Risk Modeling and Classification

EN5422/EV4238 | Fall 2023

w05_classification_1.pdf

(Week 5 – 1/2)

Contents

1 Risk Modeling Intro

1.1 Credit Card Default Data (Dafault)

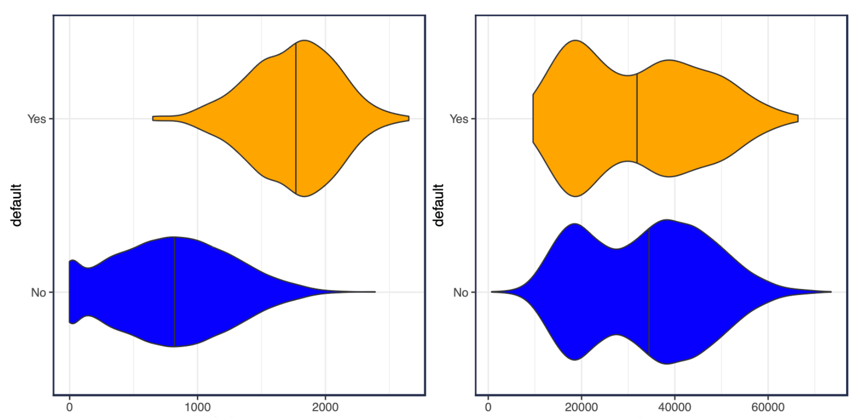
The textbook *An Introduction to Statistical Learning (ISL)* has a description of a simulated credit card default dataset. The interest is on predicting whether an individual will default on their credit card payment.

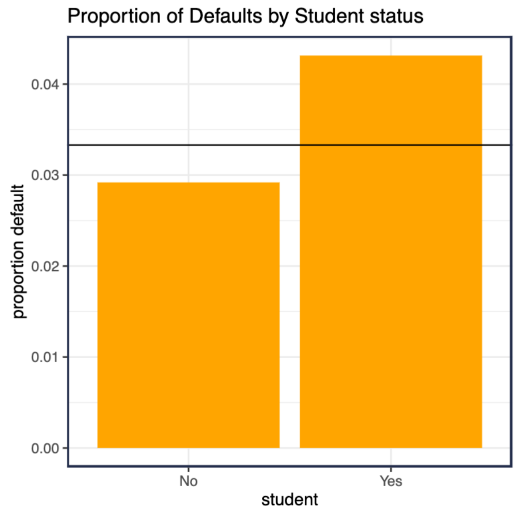
```
# The URL of the xlsx file
url = 'https://github.com/JWarmenhoven/ISLR-python/raw/4100e941914519eea18385daadc9b3dab99ca8e2/Notebooks/Data/Default.xlsx'
# Load the xlsx data
df = pd.read_excel(url, engine='openpyxl')
```

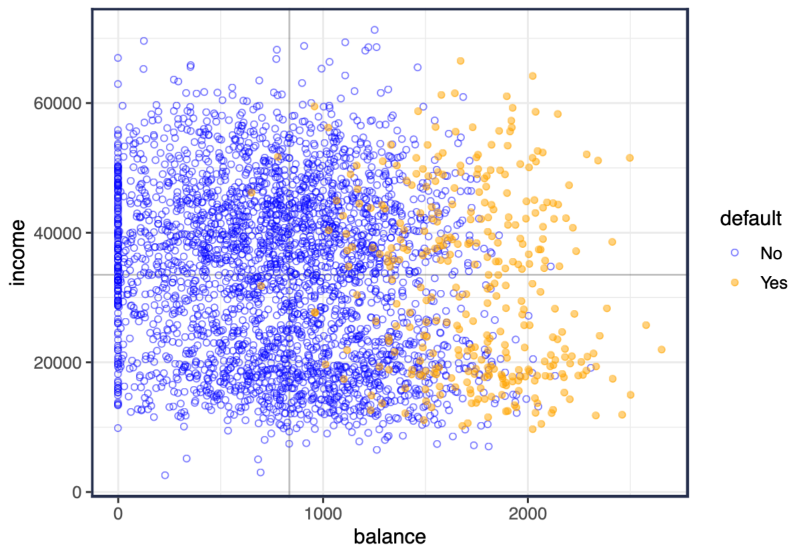
The variables are:

- *Outcome variable* is categorical (factor) Yes and No (default)
- The categorical (factor) variable (student) is either Yes or No
- The average balance a customer has after making their monthly payment (balance)
- The customer's income (income)

default	student	balance	income
No	No	729.526495	44361.625074
No	Yes	817.180407	12106.134700
No	No	1073.549164	31767.138947
No	No	529.250605	35704.493935
No	No	785.655883	38463.495879







Your Turn #1
How would you construct a model to predict the risk of default?

1.2 Set-up

- The outcome variable is categorical and denoted $G \in \mathcal{G}$.
    - Default Credit Card Example: $\mathcal{G} = \{\text{"Yes"}, \text{"No"}\}$
    - Medical Diagnosis Example: $\mathcal{G} = \{\text{"stroke"}, \text{"heart attack"}, \text{"drug overdose"}\}$
- The training data is $D = \{(X_1, G_1), (X_2, G_2), \dots, (X_n, G_n)\}$
- The optimal decision/classification is often based on the posterior probability $\Pr(G = g|\mathbf{X} = \mathrm{x})$

1.3 Binary Risk Modeling

- Classification is simplified when there are only 2 classes.
    - Many multi-class problems can be addressed by solving a set of binary classification problems (e.g., on-vs-rest).
- It is often convenient to transform the outcome variable to a binary {0, 1} variable:

$$Y_i = \begin{cases} 1 & G_i \in \mathcal{G}_1 \quad \text{(outcome of interest)} \\ 0 & G_i \in \mathcal{G}_2 \end{cases}$$

- In the **Default** data, it would be natural to set **default=Yes** to 1 and **default=No** to 0.

1.3.1 Linear Regression

- In this set-up we can run linear regression:

$$\hat{y}(\mathbf{x}) = \hat{\beta}_0 + \sum_{j=1}^{p} \hat{\beta}_j x_j$$

```python
import pandas as pd
import statsmodels.api as sm

Default = df
# Convert 'Yes'/'No' columns to 1/0, if applicable.
# For example, if 'student' is 'Yes'/'No'
Default['student'] = (Default['student'] == "Yes").astype(int)

# Create binary column (y)
Default['y'] = (Default['default'] == "Yes").astype(int)

# Now, for the regression:
X = Default[['student', 'balance', 'income']]
# Adding a constant to the model (intercept)
X = sm.add_constant(X)
y = Default['y']
```

```
# Fit the model
model = sm.OLS(y, X).fit()

# Print the summary of the regression
print(model.summary())
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0812      0.008     -9.685      0.000      -0.098      -0.065
student       -0.0103      0.006     -1.824      0.068      -0.021       0.001
balance        0.0001   3.55e-06     37.412      0.000       0.000       0.000
income      1.992e-07   1.92e-07      1.039      0.299   -1.77e-07    5.75e-07
==============================================================================
```

Your Turn #2

1. For the binary Y, what is linear regression estimating?
2. What is the *loss function* that linear regression is using?
3. How could you create a *hard classification* from the linear model?
4. Does it make sense to use linear regression for binary risk modeling and classification?

1.3.2 *k*-nearest neighbor (kNN)

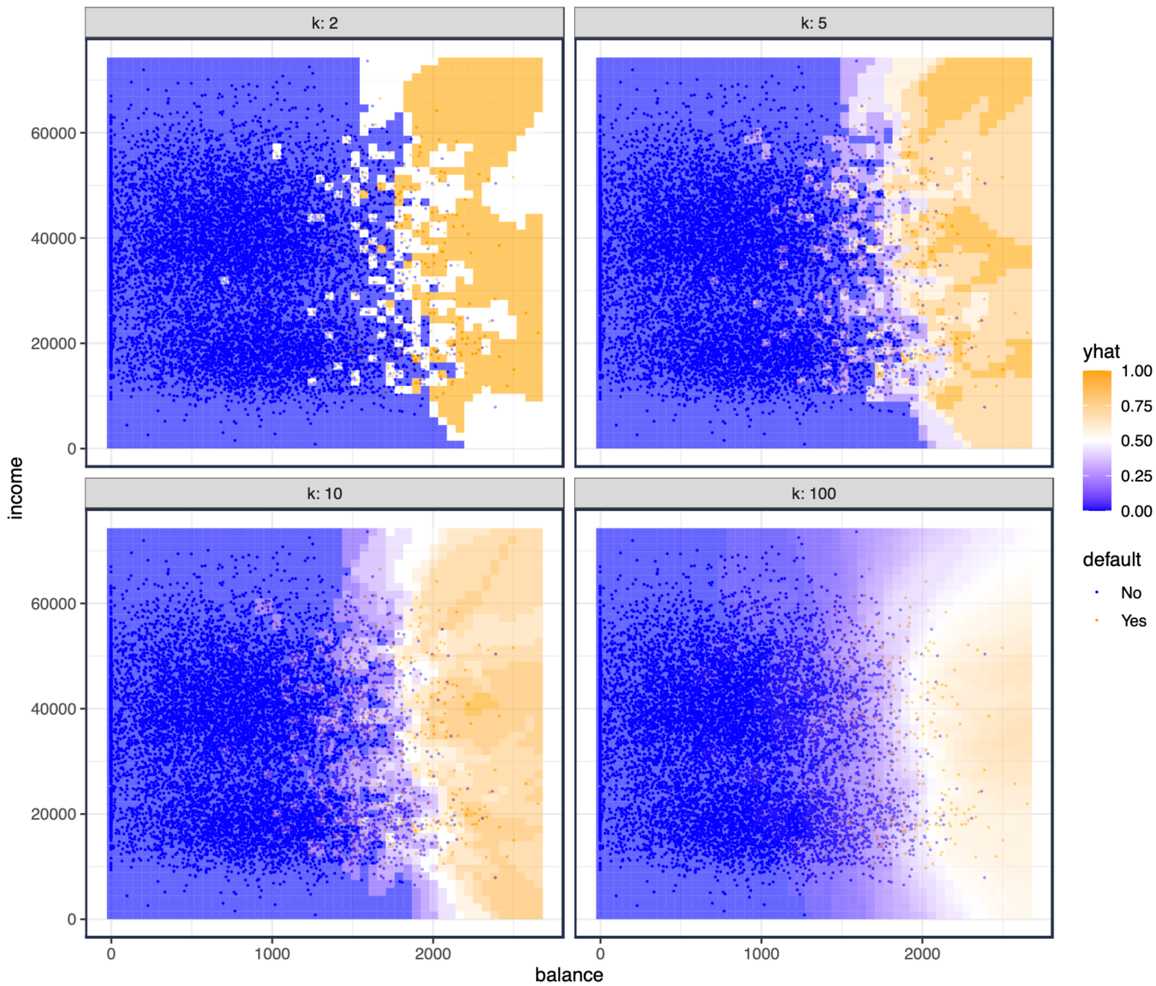
- The k-NN method is a non-parametric *local* method, meaning that to make a prediction $\hat{y}|x$, it only uses the training data in the *vicinity* of x.
    - Contrast with OLS linear regression, which uses all $\boldsymbol{X}$'s to get prediction.
- The model (for regression and binary classification) is simple to describe:

$$f_{kNN}(x;k) = \frac{1}{k}\sum_{i:x_i \in N_k(x)} y_i = \text{Avg}(y_i \mid x_i \in N_k(x))$$

- $N_k(x)$ are the set of k nearest neighbors.
- Only the k closet y's are used to generate a prediction

- When y is binary (i.e., $y \in \{0,1\}$), the kNN model estimates:

$$f_{kNN}(x;k) \approx p(x) = \Pr\,(Y = 1|\boldsymbol{X} = x)$$



Your Turn #3
The above plots show a kNN model using the *continuous* predictors of balance and income. • How could you use kNN with the categorical student predictor?

- The kNN model also has a more general description when the outcome variables is categorical $G \in \mathcal{G}$

$$f_g^{knn}(x;k) = \frac{1}{k} \sum_{i:x_i \in N_k(x)} 1(g_i = g) = \widehat{Pr}(G_i = g \mid x_i \in N_k(x))$$

- $N_k(x)$ are the set of k nearest neighbors.
- Only the k closest y's are used to generate a prediction.
- It is a *simple proportion* of the k nearest observations that are of class g.

2 Logistic Regression

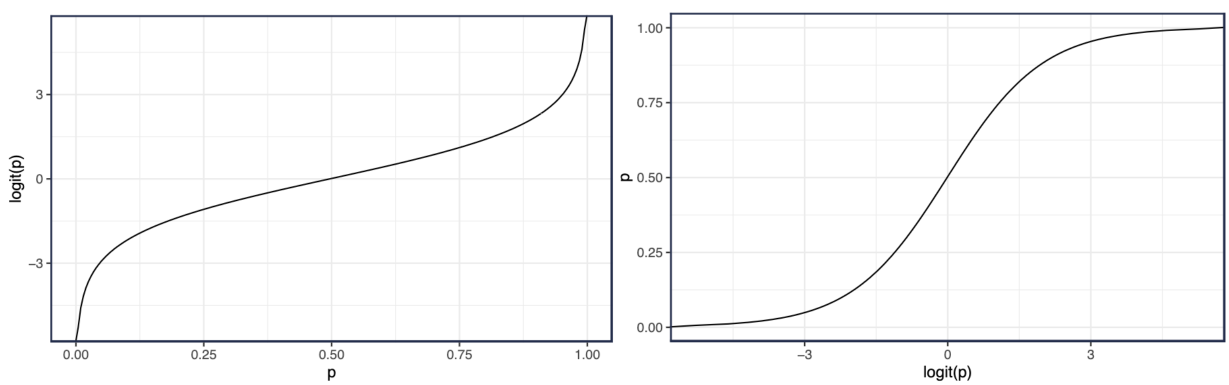
2.1 Basics

- Let $0 \le p \le 1$ be a probability.
- The log-odds of p is called the *logit*.

$$\text{logit}(p) = \log\left(\frac{p}{1-p}\right)$$

- The inverse logit is the *logistic function.* Let $f = \text{logit}(p)$, then

$$p = \frac{e^f}{1+e^f} = \frac{1}{1+e^{-f}}$$



- For binary outcome variables $Y \in \{0,1\}$, Linear Regression models

$$\text{E}[Y \mid \boldsymbol{X} = x] = \Pr(Y = 1 \mid \boldsymbol{X} = x) = \beta^{\text{T}} x$$

- Alternatively, Logistic Regression models

$$\text{logit}\Pr(Y = 1 \mid X = x) = \log\left(\frac{\Pr(Y = 1 \mid X = x)}{1 - \Pr(Y = 1 \mid X = x)}\right) = \beta^{\text{T}} x$$

And thus,

$$\Pr(Y = 1 \mid \boldsymbol{X} = x) = \hat{p}(x) = \frac{e^{\beta^{\mathrm{T}} x}}{1 + e^{\beta^{\mathrm{T}} x}} = \left(1 + e^{-\beta^{\mathrm{T}} x}\right)^{-1}$$

2.2 Estimation

- The input data for logistic regression are: $(\mathbf{x}_i, y_i)_{i=1}^{n}$ where $y_i \in \{0,1\}$, $\mathbf{x}_i = \left(x_{i0}, x_{i1}, \dots, x_{ip}\right)^{\mathrm{T}}$.
- $y_i \mid \mathbf{x}_i \sim \mathrm{Bern}\left(p_i(\beta)\right)$
    - $p_i(\beta) = \Pr(Y = 1 \mid \mathbf{X} = \mathbf{x}_i;\ \beta) = \left(1 + e^{-\beta^{\mathrm{T}} x}\right)^{-1}$
    - Where $\beta^{\mathrm{T}} \mathbf{x}_i = \mathbf{x}_i^{\mathrm{T}} \beta = \beta_0 + \sum_{j=1}^{p} x_{ij} \beta_j$
- Bernoulli Likelihood Function is:

$$L(\beta) = \prod_{i=1}^{n} p_i(\beta)^{y_i} \left(1 - p_i(\beta)\right)^{1 - y_i}$$

$$\begin{aligned}
\log L(\beta) &= \sum_{i=1}^{n} \left\{ y_i \ln p_i(\beta) + (1 - y_i) \ln\left(1 - p_i(\beta)\right) \right\} \\
&= \sum_{i=1}^{n} \begin{cases} \ln p_i(\beta) & y_i = 1 \\ \ln\left(1 - p_i(\beta)\right) & y_i = 0 \end{cases} \\
&= \sum_{i:y_i=1} \ln p_i(\beta) + \sum_{i:y_i=0} \ln(1 - p_i(\beta))
\end{aligned}$$

- The usual approach to estimating the Logistic Regression coefficients is *maximum likelihood*:

$$\hat{\beta} = \arg\max_{\beta} L(\beta) = \arg\max_{\beta} \log L(\beta)$$

- We can also view this as the coefficients that minimize the *loss function* $\ell(\beta)$, where the loss function is the negative log-likelihood:

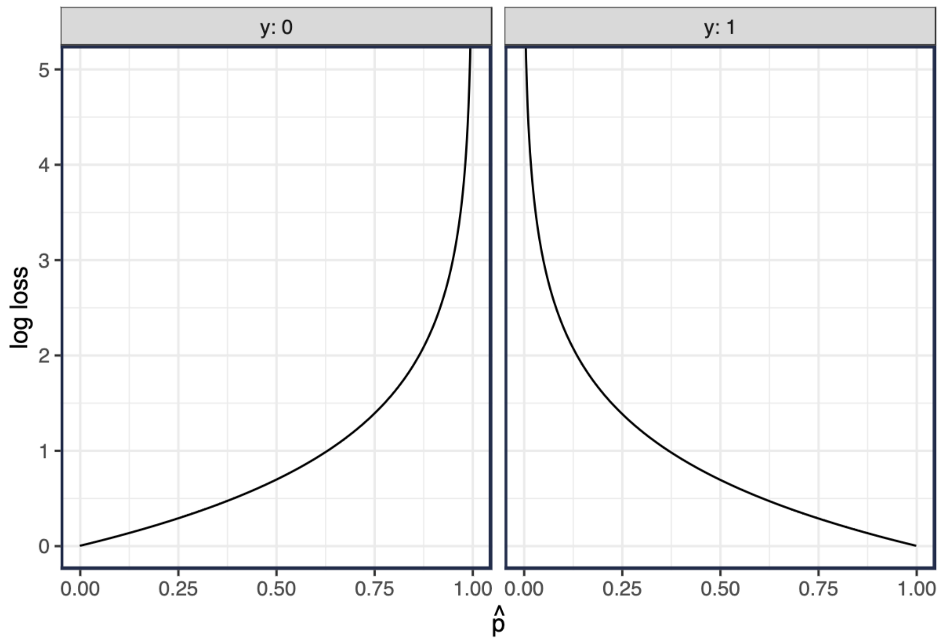
$$\hat{\beta} = \arg\min_{\beta} \ell(\beta)$$

  Using loss $\ell(\beta) = -C \log L(\beta)$ where $C > 0$ is some positive constant, e.g., $C = 1/n$

- This view facilitates *penalized logistic regression*:

$$\hat{\beta} = \arg\min_{\beta} \ell(\beta) + \lambda P(\beta)$$

Ridge Penalty	$P(\beta) = \|\beta\|_2^2 = \sum_{j=1}^{p} \left\|\beta_j\right\|^2 = \beta^{\mathrm{T}}\beta$
Lasso Penalty	$P(\beta) = \|\beta\|_1 = \sum_{j=1}^{p} \left\|\beta_j\right\|$
Best Subsets	$P(\beta) = \|\beta\|_0 = \sum_{j=1}^{p} \left\|\beta_j\right\|^0 = \sum_{j=1}^{p} 1_{\beta_j \neq 0}$
Elastic Net	$P(\beta, \alpha) = \frac{(1-\alpha)\|\beta\|_2^2}{2} + \alpha\|\beta\|_1 = \sum_{j=1}^{p} \left(\frac{(1-\alpha)\left\|\beta_j\right\|^2}{2} + \alpha\|\beta\|_1\right)$



2.3 Logistic Regression in Action

- In **Python**, logistic regression can be implemented with smf.logit() function.

```
import statsmodels.api as sm
import statsmodels.formula.api as smf

# Fit logistic regression model
model_formula = 'y ~ student + balance + income'
fit_lr = smf.logit(formula=model_formula, data=Default).fit()

# Print the summary
print(fit_lr.summary())
```

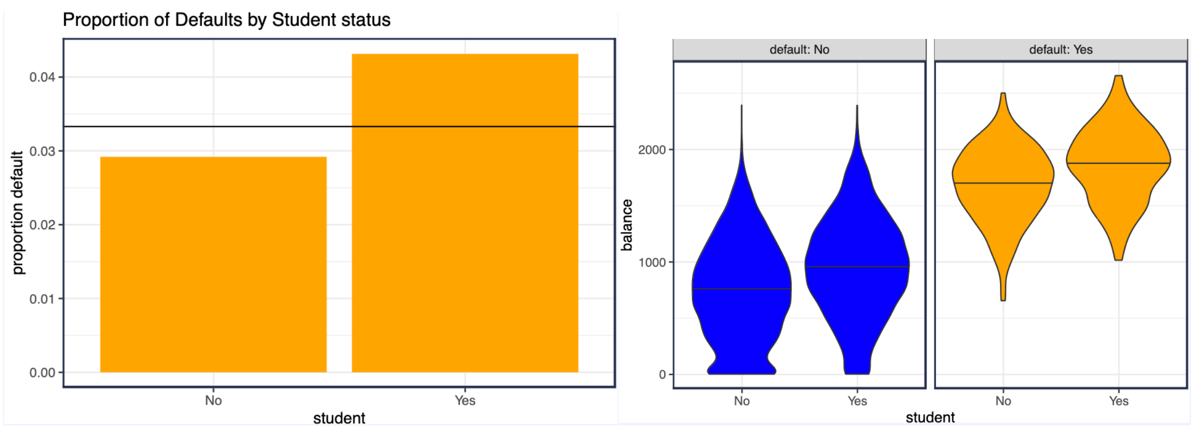
```
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept    -10.8690      0.492    -22.079      0.000     -11.834      -9.904
student       -0.6468      0.236     -2.738      0.006      -1.110      -0.184
balance        0.0057      0.000     24.737      0.000       0.005       0.006
income      3.033e-06    8.2e-06      0.370      0.712    -1.3e-05    1.91e-05
==============================================================================
```

Your Turn #4: Interpreting Logistic Regression

1. What is the estimated probability of `default` for a `Student` with a balance of $1000?

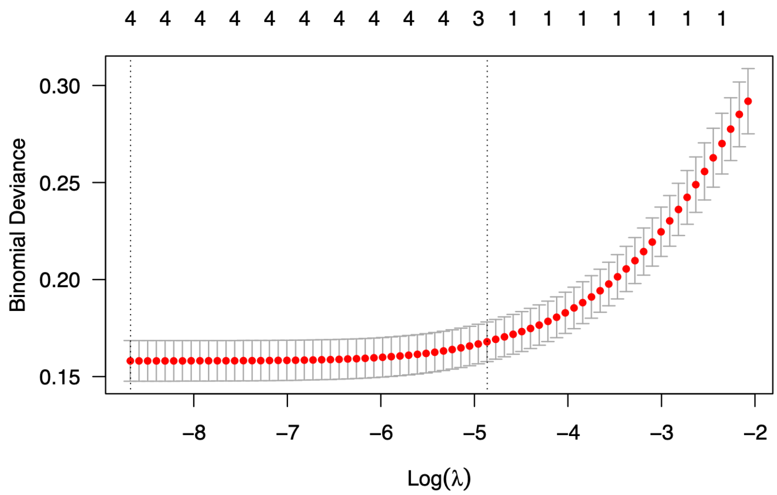
2. What is the estimated probability of `default` for a `Non-Student` with a balance of $1000?

3. Why does `student=Yes` appear to lower risk of `default`, when the plot of student status vs. default appears to increase risk?



2.3.1 Penalized Logistic Regression

- The `smf` lib can estimate logistic regression using an elastic net penalty (e.g., ridge, lasso).



term	unpenalized	lambda.min	lambda.1se
(Intercept)	-10.869	-11.056	-7.937
studentYes	-0.647	-0.299	-0.041
balance	0.006	0.006	0.004
income	0.000	0.000	0.000
studentNo	NA	0.325	0.044

2.3.2 Logistic Regression Summary

- Logistic Regression (both penalized and unpenalized) estimates a *posterior probability*, $\hat{p}(x) = \widehat{\Pr}(Y = 1 \mid X = x)$.
- This estimate is a function of estimated coefficients.

$$\hat{p}(x) = \frac{e^{\beta^{\mathrm{T}}x}}{1 + e^{\beta^{\mathrm{T}}x}} = \left(1 + e^{-\beta^{\mathrm{T}}x}\right)^{-1}$$

Your Turn #5

1. Given a person's student status, balance, and income, how could you use Logistic Regression to decide if they will default? (i.e., make a hard classification