From Consensus to Coordination of Heterogeneous Multi-agent Systems

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Disclaimer

Consensus studies form a vast field, which cannot be fully covered in a limited amount of time.

For an introduction to this field, one may refer to

> *Sync: The Emerging Science of Spontanenous Order* (2003) by Steven Strogatz

More limitations in this presentation:

- We focus on continuous-time algorithms, assuming that the communication bandwidth is sufficiently large.
- Communication graph does not change over time, and there are no communication delays.
- To emphasize conceptual ideas, results of cited papers are highly simplified.
- Citations are very limited.

One may also refer to excellent surveys:

- Olfati-Saber, Fax, & Murray, Consensus and Cooperation in Networked Multi-Agent Systems, Proc. IEEE, 2007
- Ren, Beard, & Atkins, Information Consensus in Multivehicle Cooperative Control, IEEE CSM, 2007
- Proskurnikov & Tempo, A Tutorial on Modeling and Analysis of Dynamic Social Networks, ARC, 2017
- Dörfler & Bullo, Synchronization in Complex Networks of Phase Oscillators: A Survey, AUT, 2014
- and more...

In this presentation

We are interested in the question: How does spontaneous coordination arise among many (heterogeneous) agents, without a centralized coordinator?

For coordination, at least some portion of the local variables should be in consensus across the network.

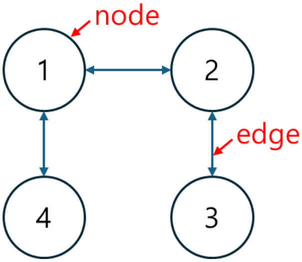
Consensus is behind the scene.

Contents of the presentation:

1. Review of consensus for identical agents
2. Review of consensus for heterogeneous agents
3. Coordination of heterogeneous agents

Preliminaries

Networked Dynamical Systems (= Multi-agent Systems)



communication graph

In this presentation,

$$w_i \equiv y_i$$

and in most cases,

$$z_i \equiv x_i \equiv w_i \equiv y_i$$

node dynamics (agent):

$$\dot{x}_i = \mathsf{f}_i(x_i, a_i, u_i), \qquad y_i = \mathsf{h}_i(x_i) \qquad\qquad i \in \mathcal{N} := \{1, \cdots, N\}$$

$$w_i = \mathsf{s}_i(x_i)$$

$$z_i = \mathsf{g}_i(x_i)$$

- x_i: state variable of agent i
- a_i: external input to agent i or attribute of agent i
- y_i: communicated variable with neighboring agents
- u_i: coupling input (signal for synchronization) given by

$$u_i = \mathcal{C}_i\left(x_i, \{y_j : j \in \mathcal{N}_i\}\right), \quad \mathcal{C}_i\text{: static or dynamic}$$

- w_i: synchronization (= consensus) variable, which is to be

$$\lim_{t\to\infty} \|w_i(t) - w_j(t)\| = 0, \quad \forall i, j$$

- z_i: variable of interest, performance output

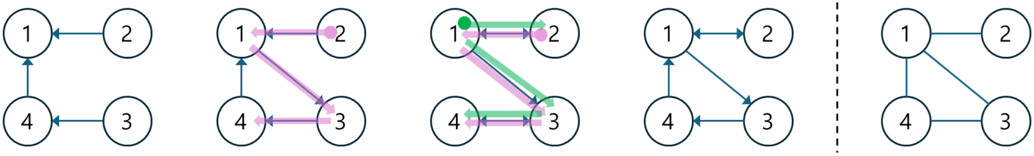
$$\lim_{t\to\infty} \|z_i(t) - z_i^*(t)\| = 0, \quad \forall i$$

where z_i^* is some desired (emergent) behavior

Graph and its connectedness

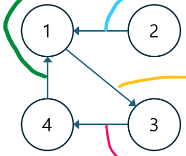
Rooted Spanning Tree (RST): a tree that reaches all the nodes from a 'root'

Directed graph | Undirected graph



$\nexists$ RST | $\exists$ RST | $\exists$ (two) RST | $\exists$ (many) RST | $\exists$ RST=connected

A graph can be equivalently represented by a matrix



$$
\begin{array}{c|c}
 & \text{from} \quad 1 \quad\quad 2 \quad\quad 3 \quad\quad 4 \\
\begin{array}{c} \text{to} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\begin{array}{c} \\ \begin{bmatrix} \square & -1 & 0 & -1 \\ 0 & \square & 0 & 0 \\ -1 & 0 & \square & 0 \\ 0 & 0 & -1 & \square \end{bmatrix} \end{array}
\end{array}
$$

Fill the diagonals so that every row sums are 0.

$$
\begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} =: L \text{ (graph Laplacian)}
$$

- 0 is always an eigenvalue of L (by construction) and $\mathrm{Re}(\lambda_i(L)) \geq 0$ (by Gershgorin disc theorem)
- Sort the eigenvalues of L as

$$0 = \mathrm{Re}(\lambda_1) \leq \mathrm{Re}(\lambda_2) \leq \cdots \leq \mathrm{Re}(\lambda_N)$$

Then, algebraic graph theory (e.g. Fiedler, 1973) tells us that

$$\mathrm{Re}(\lambda_2) > 0 \quad \Leftrightarrow \quad \exists \text{ RST in the graph}$$

If the graph is undirected (i.e., $L = L^\top$),

$$\lambda_2 > 0 \quad \Leftrightarrow \quad \text{the graph is connected.}$$

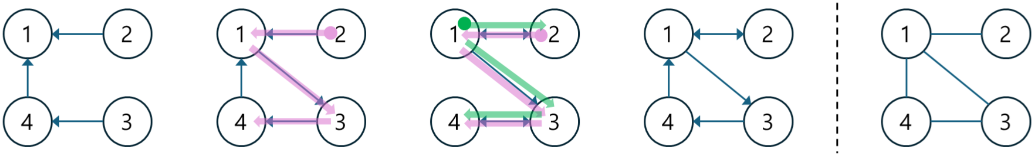
Throughout the presentation, we assume the graph contains RST!

Now, let $\theta \in \mathbb{R}^N$ be a left eigenvector of L for $\lambda_1 = 0$, i.e.,

$$\theta^\top L = 0, \qquad \text{and} \qquad \sum_{i=1}^{N} \theta_i = \theta^\top \mathbf{1}_N = 1 \ \ \text{(normalized)}.$$

Then, it is known that $\theta_i \geq 0$, and in particular,

$$\theta = \begin{bmatrix} 0 \\ + \\ 0 \\ 0 \end{bmatrix} \qquad \theta = \begin{bmatrix} + \\ + \\ 0 \\ 0 \end{bmatrix} \qquad \theta = \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix} \qquad \theta = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$



$\nexists$ RST $\quad$ $\exists$ RST $\quad$ $\exists$ (two) RST $\quad$ $\exists$ (many) RST $\quad$ $\exists$ RST=connected

A useful fact

Since $\theta^\top \mathbf{1}_N = 1$, $\exists R \in \mathbb{R}^{N\times(N-1)}$ and $Q \in \mathbb{R}^{N\times(N-1)}$ such that

$$\begin{bmatrix}\theta^\top \\ Q^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}_N & R\end{bmatrix} = I_N.$$

- If the graph is undirected (i.e., $L^\top = L$), then $\theta_i = \frac{1}{N}$ and $Q = R$.

$\therefore$ With θ, Q, and R, a similarity transformation of L yields a matrix M:

$$\begin{bmatrix}\theta^\top \\ Q^\top\end{bmatrix} L \begin{bmatrix}\mathbf{1}_N & R\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & M\end{bmatrix}$$

in which, $M = Q^\top L R \in \mathbb{R}^{(N-1)\times(N-1)}$, and $\mathrm{Re}(\lambda_l(M)) > 0, \forall l$!

MAS can be analyzed in a new coordinate under Balanced Action Property

Consider a (heterogeneous) MAS given by

$$\dot{x}_i = f_i(x_i, a_i) + u_i \qquad \in \mathbb{R}^n$$

and with $x = \mathrm{col}(x_1, \cdots, x_N)$, consider the coordinate change:

$$\begin{aligned} z_{\mathsf{o}} &= (\theta^\top \otimes I_n)x \\ z_{\mathsf{x}} &= (Q^\top \otimes I_n)x \end{aligned} \quad \Leftrightarrow \quad x = (\mathbf{1}_N \otimes I_n)z_{\mathsf{o}} + (R \otimes I_n)z_{\mathsf{x}} \qquad \text{(i.e., } x_i = z_{\mathsf{o}} + (R_i \otimes I_n)z_{\mathsf{x}}\text{)}.$$

If it holds that

$$\sum_{i=1}^{N} \theta_i u_i(t) = 0, \qquad \forall t. \qquad \textbf{(BAP: Balanced Action Property)}$$

the MAS is converted into

$$\dot{z}_{\mathsf{o}} = \sum_{i=1}^{N} \theta_i f_i(x_i, a_i) + 0 \in \mathbb{R}^n, \quad \dot{z}_{\mathsf{x}} = (Q^\top \otimes I_n) \begin{bmatrix} f_1(x_1, a_1) \\ \vdots \\ f_N(x_N, a_N) \end{bmatrix} + (Q^\top \otimes I_n)u \in \mathbb{R}^{(N-1)\times n}$$

where $u = \mathrm{col}(u_1, \cdots, u_N)$.

ox-representation of the MAS

$$\dot{x}_i = f_i(x_i, a_i) + u_i$$

$$\Updownarrow$$

$$\dot{z}_{\mathsf{o}} = \sum_{i=1}^{N} \theta_i f_i(z_{\mathsf{o}} + (R_i \otimes I_n) z_{\mathsf{x}}, a_i), \qquad z_{\mathsf{o}}(0) = \sum_{i=1}^{N} \theta_i x_i(0),$$

$$\dot{z}_{\mathsf{x}} = (Q^\top \otimes I_n) \begin{bmatrix} f_1(z_{\mathsf{o}} + (R_1 \otimes I_n) z_{\mathsf{x}}, a_1) \\ \vdots \\ f_N(z_{\mathsf{o}} + (R_N \otimes I_n) z_{\mathsf{x}}, a_N) \end{bmatrix} + (Q^\top \otimes I_n) u.$$

Recalling $x_i = z_{\mathsf{o}} + (R_i \otimes I_n) z_{\mathsf{x}}$, the quantity z_{x} is a measure of synchrony. Imagine $z_{\mathsf{x}} \equiv 0$:

- We have

$$\dot{z}_{\mathsf{o}} = \sum_{i=1}^{N} \theta_i f_i(z_{\mathsf{o}}, a_i) \quad \text{and} \quad x_i = z_{\mathsf{o}}, \quad \forall i \quad \Rightarrow \quad z_{\mathsf{o}} \text{ governs the behavior of the MAS.}$$

- Since $Q^\top \mathbf{1}_N = 0$, the first term in $\dot{z}_{\mathsf{x}}$ vanishes when $f_1(\cdot,\cdot) = f_2(\cdot,\cdot) = \cdots = f_N(\cdot,\cdot)$ and $a_1 = \cdots = a_N$, which therefore can be regarded as a measure of heterogeneity.

Consensus for Identical Node Dynamics

Network of integrators

$$\dot{x}_i = u_i \quad \text{with} \quad u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i), \quad i = 1, \cdots, N \qquad \text{(written as } \dot{x} = -(L \otimes I_n)x\text{)}$$

achieves consensus if and only if

$$-\mathrm{Re}(\lambda_2(L)) < 0.$$

- Appears in (Abelson, 1964), (Olfati-Saber & Murray, 2004), (Moreau, 2004), and more
- Because $\theta^\top L = 0$, which implies $\sum_{i=1}^{N} \theta_i \dot{x}_i = 0$,

$$\sum_{i=1}^{N} \theta_i x_i(t) = \sum_{i=1}^{N} \theta_i x_i(0) \qquad \textbf{(CP: Conservation Property)}$$

$$\therefore \qquad x_i(t) \to s(t) \quad \Rightarrow \quad \sum_{i=1}^{N} \theta_i s(t) = s(t) = \sum_{i=1}^{N} \theta_i x_i(0).$$

Under the hood: Since BAP holds, ox-representation:

$$\dot{z}_\mathsf{o} = \sum_{i=1}^{N} \theta_i f_i(z_\mathsf{o} + (R_i \otimes I_n) z_\mathsf{x}, a_i) = 0, \qquad z_o(0) = \sum_{i=1}^{N} \theta_i x_i(0)$$

$$\dot{z}_\mathsf{x} = (Q^\top \otimes I_n) \begin{bmatrix} f_1(z_\mathsf{o} + (R_1 \otimes I_n) z_\mathsf{x}, a_1) \\ \vdots \\ f_N(z_\mathsf{o} + (R_N \otimes I_n) z_\mathsf{x}, a_N) \end{bmatrix} - (Q^\top \otimes I_n)(L \otimes I_n) x = -(M \otimes I_n) z_\mathsf{x}$$

where $M = Q^\top L R$ and $-(M \otimes I_n)$ is Hurwitz.

If $L = L^\top$, then $x_i(t) \to \frac{1}{N} \sum_{i=1}^{N} x_i(0)$.

Linear node dynamics: State coupling

State coupling (Fax & Murray, TAC, 2004) (Tuna, arXiv, 2008)

$$\dot{x}_i = Ax_i + Bu_i \quad \text{with} \quad u_i = K \sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

achieves consensus if and only if

$$A - \lambda_i(L)BK \quad \text{is Hurwitz for all } i = 2, \cdots, N.$$

- Given (A, B, L), consensus problem $\Rightarrow$ robust stabilization by K against multiplicative perturbation of λ_i $\rightarrow$ called 'master stability condition' (Pecora & Carroll, PRL, 1998)
- One way to design K:

$$K = B^\top P \quad \text{with} \quad P > 0 \quad \text{from} \quad PA + A^\top P - 2(\mathrm{Re}(\lambda_2(L)))PBB^\top P < 0$$

Under the hood: BAP holds for this case, and so, ox-representation:

$$\dot{z}_{\mathsf{o}} = \sum_{i=1}^{N} \theta_i A(z_{\mathsf{o}} + (R_i \otimes I_n) z_{\mathsf{x}}) = A z_{\mathsf{o}}$$
$$\dot{z}_{\mathsf{x}} = [(I_{N-1} \otimes A) - (M \otimes BK)] z_{\mathsf{x}}$$

Coordinate change of z_{x} for converting M to its Jordan form J:

$$\dot{z}_{\mathsf{x}} = [(I_{N-1} \otimes A) - (M \otimes BK)] z_{\mathsf{x}} \quad \Leftrightarrow \quad \dot{\xi} = [(I_{N-1} \otimes A) - (J \otimes BK)] \xi$$

which is a collection of the master stability equations:

$$\dot{w} = (A - \lambda_i BK) w \quad \text{or} \quad \dot{w} = (A - \lambda_i BK) w + BK \bar{w}, \qquad i = 2, \cdots, N.$$

$\therefore$ z_{x} converges to zero iff

$$A - \lambda_i(L) BK \quad \text{is Hurwitz, } \forall i = 2, \cdots, N.$$

- Behavior of MAS: Recalling $x_i = z_{\mathsf{o}} + (R_i \otimes I_n) z_{\mathsf{x}}$,

$$x_i(t) \quad \xrightarrow{z_{\mathsf{x}}(t) \to 0} \quad z_{\mathsf{o}}(t) \quad \text{where} \quad \dot{z}_{\mathsf{o}} = A z_{\mathsf{o}}, \quad z_{\mathsf{o}}(0) = \sum_{i=1}^{N} \theta_i x_i(0)$$

Linear node dynamics: Dynamic coupling

Dynamic output coupling (Li, Duan, Chen, & Huang, TCS, 2010)

$$\dot{x}_i = Ax_i + Bu_i, \qquad y_i = Cx_i$$

with the dynamic coupling law:

$$u_i = Kv_i, \qquad \dot{v}_i = (A + BK)v_i + H \sum_{j \in \mathcal{N}_i} \left((Cv_j - y_j) - (Cv_i - y_i) \right)$$

achieves consensus if and only if

$$\begin{bmatrix} A - \lambda_i(L)HC & 0 \\ \lambda_i(L)HC & A + BK \end{bmatrix} \quad \text{is Hurwitz for all } i = 2, \cdots, N.$$

- Given (A, B, C, L), K can be designed such that $A + BK$ is Hurwitz and H as

$$H = PC^\top \quad \text{with} \quad P > 0 \quad \text{from} \quad AP + PA^\top - 2(\mathrm{Re}(\lambda_2(L)))PC^\top CP < 0.$$

Under the hood: Agent i is written containing the couping dynamics with new output Y_i:

$$\begin{bmatrix} \dot{x}_i \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ H \end{bmatrix} \left(\sum_{j \in \mathcal{N}_i} (Y_j - Y_i) \right)$$

$$Y_i = \begin{bmatrix} -C & C \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

and BAP holds for the new inputs.

$\therefore$ Consensus condition (the master stability condition) becomes

$$\begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix} - \lambda_i \begin{bmatrix} 0 \\ H \end{bmatrix} \begin{bmatrix} -C & C \end{bmatrix} = \begin{bmatrix} A & BK \\ \lambda_i HC & A+BK-\lambda_i HC \end{bmatrix} \quad \text{is Hurwitz, } i = 2, \cdots, N$$

which is similar to the matrix in the theorem.

- Behavior of MAS: $(x_i(t), v_i(t)) \xrightarrow{z_\times^x, z_\times^y \to 0} (z_\circ^x(t), z_\circ^v(t))$ where

$$\begin{bmatrix} \dot{z}_\circ^x \\ \dot{z}_\circ^v \end{bmatrix} = \begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix} \begin{bmatrix} z_\circ^x \\ z_\circ^v \end{bmatrix}, \qquad \begin{bmatrix} z_\circ^x(0) \\ z_\circ^v(0) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N \theta_i x_i(0) \\ \sum_{i=1}^N \theta_i v_i(0) \end{bmatrix}$$

Further results

Dynamic output coupling (Seo, S, & Back, AUT, 2008)

$\dot{x}_i = Ax_i + Bu_i$, $y_i = Cx_i$ can achieve consensus, if $\mathrm{Re}(\lambda_l(A)) \leq 0, \forall i$, with

$$u_i = \mathcal{H}\left(\sum_{j \in \mathcal{N}_i} (y_j - y_i)\right) \quad \text{where } \mathcal{H}(\cdot) \text{ a (low-gain) LTI dynamic system}$$

Consensus using passivity (Arcak, TAC, 2007)

$$\dot{x}_i = a(t) + \mathcal{H}_i\left(\sum_{j \in \mathcal{N}_i} \psi_{ji}(x_j - x_i)\right) \qquad (a(t)\text{: a common external input})$$

achieves consensus with $\mathcal{H}_i$: strictly passive nonlinear system, ψ_{ji}: strictly passive nonlinear function.
(Useful because many practical systems, e.g. multi-machine power system, can be interpreted to have this form.)

Consensus using partial contraction (Wang & Slotine, Bio. Cyber, 2005)

$$\dot{x}_i = f(t, x_i) + k \sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

achieves consensus if $\lambda_{\max}\left\{(\nabla_x f + \nabla_x f^\top)(t, x)\right\}$ is uniformly bounded and if k is large enough.

Under the hood of partial contraction: MAS can be written as

$$\dot{x}_i = f(t, x_i) + k \sum_{j \in \mathcal{N}_i} (x_j - x_i) - k \sum_{j=1}^{N} x_j + k \sum_{j=1}^{N} x_j, \quad i = 1, \cdots, N \tag{1}$$

Suppose $x_i(t)$'s drive two virtual systems:

$$\dot{w}_i = f(t, w_i) + k \sum_{j \in \mathcal{N}_i} (w_j - w_i) - k \sum_{j=1}^{N} w_j + k \sum_{j=1}^{N} x_j(t), \quad i = 1, \cdots, N \tag{2}$$

$$\dot{w}_0 = f(t, w_0) - kNw_0 + k \sum_{i=1}^{N} x_j(t) \tag{3}$$

System (2) can be made contractive with large k because its symmetric part of the Jacobian is

$$\frac{1}{2} \begin{bmatrix} (\nabla_w f + \nabla_w f^\top)(t, w_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\nabla_w f + \nabla_w f^\top)(t, w_N) \end{bmatrix} - k \underbrace{\left[(L \otimes I_n) + (\mathbf{1}_N \mathbf{1}_N^\top \otimes I_n) \right]}_{\text{positive definite}}$$

- $w^a := (x_1(t), \cdots, x_N(t))$ is a solution to (2) because (2) becomes (1) in this case.
- $w^b := (w_0(t), \cdots, w_0(t))$ is a solution to (2) because (2) becomes (3) in this case.

Since (2) is contractive, $\lim_{t \to \infty} \|w^a(t) - w^b(t)\| = 0$, which implies consensus.

Consensus for Heterogeneous Node Dynamics

$$\dot{x}_i = f_i(x_i) + u_i, \qquad y_i = h_i(x_i)$$

Consensus of the states is not possible unless $\exists (s(t), u_1(t), \cdots, u_N(t))$ such that

$$\dot{s}(t) = f_1(s(t)) + u_1(t) \; = \; f_2(s(t)) + u_2(t) \; = \; \cdots \; = \; f_N(s(t)) + u_N(t).$$

The differences in f_i should be compensated by u_i when $x_1 = \cdots = x_N = s$, which is difficult.

Output consensus: $\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0$

A part of the states are to be synchronized:

$$h_1(x_1(t)) = h_2(x_2(t)) = \cdots = h_N(x_N(t))$$

$$\dot{x}_1 = f_1(x_1), \;\; \dot{x}_2 = f_2(x_2), \;\; \cdots, \;\; \dot{x}_N = f_N(x_N)$$

We will look at

1. Passivity approach
2. Embedding a common internal model

Useful for handling heterogeneity caused by uncertainty

Approx. consensus: $\limsup_{t\to\infty} \|x_j(t) - x_i(t)\| \ll 1$

We will look at

1. Dynamic average consensus

   Identical node dynamics with different external inputs $\rightarrow$ get average of the inputs

2. Strong coupling

   Strong couplings suppress heterogeneity, and yield arbitrary small consensus error.

Useful not only for handling uncertainty, but also for getting a new behavior that arises in the conflict between consensus and heterogeneity

Nonlinear passive heterogeneous node dynamics

Output consensus using passivity (Chopra & Spong, Adv in Robot Contr, 2006)

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \quad y_i = h_i(x_i) \quad \text{with} \quad u_i = -\sum_{j\in\mathcal{N}_i} \psi_{ij}(y_i - y_j)$$

achieves output consensus ($\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0$, $\forall i, j$) if

- node dynamics is passive: $\exists$ positive definite, radially unbounded $V_i(x_i)$ s.t.

$$L_{f_i}V_i(x_i) = -S_i(x_i) \leq 0, \qquad L_{g_i}V_i(x_i) = h_i(x_i)^\top$$

- $L = L^\top$
- ψ_{ij} is symmetric ($\psi_{ij} = \psi_{ji}$), odd, and passive:

$$\mathsf{u}^\top \psi_{ij}(\mathsf{u}) > 0, \quad \forall \mathsf{u} \neq 0$$

- Node dynamics can be uncertain as long as it is passive.

Under the hood: With $V = \sum_{i=1}^{N} V_i$,

$$\dot{V} = -\sum_{i=1}^{N} \left(S_i(x_i) + y_i^\top \sum_{j \in \mathcal{N}_i} \psi_{ij}(y_i - y_j) \right) = -\sum_{i=1}^{N} S_i(x_i) - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} y_i^\top \psi_{ij}(y_i - y_j).$$

Since $L = L^\top$, if $y_i^\top \psi_{ij}(y_i - y_j)$ appears in the summation, $y_j^\top \psi_{ji}(y_j - y_i)$ also appears where

$$\psi_{ji}(y_j - y_i) = \psi_{ij}(y_j - y_i) = -\psi_{ij}(y_i - y_j).$$

Therefore,

$$\dot{V} = -\sum_{i=1}^{N} S_i(x_i) - \sum_{(i,j) \in \mathcal{E}_{\text{undirected}}} (y_i - y_j)^\top \psi_{ij}(y_i - y_j) \leq 0.$$

$\therefore$ $x_i(t)$ converges to the set

$$\{(x_1, \cdots, x_N) : S_1(x_1) = \cdots = S_N(x_N) = 0,\ h_1(x_1) = \cdots = h_N(x_N)\}$$

Embedding a common internal model in the coupling dynamics

$$\dot{x}_i = A_i x_i + B_i u_i, \qquad y_i = C_i x_i$$

achieves output synchronization ($\lim_{t\to\infty} \|y_i(t) - z^*(t)\| = 0, \forall i$) where

$$\dot{w} = Sw, \qquad z^* = Rw$$

by the dynamic coupling of

(Wieland, Sepulchre, & Allgöwer, AUT, 2011)

$$\dot{w}_i = Sw_i + \textstyle\sum_{j\in\mathcal{N}_i}(w_j - w_i)$$
$$\dot{\chi}_i = A_i\chi_i + B_i u_i + H_i(y_i - C_i\chi_i)$$
$$u_i = K_i(\chi_i - \Pi_i w_i) + \Gamma_i w_i$$

under the standard assumption of output regulator design on (A_i, B_i, C_i, S)

(Kim, S, & Seo, TAC, 2011)

$$\dot{w}_i = Sw_i + Q\omega_i$$
$$\dot{\omega}_i = M\omega_i + N \textstyle\sum_{j\in\mathcal{N}_i}(y_j - y_i)$$
$$\dot{\eta}_i = \Phi\eta_i + \Psi(y_i - Rw_i)$$
$$u_i = \phi\eta_i$$

if (A_i, B_i, C_i) is uncertain but has known relative degree and is mininmum phase

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \quad y_i = h_i(x_i)$$

achieves output synchronization with

$$\dot{w} = s(w), \qquad z^* = Rw$$

by the dynamic coupling of

(Isidori, Marconi, & Casadei, TAC, 2014)

$$\dot{w}_i = s(w_i) + K\sum_{j\in\mathcal{N}_i}(Rw_j - Rw_i)$$
$$\dot{\eta}_i = \psi_i(\eta_i, y_i - Rw_i)$$
$$u_i = \phi_i(\eta_i, y_i - Rw_i)$$

if (f_i, g_i, h_i) is uncertain but has known relative degree and is (weakly) minimum phase

Internal model principle and (robust) output regulation theory are utilized in the design.

Approximate consensus ($\limsup_{t\to\infty} \|x_j(t) - x_i(t)\| \ll 1$) **for heterogeneous node dynamics**

1. Dynamic average consensus

   - While consensus is an interesting phenomenon on its own, it would be even more useful to go beyond consensus, such as computing a common estimate of an unknown quantity or fusing information to make a decision. Dynamic average consensus (DAC) is one way to extend the study of consensus in this direction.
   - While average consensus can compute the (time-invariant) average of the initial conditions of each agent, the initial motivation for DAC is to compute the average of 'time-varying' attributes $a_i(t)$ for each agent i in a distributed manner.
   - See, e.g., (Kia, van Scoy, Cortes, Freeman, Lynch, & Martinez, IEEE CSM, 2019).

2. Strong coupling

Dynamic Average Consensus (Spanos, Olfati-Saber, & Murray, IFAC-WC, 2005)

$$\dot{x}_i = \dot{a}_i + u_i \quad \text{with} \quad x_i(0) = a_i(0) \quad \text{and} \quad u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

achieves approximate consensus as

$$\limsup_{t\to\infty} \|x_i(t) - \theta^\top a(t)\| \quad \propto \quad \limsup_{t\to\infty} \left\| \left(I - \frac{1}{N} 1_N 1_N^\top \right) \dot{a}(t) \right\|.$$

- $\mathcal{N}\left(I - \frac{1}{N}1_N 1_N^\top\right) = \operatorname{span}\{\mathbf{1}_N\} = \mathcal{N}(R^\top)$ (synchronization subspace)
- **Under the hood:** Since BAP holds, ox-representation:

$$\dot{z}_{\mathsf{o}} = \theta^\top \dot{a}, \qquad z_{\mathsf{o}}(0) = \theta^\top a(0) \qquad \therefore\ z_{\mathsf{o}}(t) = \theta^\top a(t)$$

$$\dot{z}_{\mathsf{x}} = -M z_{\mathsf{x}} + R^\top \dot{a} \qquad \therefore\ \text{ISS (input-to-state stable) from } R^\top \dot{a} \text{ to } z_{\mathsf{x}}$$

$$\therefore \quad \limsup_{t\to\infty} \|z_{\mathsf{x}}(t)\| \quad \propto \quad \limsup_{t\to\infty} \|R^\top \dot{a}(t)\| \quad \propto \quad \limsup_{t\to\infty} \left\| \left(I - \frac{1}{N} 1_N 1_N^\top \right) \dot{a}(t) \right\|$$

- Conservation Property holds ($\sum_{i=1}^N \theta_i x_i(t) = \sum_{i=1}^N \theta_i a_i(t)$), and so, initialization is needed.

Initialization-free Dynamic Average Consensus (Freeman, Yang, & Lynch, CDC, 2006)

$$\dot{x}_i = -\mu(x_i - a_i) + \dot{a}_i + u_i \quad \text{with} \quad u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

where $\mu > 0$, achieves approximate consensus as

$$\limsup_{t\to\infty} \|x_i(t) - \theta^\top a(t)\| \quad \propto \quad \limsup_{t\to\infty} \left\| \left(I - \frac{1}{N} 1_N 1_N^\top \right) (\mu a(t) + \dot{a}(t)) \right\|.$$

- **Under the hood:** Since BAP holds, ox-representation:

$$\begin{aligned} \dot{z}_\mathsf{o} &= -\mu(z_\mathsf{o} - \theta^\top a) + \theta^\top \dot{a} \qquad \text{i.e., } (\dot{z}_\mathsf{o} - \theta^\top \dot{a}) = -\mu(z_\mathsf{o} - \theta^\top a) \\ \dot{z}_\mathsf{x} &= -\mu(z_\mathsf{x} - Q^\top a) + Q^\top \dot{a} - M z_\mathsf{x} = -(\mu I + M) z_\mathsf{x} + Q^\top (\mu a + \dot{a}) \end{aligned}$$

- Having $\mu > 0$ introduces stability and enables forgetting initial conditions (so that initialization-free), but introduces a steady-state error even for constant input $a(t)$.
- With PI (proportional-integral) coupling, the synchronization error can be made zero when a_i's are all constants.

Strong coupling (Kim, Yang, S, Kim, & Seo, TAC, 2016) (Panteley & Loria, TAC, 2017)

$$\dot{x}_i = f_i(x_i, a_i) + u_i \quad \text{with} \quad u_i = k \textstyle\sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

where f_i is globally Lipschitz (if not, semi-global result), achieves

$$\limsup_{t \to \infty} \|x_i(t) - s(t)\| = O(1/k), \qquad \forall i, \quad \forall k > k^* \text{ (threshold)}$$

if

$$\dot{s} = \sum_{i=1}^{N} \theta_i f_i(s, a_i) \quad \text{("blended dynamics")} \quad \text{is contractive.}$$

Under the hood: Since BAP holds, ox-representation:

$$\dot{z}_{\mathsf{o}} = \sum_{i=1}^{N} \theta_i f_i(z_{\mathsf{o}} + (R_i \otimes I_n) z_{\mathsf{x}}, a_i)$$

$$\left(\frac{1}{k}\right) \dot{z}_{\mathsf{x}} = -(M \otimes I_n) z_{\mathsf{x}} + \left(\frac{1}{k}\right) (Q^\top \otimes I_n) \begin{bmatrix} f_1(z_{\mathsf{o}} + (R_1 \otimes I_n) z_{\mathsf{x}}, a_1) \\ \vdots \\ f_N(z_{\mathsf{o}} + (R_N \otimes I_n) z_{\mathsf{x}}, a_N) \end{bmatrix}$$

There are many consensus-enforcing coupling laws satisfying BAP

type of coupling	stability for blended dynamics	coupling law	method for strong coupling	consensus error
Linear coupling (Kim, Yang, S, Kim, & Seo, TAC, 2016) (Panteley & Loria, TAC, 2017) (Lee & S, AUT, 2020)	contractive, $\exists$ asymp. stable X^*	$u_i = k \sum_{j \in \mathcal{N}_i} (x_j - x_i)$	high gain k	$O(\frac{1}{k})$
PI coupling (Lee & S, AUT, 2022)	$\exists$ exp. stable X^*	$\dot{v}_i = -k \sum_{j \in \mathcal{N}_i} (x_j - x_i)$ $u_i = k \sum_{j \in \mathcal{N}_i} (x_j - x_i) + k \sum_{j \in \mathcal{N}_i} (v_j - v_i)$	high gain k **(with $L = L^\top$)**	0
Signum coupling (Franceschelli, Giua, & Pisano, TAC, 2017)	contractive, $\exists$ asymp. stable X^*	$u_i = k \sum_{j \in \mathcal{N}_i} \mathrm{sign}(x_j - x_i)$	high gain k **(with $L = L^\top$)**	0
Funnel coupling (Lee, Berger, Trenn, & S, AUT, 2023)	contractive	$u_i = \sum_{j \in \mathcal{N}_i} \psi\left(\frac{x_j - x_i}{\phi(t)}\right)$	$\lim_{s \to \pm 1} \psi(s) = \pm\infty$ **(with $L = L^\top$)**	$\limsup_{t \to \infty} \phi(t)$
Impulsive gossiping (Tanwani, S, & Teel, MTNS, 2024)	contractive	$u_i = \sum_{j \in \mathcal{N}_i, t_{ij}^* \in \mathcal{T}_{ij}} \underbrace{\delta(t - t_{ij}^*)}_{\textbf{Dirac's delta}} \cdot \frac{(x_j - x_i)}{2}$	frequent jumps	$\propto \frac{1}{\textbf{update freq}}$

Coordination of heterogeneous agents

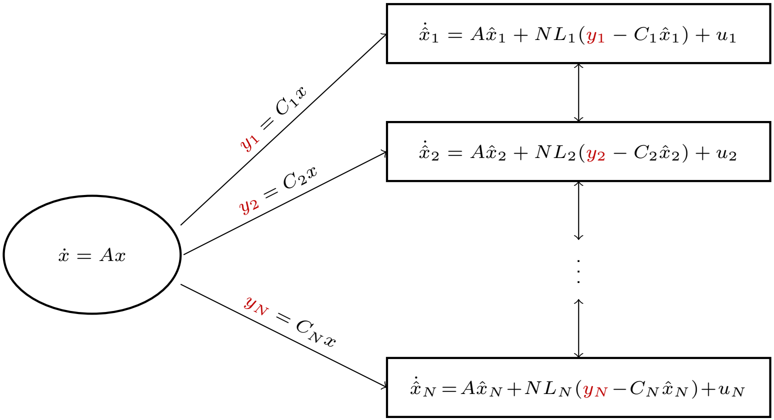
⋆ Some useful behaviors arise in the conflict between consensus and heterogeneity. ⋆

- By strong coupling, consensus is enforced against heterogeneity $\rightarrow \limsup_{t\to\infty} \|z_{\mathsf{x}}(t)\| \approx 0$.
- Then,

$$\dot{z}_{\mathsf{o}} = \sum_{i=1}^{N} \theta_i f_i(z_{\mathsf{o}} + (R_i \otimes I_n) z_{\mathsf{x}}, a_i) \quad \xrightarrow{\limsup_{t\to\infty} \|z_{\mathsf{x}}(t)\| \approx 0} \quad \dot{z}_{\mathsf{o}} \approx \sum_{i=1}^{N} \theta_i f_i(z_{\mathsf{o}}, a_i)$$

- A certain stability of $\dot{s} = \sum_{i=1}^{N} \theta_i f_i(s, a_i)$ guarantees $z_{\mathsf{o}}(t) \approx s(t)$, and also prevents the size of heterogeneity that may depend on z_{o} from growing unbounded.
- Therefore, $x_i(t) \approx s(t)$ for all future time
- The behavior $s(t)$ of the blended dynamics is an emergent one, and can be useful.
- Example (distributed optimization):
  - $\dot{x}_i = -\nabla f_i(x_i) + u_i^{\mathsf{PI}}$: heterogeneous multi-agent system with PI coupling
  - $x_i(t) \rightarrow s(t)$ where $\dot{s} = -\frac{1}{N} \sum_{i=1}^{N} \nabla f_i(s)$
  - It is the gradient descent algorithm that solves $\min_x F(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x)$.
  - Stability of the blended dynamics asks F is convex (but not for individual f_i).

Application to Distributed Observer (Kim, S, & Cho, CDC, 2016)



heterogeneous agent i:

$$\dot{\hat{x}}_i = A\hat{x}_i + NL_i(y_i - C_i\hat{x}_i) + u_i$$

blended dynamics:

$$\begin{aligned} \dot{s} &= As + \sum_{i=1}^{N}(L_i y_i - L_i C_i s) \\ &= As + L(y - Cs) \end{aligned}$$

therefore, $s(t) \to x(t)$, and so, $\hat{x}_i(t) \to x(t)$

- Even if every (A, C_i) is undetectable, so that none of the agent i is stable, the group behavior can be stable as long as $\left(A, \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}\right)$ is detectable; stability emerges.

Application to Coupled oscillators (Lee & S, CDC, 2018)

Van der Pol oscillator:

$$\begin{aligned}\dot{x}_i^{(1)} &= -x_i^{(1)} + x_i^{(2)} \\ \dot{x}_i^{(2)} &= (1 - \mu_i(x_i^{(1)} \cdot x_i^{(1)} - 1))(-x_i^{(1)} + x_i^{(2)}) - \nu_i x_i^{(1)} + u_i\end{aligned}$$

has a stable limit cycle (when $u_i \equiv 0$) if and only if both $\mu_i > 0$ and $\nu_i > 0$. With the coupling

$$u_i = k \sum_{j \in \mathcal{N}_i} (x_j^{(2)} - x_i^{(2)}), \qquad k \gg 1$$

the group of Van der Pol oscillators has a stable limit cycle if and only if

$$\textstyle\sum_{i=1}^N \theta_i \mu_i > 0 \quad \text{and} \quad \sum_{i=1}^N \theta_i \nu_i > 0$$

because their blended dynamics is

$$\begin{aligned}\dot{s}^{(1)} &= -s^{(1)} + s^{(2)} \\ \dot{s}^{(2)} &= \left(1 - \left(\textstyle\sum_{i=1}^N \theta_i \mu_i\right)(s^{(1)} \cdot s^{(1)} - 1)\right)(-s^{(1)} + s^{(2)}) - \left(\textstyle\sum_{i=1}^N \theta_i \nu_i\right) s^{(1)}\end{aligned}$$

Application to "f-consensus" $(x_i(t) \to f(a_1, \cdots, a_N), \forall i)$

Example: Distributed median solver (Franceschelli, Giua, & Pisano, TAC, 2017) (Lee, Kim, & S, TAC, 2020)

$$\dot{x}_i = \text{sign}(a_i - x_i) + \underbrace{u_i,}_{\substack{\text{consensus enforcing} \\ \text{(e.g. signum, PI)}}} \quad \Rightarrow \quad \lim_{t\to\infty} x_i(t) = \text{median}(a_1, \cdots, a_N)$$

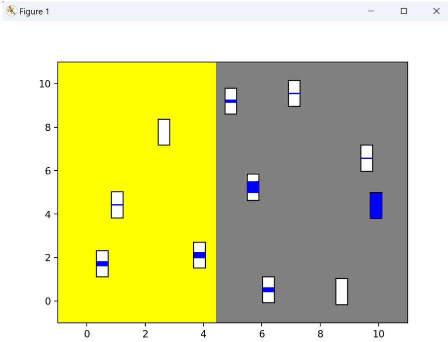
Under the hood: Blended dynamics under undirected graph is

$$\dot{s} = \frac{1}{N}\sum_{i=1}^{N} \text{sign}(a_i - s) \qquad \text{which is the gradient descent for } s^* = \arg\min_s \frac{1}{N}\sum_{i=1}^{N} |s - a_i|$$

and the median is known to be the minimizing value of l_1 norm.

- Available in the literature are other distributed algorithms for computing least square, maximum/minimum, mode, network size, ... (Cortés, Mou, Morse, Liu, Anderson, Chao, ...)

Example: Spontaneous order



Under the hood: Identical structure with different parameters μ_i and ν_i and input a_i:

$$\dot{x}_i^{(1)} = -x_i^{(1)} + x_i^{(2)}$$

$$\dot{x}_i^{(2)} = (1 - \mu_i(x_i^{(1)} \cdot x_i^{(1)} - 1))(-x_i^{(1)} + x_i^{(2)}) - \nu_i x_i^{(1)} + \frac{k_a}{\varphi(x_i^{(3)}, \mu_i)} \sum_{j \in \mathcal{N}_i} (x_j^{(2)} - x_i^{(2)})$$

$$\dot{x}_i^{(3)} = \operatorname{sign}(a_i - x_i^{(3)}) + k_b \sum_{j \in \mathcal{N}_i} (x_j^{(3)} - x_i^{(3)})$$

where $\varphi(a, b)$ is large if $a \approx b$, and small otherwise; $a_i = 1$ in sunlight and $a_i = 0$ in shade

- By introducing multiple layers, complex behavior can emerge even if each layer performing a simple task.

Summary

We have reviewed several contributions on consensus for identical and heterogeneous multi-agent systems, and we asserted that consensus is the fundamental cornerstone of spontaneous order. In particular, new emergent behavior may arise when (approximate) consensus is enforced against heterogeneity, which can be utilized as a building block for more complex behavior.

"Consensus may be a tool for harnessing heterogeneity towards emergent order."

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

and $x_1(0) \neq x_2(0) \neq \dots$

$$x_i(t) \to s(t) = \sum_{i=1}^{N} \theta_i x_i(0)$$

$$\dot{x}_i = f_i(x_i) + k \sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

and $f_1 \neq f_2 \neq \dots$

$$x_i(t) \xrightarrow{\text{approx}} s(t) \text{ where } \dot{s} = \sum_{i=1}^{N} \theta_i f_i(s)$$

Slides and simulation files are at `https://hyungbo.github.io/cssdays2024`.