

# Program Reasoning

## 7. Hoare Logic

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# The Story So Far

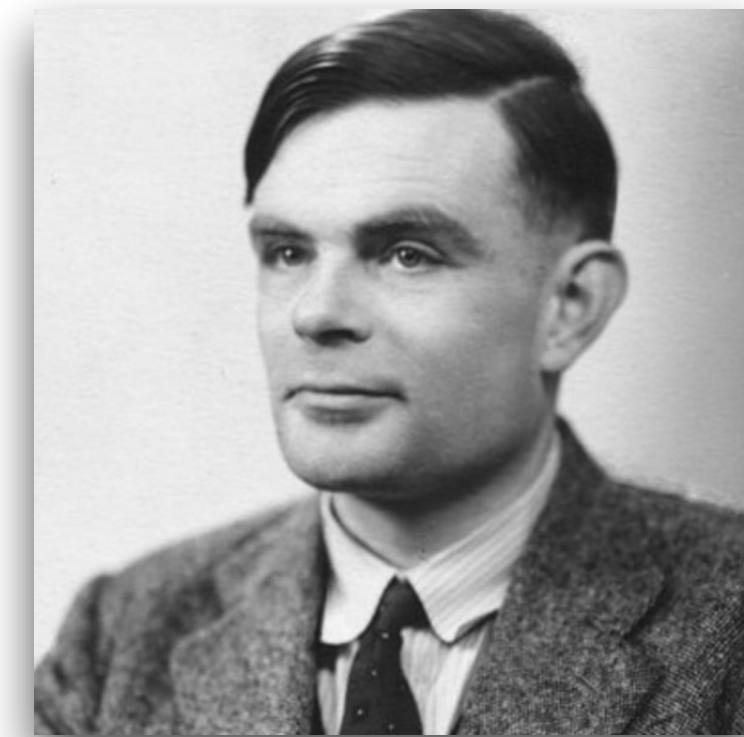
- Mechanization of logic and mathematics



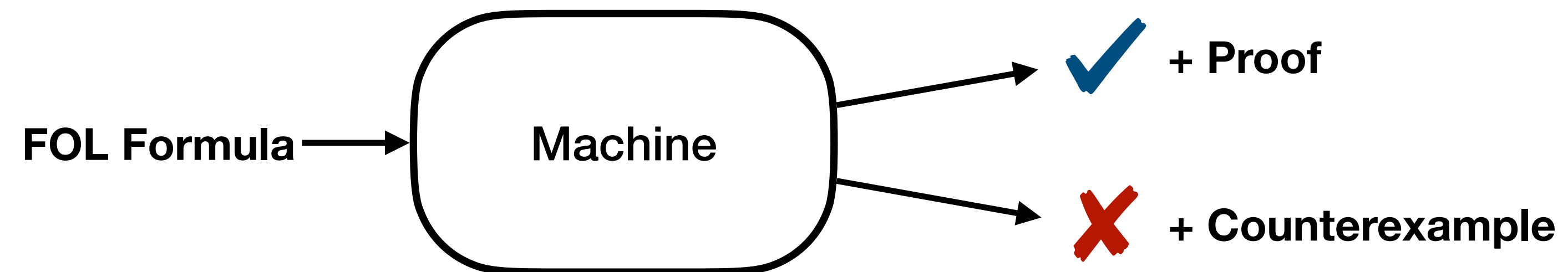
**“Is the following machine possible?”**  
- D. Hilbert, 1930



**“NO!”**  
-K. Gödel, 1931

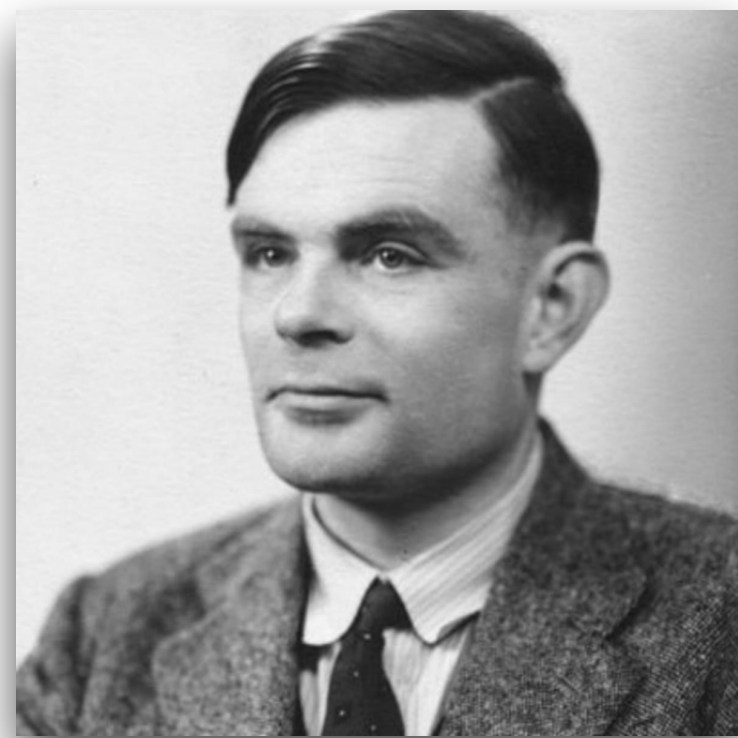


**“NO!”**  
- A. Turing, 1936

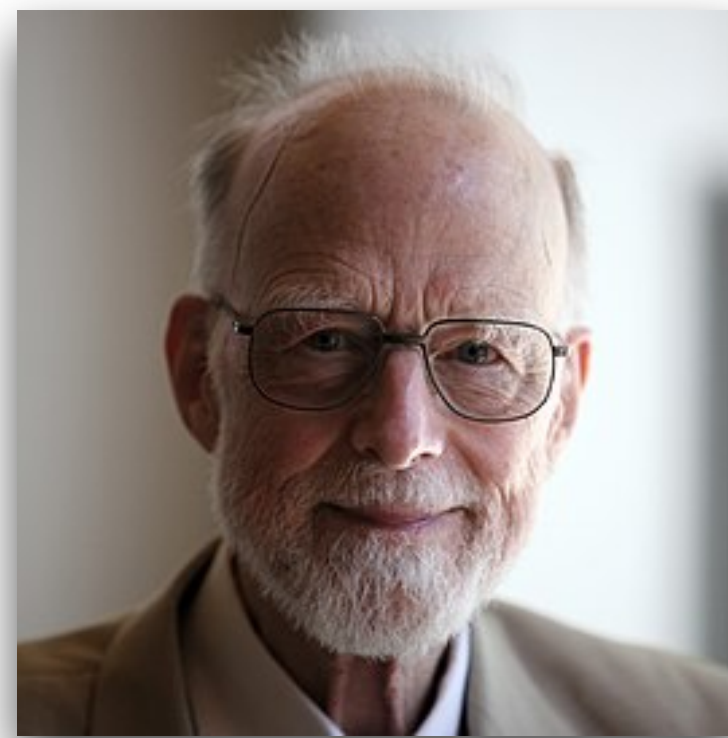


# Never Ending Story

- Reasoning about programs



**“How to check a program is correct?”**  
- A. Turing, 1949



**“Hoare Logic”**  
- T. Hoare, 1969

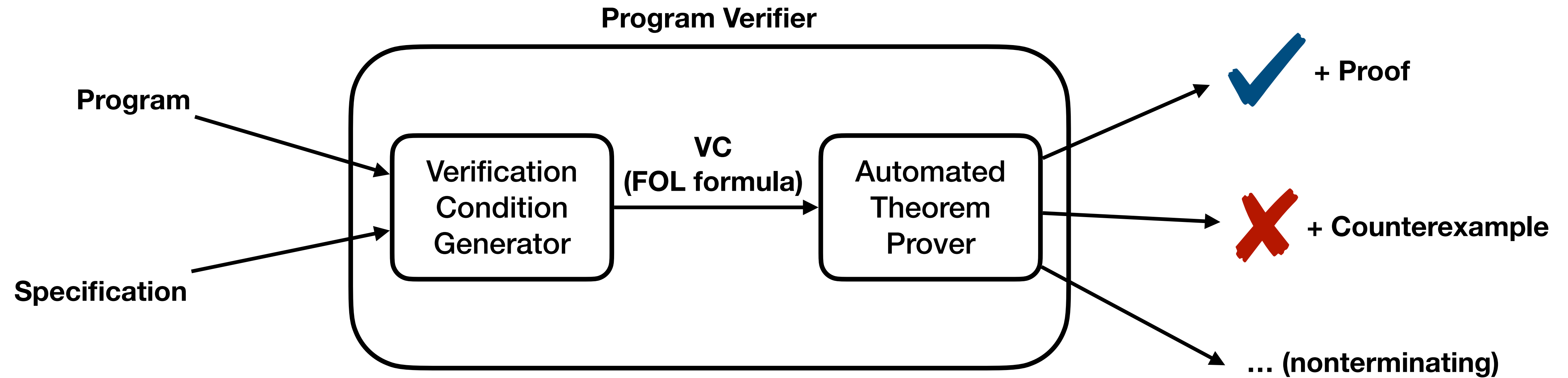


**“Weakest Precondition Calculus”**  
- E. Dijkstra, 1975

# Program Verification

- Specifying and proving properties of programs
- Specification: precise statement of properties that a program should exhibit in FOL
- Partial correctness properties: certain states cannot ever occur during the execution
  - “Bad things never happen” (e.g., integer overflow, buffer overflow, deadlock, etc)
  - Proof by inductive assertion method
- Total correctness properties: certain states are eventually reached during the execution
  - “Good things will eventually happen” (e.g., termination, fairness)
  - Proof by ranking function method

# Overview





# Specification

- Typically embedded into program text as program annotations
- An annotation is a FOL formula  $F$
- An annotation  $F$  at location  $L$  asserts that  $F$  is true whenever program control reaches  $L$
- Three types of annotations:
  - Function specification
  - Loop invariant
  - Assertion

# Function Specification

- A pair of annotations: precondition and postcondition
- Precondition (전제조건): a formula whose free variables include only the formal parameters
  - “What should be true upon entering the function?”
- Postcondition (사후조건): a formula whose free variables include only the formal parameters and the return value
  - “What is the relationship between the input and output?”

# Example: Linear Search

- What would be the pre and post conditions?

**@pre:**  $0 \leq l \wedge u < |a|$

**@post:**  $rv \leftrightarrow \exists i. l \leq i \leq u \wedge a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {  
    for (int i := l; i <= u; i := i + 1) {  
        if (a[i] = e) return true;  
    }  
    return false;  
}
```

- BTW, is this nontrivial precondition (a formula other than  $\top$ ) is always acceptable?
  - In terms of the software engineering practice (e.g., public API)



# Example: More Robust Linear Search

- What would be the pre and post conditions?

@pre:  $\top$

@post:  $rv \leftrightarrow \exists i. 0 \leq l \leq i \leq u < |a| \wedge a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {  
    if (l < 0 || u >= |a|) return false;  
    for (int i := l; i <= u; i := i + 1) {  
        if (a[i] = e) return true;  
    }  
    return false;  
}
```

# Example: Binary Search

- What would be the pre and post conditions?

**@pre:**  $0 \leq l \wedge u < |a| \wedge \text{sorted}(a, l, u)$

**@post:**  $rv \leftrightarrow \exists i. l \leq i \leq u \wedge a[i] = e$

```
bool BinarySearch(int[] a, int l, int u, int e) {  
    if (l > u) return false;  
    int m := (l + u) / 2;  
    if (a[m] = e) return true;  
    else if (a[m] < e) return BinarySearch(a, m + 1, u, e)  
    else return BinarySearch(a, l, m - 1, e)  
}
```

- The sorted predicate is defined in the combined theory of integers and arrays:

$$\text{sorted}(a, l, u) \iff \forall i, j. l \leq i \leq j \leq u \rightarrow a[i] \leq a[j]$$

# Example: Bubble Sort

- What would be the pre and post conditions?

@pre:  $T$

@post:  $\text{sorted}(rv, 0, |rv| - 1)$

```
bool BubbleSort(int[] a0) {  
    int a[] := a0;  
    for (int i := |a| - 1; i > 0; i := i - 1) {  
        for (int j := 0; j < i; j := j + 1) {  
            if (a[j] > a[j + 1]) {  
                int t := a[j];  
                a[j] := a[j + 1];  
                a[j + 1] := t;  
            }  
        }  
    }  
    return a;  
}
```



# Loop Invariants

- Each loop has an annotation called loop invariant

```
while
  @ $F$ 
  ( $\langle condition \rangle$ ) {
     $\langle body \rangle$ 
  }
```

- The assertion  $F$  must hold at the beginning of every iteration
  - $F \wedge \langle condition \rangle$  holds on entering the body
  - $F \wedge \neg \langle condition \rangle$  holds when exiting the loop
- Why are loop invariants needed?

# Example: Linear Search

- What would be the loop invariant?

**@pre:**  $0 \leq l \wedge u < |a|$

**@post:**  $rv \leftrightarrow \exists i. l \leq i \leq u \wedge a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {  
    int i := l;  
    while  
        @L :  $l \leq i \wedge (\forall j. l \leq j < i \rightarrow a[j] \neq e)$   
        (i <= u) {  
        if (a[i] = e) return true;  
        i := i + 1;  
    }  
    return false;  
}
```

# Assertions

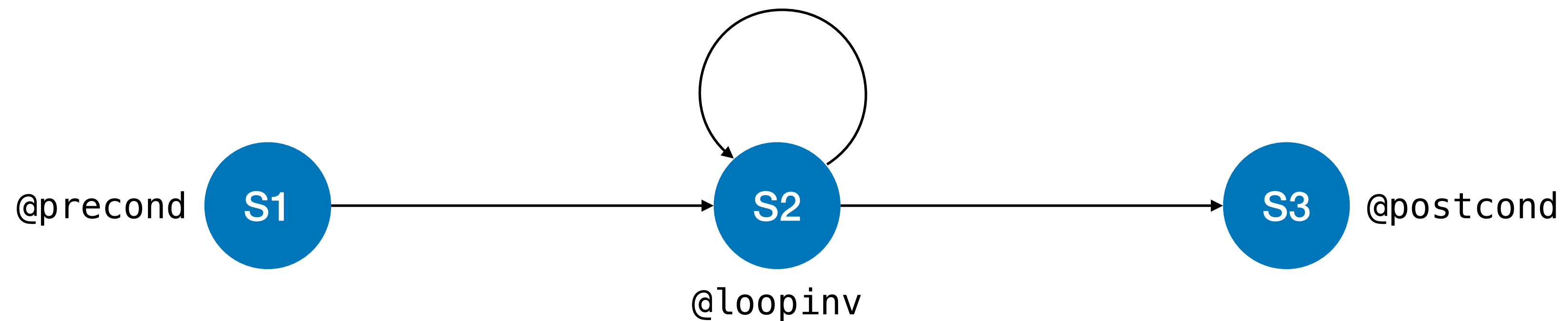
- Allows programmers to provide a formal comment
- Runtime assertions: a special class of assertions
  - E.g., division by 0, null dereference, etc
- Example: linear search with runtime assertions

```
bool LinearSearch(int[] a, int l, int u, int e) {  
    int i := l;  
    while (i <= u) {  
        @ 0 ≤ i < |a|  
        if (a[i] = e) return true;  
        i := i + 1;  
    }  
    return false;  
}
```



# Inductive Assertion Method

- Proof technique for partial correctness of programs
- Idea: derive verification conditions from a function given annotations
- Example:

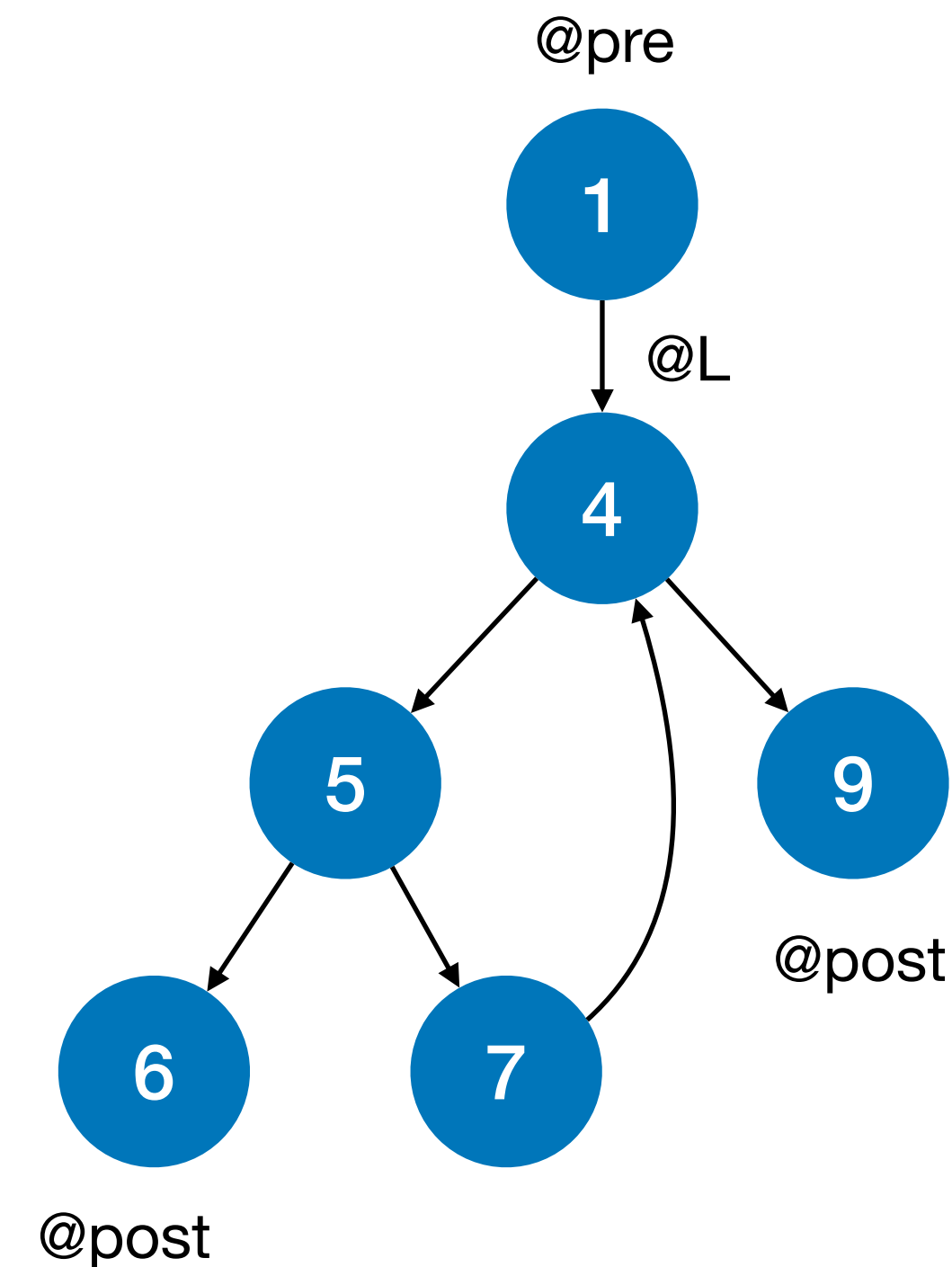


# Hoare Triple

- Partial correctness specified using Hoare triple:  $\{P\} S \{Q\}$ 
  - $S$  : program fragment
  - $P$  : precondition
  - $Q$  : postcondition
- Meaning of Hoare triple:
  - If  $S$  is executed in a state satisfying  $P$  and if the execution of  $S$  terminates
  - Then, the program state after  $S$  terminates satisfies  $Q$

# Example

```
@pre:  $0 \leq l \wedge u < |a|$ 
@post:  $rv \leftrightarrow \exists i. l \leq i \leq u \wedge a[i] = e$ 
bool LinearSearch(int[] a, int l, int u, int e) {
1:   int i := l;
2:   while
3:     @L:  $l \leq i \wedge (\forall j. l \leq j < i \rightarrow a[j] \neq e)$ 
4:     (i <= u) {
5:       if (a[i] = e)
6:         return true;
7:       i := i + 1
8:     }
9:   return false;
}
```



- What conditions should be proven?

$\{\text{@pre}\} S1 \{\text{@L}\} \quad \{\text{@L}\} S4; S5; S6 \{\text{@post}\} \quad \{\text{@L}\} S4; S5; S7 \{\text{@L}\} \quad \{\text{@L}\} S4; S9 \{\text{@post}\}$

- What about  $\{\text{@L}\} S4; S5; S7; S4; S5; S6 \{\text{@post}\}$  ? Why?

# Hoare Logic

- A logic to prove the validity of Hoare triple
- A set of logical rules for reasoning about the partial correctness of programs
- In this lecture, we assume the following simple imperative language

$$\begin{array}{l} S \rightarrow \text{skip} \\ \quad | \ x := E \\ \quad | \ S; S \\ \quad | \ \text{if } E \text{ then } S \text{ else } S \\ \quad | \ \text{while } E \text{ do } S \end{array}$$

# Example

- Which one is valid?
  - $\{x = 0\} \ x \ := \ x + 1 \ \{x = 1\}$
  - $\{x = 0 \wedge y = 1\} \ x \ := \ x + 1 \ \{x = 1 \wedge y = 2\}$
  - $\{x = 0\} \ x \ := \ x + 1 \ \{x = 1 \vee y = 2\}$
  - $\{x = 0\} \ \text{while true do } x \ := \ 0 \ \{x = 1\}$

# Hoare Rules (1)

- Rule for skip  $\frac{}{\{P\} \text{ skip } \{P\}}$
- Rule for assignment  $\frac{}{\{Q[E/x]\} x := E \{Q\}}$ 
  - Intuition: revert to the state before the assignment
  - Example:

$$\{\text{true}\} x := 1 \{x = 1\}$$

$$\{x + 1 > 0\} x := x + 1 \{x > 0\}$$

$$\{y = 1\} x := y \{x = 1\}$$

$$\{\text{false}\} x := y + 3 \{y = 0 \wedge x = 12\}$$



# Hoare Rules (2)

- Rule for precondition strengthening 
$$\frac{\{P'\} S \{Q\} \quad P \implies P'}{\{P\} S \{Q\}}$$

- Example:

$$\frac{\frac{\{y > 0[x/y]\} y := x \{y > 0\}}{\{x > 0\} y := x \{y > 0\}} \quad x = 2 \implies x > 0}{\{x = 2\} y := x \{y > 0\}}$$

- Rule for postcondition weakening 
$$\frac{\{P\} S \{Q'\} \quad Q' \implies Q}{\{P\} S \{Q\}}$$

- Example:

$$\frac{\dots}{\{true\} S \{x = y \wedge z = 2\}} \quad x = y \wedge z = 2 \implies x = y}{\{true\} S \{x = y\}}$$

# Hoare Rules (3)

- Rule for composition 
$$\frac{\{P\} S_1 \{Q\} \quad \{Q\} S_2 \{R\}}{\{P\} S_1; S_2 \{R\}}$$

$$\frac{\frac{\{x = 2[2/x]\} x := 2 \{x = 2\}}{\{\text{true}\} x := 2 \{x = 2\}} \quad \frac{\{x = 2 \wedge y = 2[x/y]\} y := x \{x = 2 \wedge y = 2\}}{\{x = 2\} y := x \{x = 2 \wedge y = 2\}}}{\{\text{true}\} x := 2; y := x \{x = 2 \wedge y = 2\}}$$

- Rule for if statement 
$$\frac{\{P \wedge E\} S_1 \{Q\} \quad \{P \wedge \neg E\} S_2 \{Q\}}{\{P\} \text{ if } E \text{ then } S_1 \text{ else } S_2 \{Q\}}$$

$$\frac{\frac{\{y \geq 0[x/y]\} y := x \{y \geq 0\} \quad x > 0 \implies x \geq 0}{\{x > 0\} y := x \{y \geq 0\}} \quad \frac{\{y \geq 0[-x/y]\} y := -x \{y \geq 0\}}{\{x \leq 0\} y := -x \{y \geq 0\}}}{\{\text{true}\} \text{ if } x > 0 \text{ then } y := x \text{ else } y := -x \{y \geq 0\}}$$

# Hoare Rules (4)

- Rule for loop 
$$\frac{\{P \wedge E\} S \{P\}}{\{P\} \text{ while } E \text{ do } S \{P \wedge \neg E\}}$$
- $$\dots$$
- $$\frac{\{x \leq n \wedge x < n\} x := x + 1 \{x \leq n\}}{\{x \leq n\} \text{ while } x < n \text{ do } x := x + 1 \{x = n\}}$$

# Loop Invariant

- Challenge: impossible to know how many times a given loop iterates
- How to prove the partial correctness of a loop within finite time?
- Analogy: mathematical induction
- Loop invariant  $I$  satisfies the following properties:
  - $I$  holds initially before the loop
  - $I$  holds after each iteration of the loop

- Example

```
i := 0; sum := 0; n := 10;
while (i < n) { // loop invariants?
    i := i + 1;
    sum := sum + i;
}
```

# Inductive Invariant

- Not all invariants are provable
- Example:

```
i := 5;  
while (i > 1) { // invariant: i > 0  
    i := i - 2;  
}  
assert(i = 1);
```

$$\frac{\{P \wedge E\} S \{P\}}{\{P\} \text{ while } E \text{ do } S \{P \wedge \neg E\}}$$

- Inductive invariant: invariant we can prove using induction
- Challenge: finding inductive loop invariants
  - Practice: human, static analysis, machine learning, etc

# Automatically Proving Partial Correctness

- $\{P\} S \{Q\}$ : Given the precondition satisfied, the postcondition is satisfied after the execution (if it terminates)
- Assumption: loop invariants are given by an oracle
  - Oracle: human, static analysis, machine learning, etc
- How to automatically prove correctness?
- Idea: deriving verification conditions (VCs) and check the validity



# Verification Condition

- A FOL formula  $F$  such that the program is correct iff  $F$  is valid
- Automatically proving partial correctness
  - Generating VCs from a program + checking the validity of VCs by a theorem prover
- Two ways to generate verification conditions
  - Forward: starting from prediction, generate formulas to prove postcondition (strongest postconditions)
  - Backward: starting from postcondition, generate formulas to prove precondition (weakest preconditions)

# Weakest Liberal Preconditions

- Goal: verify Hoar triple  $\{P\} S \{Q\}$
- Weakest liberal precondition  $wlp(S, Q)$  [Dijkstra75]
  - Weakest: most general condition that guarantees  $Q$  will hold after  $S$  in any execution
  - Liberal: we do not care about termination
- Proof of the Hoar triple  $\{P\} S \{Q\}$ :  $P \rightarrow wlp(S, Q)$
- Example:  $\{y \geq 10\} x := y + 1 \{x \geq 0\}$

# Weakest Precondition Calculus

- Inductively define  $wlp$  following Hoare rules
- $wlp(x := E, Q) = Q[E/x]$
- $wlp(s_1; s_2, Q) = wlp(s_1, wlp(s_2, Q))$
- $wlp(\text{if } E \text{ then } s_1 \text{ else } s_2, Q) = E \rightarrow wlp(s_1, Q) \wedge \neg E \rightarrow wlp(s_2, Q)$
- $wlp(\text{while } E \text{ do } S, Q) = I \wedge \forall y_1, \dots, y_k. ((E \wedge I \rightarrow wlp(S, I)) \wedge (\neg E \wedge I \rightarrow Q))[y_i/x_i]$ 
  - Assumption: an inductive invariant  $I$  is provided
  - $x_i$  is an assigned variable in  $S$  and  $y_i$  is a fresh variable

# Example (1)

- $S : x := y + 1; \text{if } x > 0 \text{ then } z := 1 \text{ else } z := -1$ 
  - $wlp(S, z > 0)?$
  - $wlp(S, z \leq 0)?$
  - $\{y > -1\} S \{z > 0\}?$
  - $\{y = -2\} S \{z < 0\}?$

# Example (2)

- Prove the assertion when  $I : i \geq 0 \wedge \text{odd}(i)$  using  
 $wlp(\text{while } E \text{ do } S, Q) = I \wedge \forall y_1, \dots, y_k. ((E \wedge I \rightarrow wlp(S, I)) \wedge (\neg E \wedge I \rightarrow Q))[y_i/x_i]$

```
// @pre: T
i := 5;
while (i > 1) {
    i := i - 2;
}
assert(i = 1);
```

**(1):**  $i \geq 0 \wedge \text{odd}(i)$

**(2):**  $\forall i'. (i' > 1) \wedge (i' \geq 0 \wedge \text{odd}(i')) \rightarrow (i' \geq 2) \wedge \text{odd}(i')$

**(3):**  $\forall i'. (i' \leq 1) \wedge (i' \geq 0 \wedge \text{odd}(i')) \rightarrow i' = 1$

# Example (3)

- Prove the assertion when  $I : x \geq 0 \wedge y = 5$  using  
 $wlp(\text{while } E \text{ do } S, Q) = I \wedge \forall y_1, \dots, y_k. ((E \wedge I \rightarrow wlp(S, I)) \wedge (\neg E \wedge I \rightarrow Q))[y_i/x_i]$

```
// @pre: T
x := 0;
y := 5;
while (x < 10) {
    x := x + 1;
}
assert(y = 5);
```

(1):  $x \geq 0 \wedge y = 5$

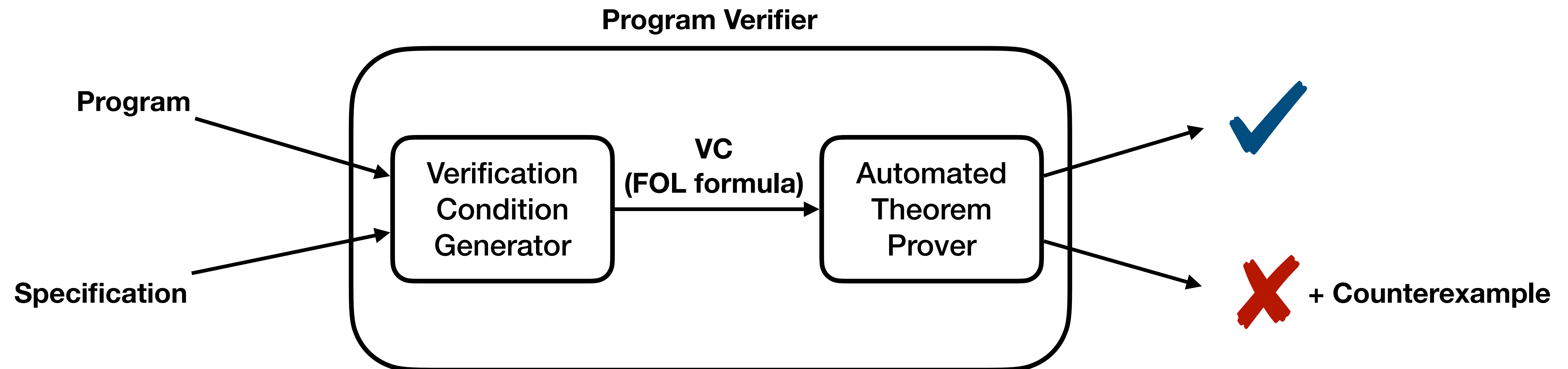
(2):  $\forall x'. (0 \leq x' < 10) \wedge (y = 5) \rightarrow x' \geq -1 \wedge y = 5$

(3):  $\forall x'. (x' \geq 10) \rightarrow y = 5$



# Verification of Hoare Triple

- Validity of  $\{P\} S \{Q\}$
- Verification condition:  $P \rightarrow wlp(S, Q)$



# Summary

- Hoare triple: specifications for partial correctness  $\{P\} S \{Q\}$
- Hoare logic: a logic to prove the validity of Hoare triple
  - Proof rules for each program command
- Verification condition is valid iff the Hoare triple is valid
- Automated program verification: check whether the VC is valid using theorem provers