I-Form

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# Fractional Factorial Designs

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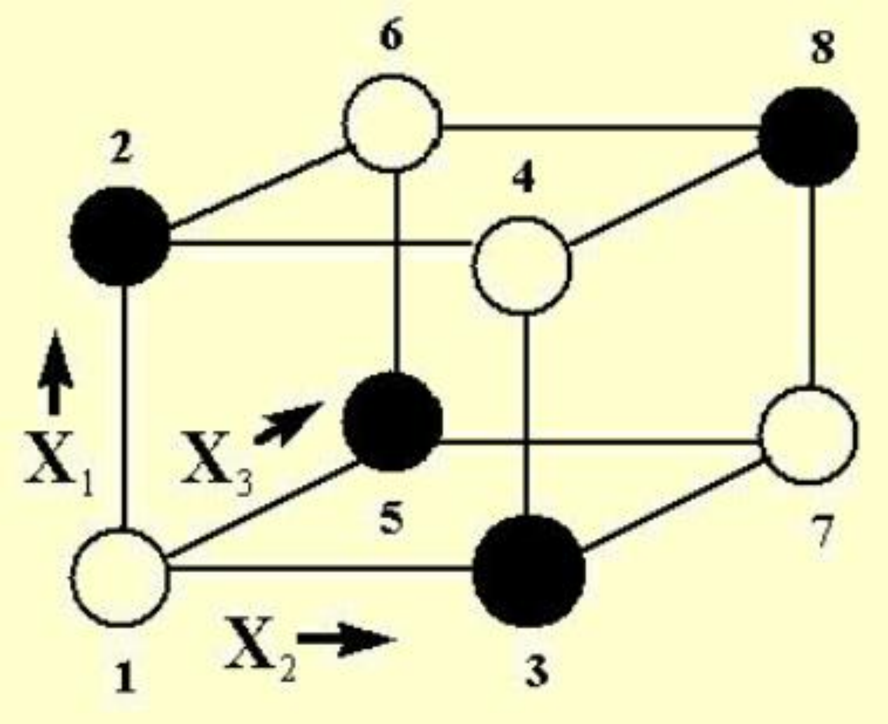
NIST/SEMATECH e-Handbook of Statistical Methods

http://www.itl.nist.gov/div898/handbook/

# Full Factorial Design

$2^3$ design=eight runs (not counting replications or center points)




| | $X_1$ | $X_2$ | $X_3$ | $Y$ |
|---|---|---|---|---|
| 1 | -1 | -1 | -1 | $y_1 = 33$ |
| 2 | +1 | -1 | -1 | $y_2 = 63$ |
| 3 | -1 | +1 | -1 | $y_3 = 41$ |
| 4 | +1 | +1 | -1 | $Y_4 = 57$ |
| 5 | -1 | -1 | +1 | $y_5 = 57$ |
| 6 | +1 | -1 | +1 | $y_6 = 51$ |
| 7 | -1 | +1 | +1 | $y_7 = 59$ |
| 8 | +1 | +1 | +1 | $y_8 = 53$ |

**Runs in Standard Order**

# Full Factorial Design

$2^3$ design=eight runs (not counting replications or center points)

We can compute all ‘effects’: main effects, two-way interaction effects, etc.

The main effect estimate $c_1$ of factor $X_1$ is:

$$c_1 = (y_2 + y_4 + y_6 + y_8)/4 - (y_1 + y_3 + y_4 + y_5)/4$$

| | $X_1$ | $X_2$ | $X_3$ | $Y$ |
|---|---|---|---|---|
| **1** | -1 | -1 | -1 | $y_1 = 33$ |
| **2** | +1 | -1 | -1 | $y_2 = 63$ |
| **3** | -1 | +1 | -1 | $y_3 = 41$ |
| **4** | +1 | +1 | -1 | $Y_4 = 57$ |
| **5** | -1 | -1 | +1 | $y_5 = 57$ |
| **6** | +1 | -1 | +1 | $y_6 = 51$ |
| **7** | -1 | +1 | +1 | $y_7 = 59$ |
| **8** | +1 | +1 | +1 | $y_8 = 53$ |

# Why and How?

A full factorial design runs up a very large resource quickly.

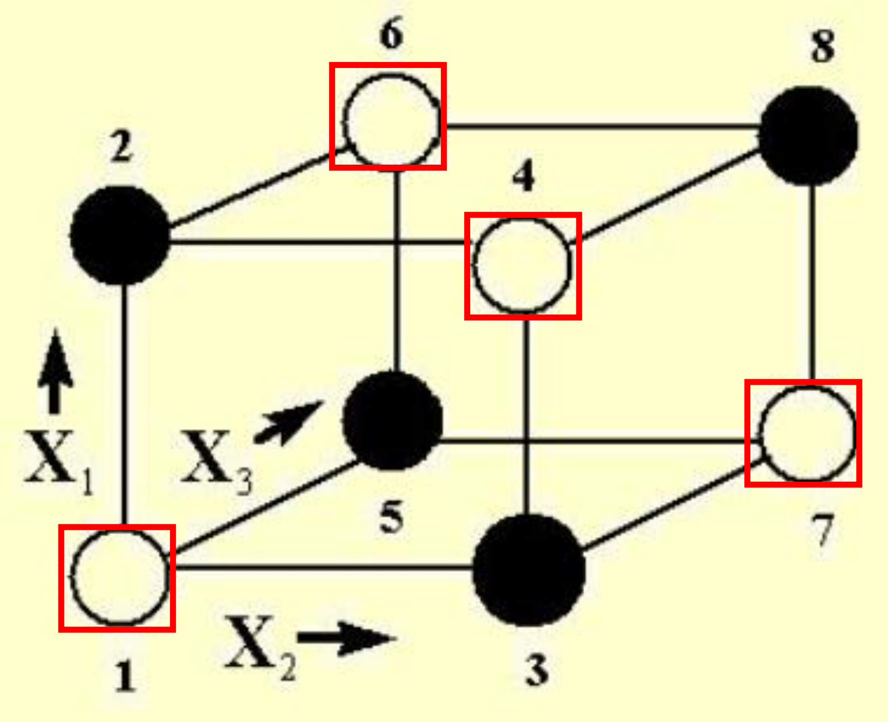
A two-level full factorial design with six factors requires $2^6$=64 runs.

Fractional factorial design selects a fraction (½, ¼, etc.) of the runs specified by the full factorial design.

Which runs to make, and which runs to leave?

# Fractional Factorial Design

$2^{3-1}$ design=four runs {1, 4, 6, 7}




| | $X_1$ | $X_2$ | $X_3$ | $Y$ |
|---|---|---|---|---|
| 1 | -1 | -1 | -1 | $y_1 = 33$ |
| 2 | +1 | -1 | -1 | $y_2 = 63$ |
| 3 | -1 | +1 | -1 | $y_3 = 41$ |
| 4 | +1 | +1 | -1 | $Y_4 = 57$ |
| 5 | -1 | -1 | +1 | $y_5 = 57$ |
| 6 | +1 | -1 | +1 | $y_6 = 51$ |
| 7 | -1 | +1 | +1 | $y_7 = 59$ |
| 8 | +1 | +1 | +1 | $y_8 = 53$ |

**Runs in Standard Order**

# Fractional Factorial Design

$2^{3-1}$ design=four runs {1, 4, 6, 7}

We can compute all main effects.

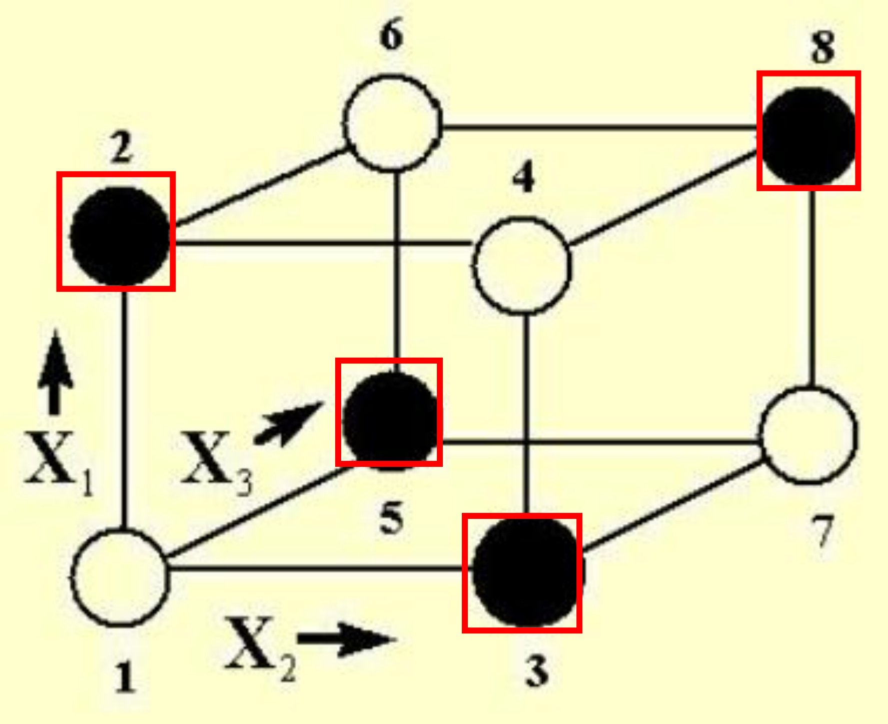
The main effect estimate $c_1$ of factor $X_1$ is:

$$c_1 = (y_4 + y_6)/2 - (y_1 + y_7)/2$$

| | $X_1$ | $X_2$ | $X_3$ | $Y$ |
|---|---|---|---|---|
| 1 | -1 | -1 | -1 | $y_1 = 33$ |
| 2 | +1 | -1 | -1 | $y_2 = 63$ |
| 3 | -1 | +1 | -1 | $y_3 = 41$ |
| 4 | +1 | +1 | -1 | $Y_4 = 57$ |
| 5 | -1 | -1 | +1 | $y_5 = 57$ |
| 6 | +1 | -1 | +1 | $y_6 = 51$ |
| 7 | -1 | +1 | +1 | $y_7 = 59$ |
| 8 | +1 | +1 | +1 | $y_8 = 53$ |

# Fractional Factorial Design

$2^{3-1}$ design=four runs {2, 3, 5, 8}




| | $X_1$ | $X_2$ | $X_3$ | $Y$ |
|---|---|---|---|---|
| 1 | -1 | -1 | -1 | $y_1 = 33$ |
| 2 | +1 | -1 | -1 | $y_2 = 63$ |
| 3 | -1 | +1 | -1 | $y_3 = 41$ |
| 4 | +1 | +1 | -1 | $Y_4 = 57$ |
| 5 | -1 | -1 | +1 | $y_5 = 57$ |
| 6 | +1 | -1 | +1 | $y_6 = 51$ |
| 7 | -1 | +1 | +1 | $y_7 = 59$ |
| 8 | +1 | +1 | +1 | $y_8 = 53$ |

**Runs in Standard Order**

An experiment calls for running three factors, Pressure (P), Table speed (T), and Down force (D), each at a 'high' and a 'low' setting, on a production tool to determine which has the greatest effect on product uniformity. Interaction effects are considered negligible.

(1) Uniformity measurement error requires that at least two separate runs (replications) be made at each process setting.

(2) Several 'standard setting' runs (centerpoint runs) need be made at regular intervals during the experiment to monitor process drift.

(3) No more than 15 runs can be planned.

A $2^{3-1}$ design replicated twice requires 8 runs; of the 7 spare runs, 3 to 5 can be used for centerpoint runs, and the rest saved for backup in case something goes wrong with any run.

| | Pattern | P | T | D=P*T<br>D | Center Point |
|---|---|---|---|---|---|
| 1 | 000 | 0 | 0 | 0 | 1 |
| 2 | +-- | +1 | -1 | -1 | 0 |
| 3 | -+- | -1 | +1 | -1 | 0 |
| 4 | 000 | 0 | 0 | 0 | 1 |
| 5 | +++ | +1 | +1 | +1 | 0 |
| 6 | --+ | -1 | -1 | +1 | 0 |
| 7 | 000 | 0 | 0 | 0 | 1 |
| 8 | +-- | +1 | -1 | -1 | 0 |
| 9 | --+ | -1 | -1 | +1 | 0 |
| 10 | 000 | 0 | 0 | 0 | 1 |
| 11 | +++ | +1 | +1 | +1 | 0 |
| 12 | -+- | -1 | +1 | -1 | 0 |
| 13 | 000 | 0 | 0 | 0 | 1 |

Replicated $2^{3-1}$ in randomized run order, with five centerpoint runs ('000') interspersed among the runs.

# Constructing $2^{3-1}$ Design

| | $X_1$ | $X_2$ |
|---|---|---|
| 1 | -1 | -1 |
| 2 | +1 | -1 |
| 3 | -1 | +1 |
| 4 | +1 | +1 |

| | $X_1$ | $X_2$ | $X1*X_2$ |
|---|---|---|---|
| 1 | -1 | -1 | +1 |
| 2 | +1 | -1 | -1 |
| 3 | -1 | +1 | -1 |
| 4 | +1 | +1 | +1 |

$$X_3 = X_1 X_2$$

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| 1 | -1 | -1 | +1 |
| 2 | +1 | -1 | -1 |
| 3 | -1 | +1 | -1 |
| 4 | +1 | +1 | +1 |

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| 1 | -1 | -1 | -1 |
| 2 | +1 | -1 | +1 |
| 3 | -1 | +1 | +1 |
| 4 | +1 | +1 | -1 |

$$X_3 = -X_1 X_2$$

# Confounding

One price we pay for using the column $X_1 * X_2$ to obtain column $X_3$ is our inability to obtain an estimate of the interaction effect for $X_1 * X_2$ (i.e., $c_{12}$) that is separate from an estimate of the main effect for $X_3$ (i.e., $c_3$).

In the $2^{3-1}$ design, our computation of $c_3$ is in fact a computation of $c_3 + c_{12}$; we assume that $c_{12}$ is small compared to $c_3$.

If the desired effects (e.g., $c_3$) are only confounded with non-significant interactions (e.g., $c_{12}$), then we are OK.

# Confounding

A short way of writing '$X_3$=$X_1$*$X_2$' is: '3 = 12' (similarly 3 = -12 refers to $X_3$=-$X_1$*$X_2$). Any column multiplied by itself gives the identity column of all 1's.

Multiply both sides of 3=12 by 3 and obtain I=123.

Playing around with this 'algebra', we have

$$1I=1123 \rightarrow 1=23,$$

$$2I=2123 \rightarrow 2=13,$$

$$3I=3123 \rightarrow 3=12.$$

The complete confounding pattern is {1=23, 2=13, 3=12, I=123}; that is, all the main effects are confounded with two-way interactions.

# Confounding

I=123 is called a ***design generator*** for the dark-shaded $2^{3-1}$ design. Equally, I=-123 is the design generator for the light-shaded $2^{3-1}$ design.

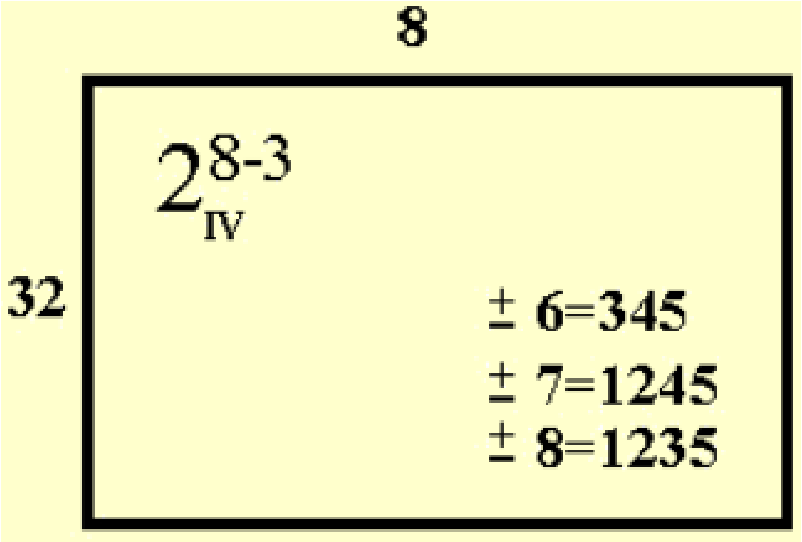
We can replace any design generator by its negative counterpart and have an equivalent, but different fractional design.

A $2^{k-p}$ fractional factorial needs $p$ generators, which define how the $p$ additional columns to be added to the $2^k$ full factorial design.

Generators are typically written in "I = ... " form.

# Another Example

$2^{8-3}$ design=32 runs (not including centerpoint runs)




{I=+3456; I=+12457; I=+12358}

Write down a full factorial design in standard order for k-p=5 factors.

Add a column for factor 6, using 6 = 345 (or 6 = -345).
Add a column for factor 7, using 7 = 1245 (or 7 = -1245).
Add a column for factor 8, using 8 = 1235 (or 8 = -1235).

When running the experiment, we need to randomize the order.

# Another Example

The set of all design generators for a fractional design, including all new generators that can be formed as products of original generators, is called a ***defining relation***.

There are seven ‘words’ in the defining relation for the $2^{8-3}$ design: {I=3456, =12457, =12358, =12367, =12468, =3478, =5678}.

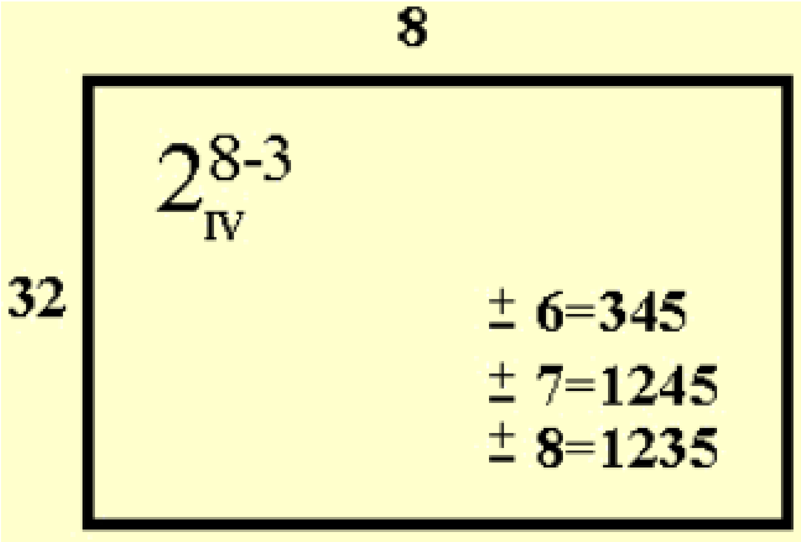
There are ($2^{p}$-1) words in the defining relation for a $2^{k-p}$ fractional factorial. The length of the shortest word in the defining relation is called the ***resolution*** of the design.

The $2^{8-3}$ design has resolution four, written as $2_{IV}^{8-3}$.

The $2^{3-1}$ design has resolution three, written as $2_{III}^{3-1}$

# Another Example

$2^{8-3}$ design=32 runs (not including centerpoint runs)




{I=3456, =12457, =12358, =12367, =12468, =3478, =5678}

The confounding pattern for two-way interactions is

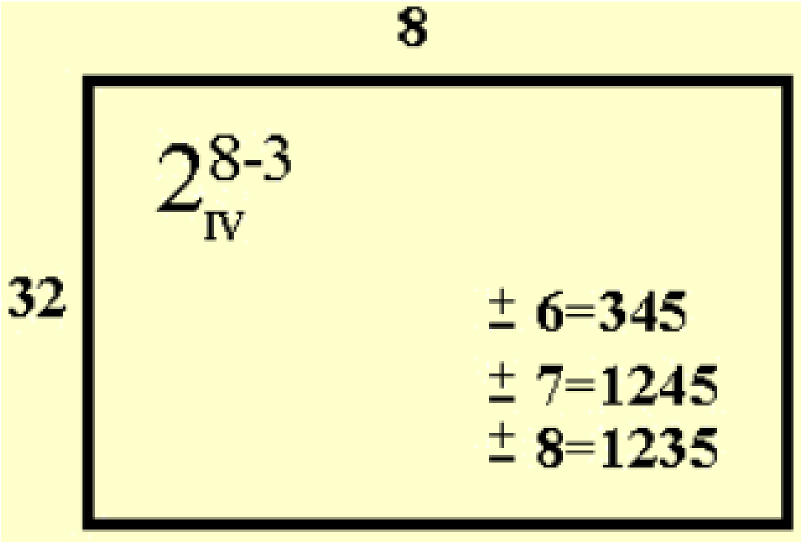
34 = 56 = 78
35 = 46
36 = 45
37 = 48
38 = 47
57 = 68
58 = 67

15 two-way interactions are confounded

For example, 34=343456=56.

# Another Example

$2^{8-3}$ design=32 runs (not including centerpoint runs)



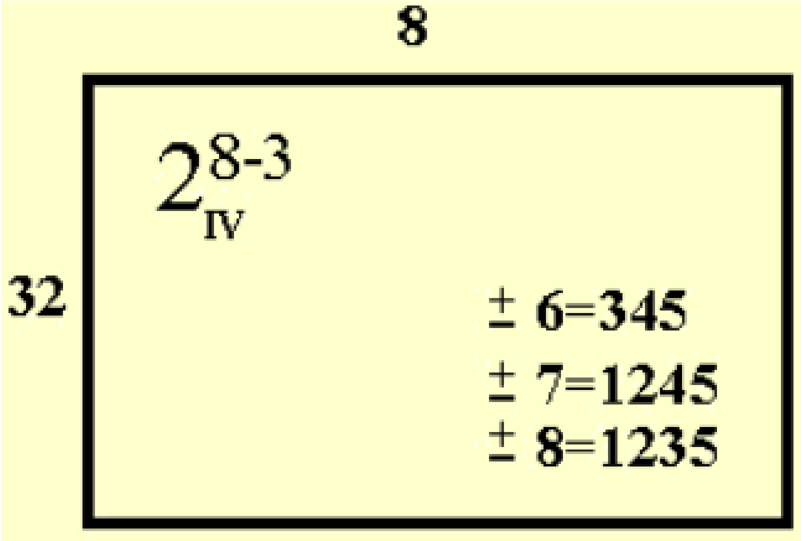

{I=3456, =12457, =12358, =12367, =12468, =3478, =5678}

15 two-way interactions are confounded in pairs or in a group of three. The remaining 28-15=13 two-way interactions are confounded with higher-way interactions (which are generally assumed to be negligible).

Factors "1" and "2" never appear in a length-4 word in the defining relation. Hence, all interactions involving "1" or "2" are clear of confounding with any other two-way interaction.

# Another Example

$2^{8-3}$ design=32 runs (not including centerpoint runs)



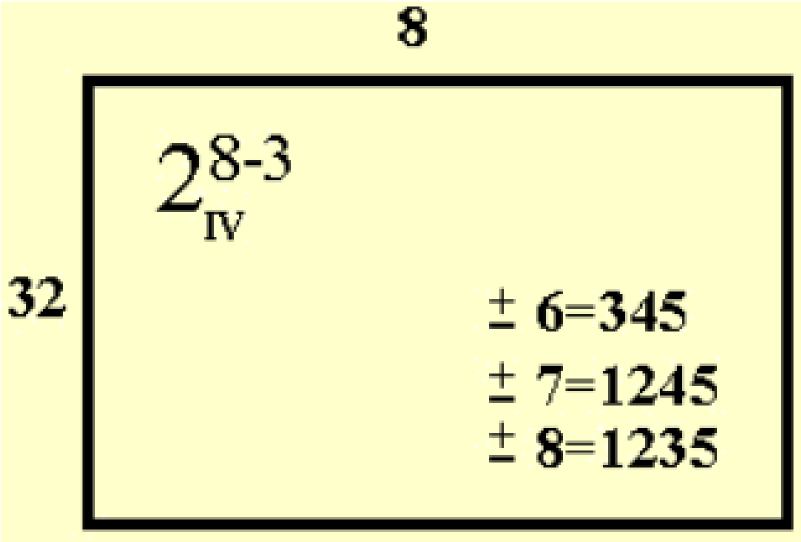

{I=3456, =12457, =12358, =12367, =12468, =3478, =5678}

If two factors are suspected of having significant interactions, to avoid having them confounded, they should **not** appear in length-4 generators.

For a given k and p, ***minimum aberration*** fractional factorial designs <u>maximize the resolution</u> and <u>minimize the number of short words</u> in the defining relation (which minimizes two-way confounding).

# Another Example

$2^{8-3}$ design=32 runs (not including centerpoint runs)




{I=3456, =12457, =12358, =12367, =12468, =3478, =5678}

There are other $2_{IV}^{8-3}$ fractional designs that can be derived, with different choices of design generators for the "6", "7" and "8" factors.

However, they are either equivalent or inferior to the given fraction.

For example, the design with generators 6 = 12345, 7 = 135, and 8 = 245 has five length-four words in the defining relation; this design would confound 23 out of 28 two-way interactions.

# Another Example

With a resolution V design,

- main effects would be confounded with four- and higher-way interactions, and
- two-way interactions would be confounded with three-way interactions.

A resolution V design is "better" than a resolution IV design (yet requiring more runs), because we have less-severe confounding pattern in the 'V' than in the ‘IV' situation; higher-way interactions are less likely to be significant than low-way interactions.

# Design Resolution Summary

- Resolution III Designs: Main effects are confounded with two-way interactions.
- Resolution IV Designs: No main effects are confounded with two-way interactions, but two-way interactions are confounded with each other.
- Resolution V Designs: No main effect or two-way interaction is confounded with any other main effect or two-way interaction, but two-way interactions are confounded with three-way interactions.

# Centerpoint Runs

We add centerpoint runs interspersed among the experimental setting runs for two purposes:

- To provide a measure of process stability and inherent variability.
- To check for curvature.

Centerpoint runs should begin and end the experiment and should be dispersed as evenly as possible throughout the design matrix.

Randomized, replicated $2^3$ full factorial design matrix with centerpoint control runs added

| | Random Order | Standard Order | SPEED | FEED | DEPTH |
|---|---|---|---|---|---|
| 1 | not applicable | not applicable | 0 | 0 | 0 |
| 2 | 1 | 5 | -1 | -1 | 1 |
| 3 | 2 | 15 | -1 | 1 | 1 |
| 4 | 3 | 9 | -1 | -1 | -1 |
| 5 | 4 | 7 | -1 | 1 | 1 |
| 6 | 5 | 3 | -1 | 1 | -1 |
| 7 | 6 | 12 | 1 | 1 | -1 |
| 8 | 7 | 6 | 1 | -1 | 1 |
| 9 | 8 | 4 | 1 | 1 | -1 |
| 10 | not applicable | not applicable | 0 | 0 | 0 |
| 11 | 9 | 2 | 1 | -1 | -1 |
| 12 | 10 | 13 | -1 | -1 | 1 |
| 13 | 11 | 8 | 1 | 1 | 1 |
| 14 | 12 | 16 | 1 | 1 | 1 |
| 15 | 13 | 1 | -1 | -1 | -1 |
| 16 | 14 | 14 | 1 | -1 | 1 |
| 17 | 15 | 11 | -1 | 1 | -1 |
| 18 | 16 | 10 | 1 | -1 | -1 |
| 19 | not applicable | not applicable | 0 | 0 | 0 |

# Ref

*Statistics for Experimenters* by G.E.P. Box, W.G. Hunter, and J.S. Hunter (New York, John Wiley & Sons, 1978).

- Screen Designs
- Response Surface Designs
- Foldover Designs
- Mixed-Level Designs

# Screening Designs

We refer to a design as a ***screening design***, if its primary purpose is to identify significant main effects rather than interaction effects.

Screening designs are typically of resolution III.

In designs of resolution IV, main effects are confounded with three-way interactions. This is better from the confounding viewpoint, but the designs require more runs than a resolution III design.

# Screening Designs

*R.L. Plackett and J.P. Burman, “The Design of Optimal Multifactorial Experiments”, Biometrika (vol. 33), 1946*.

PB designs are efficient for detecting large main effects, assuming all interactions are negligible.

- 12 runs for up to 11 factors;
- 20-runs for up to 19 factors;
- 24-runs for up to 23 factors;
- 28-runs for up to 27 factors.

| | Pattern | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | +++++++++++ | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 |
| 2 | -+-+++---+- | -1 | +1 | -1 | +1 | +1 | +1 | -1 | -1 | -1 | +1 | -1 |
| 3 | --+-+++---+ | -1 | -1 | +1 | -1 | +1 | +1 | +1 | -1 | -1 | -1 | +1 |
| 4 | +--+-+++--- | +1 | -1 | -1 | +1 | -1 | +1 | +1 | +1 | -1 | -1 | -1 |
| 5 | -+--+-+++-- | -1 | +1 | -1 | -1 | +1 | -1 | +1 | +1 | +1 | -1 | -1 |
| 6 | --+--+-+++- | -1 | -1 | +1 | -1 | -1 | +1 | -1 | +1 | +1 | +1 | -1 |
| 7 | ---+--+-+++ | -1 | -1 | -1 | +1 | -1 | -1 | +1 | -1 | +1 | +1 | +1 |
| 8 | +---+--+-++ | +1 | -1 | -1 | -1 | +1 | -1 | -1 | +1 | -1 | +1 | +1 |
| 9 | ++---+--+-+ | +1 | +1 | -1 | -1 | -1 | +1 | -1 | -1 | +1 | -1 | +1 |
| 10 | +++---+--+- | +1 | +1 | +1 | -1 | -1 | -1 | +1 | -1 | -1 | +1 | -1 |
| 11 | -+++---+--+ | -1 | +1 | +1 | +1 | -1 | -1 | -1 | +1 | -1 | -1 | +1 |
| 12 | +-+++---+-- | +1 | -1 | +1 | +1 | +1 | -1 | -1 | -1 | +1 | -1 | -1 |

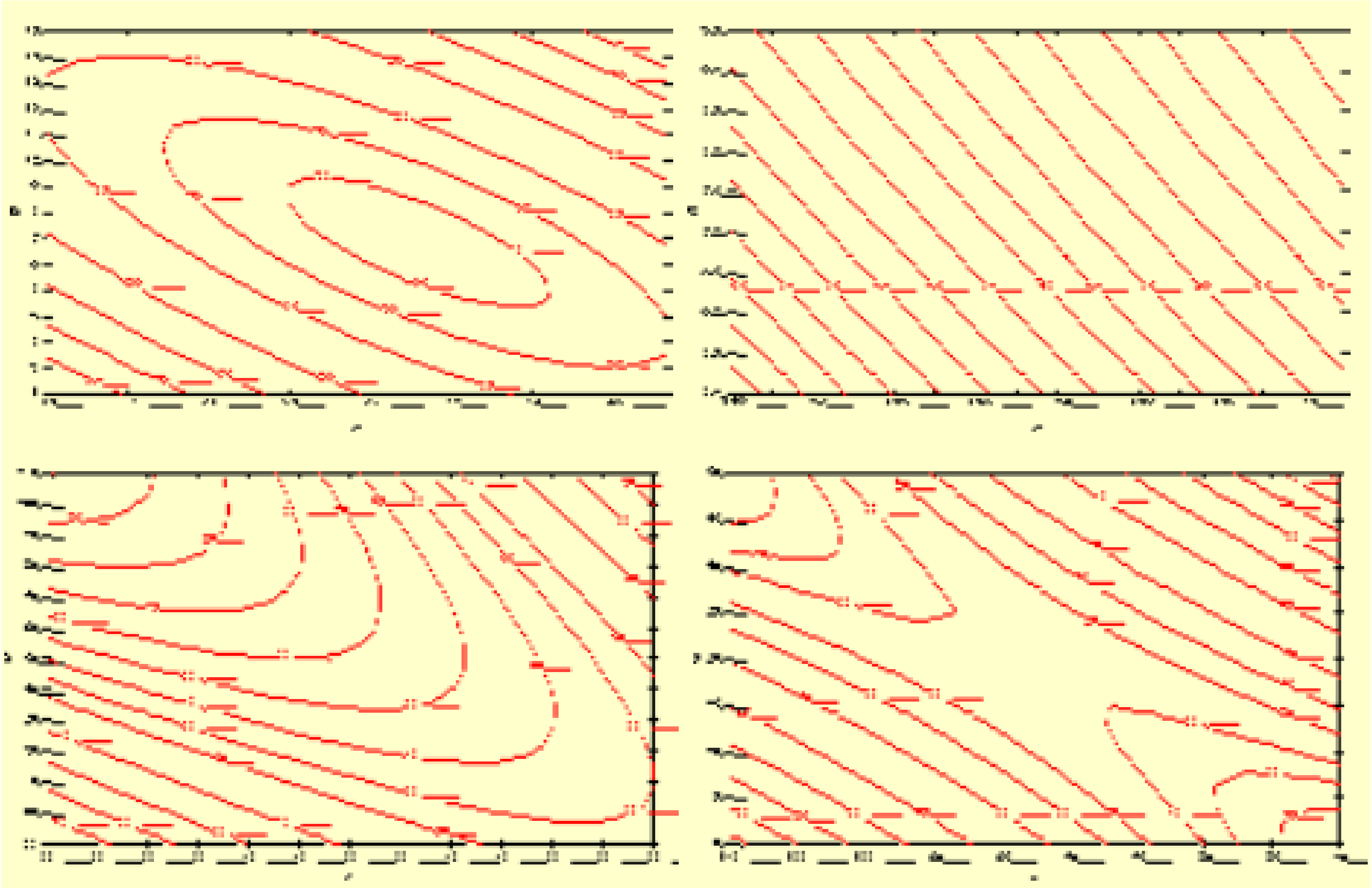
# Response Surface Designs

A complete description of the process behavior might require a quadratic or cubic model:

- Quadratic: $\hat{y} = \beta_0 + \beta_1 x_1 + \cdots + \beta_{12} x_1 x_2 + \cdots + \beta_{11} x_1^2 + \cdots;$
- Cubic: $\hat{y} =$ Quadratic $+ \beta_{123} x_1 x_2 x_3 + \beta_{112} x_1^2 x_2 + \cdots + \beta_{111} x_1^3 + \cdots;$

Response surface method (RSM) designs allow us to estimate interaction and even quadratic effects, and therefore give us an idea of the (local) shape of the response surface.

# Response Surface Designs



General quadratic surface types:

| | |
|---|---|
| Peak | Hillside |
| Rising Ridge | Saddle |

# Response Surface Designs

If a response behaves as linear, the design matrix to quantify that behavior need only contain factors with two levels -- low and high.

If a response behaves as quadratic, the minimum number of levels required for a factor to quantify that behavior is three.

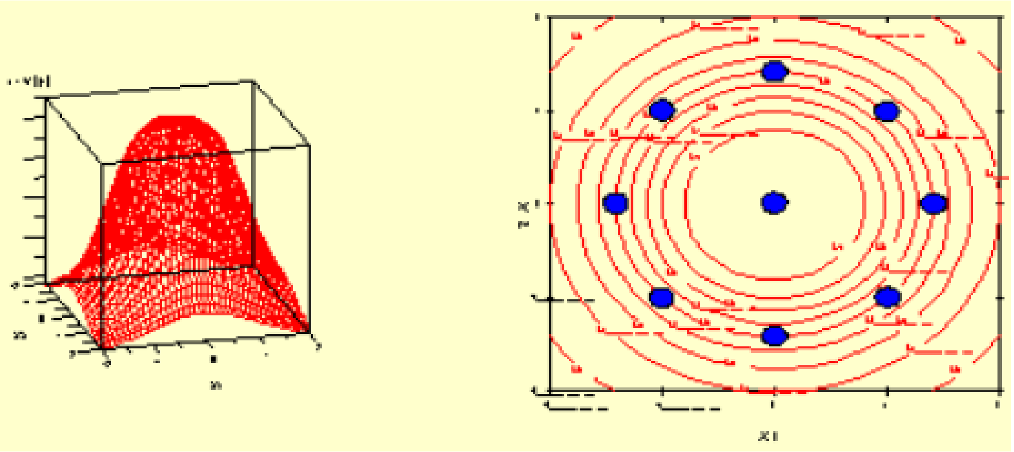
- A two-level experiment with center points can detect, but not estimate, quadratic effects.

A solution to creating a design matrix that permits the estimation of simple curvature would be to use a three-level factorial design.

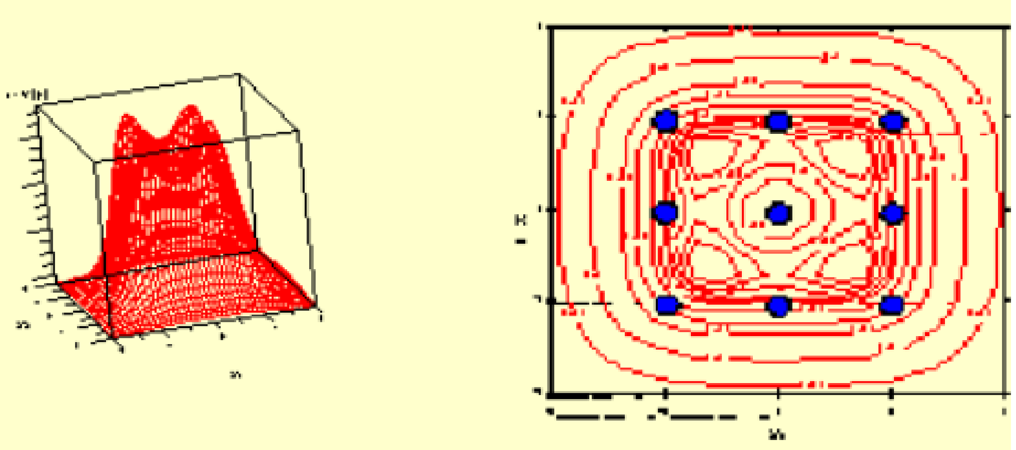
# Response Surface Designs

The confounding structure for three-level fractional factorial designs is considerably more complex and harder to define than in the two-level case.

Additionally, three-level factorial designs suffer a major flaw in their lack of 'rotatability.'



Information function of a rotatable $2^2$ design.



Information function of a $3^2$ design.

# Response Surface Designs

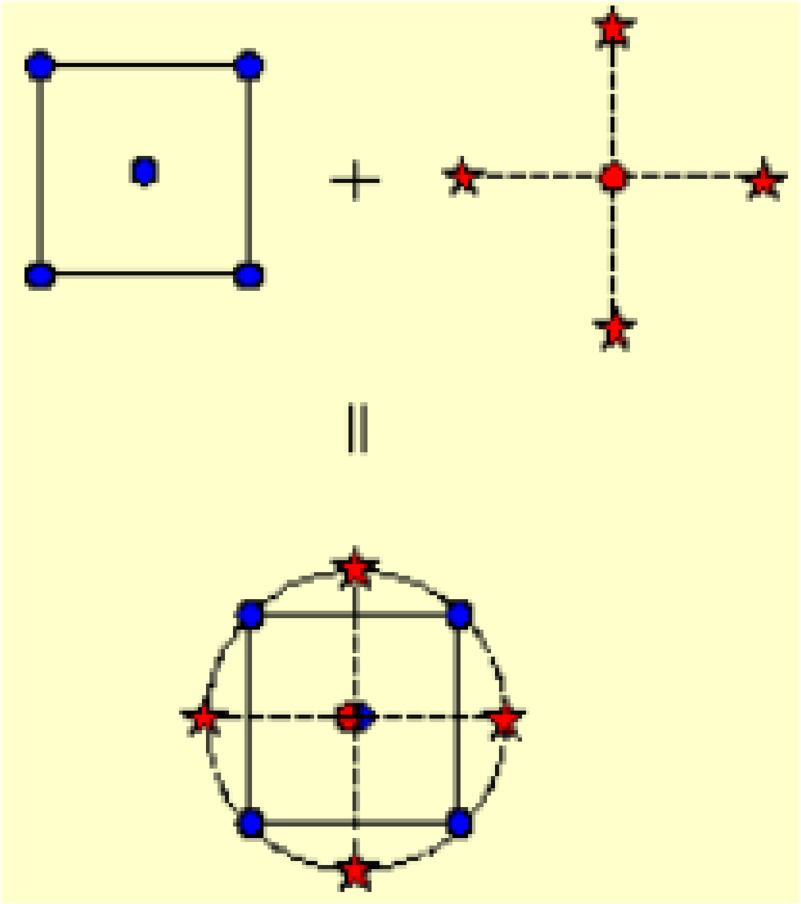
Central Composite (CC) Designs

CC designs start with a factorial or fractional factorial design (with center points) and add "star" points (2*k) to estimate curvature.

If the distance from the center of the design space to each factorial point is 1 unit, the distance from the center to a star point is $\alpha$.

To maintain rotatability,

$\alpha$=[number of fractional factorial runs]$^{1/4}$.



# Response Surface Designs

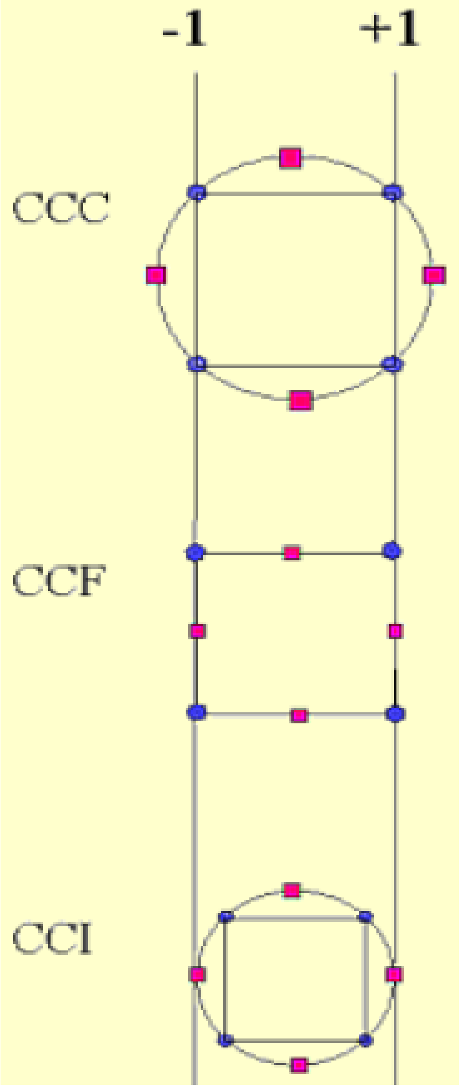
## Central Composite (CC) Designs

| Number of Factors | Factorial Portion | Scaled Value for $\alpha$ Relative to $\pm 1$ |
|---|---|---|
| 2 | $2^2$ | $2^{2/4} = 1.414$ |
| 3 | $2^3$ | $2^{3/4} = 1.682$ |
| 4 | $2^4$ | $2^{4/4} = 2.000$ |
| 5 | $2^{5-1}$ | $2^{4/4} = 2.000$ |
| 5 | $2^5$ | $2^{5/4} = 2.378$ |
| 6 | $2^{6-1}$ | $2^{5/4} = 2.378$ |
| 6 | $2^6$ | $2^{6/4} = 2.828$ |

Determining α for Rotatability

# Response Surface Designs

| CC Design Type | Terminology | Comments |
|---|---|---|
| Circumscribed | CCC | The star points are at distance α from the center, where α depends on the properties desired for the design. Augmenting an existing factorial or resolution V design can produce this design. |
| Face Centered | CCF | The star points are at distance α=1 from the center. Augmenting an existing factorial or resolution V design can produce this design. |
| Inscribed | CCI | When the limits specified for factor settings are truly limits, the CCI design uses the factor settings as the star points and creates a (fractional) factorial design within those limits (that is, a CCI design is a scaled down CCC design with each factor level of the CCC design divided by $\alpha$). |



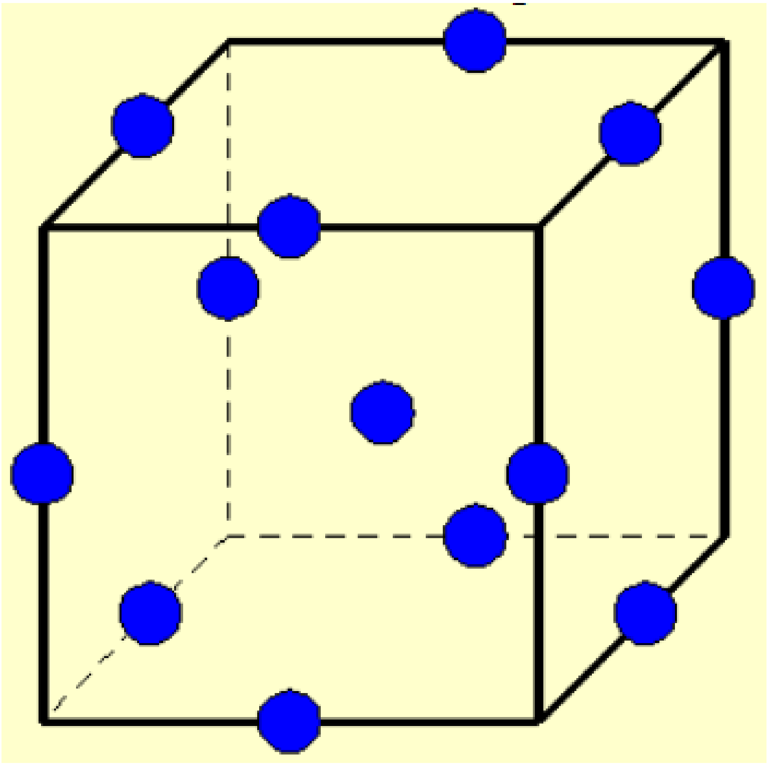
# Response Surface Designs

Box-Behnken (BB) designs

BB designs do not contain an embedded factorial or fractional factorial design.

The treatment combinations are at the midpoints of edges of the design space and at the center.

BB designs are rotatable (or near rotatable) and require 3 levels of each factor.



Comparing CC and BB designs for three factors.

| CCC (CCI) | | | | CCF | | | | Box-Behnken | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Rep | $X_1$ | $X_2$ | $X_3$ | Rep | $X_1$ | $X_2$ | $X_3$ | Rep | $X_1$ | $X_2$ | $X_3$ |
| 1 | -1 | -1 | -1 | 1 | -1 | -1 | -1 | 1 | -1 | -1 | 0 |
| 1 | +1 | -1 | -1 | 1 | +1 | -1 | -1 | 1 | +1 | -1 | 0 |
| 1 | -1 | +1 | -1 | 1 | -1 | +1 | -1 | 1 | -1 | +1 | 0 |
| 1 | +1 | +1 | -1 | 1 | +1 | +1 | -1 | 1 | +1 | +1 | 0 |
| 1 | -1 | -1 | +1 | 1 | -1 | -1 | +1 | 1 | -1 | 0 | -1 |
| 1 | +1 | -1 | +1 | 1 | +1 | -1 | +1 | 1 | +1 | 0 | -1 |
| 1 | -1 | +1 | +1 | 1 | -1 | +1 | +1 | 1 | -1 | 0 | +1 |
| 1 | +1 | +1 | +1 | 1 | +1 | +1 | +1 | 1 | +1 | 0 | +1 |
| 1 | -1.682 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | 0 | -1 | -1 |
| 1 | 1.682 | 0 | 0 | 1 | +1 | 0 | 0 | 1 | 0 | +1 | -1 |
| 1 | 0 | -1.682 | 0 | 1 | 0 | -1 | 0 | 1 | 0 | -1 | +1 |
| 1 | 0 | 1.682 | 0 | 1 | 0 | +1 | 0 | 1 | 0 | +1 | +1 |
| 1 | 0 | 0 | -1.682 | 1 | 0 | 0 | -1 | 3 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1.682 | 1 | 0 | 0 | +1 | | | | |
| 6 | 0 | 0 | 0 | 6 | 0 | 0 | 0 | | | | |
| Total Runs = 20 | | | | Total Runs = 20 | | | | Total Runs = 15 | | | |

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| Design Type | Comment |
|---|---|
| CCC | CCC designs provide high quality predictions over the entire design space but require factor settings outside the range of the factors in the factorial part. |
| CCI | CCI designs use only points within the factor ranges originally specified, but do not provide the same high-quality prediction over the entire space compared to the CCC. |
| CCF | CCF designs provide relatively high-quality predictions over the entire design space and do not require using points outside the original factor range. However, they give poor precision for estimating pure quadratic coefficients. |
| Box-Behnken | BB designs require fewer treatment combinations than a CC design in case of 3 or 4 factors. The BB design is rotatable (or nearly) but it contains regions of poor prediction quality like the CCI. |

# Response Surface Designs

An important point to take into account when choosing a RSM design is the possibility of running the design in blocks.

If an investigator has run either a $2^k$ full factorial or a $2^{k-p}$ fractional factorial design of at least resolution V, augmentation of that design to a CC design is easily accomplished by adding an additional block of star and centerpoint runs.

If the factorial experiment indicated curvature, this composite augmentation is the best follow-up option.

# Foldover Designs

Fractional factorial designs confound main effects with certain interactions.

Run a few additional treatment combinations to remove certain confounding structures:

- Mirror-image foldover design (to de-alias all the main effect estimates from two-way interactions).
- Alternative foldover designs (to break up specific confounding patterns).

# Mirror-Image Foldover Design

A mirror-image foldover design is to build a resolution IV design from a resolution III design.

It is obtained by reversing the signs of all the columns of the original design matrix.

The augmented design can estimate all main effects clear of any two-way interaction.

# Mirror-Image Foldover Design

| run | $X_1$ | $X_2$ | $X_3$ | $X_4 = X_1X_2$ | $X_5 = X_1X_3$ |
|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | +1 | +1 |
| 2 | +1 | -1 | -1 | -1 | -1 |
| 3 | -1 | +1 | -1 | -1 | +1 |
| 4 | +1 | +1 | -1 | +1 | -1 |
| 5 | -1 | -1 | +1 | +1 | -1 |
| 6 | +1 | -1 | +1 | -1 | +1 |
| 7 | -1 | +1 | +1 | -1 | -1 |
| 8 | +1 | +1 | +1 | +1 | +1 |

| run | $X_1$ | $X_2$ | $X_3$ | $X_4 = -X_1X_2$ | $X_5 = -X_1X_3$ |
|---|---|---|---|---|---|
| 9 | +1 | +1 | +1 | -1 | -1 |
| 10 | -1 | +1 | +1 | +1 | +1 |
| 11 | +1 | -1 | +1 | +1 | -1 |
| 12 | -1 | -1 | +1 | -1 | +1 |
| 13 | +1 | +1 | -1 | -1 | +1 |
| 14 | -1 | +1 | -1 | +1 | -1 |
| 15 | +1 | -1 | -1 | +1 | +1 |
| 16 | -1 | -1 | -1 | -1 | -1 |

The defining relation is I = 124 = 135 = 2345.
Every main effect is confounded with at least one two-way interaction.

Increase the resolution to IV by adding on 8 runs from the mirror-image foldover design with the defining relation I = -124 = -135 = 2345.

# Mirror-Image Foldover Design

| run | $X_1$ | $X_2$ | $X_3$ | $X_4 = X_1X_2$ | $X_5 = X_1X_3$ | $X_6 = X_2X_3$ | $X_7 = X_1X_2X_3$ |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | +1 | +1 | +1 | -1 |
| 2 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| 3 | -1 | +1 | -1 | -1 | +1 | -1 | +1 |
| 4 | +1 | +1 | -1 | +1 | -1 | -1 | -1 |
| 5 | -1 | -1 | +1 | +1 | -1 | -1 | +1 |
| 6 | +1 | -1 | +1 | -1 | +1 | -1 | -1 |
| 7 | -1 | +1 | +1 | -1 | -1 | +1 | -1 |
| 8 | +1 | +1 | +1 | +1 | +1 | +1 | +1 |

| run | $X_1$ | $X_2$ | $X_3$ | $X_4 = -X_1X_2$ | $X_5 = -X_1X_3$ | $X_6 = -X_2X_3$ | $\boxed{X_7 = X_1X_2X_3}$ |
|---|---|---|---|---|---|---|---|
| 1 | +1 | +1 | +1 | -1 | -1 | -1 | +1 |
| 2 | -1 | +1 | +1 | +1 | +1 | -1 | -1 |
| 3 | +1 | -1 | +1 | +1 | -1 | +1 | -1 |
| 4 | -1 | -1 | +1 | -1 | +1 | +1 | +1 |
| 5 | +1 | +1 | -1 | -1 | +1 | +1 | -1 |
| 6 | -1 | +1 | -1 | +1 | -1 | +1 | +1 |
| 7 | +1 | -1 | -1 | +1 | +1 | -1 | +1 |
| 8 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |

$2_{III}^{7-4}$, all the main effects are confounded with two- and higher-way interaction terms.
The defining relation is
I = 124 = 135 = 236 = 347 = 257 = 167 = 456 =…

Ignore all interactions with 3 or more factors:

| | |
|---|---|
| 1 = 24 = 35 = 67 | 1 = -24 = -35 = -67 |
| 2 = 14 = 36 = 57 | 2 = -14 = -36 =- 57 |
| 3 = 15 = 26 = 47 | 3 = -15 = -26 = -47 |
| 4 = 12 = 37 = 56 | 4 = -12 = -37 = -56 |
| 5 = 13 = 27 = 46 | 5 = -13 = -27 = -46 |
| 6 = 17 = 23 = 45 | 6 = -17 = -23 = -45 |
| 7 = 16 = 25 = 34 | 7 = -16 = -25 = -34 |

# Alternative Foldover Designs

By reversing ALL the columns, we can de-alias ALL the main-effect estimates from two-way interactions.

What if we want to estimate certain two-way interactions?

Reverse a single factor column to obtain de-aliased two-way interactions for that one factor (works for any resolution III or IV design).

# Alternative Foldover Designs

| run | $X_1$ | $X_2$ | $X_3$ | $X_4 = X_1X_2$ | $X_5 = X_1X_3$ | $X_6 = X_2X_3$ | $X_7 = X_1X_2X_3$ |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | +1 | +1 | +1 | -1 |
| 2 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| 3 | -1 | +1 | -1 | -1 | +1 | -1 | +1 |
| 4 | +1 | +1 | -1 | +1 | -1 | -1 | -1 |
| 5 | -1 | -1 | +1 | +1 | -1 | -1 | +1 |
| 6 | +1 | -1 | +1 | -1 | +1 | -1 | -1 |
| 7 | -1 | +1 | +1 | -1 | -1 | +1 | -1 |
| 8 | +1 | +1 | +1 | +1 | +1 | +1 | +1 |

| run | $X_1$ | $X_2$ | $X_3$ | $X_4 = -X_1X_2$ | $X_5 = X_1X_3$ | $X_6 = X_2X_3$ | $X_7 = X_1X_2X_3$ |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | +1 | +1 | -1 |
| 2 | +1 | -1 | -1 | +1 | -1 | +1 | +1 |
| 3 | -1 | +1 | -1 | +1 | +1 | -1 | +1 |
| 4 | +1 | +1 | -1 | -1 | -1 | -1 | -1 |
| 5 | -1 | -1 | +1 | -1 | -1 | -1 | +1 |
| 6 | +1 | -1 | +1 | +1 | +1 | -1 | -1 |
| 7 | -1 | +1 | +1 | +1 | -1 | +1 | -1 |
| 8 | +1 | +1 | +1 | -1 | +1 | +1 | +1 |

$2_{III}^{7-4}$, all the main effects are confounded with two- and higher-way interaction terms.

The defining relation is

I = 124 = 135 = 236 = 347 = 257 = 167 = 456 =…

De-confound the $X_4$ factor only:

| | |
|---|---|
| 1 = 24 = 35 = 67 | 1 = -24 = -35 = -67 |
| 2 = 14 = 36 = 57 | 2 = -14 = -36 = -57 |
| 3 = 15 = 26 = 47 | 3 = -15 = -26 = -47 |
| 4 = 12 = 37 = 56 | 4 = -12 = -37 = -56 |
| 5 = 13 = 27 = 46 | 5 = -13 = -27 = -46 |
| 6 = 17 = 23 = 45 | 6 = -17 = -23 = -45 |
| 7 = 16 = 25 = 34 | 7 = -16 = -25 = -34 |

# Alternative Foldover Designs

| run | $X_1$ | $X_2$ | $X_3$ | $X_4 = X_1X_2$ | $X_5 = X_1X_3$ | $X_6 = X_2X_3$ | $X_7 = X_1X_2X_3$ |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | +1 | +1 | +1 | -1 |
| 2 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| 3 | -1 | +1 | -1 | -1 | +1 | -1 | +1 |
| 4 | +1 | +1 | -1 | +1 | -1 | -1 | -1 |
| 5 | -1 | -1 | +1 | +1 | -1 | -1 | +1 |
| 6 | +1 | -1 | +1 | -1 | +1 | -1 | -1 |
| 7 | -1 | +1 | +1 | -1 | -1 | +1 | -1 |
| 8 | +1 | +1 | +1 | +1 | +1 | +1 | +1 |

| run | $X_1$ | $X_2$ | $X_3$ | $X_4 = -X_1X_2$ | $X_5 = X_1X_3$ | $X_6 = X_2X_3$ | $X_7 = X_1X_2X_3$ |
|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | +1 | +1 | -1 |
| 2 | +1 | -1 | -1 | +1 | -1 | +1 | +1 |
| 3 | -1 | +1 | -1 | +1 | +1 | -1 | +1 |
| 4 | +1 | +1 | -1 | -1 | -1 | -1 | -1 |
| 5 | -1 | -1 | +1 | -1 | -1 | -1 | +1 |
| 6 | +1 | -1 | +1 | +1 | +1 | -1 | -1 |
| 7 | -1 | +1 | +1 | +1 | -1 | +1 | -1 |
| 8 | +1 | +1 | +1 | -1 | +1 | +1 | +1 |

$X_1 + X_3X_5 + X_6X_7$

$X_2 + X_3X_6 + X_5X_7$

$X_3 + X_1X_5 + X_2X_6$

$\boxed{X_4}$

$X_5 + X_1X_3 + X_2X_7$

$X_6 + X_2X_3 + X_1X_7$

$X_7 + X_2X_5 + X_1X_6$

$X_1X_4$

$X_2X_4$

$X_3X_4$

$X_4X_5$

$X_4X_6$

$X_4X_7$

$X_1X_2 + X_3X_7 + X_5X_6$

The 16 runs allow estimating 14 effects.

It permits estimation of the $X_4$ effect **and** the six two-way interaction terms involving $X_4$.

# Mixed-Level Designs

Montgomery (1991):

*It is our belief that the two-level factorial and fractional factorial designs should be the cornerstone of industrial experimentation for product and process development and improvement.*

*There are, however, some situations in which it is necessary to include a factor (or a few factors) that have more than two levels.*

# Mixed-Level Designs

Generate, from a $2^3$ design, a design with one 2-level factor A and one 3-level factor X.

| Two-Level | | Three-Level |
|---|---|---|
| B | C | X |
| -1 | -1 | $x_1$ |
| +1 | -1 | $x_2$ |
| -1 | +1 | $x_2$ |
| +1 | +1 | $x_3$ |

| | A | $X_L$ | $X_L$ | $AX_L$ | $AX_L$ | $X_Q$ | $AX_Q$ | TRT | MNT |
|---|---|---|---|---|---|---|---|---|---|
| **Run** | **A** | **B** | **C** | **AB** | **AC** | **BC** | **ABC** | **A** | **X** |
| 1 | -1 | -1 | -1 | +1 | +1 | +1 | -1 | Low | Low |
| 2 | +1 | -1 | -1 | -1 | -1 | +1 | +1 | High | Low |
| 3 | -1 | +1 | -1 | -1 | +1 | -1 | +1 | Low | Medium |
| 4 | +1 | +1 | -1 | +1 | -1 | -1 | -1 | High | Medium |
| 5 | -1 | -1 | +1 | +1 | -1 | -1 | +1 | Low | Medium |
| 6 | +1 | -1 | +1 | -1 | +1 | -1 | -1 | High | Medium |
| 7 | -1 | +1 | +1 | -1 | -1 | +1 | -1 | Low | High |
| 8 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | High | High |

# Mixed-Level Designs

Generate, from a $2^3$ design, a resolution III design with four 2-level factors and one 3-level factor.

{A, AB, AC, ABC} are used for the two-level factors.

BC cannot be used because it contains the quadratic effect of the three-level factor X.

| | A | $X_L$ | $X_L$ | $AX_L$ | $AX_L$ | $X_Q$ | $AX_Q$ | TRT | MNT |
|---|---|---|---|---|---|---|---|---|---|
| **Run** | **A** | **B** | **C** | **AB** | **AC** | **BC** | **ABC** | **A** | **X** |
| 1 | -1 | -1 | -1 | +1 | +1 | +1 | -1 | Low | Low |
| 2 | +1 | -1 | -1 | -1 | -1 | +1 | +1 | High | Low |
| 3 | -1 | +1 | -1 | -1 | +1 | -1 | +1 | Low | Medium |
| 4 | +1 | +1 | -1 | +1 | -1 | -1 | -1 | High | Medium |
| 5 | -1 | -1 | +1 | +1 | -1 | -1 | +1 | Low | Medium |
| 6 | +1 | -1 | +1 | -1 | +1 | -1 | -1 | High | Medium |
| 7 | -1 | +1 | +1 | -1 | -1 | +1 | -1 | Low | High |
| 8 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | High | High |

# Mixed-Level Designs

Generate, from a $2^4$ design, a design with two 2-level factors and one 4-level factor.

Combine the -1 and 1 patterns for the A and B factors to form the levels of the 4-level factor X.

| Run | (A | B) | = X | C | D |
|---|---|---|---|---|---|
| 1 | -1 | -1 | $x_1$ | -1 | -1 |
| 2 | +1 | -1 | $x_2$ | -1 | -1 |
| 3 | -1 | +1 | $x_3$ | -1 | -1 |
| 4 | +1 | +1 | $x_4$ | -1 | -1 |
| 5 | -1 | -1 | $x_1$ | +1 | -1 |
| 6 | +1 | -1 | $x_2$ | +1 | -1 |
| 7 | -1 | +1 | $x_3$ | +1 | -1 |
| 8 | +1 | +1 | $x_4$ | +1 | -1 |
| 9 | -1 | -1 | $x_1$ | -1 | +1 |
| 10 | +1 | -1 | $x_2$ | -1 | +1 |
| 11 | -1 | +1 | $x_3$ | -1 | +1 |
| 12 | +1 | +1 | $x_4$ | -1 | +1 |
| 13 | -1 | -1 | $x_1$ | +1 | +1 |
| 14 | +1 | -1 | $x_2$ | +1 | +1 |
| 15 | -1 | +1 | $x_3$ | +1 | +1 |
| 16 | +1 | +1 | $x_4$ | +1 | +1 |

# Mixed-Level Designs

| Run | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 2 | 1 | 1 | 2 | 2 | 3 | 3 |
| 5 | 1 | 2 | 2 | 2 | 3 | 3 | 1 | 1 |
| 6 | 1 | 2 | 3 | 3 | 1 | 1 | 2 | 2 |
| 7 | 1 | 3 | 1 | 2 | 1 | 3 | 2 | 3 |
| 8 | 1 | 3 | 2 | 3 | 2 | 1 | 3 | 1 |
| 9 | 1 | 3 | 3 | 1 | 3 | 2 | 1 | 2 |
| 10 | 2 | 1 | 1 | 3 | 3 | 2 | 2 | 1 |
| 11 | 2 | 1 | 2 | 1 | 1 | 3 | 3 | 2 |
| 12 | 2 | 1 | 3 | 2 | 2 | 1 | 1 | 3 |
| 13 | 2 | 2 | 1 | 2 | 3 | 1 | 3 | 2 |
| 14 | 2 | 2 | 2 | 3 | 1 | 2 | 1 | 3 |
| 15 | 2 | 2 | 3 | 1 | 2 | 3 | 2 | 1 |
| 16 | 2 | 3 | 1 | 3 | 2 | 3 | 1 | 2 |
| 17 | 2 | 3 | 2 | 1 | 3 | 1 | 2 | 3 |
| 18 | 2 | 3 | 3 | 2 | 1 | 2 | 3 | 1 |

Taguchi $L_{18}$ orthogonal array design: a $2 \times 3^{7-5}$ fractional factorial mixed-level design.

1 factor at two levels, and 7 factors at three levels.