Grover's search algorithm

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Grover's search algorithm (GSA)

also known as the **quantum search algorithm**. It was devised by Lov Grover in 1996. GSA won renown as the second major algorithm proposed for quantum computing (after Shor's 1994 algorithm), and in 2017 was finally implemented in a scalable physical quantum system.

Unstructured Search

f work as a "Black box" function:

$$f : \{0,1\}^n \to \{0,1\}^n$$

$$\exists! \omega : f(\omega) = a$$

$$f_\omega(x) = \delta_{x=\omega}$$

Lov Kumar Grover
(born 1961)

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Creating an Oracle

$$U_f \left|x\right\rangle \left|q\right\rangle = \left|x\right\rangle \left|q \oplus f(x)\right\rangle$$

Our oracle can then be described as:

$$U_\omega \left|x\right\rangle = (-1)^{f(x)} \left|x\right\rangle \quad \Longleftrightarrow \quad U_\omega = I - 2\left|\omega\right\rangle\left\langle\omega\right|$$

$$U_\omega \left|x\right\rangle = \left|x\right\rangle - 2\left|\omega\right\rangle\left\langle\omega|x\right\rangle$$

$$U_\omega \left|x\right\rangle = \left|x\right\rangle - 2\left|\omega\right\rangle\left\langle\omega|x\right\rangle = \left|x\right\rangle \qquad \left\langle\omega|x\right\rangle = 0$$

$$U_\omega \left|\omega\right\rangle = \left|\omega\right\rangle - 2\left|\omega\right\rangle\left\langle\omega|\omega\right\rangle = -\left|\omega\right\rangle$$

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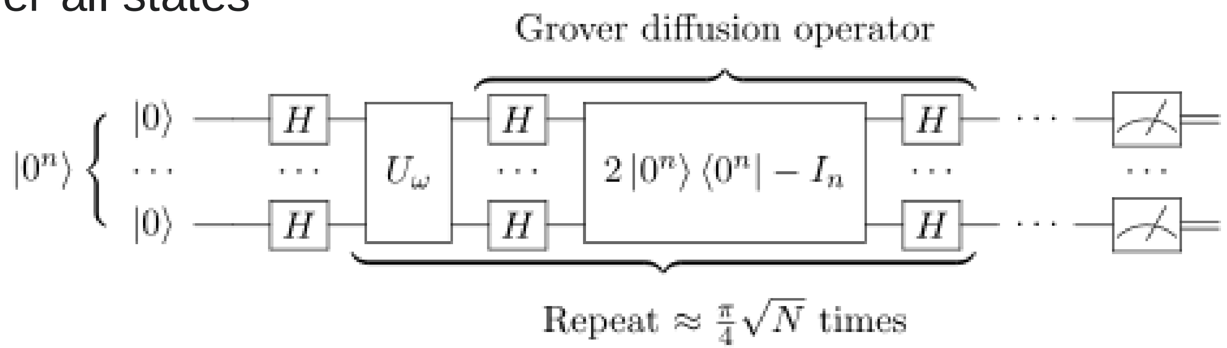
Grover iteration

The steps of Grover's algorithm are given as follows:

1. Initialize the system to the uniform superposition over all states

$$|s\rangle = H|0\rangle^n = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle$$



2. Perform the following "Grover iteration" T times

   2.1 Apply the operator U_ω

   2.2 Apply the *Grover diffusion* operator $U_s = 2|s\rangle\langle s| - I$

3. Measure the resulting quantum state in the computational basis.

Glover's algorithm is iterative. Each iteration is defined as applying R_{grov} operator to the current state of the system

$$R_{grov} = U_s U_\omega$$

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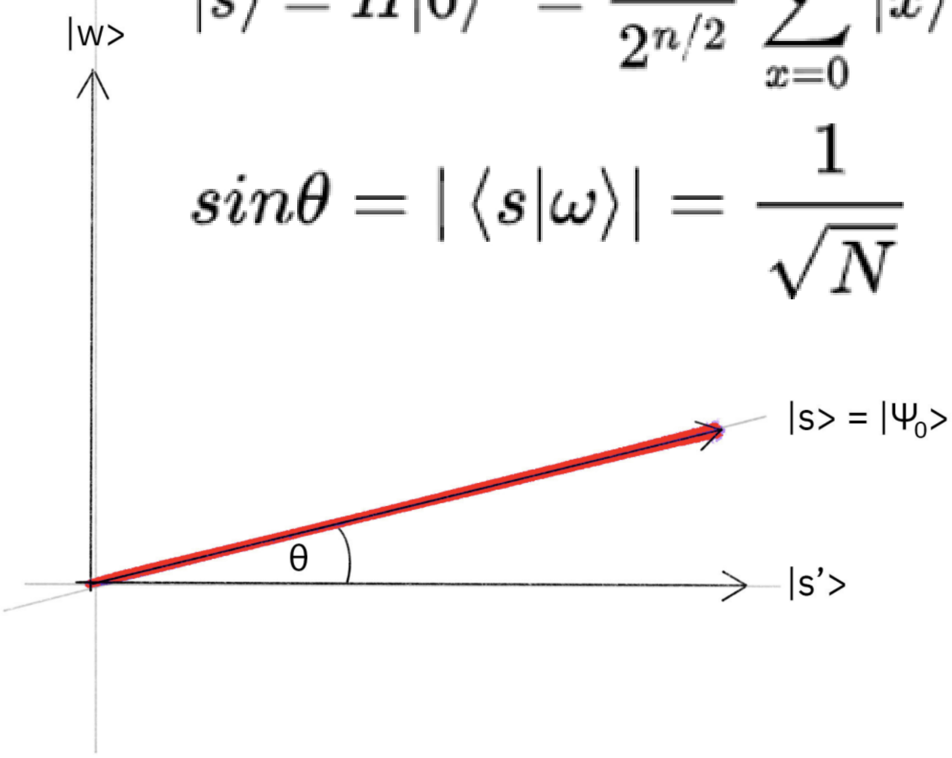
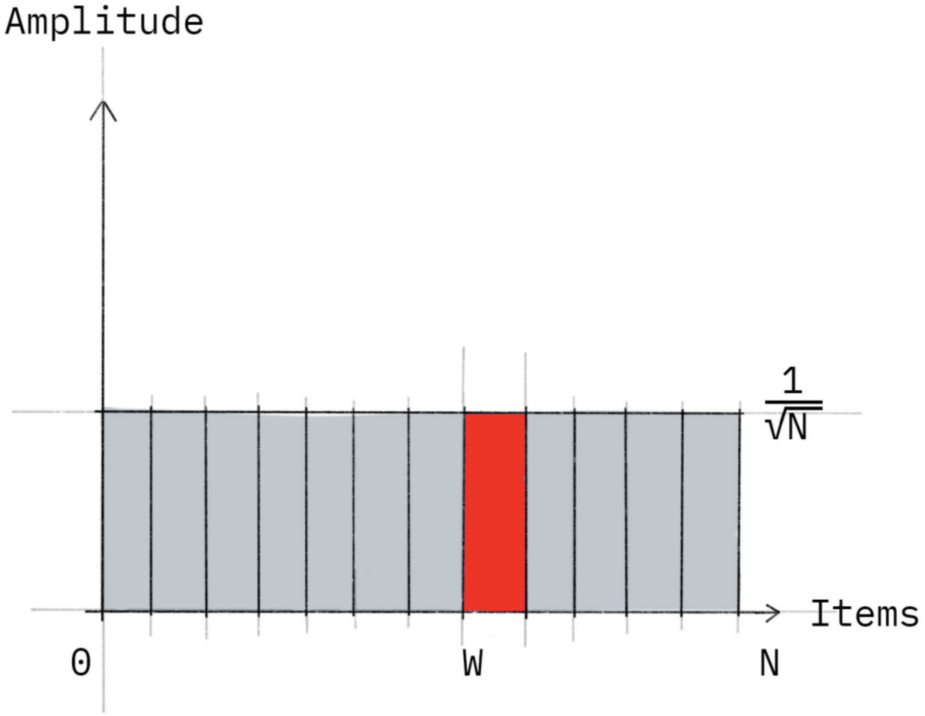
Step 1

$$|s\rangle = H|0\rangle^n = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle$$

$$sin\theta = |\langle s|\omega\rangle| = \frac{1}{\sqrt{N}}$$

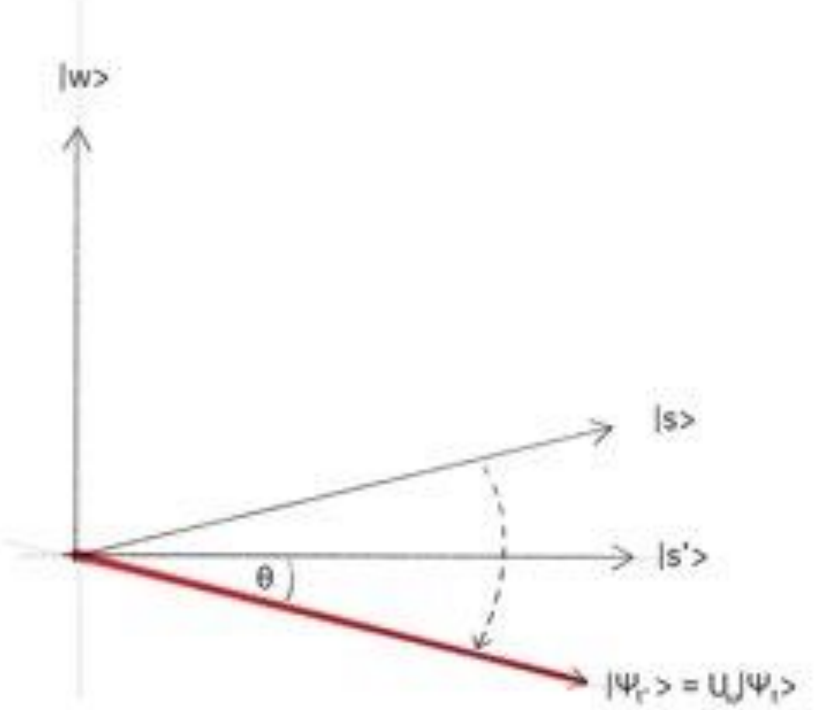


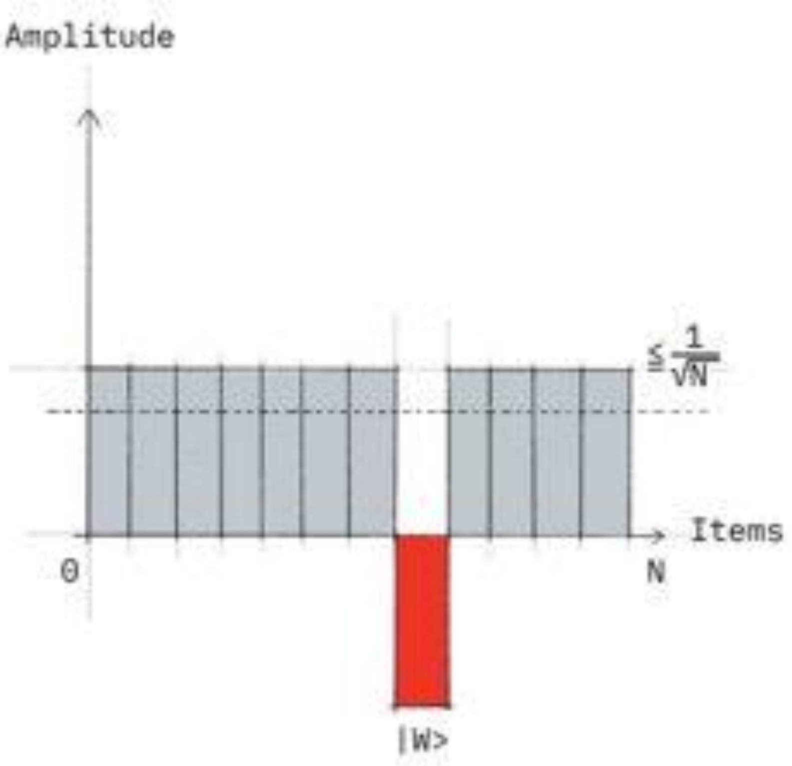


Step 2

$$U_\omega |x\rangle = (-1)^{f(x)} |x\rangle \iff U_\omega = I - 2|\omega\rangle\langle\omega|$$

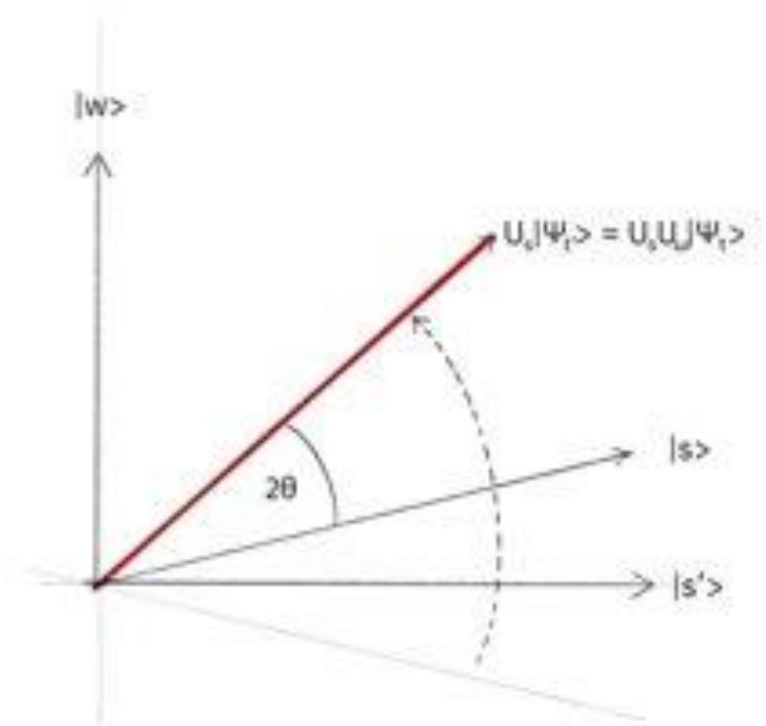
Apply the phase inversion operator U_ω

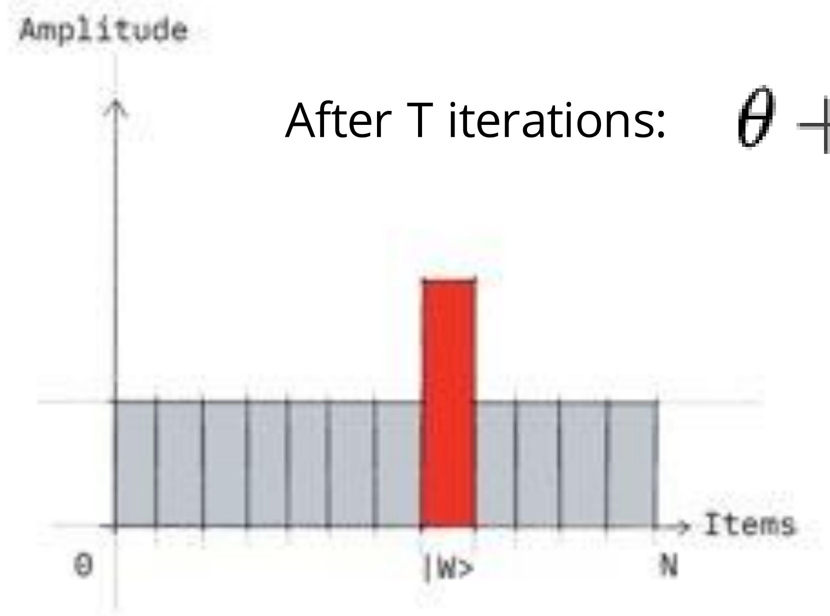




Step 3

Apply the *Grover diffusion* operator (Inversion about the mean)





After T iterations: $\theta + 2\theta \cdot T = \frac{2}{\pi}$

$$\theta + 2T\theta = \frac{2}{\pi} \iff T = \frac{\pi}{4\theta} - \frac{1}{2}$$

$$T = \frac{\pi\sqrt{N}}{4} - \frac{1}{2} \approx \frac{\pi\sqrt{2^n}}{4}$$

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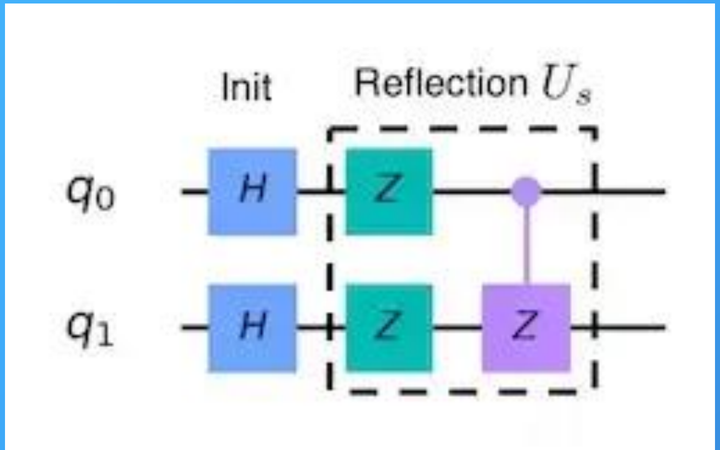
Two qubits case

$$\theta = arcsin|\langle s|\omega\rangle| = arcsin\frac{1}{2} = \frac{\pi}{6}$$

$$U_\omega U_s|s\rangle = sin\theta_T|\omega\rangle + cos\theta_T|s'\rangle$$

$$\theta_T = (2T+1)\theta \qquad T = 1$$

$$U_s|s\rangle = U_s\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle)$$



Reflection scheme

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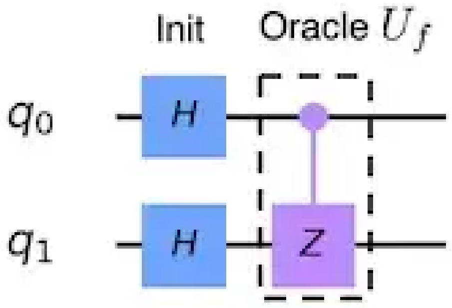
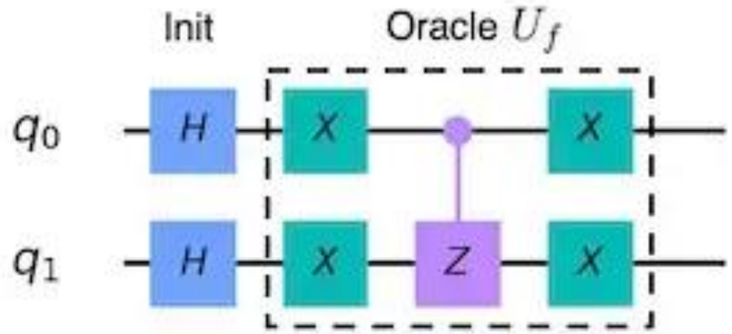
Oracle for two qubits case

Oracle for $|w\rangle = |11\rangle$

$$U_\omega |s\rangle = U_\omega \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

State $|11\rangle$ inverted negative





Oracle for $|w\rangle = |00\rangle$

$$U_\omega |s\rangle = U_\omega \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(-|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

State $|00\rangle$ inverted negative

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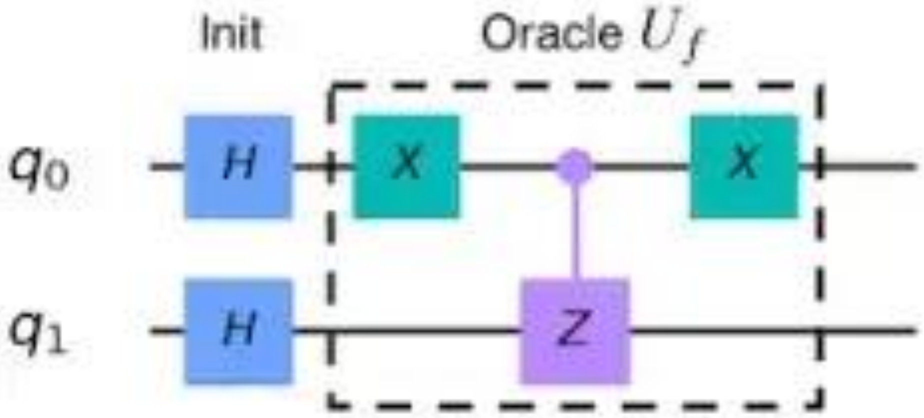
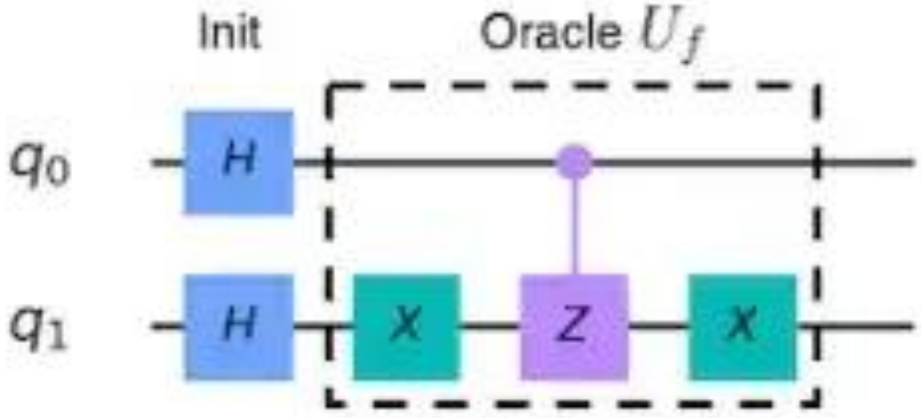
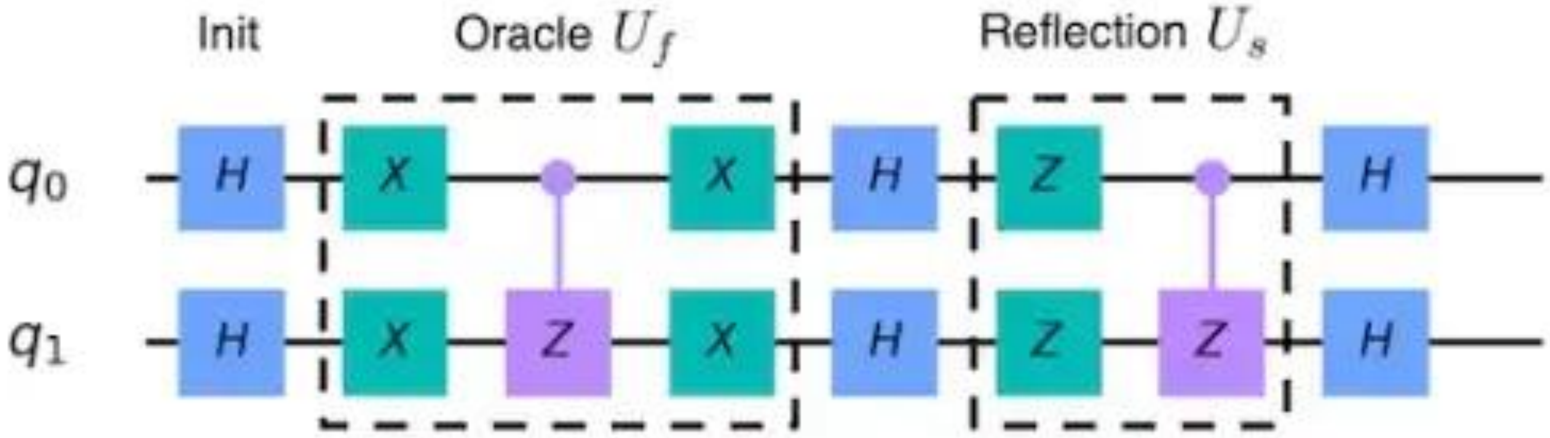
Oracle for |w⟩ = |01⟩ (left) and |w⟩ = |10⟩ (right)







Full scheme for |w⟩ = |00⟩

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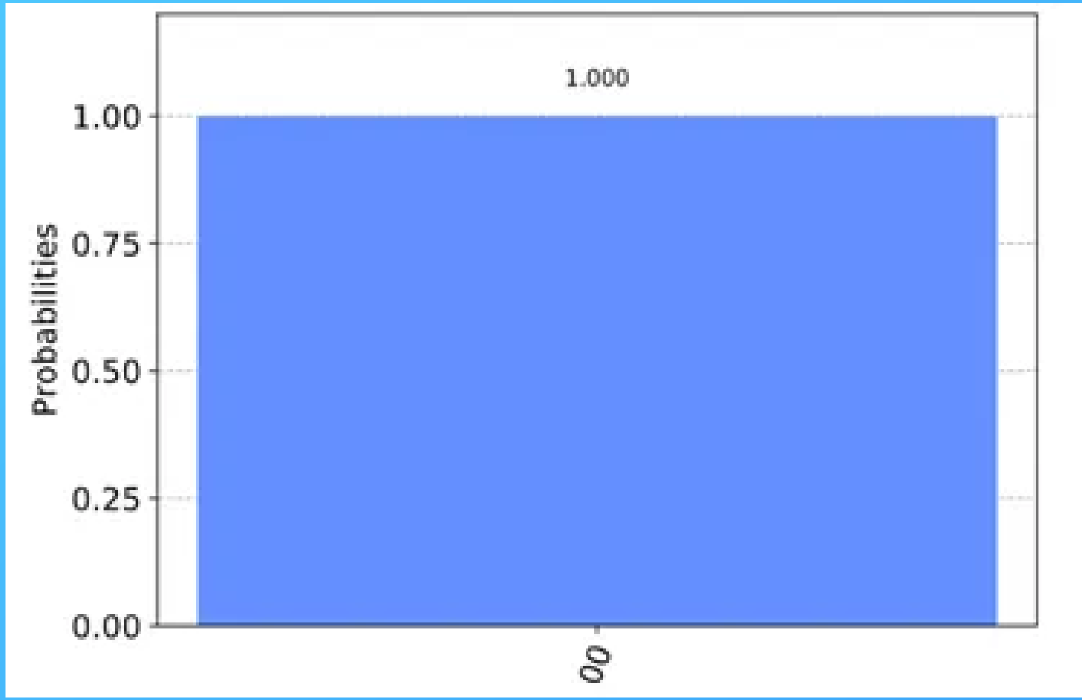
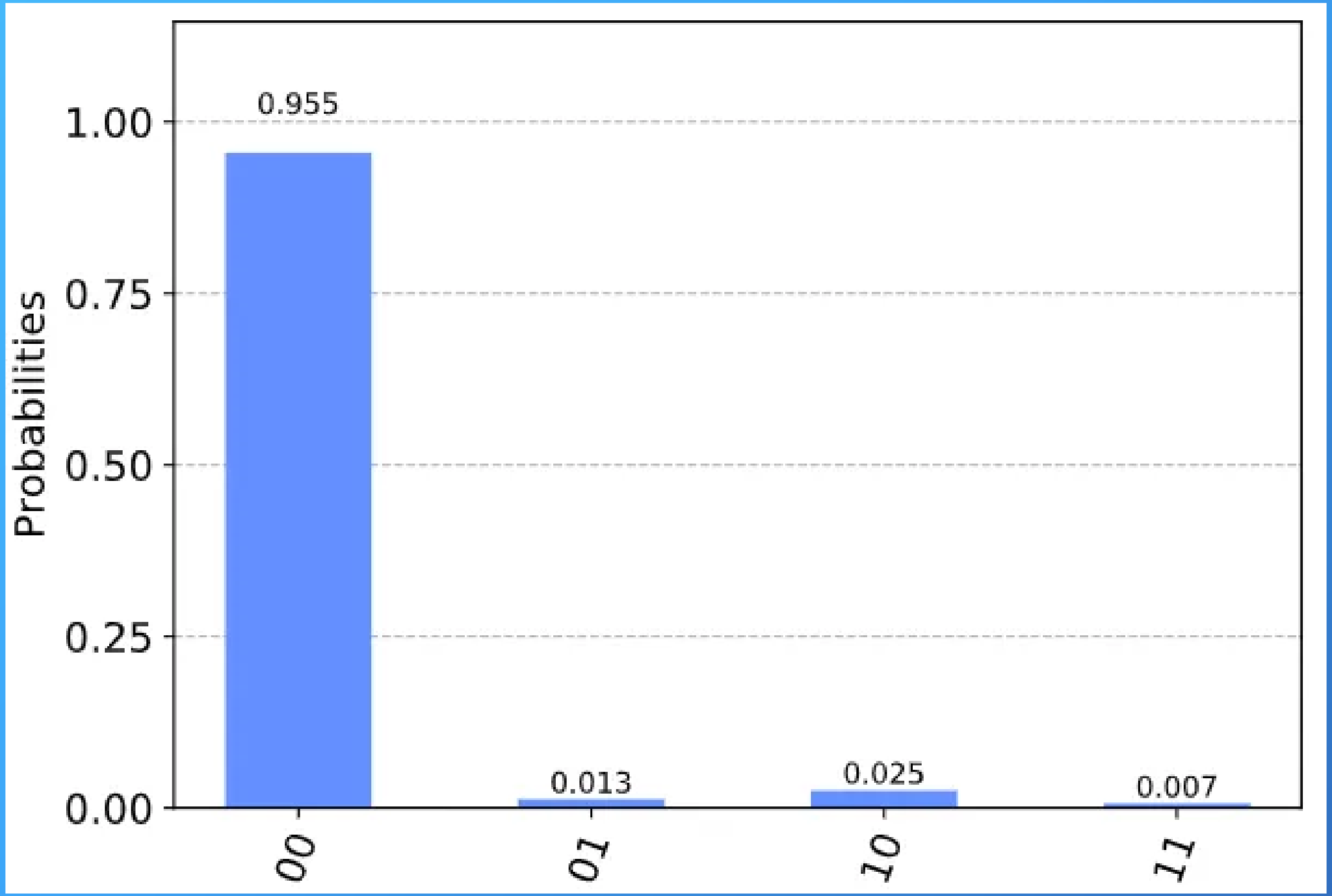
Result for $|w\rangle = |00\rangle$ case

Quantum simulator





Real quantum device

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Extensions and variants

- **Multiple matching entries**

If, instead of 1 matching entry, there are *k* matching entries, the same algorithm works, but the number of $O((N/k)^{½})$ iterations must be instead of $O(N^{1/2})$.

A version of this algorithm is used in order to solve the collision problem.

- **Quantum partial search**

A modification of Grover's algorithm called quantum partial search was described by Grover and Radhakrishnan in 2004.

In partial search, one is not interested in finding the exact address of the target item, only the first few digits of the address.

- **Finding the extremum of an integer function**

- **Structural search algorithm (Farhi, Gutman)**

- **Search for matching tapes**

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Search for sequence in the text

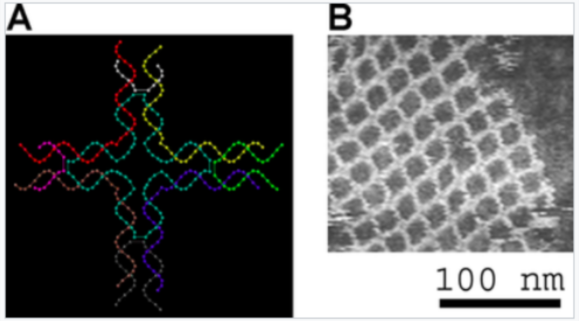
nucleic acids 6/23

and database theory.[173] String searching or mat ... e of letters inside a larger sequence of letters, were developed to search for specific s... e aligned with other DNA sequences to identify homologous sequences and locate the specific mutations that make them distinct. These techniques, especially multiple sequence alignment, are used in studying phylogenetic relationships and protein function.[175] Data sets representing entire genomes' worth of DNA sequences, such as those produced by the Human Genome Project, are difficult to use without the annotations that identify the locations of genes and regulatory elements on each chromosome. Regions of DNA sequence that have the characteristic patterns associated with protein- or RNA-coding genes can be identified by gene finding algorithms, which allow researchers to predict the presence of particular gene products and their possible functions in an organism even before they have been isolated experimentally.[176] Entire genomes may also be compared, which can shed light on the evolutionary history of particular organism and permit the examination of complex evolutionary events.

DNA nanotechnology

Further information: DNA nanotechnology

DNA nanotechnology uses the unique molecular recognition properties of DNA and other nucleic acids to create self-assembling branched DNA complexes with useful properties.[178] DNA is thus used as a structural material rather than as a carrier of biological information. This has led to the creation of two-dimensional periodic lattices (both tile-based and using the *DNA origami* method) and three-dimensional structures in the shapes of polyhedra.[179] Nanomechanical devices and algorithmic self-assembly have also been demonstrated,[180] and these DNA structures have been used to template the arrangement of other molecules such as gold nanoparticles and streptavidin proteins.[181] DNA and other nucleic acids are the basis of aptamers, synthetic oligonucleotide ligands for specific target molecules used in a range of biotechnology and biomedical applications.[182]



The DNA structure at left (schematic shown) will self-assemble into the structure visualized by atomic force microscopy at right. DNA nanotechnology is the field that seeks to design nanoscale structures using the molecular recognition properties of DNA molecules.[177]

History and anthropology

Further information: Phylogenetics and Genetic genealogy

Text length N,
sequence length M

Classical algorithm: O(N+M)

GSA based algorithm: $O(N^{1/2} + M^{1/2})$

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Discrete optimization

N questions with "yes" or "no" answers: $\sigma_i = \pm 1, i = 1, \ldots, N$

Objective function: $f = \sum_i A_i \sigma_i + \sum_{i,j} B_{i,j} \sigma_i \sigma_j$

Classical algorithm

O(2^N)

Quantum algorithm

$$g(\{\sigma_i\}, x) = \begin{cases} 1, & x > f \\ 0, & x \leq f \end{cases}$$

O(2$^{N/2}$)

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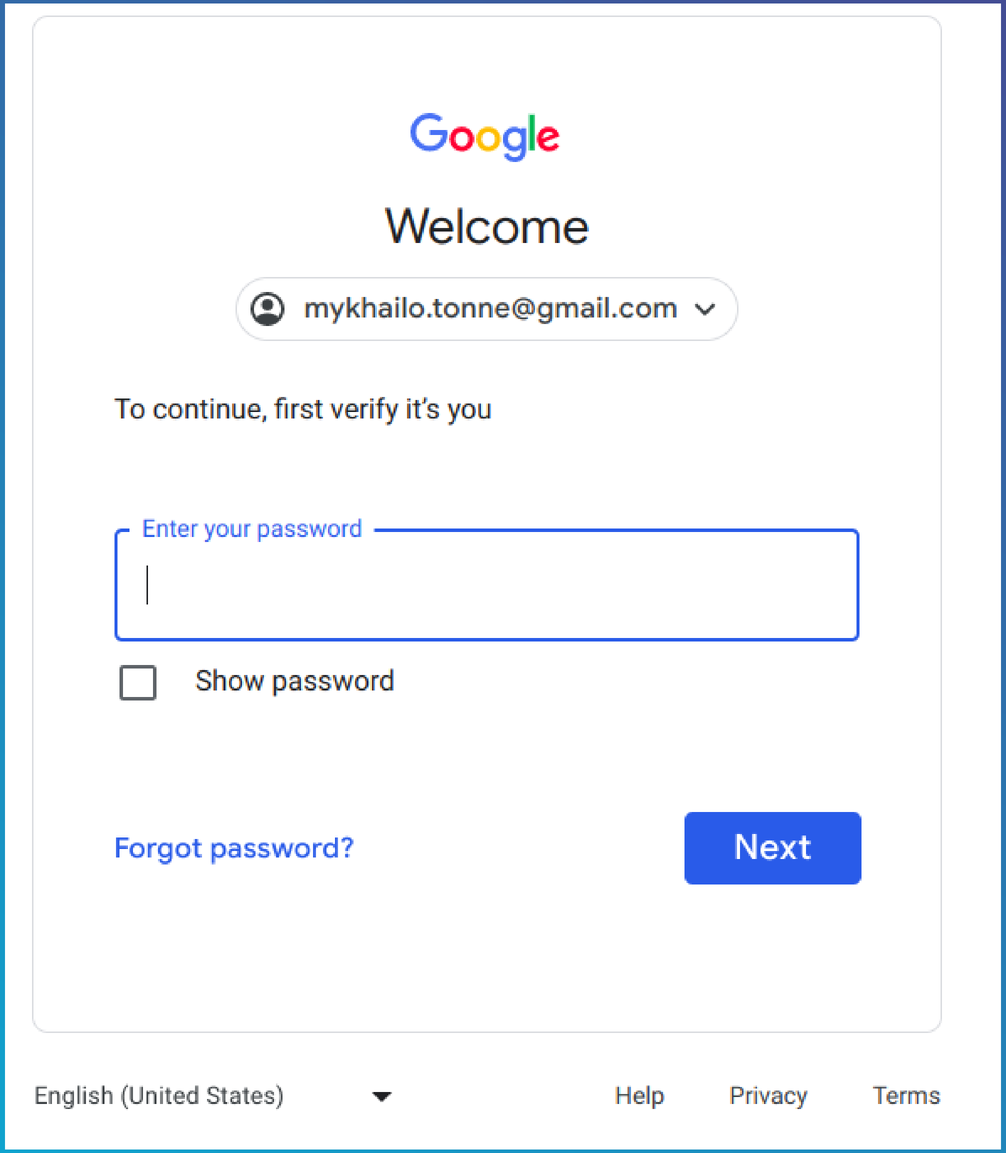
Does Google know your password?



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