ . Since  $\text{tr}(X) = \sum_i \lambda_i = 0$ , we have

$$\sum_{i=1}^r p_i = \sum_{j=1}^s q_j =: t.$$

Hence

$$\|X\|_1 = \sum_i |\lambda_i| = \sum_{i=1}^r p_i + \sum_{j=1}^s q_j = 2t.$$

Also,

$$\|X\|_2^2 = \sum_i \lambda_i^2 = \sum_{i=1}^r p_i^2 + \sum_{j=1}^s q_j^2 \leq \left(\sum_{i=1}^r p_i\right)^2 + \left(\sum_{j=1}^s q_j\right)^2 = t^2 + t^2 = 2t^2, \quad (1)$$

where we used the elementary inequality  $\sum a_i^2 \leq (\sum a_i)^2$  for nonnegative  $a_i$ . Therefore  $\|X\|_2 \leq \sqrt{2}t = \|X\|_1 / \sqrt{2}$ .  $\square$

**Lemma 2.** *Let  $Y$  be an operator on an  $N$ -dimensional Hilbert space. Then*

$$\|Y\|_1 \leq \sqrt{N} \|Y\|_2.$$

*Proof.* Let  $s_1, \dots, s_N$  be the singular values of  $Y$  (padding with zeros if necessary). Then  $\|Y\|_1 = \sum_i s_i$  and  $\|Y\|_2 = \sqrt{\sum_i s_i^2}$ . By Cauchy–Schwarz,

$$\left(\sum_{i=1}^N s_i\right)^2 \leq N \sum_{i=1}^N s_i^2, \quad (2)$$

which gives the claim.  $\square$

**Lemma 3.** *Let  $\mathcal{H}, \mathcal{K}$  be finite-dimensional and let  $\Theta$  denote transposition on  $\mathcal{H}$ . Then for all  $X \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ ,*

$$\|(\Theta \otimes \text{id})(X)\|_2 = \|X\|_2.$$

*Proof.* In the computational basis on  $\mathcal{H}$ , the transposition map is an isometry for the Hilbert–Schmidt inner product:

$$\langle A, B \rangle_{\text{HS}} := \text{tr}(A^\dagger B), \quad \text{tr}((A^\top)^\dagger B^\top) = \text{tr}(A^\dagger B).$$

Applying this entrywise on  $\mathcal{H}$  and trivially on  $\mathcal{K}$  yields

$$\text{tr}\left(\left((\Theta \otimes \text{id})(X)\right)^\dagger (\Theta \otimes \text{id})(X)\right) = \text{tr}(X^\dagger X),$$

i.e.  $\|(\Theta \otimes \text{id})(X)\|_2 = \|X\|_2$ .  $\square$

## 2.2 Proof of Theorem 1

*Proof of Theorem 1.* Let  $\Phi := \text{id} - T$ . Since  $T$  is CPTP, the map  $\Phi$  is Hermiticity-preserving and trace-annihilating:

$$\text{tr}(\Phi(Z)) = \text{tr}(Z) - \text{tr}(T(Z)) = 0 \quad \forall Z \in \mathcal{L}(\mathcal{H}).$$

Fix an ancilla  $\mathcal{K} \cong \mathbb{C}^d$ . Let  $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathcal{K})$  and define

$$X := (\Phi \otimes \text{id})(\rho) \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K}). \quad (3)$$

Then:

- $X$  is Hermitian, since  $\Phi$  is Hermiticity-preserving and  $\rho$  is Hermitian.
- $X$  is traceless:

$$\text{tr}(X) = \text{tr}((\Phi \otimes \text{id})(\rho)) = \text{tr}(\Phi(\text{tr}_{\mathcal{K}} \rho)) = 0,$$

because  $\Phi$  is trace-annihilating.

- $(\Theta \circ \Phi \otimes \text{id})(\rho) = (\Theta \otimes \text{id})(X)$ .

We now bound  $\|(\Theta \otimes \text{id})(X)\|_1$  in terms of  $\|X\|_1$ . The operator  $(\Theta \otimes \text{id})(X)$  acts on  $\mathcal{H} \otimes \mathcal{K}$ , which has dimension  $N = d^2$ . By Lemma 2,

$$\|(\Theta \otimes \text{id})(X)\|_1 \leq \sqrt{d^2} \|(\Theta \otimes \text{id})(X)\|_2 = d \|(\Theta \otimes \text{id})(X)\|_2.$$

By Lemma 3,  $\|(\Theta \otimes \text{id})(X)\|_2 = \|X\|_2$ , hence

$$\|(\Theta \otimes \text{id})(X)\|_1 \leq d \|X\|_2.$$

Finally, since  $X$  is traceless Hermitian, Lemma 1 gives

$$\|X\|_2 \leq \frac{1}{\sqrt{2}} \|X\|_1.$$

Combining these inequalities yields

$$\|(\Theta \otimes \text{id})(X)\|_1 \leq \frac{d}{\sqrt{2}} \|X\|_1.$$

Substituting back  $X = (\Phi \otimes \text{id})(\rho)$  gives, for every  $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathcal{K})$ ,

$$\|(\Theta \circ \Phi \otimes \text{id})(\rho)\|_1 \leq \frac{d}{\sqrt{2}} \|(\Phi \otimes \text{id})(\rho)\|_1.$$

Taking the supremum over  $\rho$  proves

$$\|\Theta \circ \Phi\|_{\diamond} \leq \frac{d}{\sqrt{2}} \|\Phi\|_{\diamond}.$$

Thus

$$\|\Theta \circ (\text{id} - T)\|_{\diamond} \leq \frac{d}{\sqrt{2}} \|\text{id} - T\|_{\diamond}.$$

Since  $\|\Theta\|_{\diamond} = d$ , this is equivalent to

$$\|\Theta \circ (\text{id} - T)\|_{\diamond} \leq \frac{1}{\sqrt{2}} \|\Theta\|_{\diamond} \|\text{id} - T\|_{\diamond},$$

completing the proof.  $\square$

**Remark 1.** The constant  $1/\sqrt{2}$  is universal (independent of  $d$  and  $T$ ) and arises from the fact that  $X = ((\text{id} - T) \otimes \text{id})(\rho)$  is always traceless because  $\text{id} - T$  is trace-annihilating.

### 2.3 Positive gap in finite dimension

Regrettably, the inequality in Theorem 1 is still not tight for any finite  $d$ , except in the trivial case  $T = \text{id}$ , where both LHS and RHS become zero. Fix any nonzero channel difference  $\Phi = \text{id} - T \neq 0$  (equivalently  $T \neq \text{id}$ ). Rewrite the pointwise inequality (for every  $\rho$ ):

$$\underbrace{\|(\Theta \circ \Phi \otimes \text{id})(\rho)\|_1}_{=:L(\rho)} \leq \frac{d}{\sqrt{2}} \underbrace{\|(\Phi \otimes \text{id})(\rho)\|_1}_{=:R(\rho)}. \quad (4)$$

Taking suprema gives  $\sup_{\rho} L(\rho) \leq \frac{d}{\sqrt{2}} \sup_{\rho} R(\rho)$ . In finite dimension, the suprema are maxima, so equality

$$\sup_{\rho} L(\rho) = \frac{d}{\sqrt{2}} \sup_{\rho} R(\rho)$$

can only happen if there exists some  $\rho_{\star}$  such that  $R(\rho_{\star}) = \sup_{\rho} R(\rho)$  and the pointwise inequality Eq. (4) is tight at  $\rho_{\star}$ , i.e.,  $L(\rho_{\star}) = \frac{d}{\sqrt{2}} R(\rho_{\star})$ . Let  $X_{\star} := (\Phi \otimes \text{id})(\rho_{\star})$ . Since  $\Phi \neq 0$  and  $\rho_{\star}$  maximizes  $R$ , we have  $X_{\star} \neq 0$ . Now, the chain of inequalities shows

$$\|(\Theta \otimes \text{id})(X_{\star})\|_1 \leq d \|X_{\star}\|_2 \leq \frac{d}{\sqrt{2}} \|X_{\star}\|_1.$$

So equality in Theorem 1 forces equality in Lemmas 1 and 2 at some nonzero  $X_{\star}$ . But the equality conditions of Lemmas 1 and 2 are incompatible (unless  $X = 0$ ).

**Equality condition for Lemma 1.** For a nonzero traceless Hermitian  $X$ , equality in Lemma 1 holds *iff*  $X$  has exactly two nonzero eigenvalues,  $+t$  and  $-t$ . This follows from Eq. (1), where equality requires all cross terms to vanish. Equivalently,

$$\text{rank}(X_{\star}) = 2 \quad (\text{for nonzero equality cases}). \quad (5)$$

**Equality condition for Lemma 2.** Equality in Lemma 2 holds *iff*  $s$  in Eq. (2) is proportional to  $(1, \dots, 1)$ , i.e., all singular values are equal. If  $Y \neq 0$ , that forces all  $N$  singular values to be the same positive number, hence

$$\text{rank}((\Theta \otimes \text{id})(X_{\star})) = \text{rank}(Y_{\star}) = N = d^2. \quad (6)$$

To proceed, we need the following definitions and identities.

**Definition 2.1** (Column-major vectorization). For a matrix  $A \in \mathbb{C}^{d \times d}$  with entries  $A_{ij} = \langle i|A|j\rangle$ , define

$$|\text{vec}(A)\rangle := \sum_{i=1}^d \sum_{j=1}^d A_{ij} |i\rangle_{\mathcal{H}} \otimes |j\rangle_{\mathcal{K}}.$$

**Definition 2.2.** The swap operator  $F \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$  is defined by:

$$F(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle.$$

**Lemma 4.** For any operator  $Z \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ ,  $\text{tr}_{\mathcal{H}}((T \otimes \text{id})(Z)) = \text{tr}_{\mathcal{H}}(Z)$ .

*Proof.* We can write  $Z$  as

$$Z = \sum_j A_j \otimes B_j,$$

where  $A_j \in \mathcal{L}(\mathcal{H})$  and  $B_j \in \mathcal{L}(\mathcal{K})$ . There exists a set of Kraus operators  $\{E_k\}$  acting on  $\mathcal{H}$  such that

$$T(X) = \sum_k E_k X E_k^{\dagger}, \quad \sum_k E_k^{\dagger} E_k = \text{id}.$$

By linearity we have

$$(T \otimes \text{id})Z = \sum_j T(A_j) \otimes B_j = \sum_j \left( \sum_k E_k A_j E_k^\dagger \right) \otimes B_j.$$

Now we take the partial trace  $\text{tr}_{\mathcal{H}}$ :

$$\begin{aligned} \text{tr}_{\mathcal{H}}((T \otimes \text{id})Z) &= \sum_j \text{tr}_{\mathcal{H}} \left( \sum_k E_k A_j E_k^\dagger \otimes B_j \right) = \sum_j \left( \sum_k \text{tr}(E_k A_j E_k^\dagger) \right) B_j \\ &= \sum_j \text{tr} \left( \left( \sum_k E_k^\dagger E_k \right) A_j \right) B_j = \sum_j \text{tr}(A_j) B_j. \end{aligned}$$

Meanwhile, by the definition of the partial trace on the original operator  $Z = \sum_j A_j \otimes B_j$ ,

$$\text{tr}_{\mathcal{H}}(Z) = \sum_j \text{tr}(A_j) B_j.$$

Thus,

$$\text{tr}_{\mathcal{H}}((T \otimes \text{id})(Z)) = \text{tr}_{\mathcal{H}}(Z).$$

□

**Corollary 1.** For  $X$  in Eq. (3),  $\text{tr}_{\mathcal{H}}(X) = 0$ .

*Proof.* From Lemma 4 we have

$$\text{tr}_{\mathcal{H}}(X) = \text{tr}_{\mathcal{H}}((\Phi \otimes \text{id})(\rho)) = \text{tr}_{\mathcal{H}}(\rho - (T \otimes \text{id})(\rho)) = \text{tr}_{\mathcal{H}}(\rho) - \text{tr}_{\mathcal{H}}(\rho) = 0.$$

□

**Lemma 5.** For any  $M \in \mathbb{C}^{d \times d}$  on  $\mathcal{H}$  and  $N \in \mathbb{C}^{d \times d}$  on  $\mathcal{K}$ ,

$$(M \otimes N)|\text{vec}(A)\rangle = |\text{vec}(MAN^\top)\rangle.$$

**Lemma 6.** For any  $X, Y \in \mathbb{C}^{d \times d}$ ,

$$F(X \otimes Y) = (Y \otimes X)F.$$

**Lemma 7.**  $(\Theta \otimes \text{id})(|\text{vec}(A)\rangle\langle\text{vec}(B)|) = (\text{id} \otimes A^\top)F(\text{id} \otimes B^*)$ .

Since  $X_\star$  is Hermitian of rank 2 and traceless, we can write its spectral decomposition as

$$X_\star = t(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|),$$

for some orthogonal  $|\psi\rangle, |\phi\rangle \in \mathcal{H} \otimes \mathcal{K}$  and some  $t > 0$ . From Corollary 1 we have  $\text{tr}_{\mathcal{H}}(X_\star) = 0$ , which implies

$$\text{tr}_{\mathcal{H}} |\psi\rangle\langle\psi| = \text{tr}_{\mathcal{H}} |\phi\rangle\langle\phi|.$$

By Uhlmann's theorem [3], there exists a unitary  $U$  on  $\mathcal{H}$  such that

$$|\phi\rangle = (U \otimes \text{id})|\psi\rangle.$$

Now choose  $A$  such that

$$|\psi\rangle = |\text{vec}(A)\rangle.$$

Then, using Lemma 5,

$$|\phi\rangle = (U \otimes \text{id})|\text{vec}(A)\rangle = |\text{vec}(UA)\rangle.$$

By linearity,

$$Y_\star = t((\Theta \otimes \text{id})(|\psi\rangle\langle\psi|) - (\Theta \otimes \text{id})(|\phi\rangle\langle\phi|)).$$

Using Lemma 7 we get

$$(\Theta \otimes \text{id})(|\psi\rangle\langle\psi|) = (\text{id} \otimes A^\top)F(\text{id} \otimes A^*)$$

and

$$\begin{aligned} (\Theta \otimes \text{id})(|\phi\rangle\langle\phi|) &= (\text{id} \otimes (UA)^\top)F(\text{id} \otimes (UA)^*) \\ &= (\text{id} \otimes A^\top U^\top)F(\text{id} \otimes U^* A^*) \\ &= (\text{id} \otimes A^\top)(\text{id} \otimes U^\top)F(\text{id} \otimes U^*)(\text{id} \otimes A^*). \end{aligned}$$

Therefore,

$$\begin{aligned} Y_\star &= t(\text{id} \otimes A^\top) [F - (\text{id} \otimes U^\top)F(\text{id} \otimes U^*)] (\text{id} \otimes A^*) \\ &= t(\text{id} \otimes A^\top)F(\text{id} - U^\top \otimes U^*)(\text{id} \otimes A^*), \end{aligned}$$

where the second equality follows from Lemma 6. From submultiplicativity of rank under multiplication,

$$\text{rank}(Y_\star) \leq \text{rank}(\text{id} - U^\top \otimes U^*).$$

Now diagonalize  $U$  as  $U = VDV^\dagger$  with  $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ ,  $|\lambda_i| = 1$ . Then

$$U^\top = V^* D V^\top, \quad U^* = V^* D^* V^\top,$$

so

$$U^\top \otimes U^* = (V^* \otimes V^*)(D \otimes D^*)(V^\top \otimes V^\top).$$

Thus  $U^\top \otimes U^*$  is similar to  $D \otimes D^*$ , whose eigenvalues are  $\lambda_i \lambda_j^*$ . In particular, for every  $i$ ,

$$\lambda_i \lambda_i^* = 1,$$

so

$$\dim \ker(\text{id} - U^\top \otimes U^*) \geq d \quad \Rightarrow \quad \text{rank}(\text{id} - U^\top \otimes U^*) \leq d^2 - d.$$

Hence

$$\text{rank}(Y_\star) \leq d^2 - d < d^2,$$

which contradicts Eq. (6).

### 3 Discussion

The result in Section 2.3 leaves us with the following stronger open problem:

**Problem 1.** *Does there exist an absolute constant  $\alpha < 1/\sqrt{2}$  for Theorem 1?*

### References

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