A Topological Hierarchy for Functions on Triangulated Surfaces

Authors: Peer-Timo Bremer, Herbert Edelsbrunner, Bernd Hamann and Valerio Pascucci

Presenter: Yansong Yu

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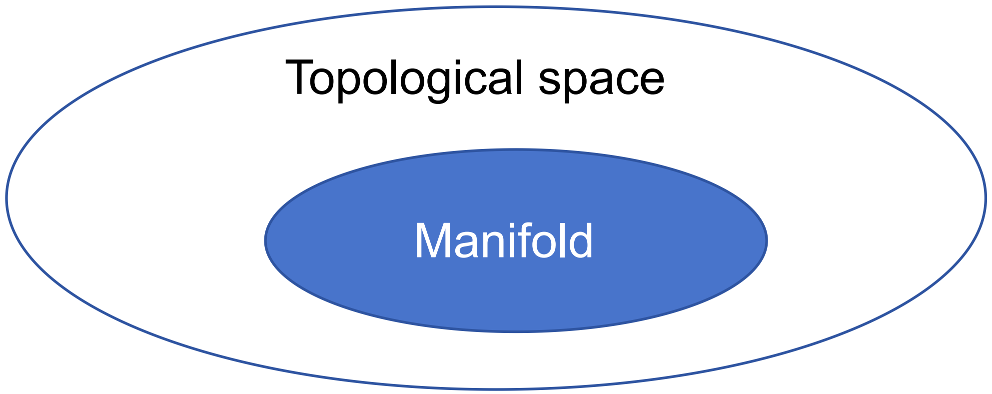
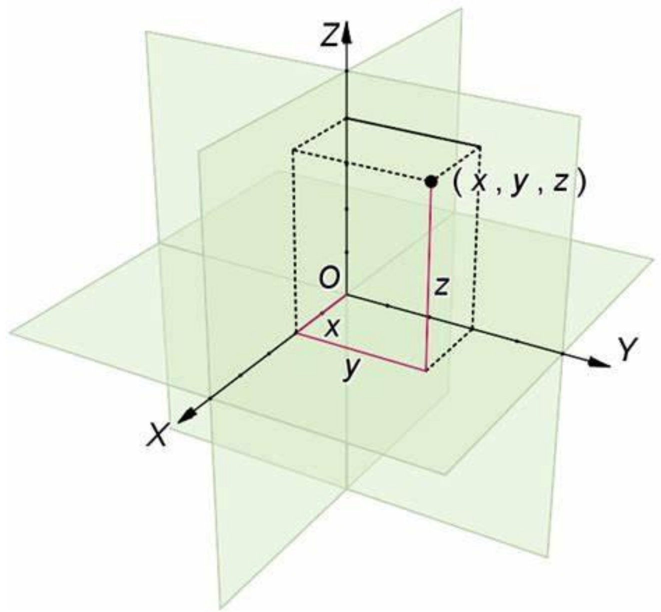
Backgrounds

- Manifold
- Morse functions
- Morse-Smale Complexes
- Piecewise linear functions
- Persistence

Manifold

- In mathematics, a manifold is a topological space that locally **resembles** Euclidean space near each point.
- More precisely, an n-dimensional manifold (n-manifold for short), is a **topological space** with the property that **each point** has a **neighborhood** that is **homeomorphic** to an **open subset** of n-dimensional Euclidean space.



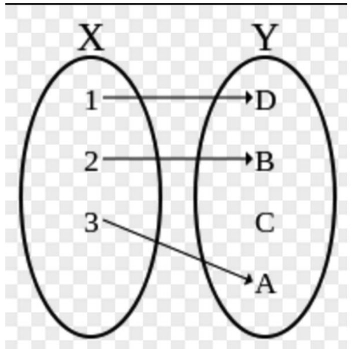


Manifold

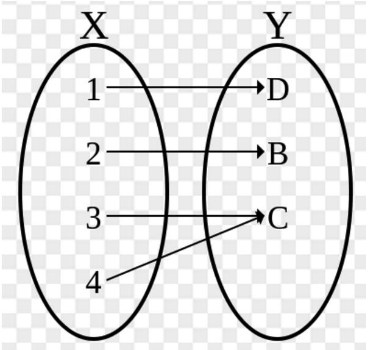
- **Topological space:** {X, T}
  - X is a set
  - T is a subset of P(X)
- T must meet the requirements:
  - Φ in T.
  - X in T.
  - The intersection of a finite number of sets in T is also in T.
  - The union of an arbitrary number of sets in T is also in T.
- Topological space: a set X paired with T (a topology of X).
- Example:
  - $X=R^1$, T={U | U is the union of Open intervals}, T is the standard topology of X
  - $X=R^2$, T={U | U is the union of Open neighbourhoods}, T is the standard topology of X
  - X={1, 2, 3}, then T_1={Φ, X}, T2={Φ, X, {1}, {2}, {1, 2}}, T=P(X)...

Manifold

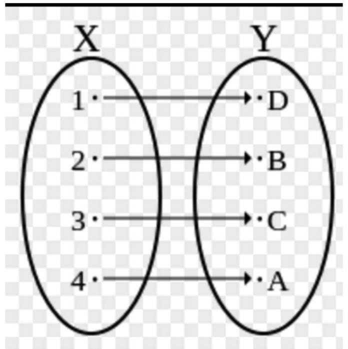
- **Homeomorphism:**
- A function f: $X \rightarrow Y$ between two topological spaces is a **homeomorphism** if it has the following properties:
  - f is a bijection (each points in Y have a preimage in X **AND** preimages are different with each other).
  - f is continuous.
  - f^{-1} is continuous.
- If this function can be found, then X and Y are **homeomorphic.**



one-to-one/injection



onto/surjection



bijection

Manifold

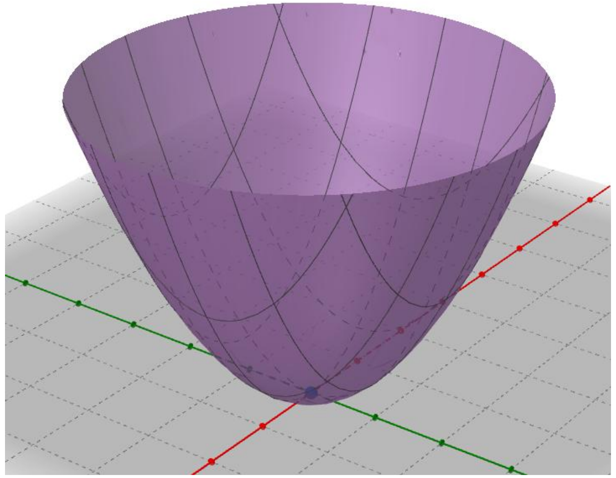
- **Open set:**
  - Each point in this set is neighborhood point, in other words, each point is inside the set (not on the boundary).
- **Examples:**
  - (a, b) is an open set, $[a, b]$, $(a, b]$, $[a, b)$ are not.
  - $\{(x, y) \mid x^2 + y^2 < 1\}$ is an open set, $\{(x, y) \mid x^2 + y^2 <= 1\}$ is not.

Morse functions

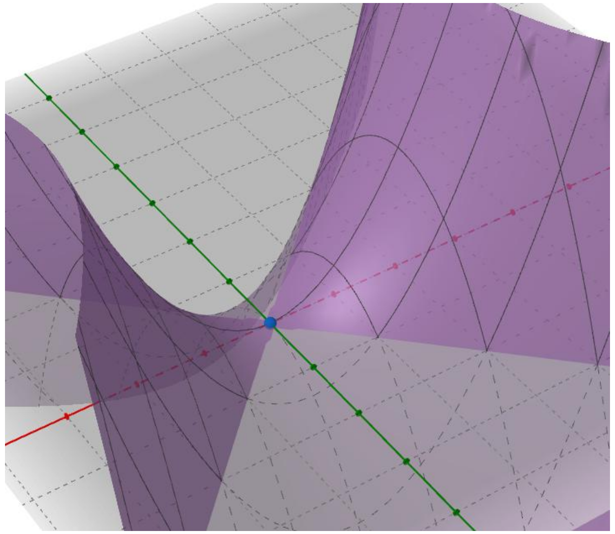
- Morse function is a type of function defined on a manifold M.
- Critical points: For an arbitrary M, take a local coordinate of a point, and calculate its partial derivatives, if **0, then it is critical point.**
- In 2D case, there are 3 types of critical points: max, min, saddle.
- In this paper, the **morse function** is a **real valued function** defined on a 2D **compact manifold without boundary (not sure but from the paper) following the rule**: All of the critical points P of it have non-singular Hessian matrix (Det(H_P) ≠ 0) and have pairwise different function values. **When Det(H_p)=0, we call this a degenerated case.**
- **Morse Lemma**: It is possible to construct a local coordinate system such that f has the form $f(x_1, x_2) = \pm x_1{}^2 \pm x_2{}^2$ in a neighborhood of a non-degenerate critical point.

Morse function

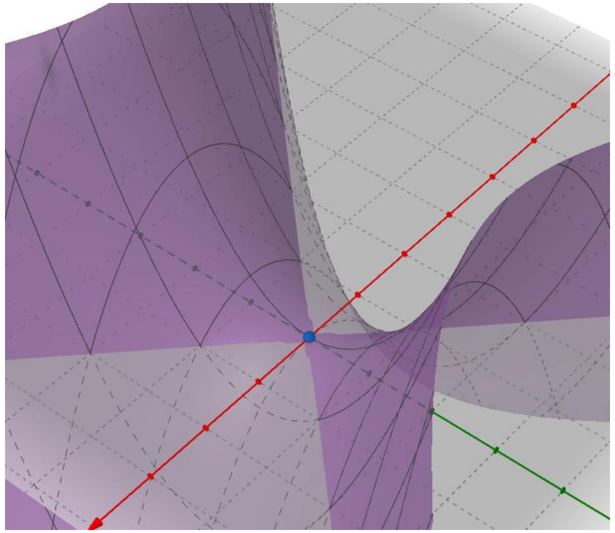
Follow the Morse Lemma, the local representation of f at the neighbourhood of non-degenerated critical points are shown below.



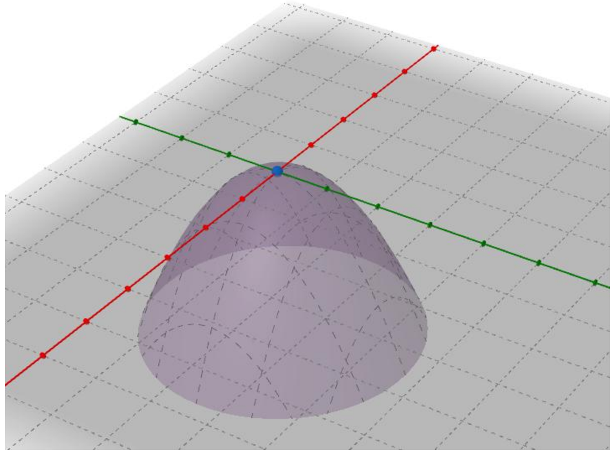
$f(x_1, x_2)=x_1^2+x_2^2$



$f(x_1, x_2)=x_1^2-x_2^2$



$f(x_1, x_2)=-x_1^2+x_2^2$

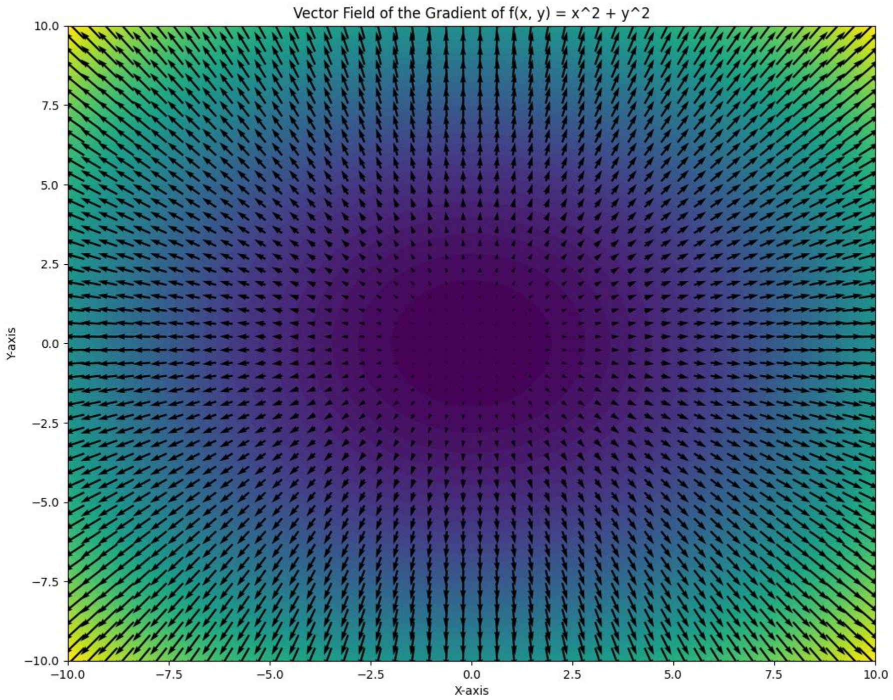


$f(x_1, x_2)=-x_1^2-x_2^2$

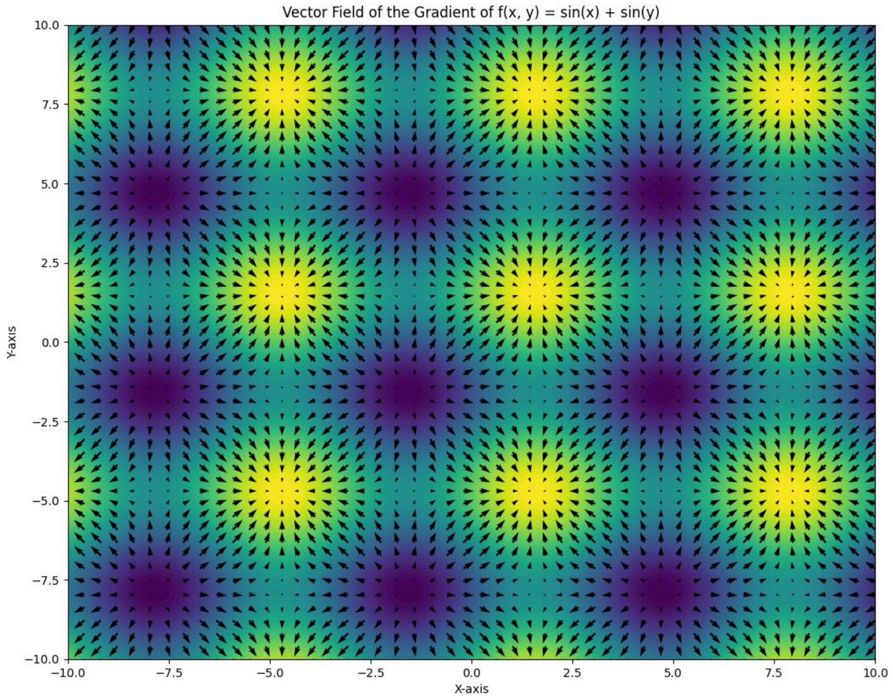
critical index = the number of the “-” in the local representation

Morse-Smale Complexes

- The gradient of a 2D function, for example, $f(x, y) = x^2 + y^2$ and $f(x, y) = \sin(x) + \sin(y)$, form vector fields.



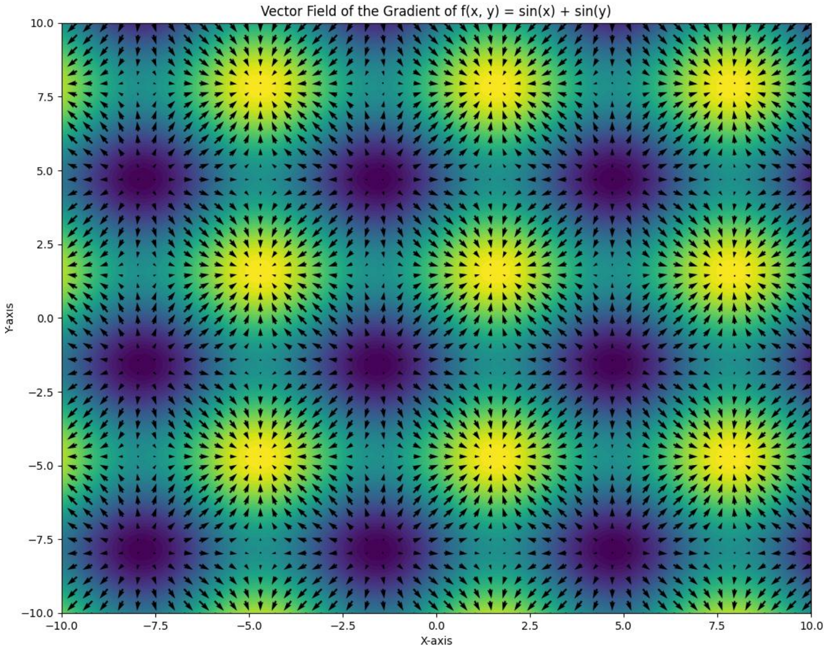
$\nabla f = (2x, 2y)$



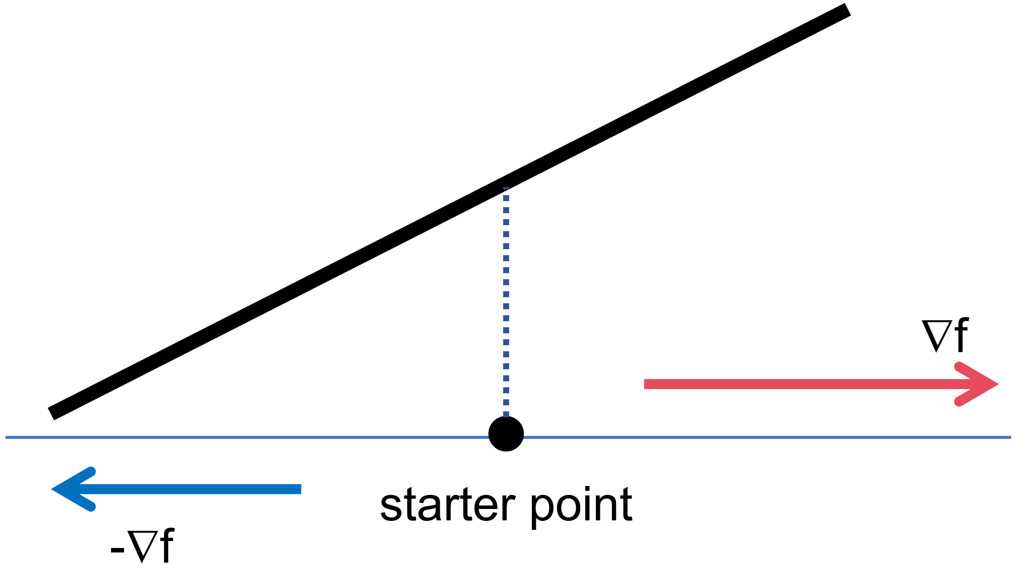
$\nabla f = (\cos(x), \cos(y))$

Morse-Smale Complexes

- From a gradient field like this, we start from a point whose gradient is not **0, then we follow the** ∇f and -∇f, then we can get a **integral line** whose two ends should both be critical points (and technically, the line doesn't contain either of the critical points).
- Since the line follow the gradient, so the line is monotonic, so the 2 criticals are not the same.

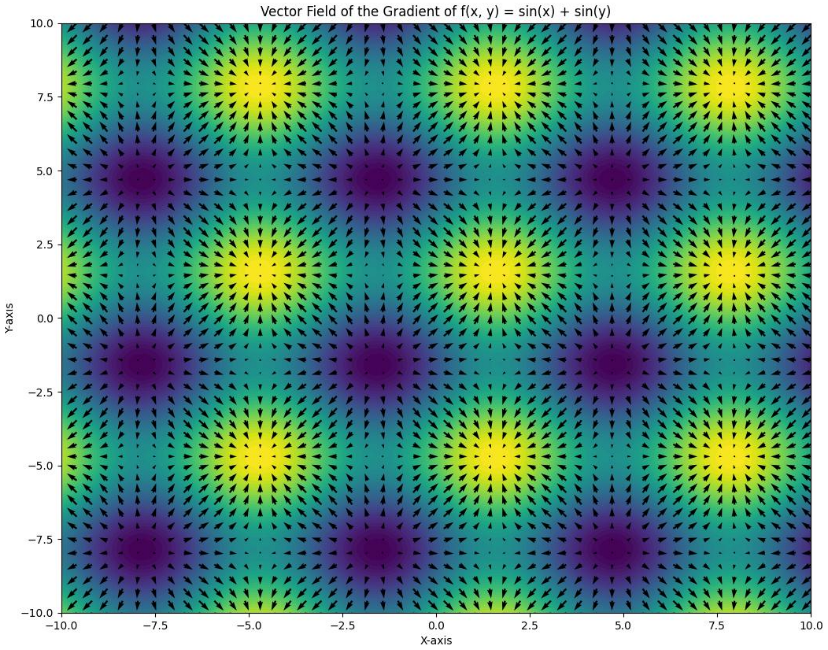


$\nabla f = (\cos(x), \cos(y))$

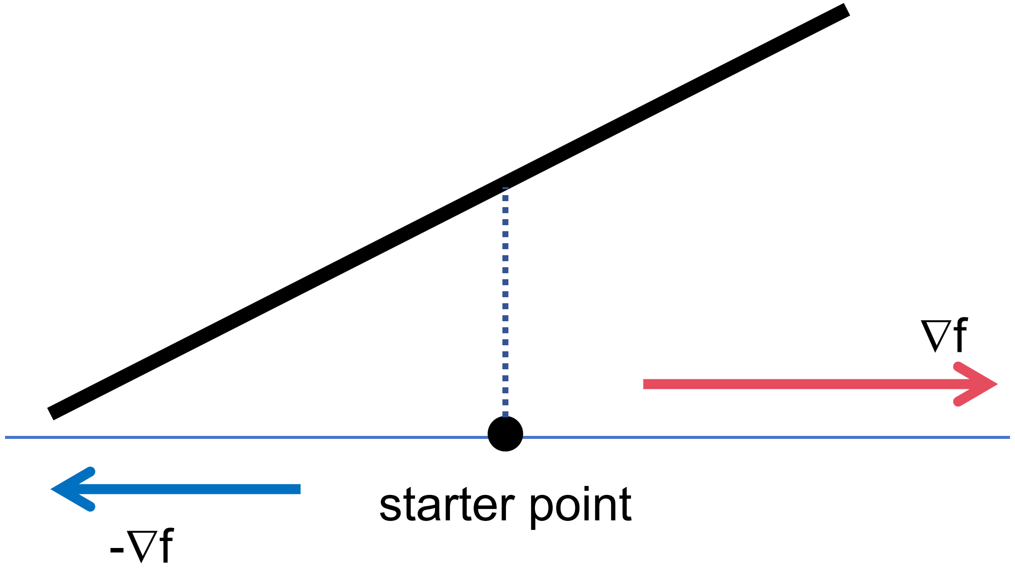


Morse-Smale Complexes

- We can trace multiple integral lines in the function domain (a 2-manifold), and after trace all the integral lines, the integral lines will **cover the entire domain**, except for the criticals. **So basically, we can see three elements which are maximums, minimums, saddles and all integral lines.**

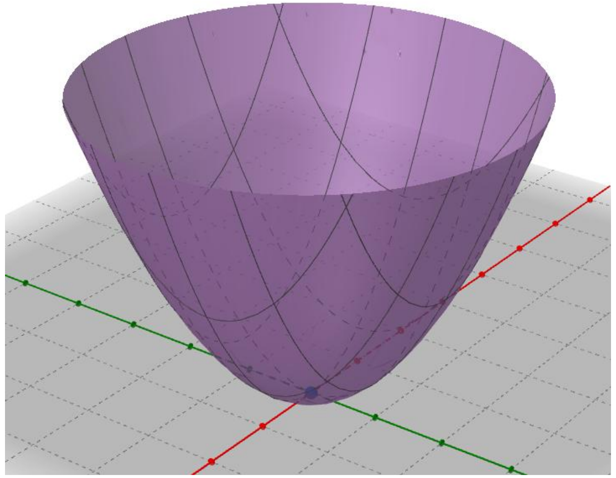


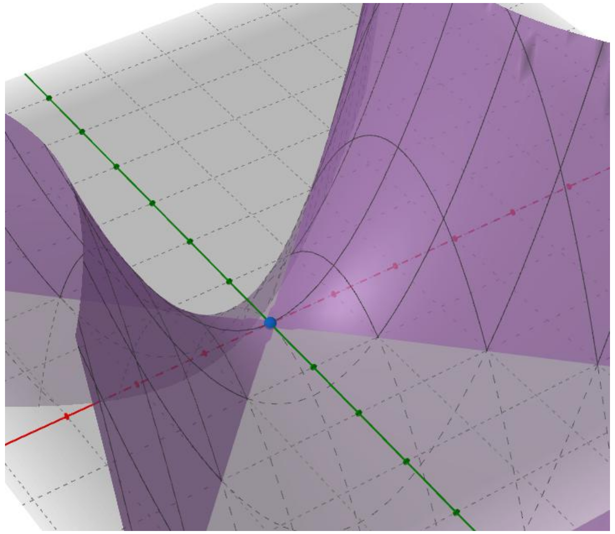
$\nabla f = (\cos(x), \cos(y))$

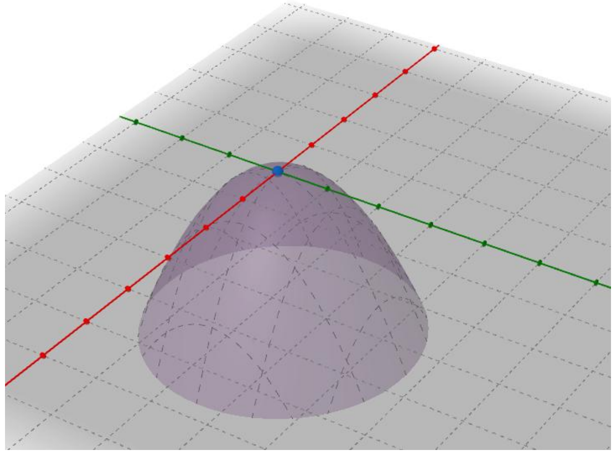


Morse-Smale Complexes

- Here are two terms, in Morse-Smale Complexes.
  - Ascending manifold (A)
  - Descending manifold (D)
- This two terms are used to describe two components for a critical point. For example, in 2D case, we have a critical point **a**, then, follow the ∇f:
  - A(**a**) = {**a**}U{all integral lines starting from **a**}
  - D(**a**) = {**a**}U{all integral lines ending at **a**}

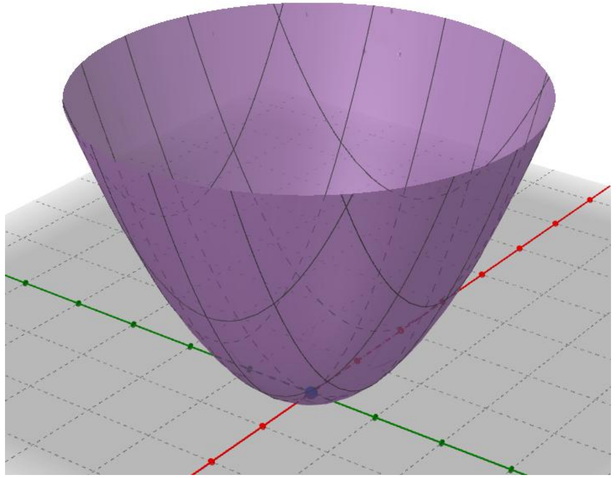
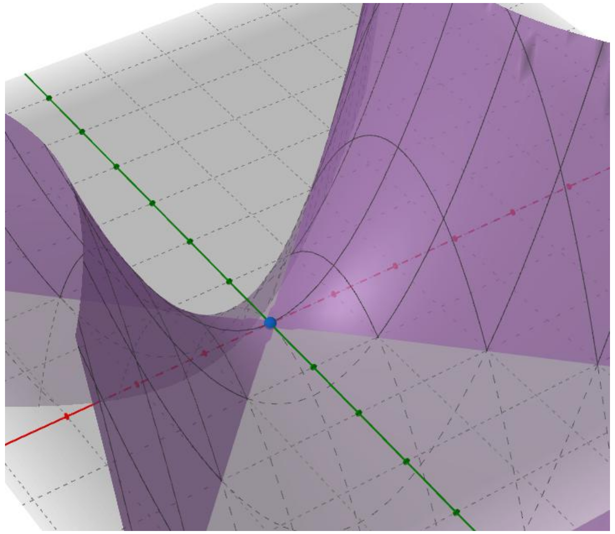
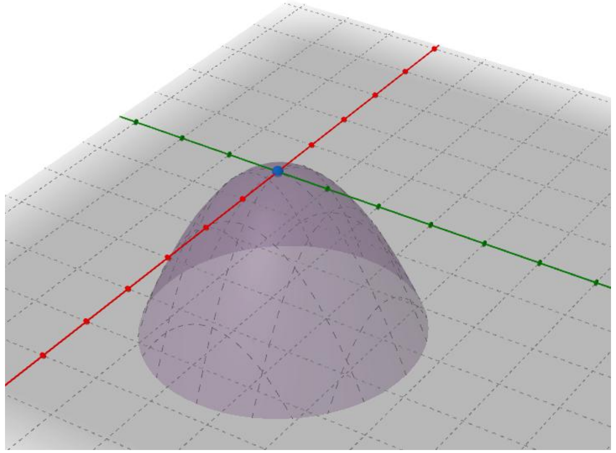






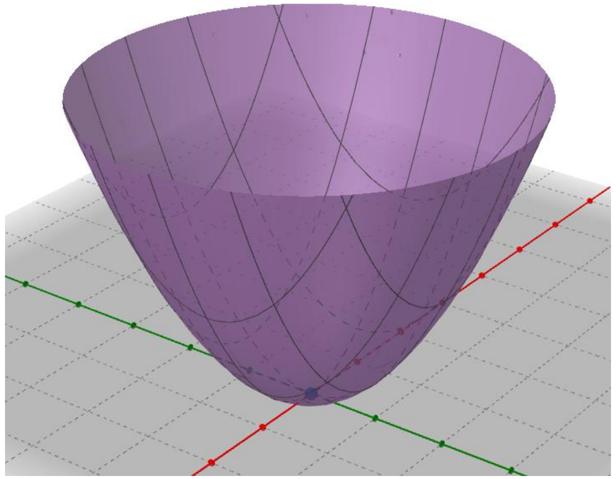
Morse-Smale Complexes

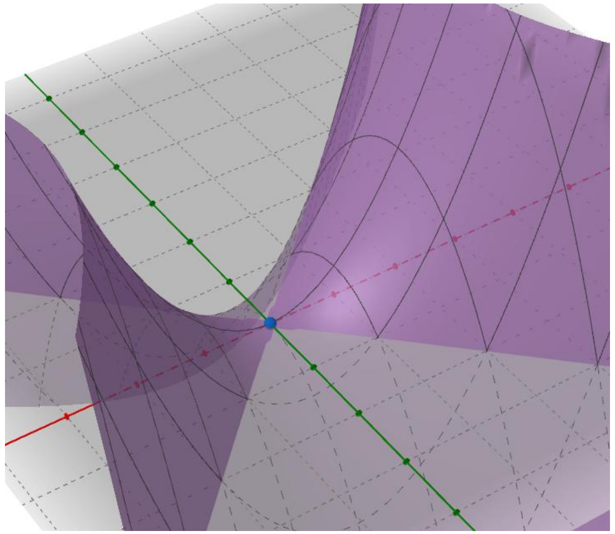
- For a continuous function, the integral lines traced will only intersect at critical points.
- Specifically, same types of integral lines can only intersect at critical points (max, min and saddle), integral lines of different types can only intersect (or cross) at saddles.

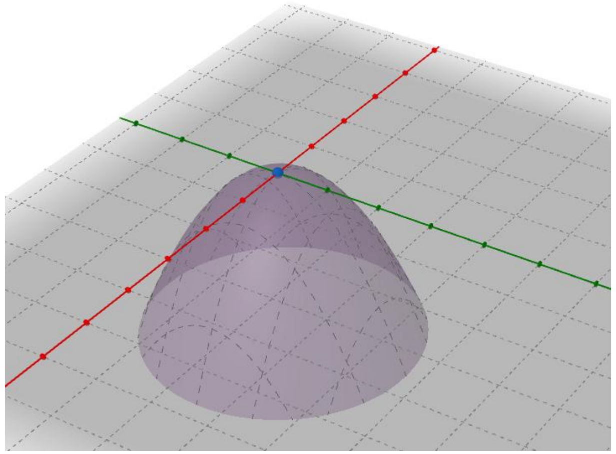




Morse-Smale Complexes

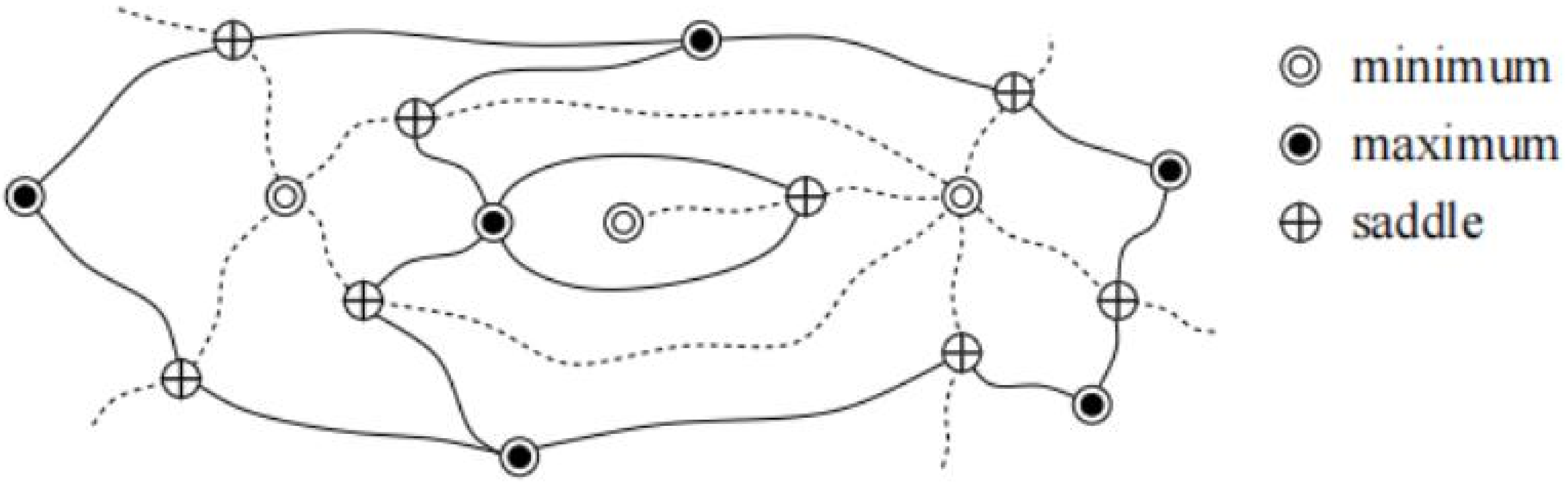
- We have all the integral lines that make up the **ascending and descending manifold in the whole domain of function f(x, y)**, and every critical points. Based on the **non-degeneracy** prerequisit, **the ascending and descending manifold can only intersect transversally (which means they cross at saddle points).**
- **Then the Morse-Smale complex is constructed through the overlay of A and D.**







Morse-Smale Complexes



Piecewise linear functions

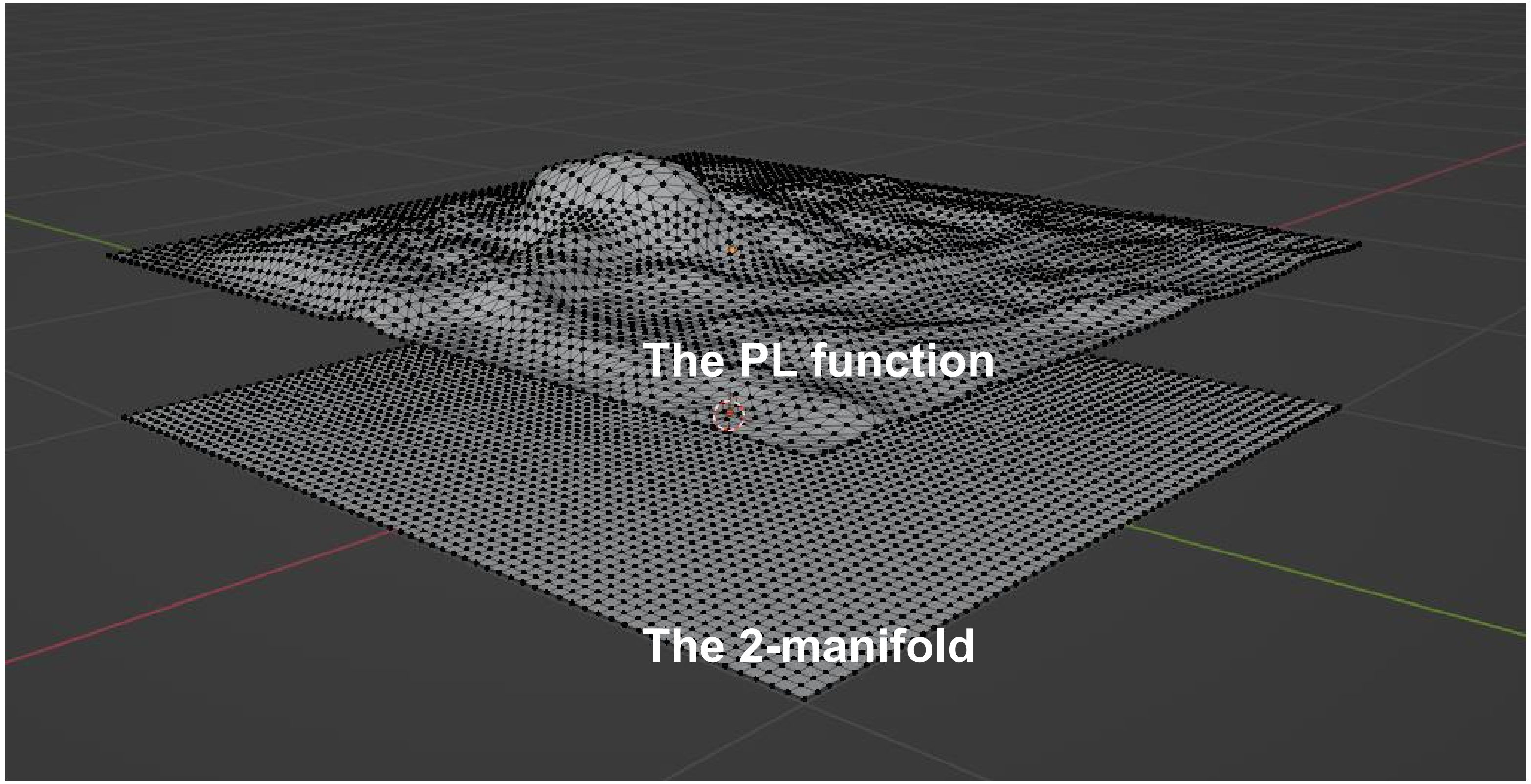
- Function defined on a finite set of points spread out over a manifold.

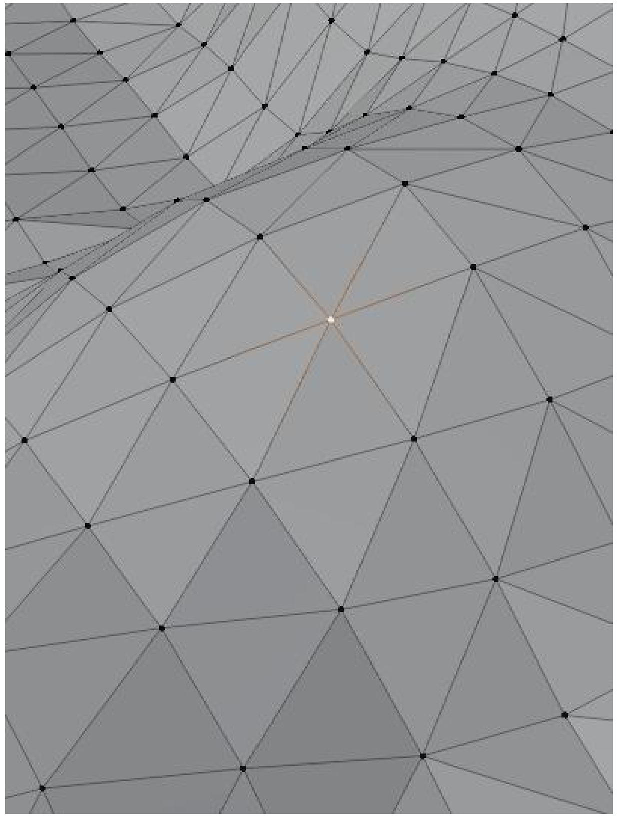
Not Smooth

- So each point has a function value attached to it. And we can connect the points to form a trianglular mesh **K** (2-manifold), and we can use **piecewise linear interpolation** for function values on edges and triangles.
- Assumption: this PL function has pairwise different values.

Piecewise linear functions

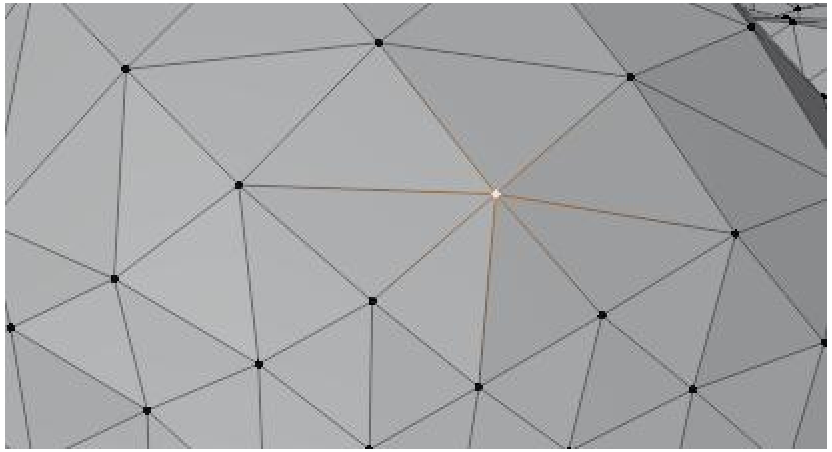
- **Star:** The star of a vertex u consists of all simplices (vertices, edges and triangles) that contain u, and the link consists of all faces of simplices in the star that are disjoint from u.





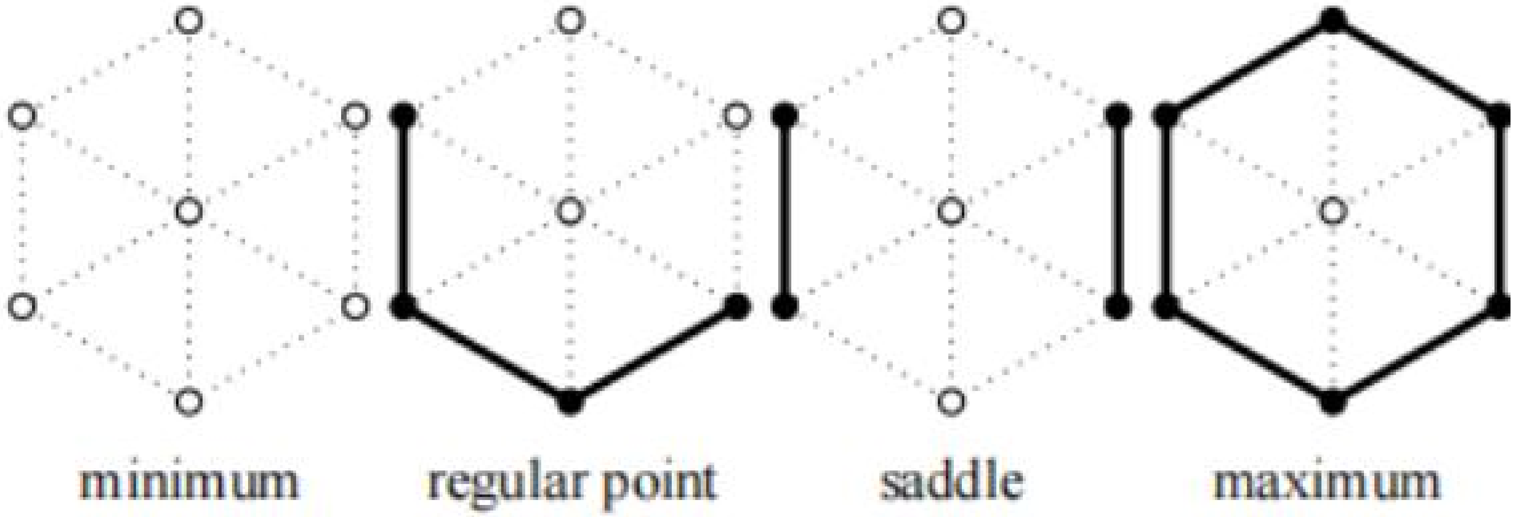
Piecewise linear functions

- Lower star: The lower star contains all simplices in the star for which u is the highest vertex.
- Upper star: The upper star contains all simplices in the star for which u is the lowest vertex.
- Lower link: The lower link contains all simplices in the link whose endpoints are lower than u.
- Upper link: The upper link contains all simplices in the link whose endpoints are higher than u.



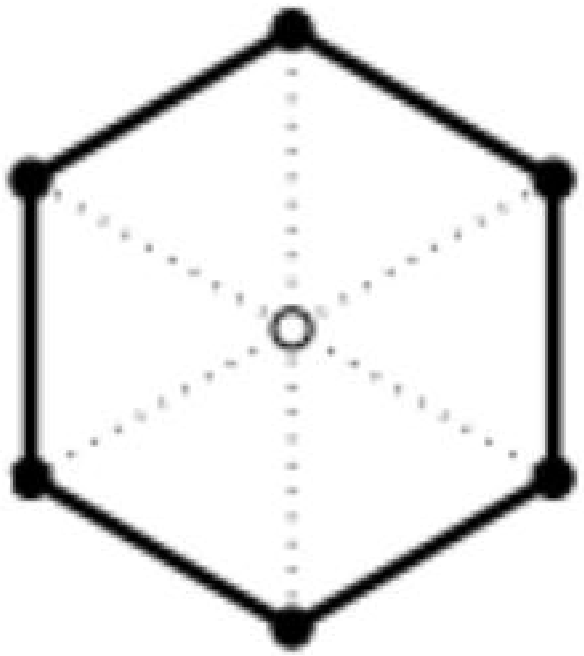
Piecewise linear functions

- We can use the connected component of lower link and upper link to decide the type of a point.

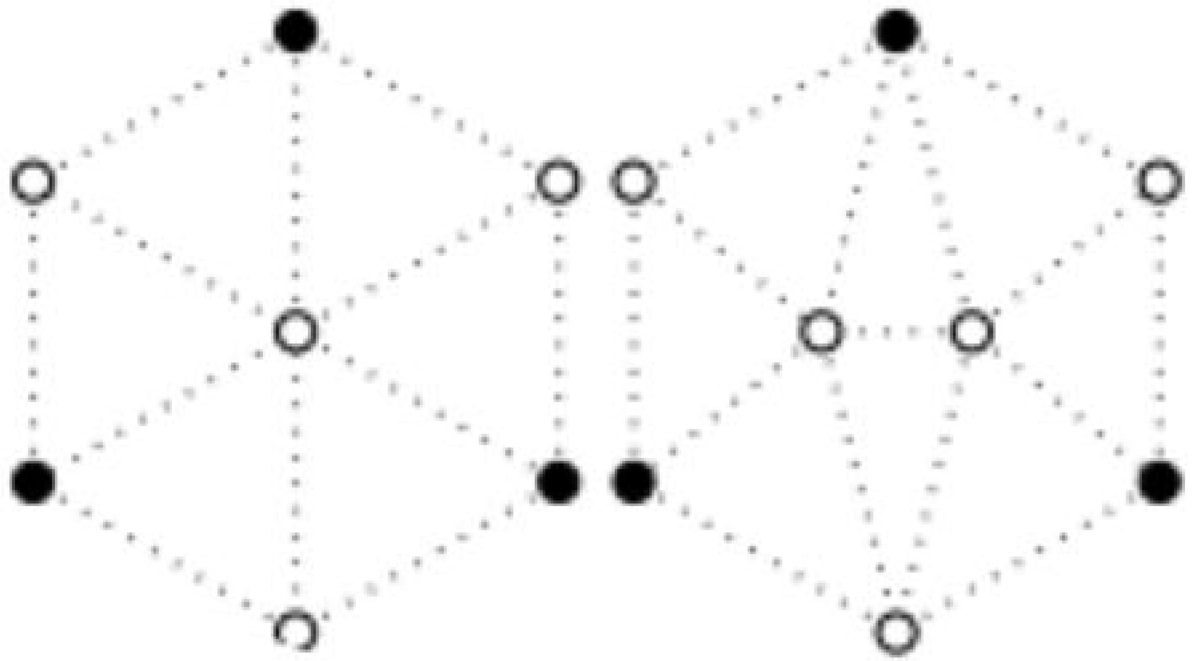


Piecewise linear functions

- We can use the connected component of lower link and upper link to decide the type of a point.



1-fold saddle



2-fold saddle

Persistence

- Also we can call it **importance. It is a measure to tell how important a pair of critical points is.**
- The persistence of u and of v (both critical points):

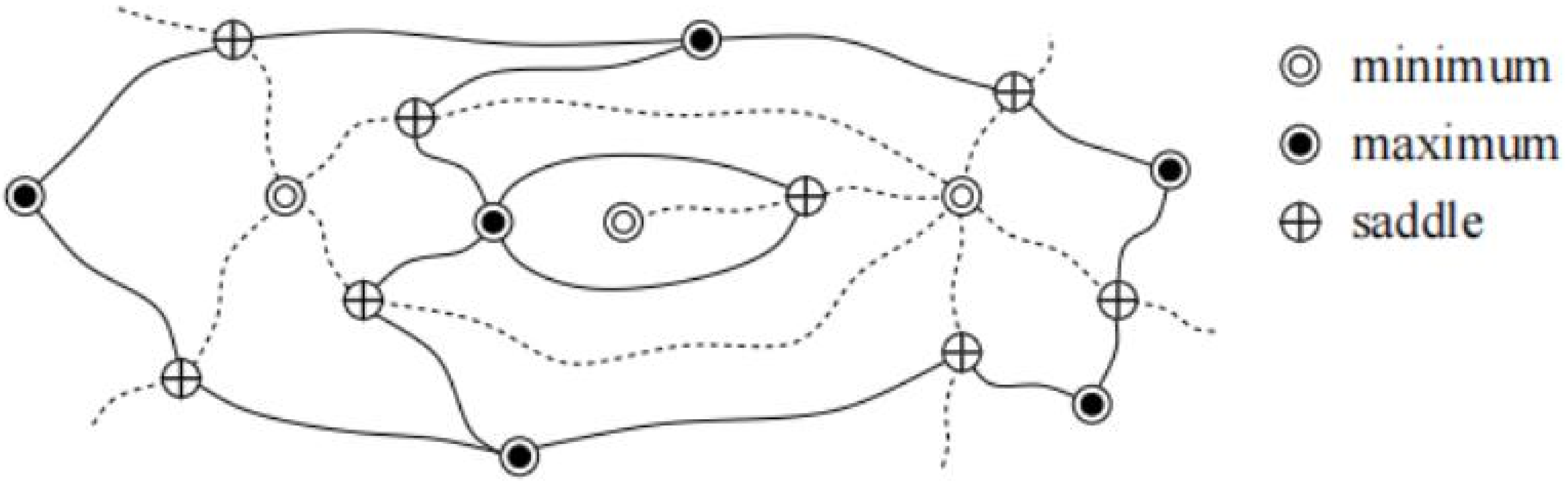
$$p = f(v) - f(u)$$

Method

- Morse-Smale complex -- preprocess and construct a MSC
- Hierarchy -- improve the performance
- Geometric approximation -- refine each level of hierarchy
- Remeshing -- remesh each MSC cell for better rendering

Morse-Smale complex

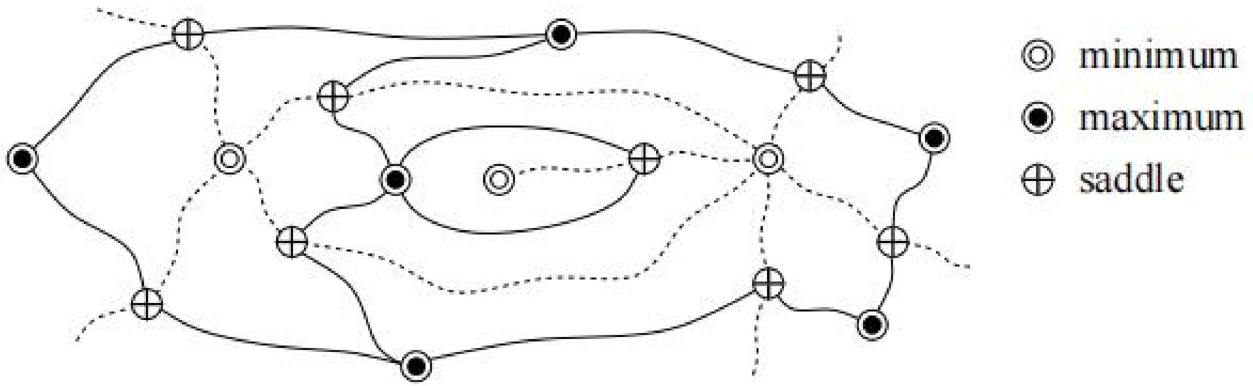
- In the paper, this is called a preprocessing stage.
- In this stage, based on a PL function dataset, a Morse-Smale complex is constructed.



Morse-Smale complex

Let $T = \{F, E, V\}$ be the triangulation of $\mathbb{M}$;
initialize the MS complex, $M = \emptyset$;
initialize the sets of paths and cells, $P = C = \emptyset$;
initialize SoS to direct the edges of T;
$S =$ FINDSADDLES(T);
$S =$ SPLITMULTIPLESADDLES(T);
SORTBYHEIGHT(S);
forall $s \in S$ in ascending order do
  COMPUTEASCENDINGPATH(P)
endfor;
forall $s \in S$ in descending order do
  COMPUTEDESCENDINGPATH(P)
endfor;
while there exists untouched $f \in F$ do
  GROWREGION(f, p_0, p_1, p_2, p_3);
  CREATEMORSECELL(C, p_0, p_1, p_2, p_3)
endwhile;
$M =$ CONNECTMORSECELLS(C).

Explanation:
1. SoS (**simulation of simplicity**) here makes the PL function a Morse function;
2. **SplitMultipleSaddles** means we make all the saddles be 1-fold saddle.

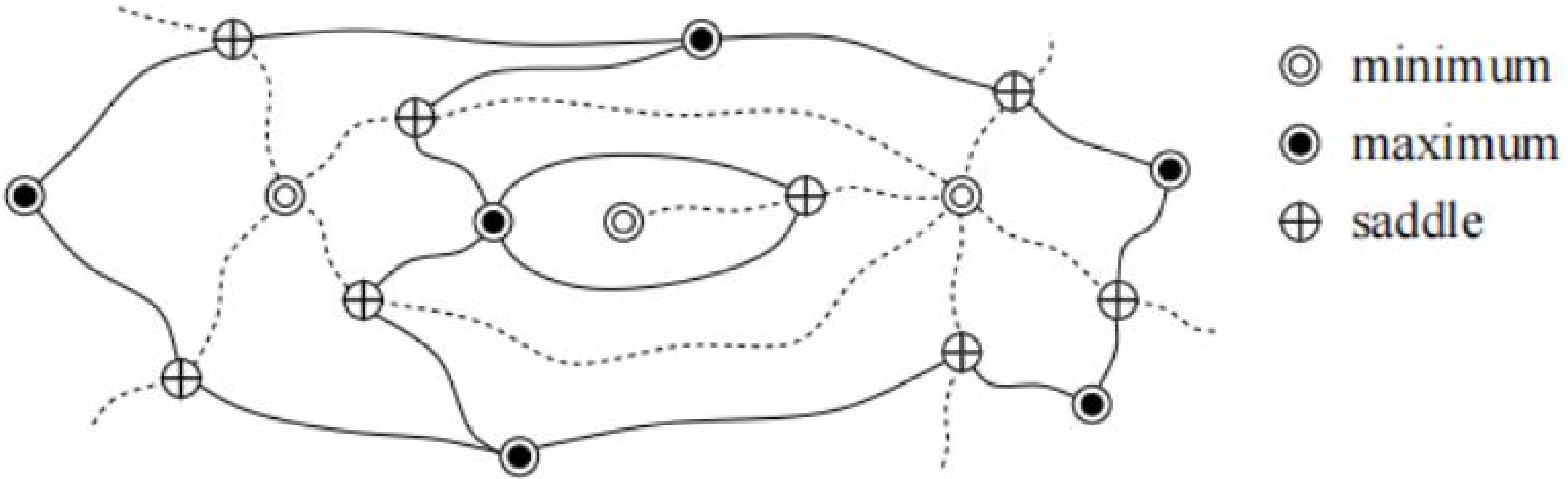


Hierarchy

- Their main objective is the design of a hierarchical data structure that supports adaptive coarsening and refinement of the data (MSC and PL function). In this section, they describe such a data structure and discuss how to use it.
  - **Diagonals and diamonds**
  - Cancellations
  - Node removal
  - Independent cancellations

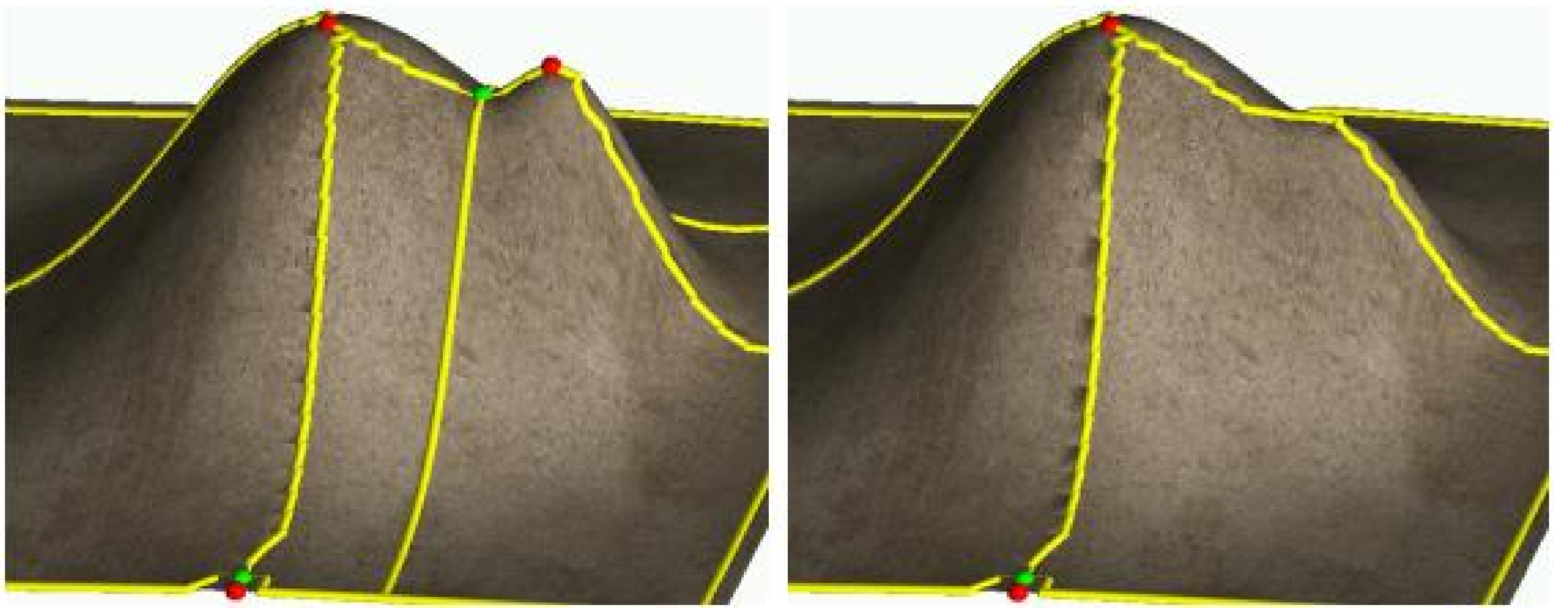
Hierarchy

- Diagonals and diamonds:



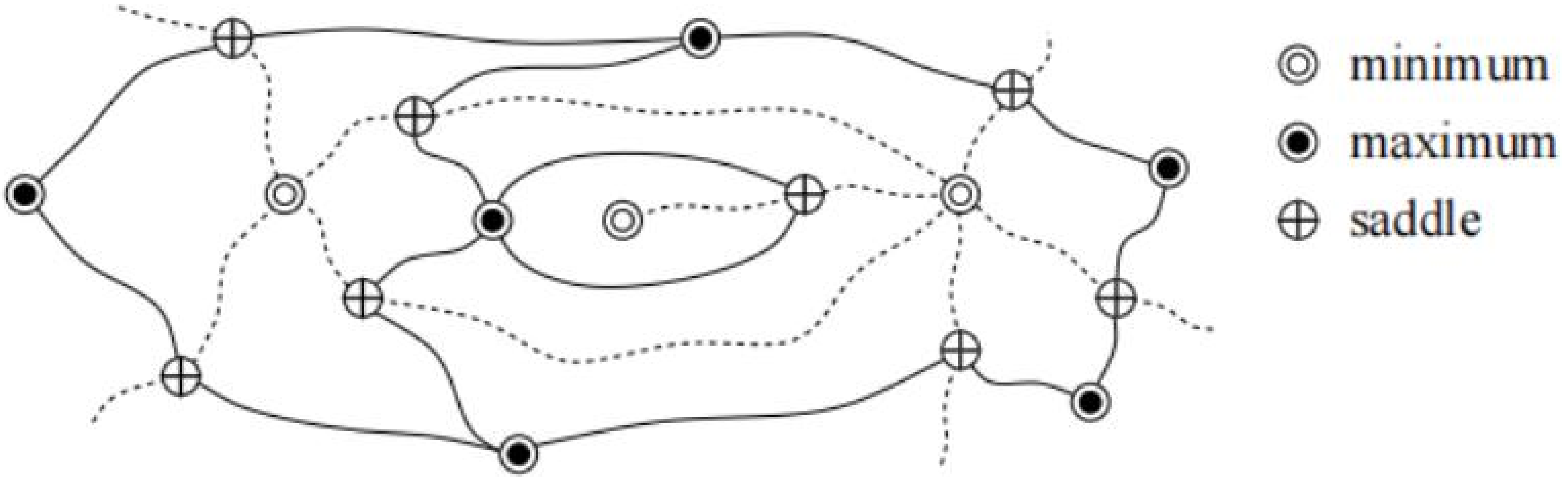
Hierarchy

- Cancellation: delete a pair of critical points
- The pair of critical points needs to be adjacent in MSC, so the only combinations are:
  - <saddle, max>
  - <saddle, min>



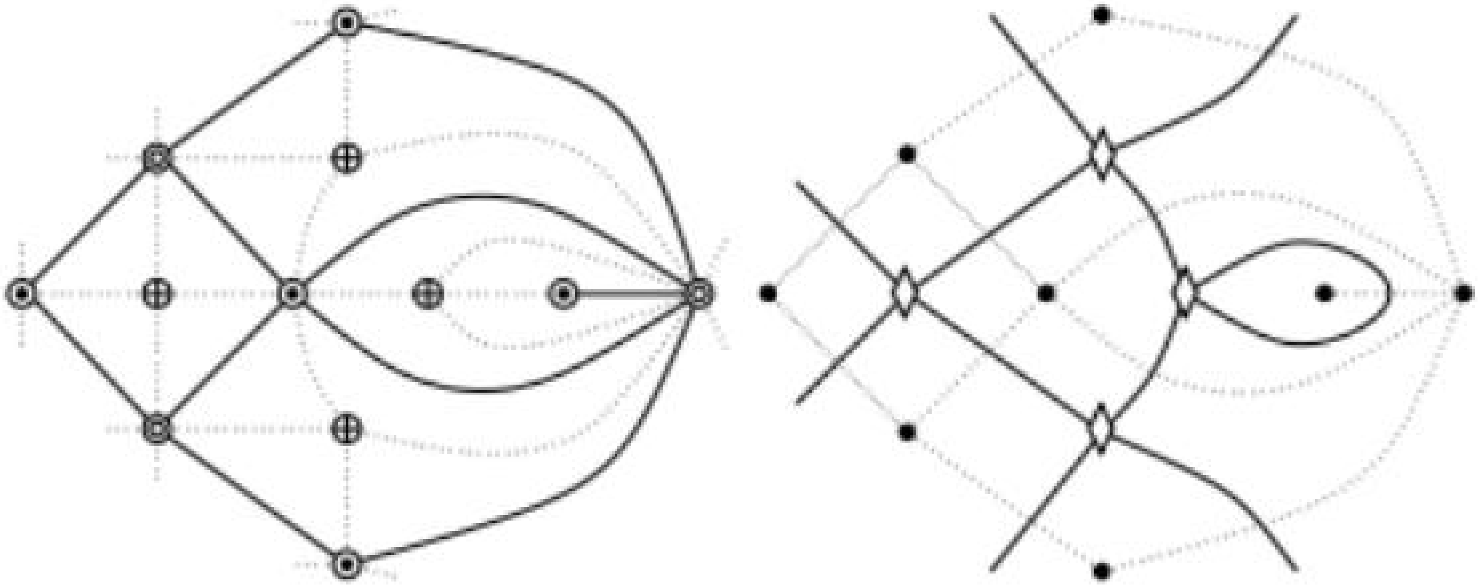
Hierarchy

- Cancellation: delete a pair of critical points
- To perform a cancellation, we first look at a saddle, then if we want to delete saddle u and a maximun, then we check out the two maximum v and w connected to this saddle.
- The requirement is that: if we want to delete <u, v>, then v≠w and f(v) < f(w)



Hierarchy

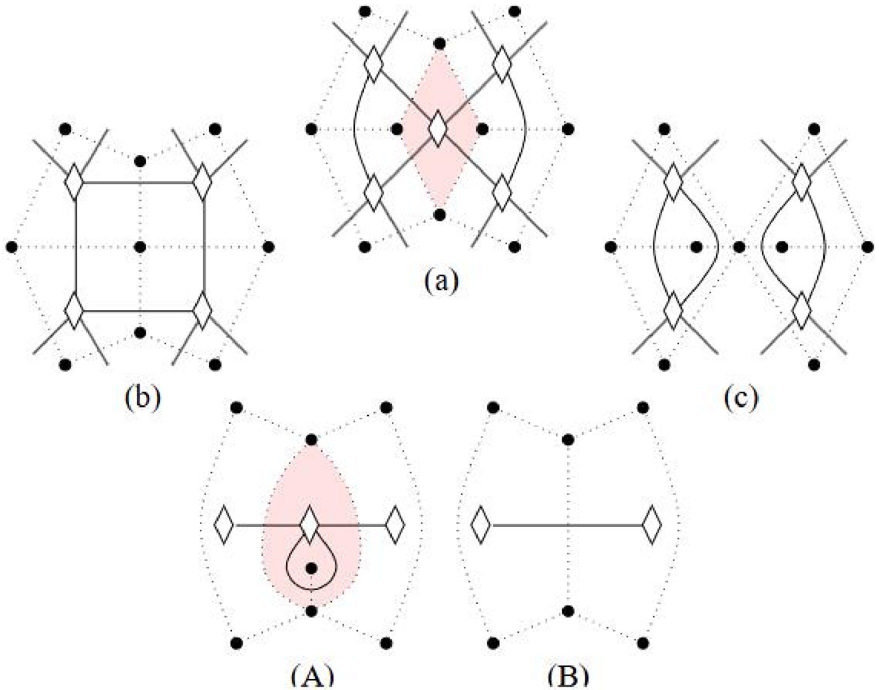
- Node removal
- From figure (a) and figure (b), they constructed a high level data structure for the cancellation operation.



(a) (b)

Hierarchy

- Node removal
- From figure (a) and figure (b), they constructed a high level data structure for the cancellation operation.

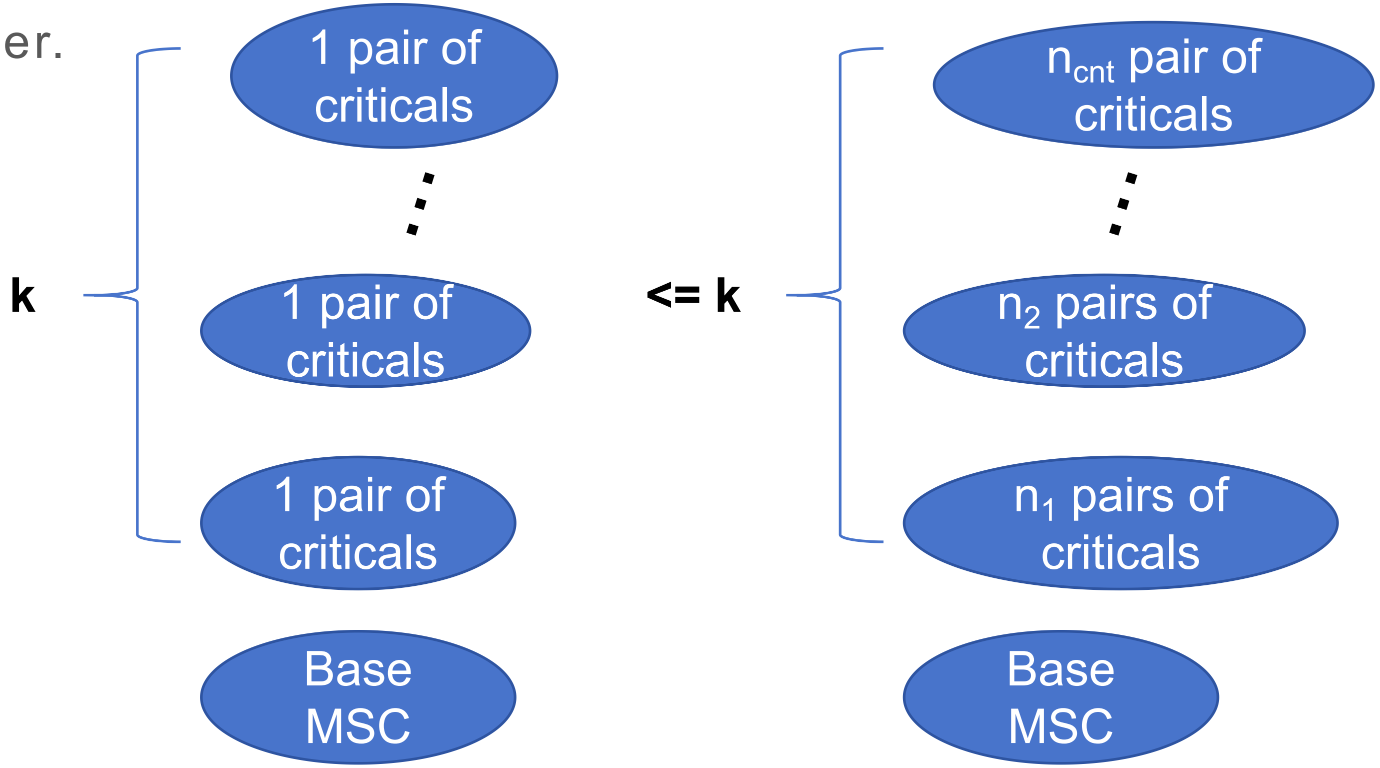


Hierarchy

- They construct the hierarchy by repeated cancellations.
- Critical pairs: $\langle s_1, v_1 \rangle$, $\langle s_2, v_2 \rangle$, $\langle s_3, v_3 \rangle$, ..., $\langle s_k, v_k \rangle$ (ordered in ascending persistence)
- For a layer of MSC and PL in the hierarchy, e.g. the j-th layer is after the cancellation of critical pairs $\langle s_1, v_1 \rangle$ to $\langle s_j, v_j \rangle$. So the 0-th layer is the original MSC and PL.

Independent cancellations

- Two independent cancenllations A and B means that A and B does not share any vertices, they are interchangable, and A and B can be done simultaneously.
- For example, there are k pairs of criticals, then the initial hierarchy is on the left having depth=k+1, and if independent pairs are deleted at once to form new layers, the hierarchy will be shallower.



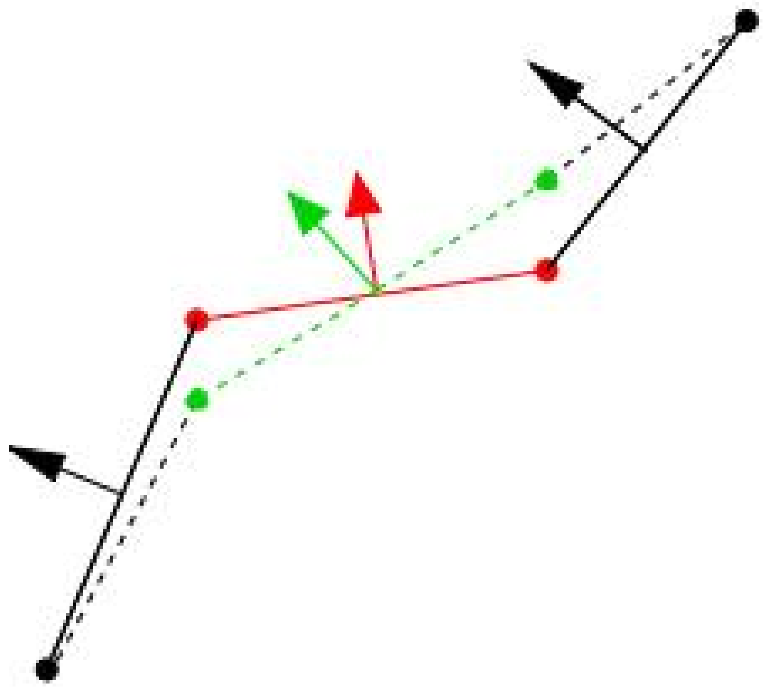
Geometric approximation

- This is the process after each cancellation. After the cancellation, two criticals are removed, so the area contain the two criticals removed should be smoothed to monotonic and all the integral lines affected by the removal of these two criticals should be recomputed and smoothed.

1. Find all paths affected by a cancellation;
2. use the gradient smoothing to geometrically remove the canceled critical points;
3. smooth the old regions until they are monotonic; (standard Laplacian smoothing)
4. erase the paths and re-compute new paths using the new geometry;
5. use one-dimensional gradient smoothing to force the new paths to comply with the constraints; and
6. smooth the new regions until all points are regular

Geometric approximation

- This is the process after each cancellation. After the cancellation, two criticals are removed, so the area contain the two criticals removed should be smoothed to monotonic and all the integral lines affected by the removal of these two criticals should be recomputed and smoothed.

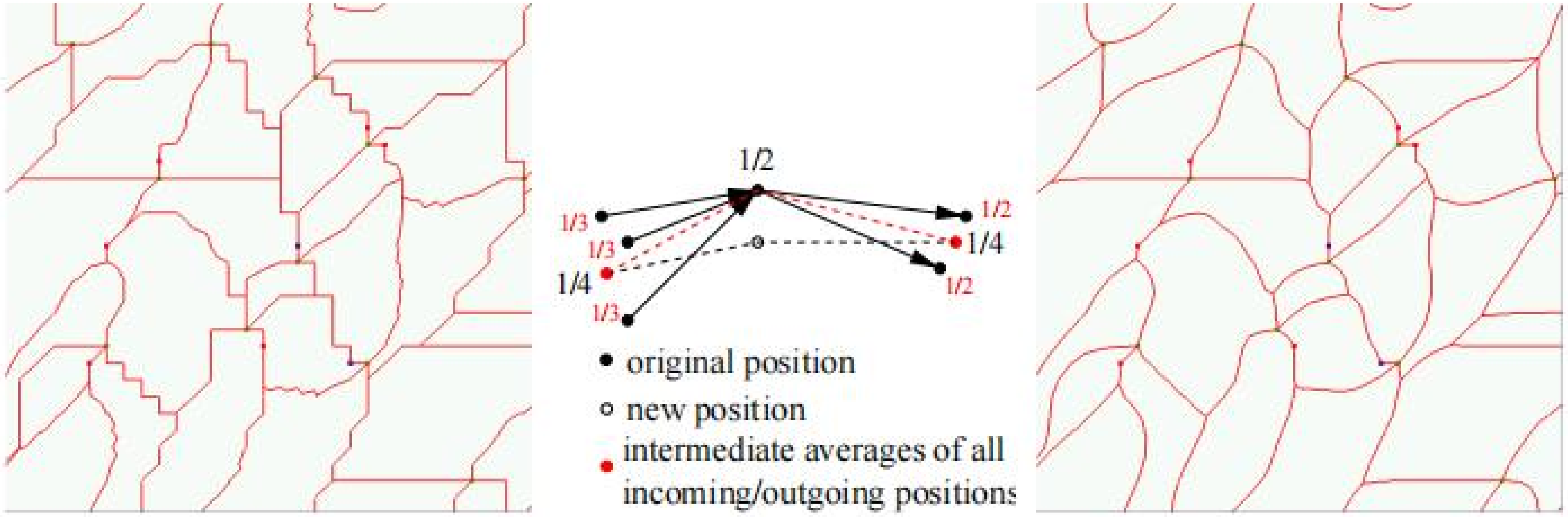


Remeshing

- This steps purpose is for better integration into the rendering pipeline. Each Morse-smale region (patch) are triangulated with regular structure.
    - Path smoothing
    - Parametrization

Remeshing

- Path smoothing: smoothing vertices and junction points on paths except for criticals.
- For path vertices smoothing, **I think they use gradient smoothing (paper does not mention)**, and for junction points, they use a modified laplacian smoothing.



(a) (b) (c)

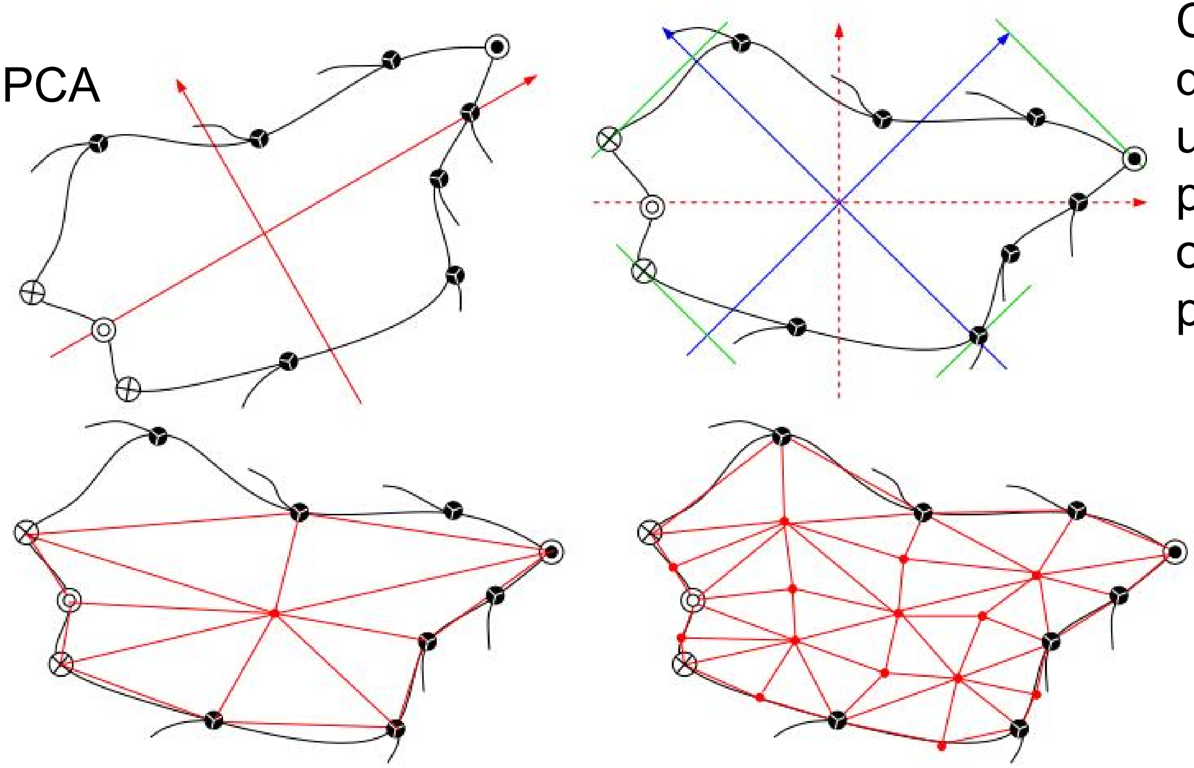
Remeshing

- Parametrization: This step is for constructing a regular structure for a MSC region (patch).
  - Map the boundary of the region to the boundary of one or multiple unit squares (param space)
  - Map the interior of the region to the param space (each vertex can be expressed as a convex combination of its neighbors. The coefficients in this combination can be computed by solving a sparse linear system).

Remeshing

- After parametrization, they sample the parameter space on a uniform grid and use its preimage on M as a new mesh for the region.

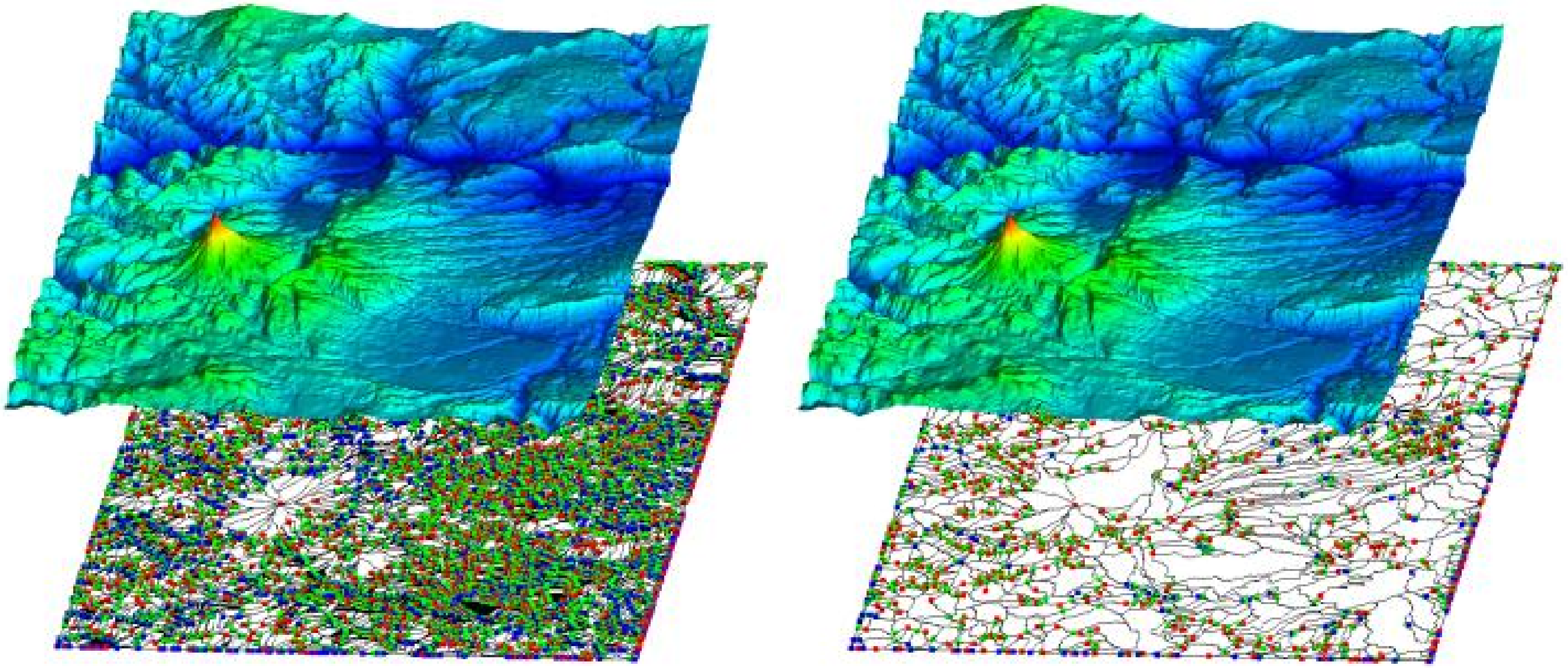
PCA



Compute projected length to determine the corners, then using arc-length parametrization to determine other boundary points' param coordinates

Preimages in M of the regular grid sampled in param space using mid-point subdivision

Results



1. (Left) Original The Dalles data set containing 24,617 critical points.
2. (Right) Same data with 2,144 critical points after removing all topological features with persistence less than 0.1% of height range.

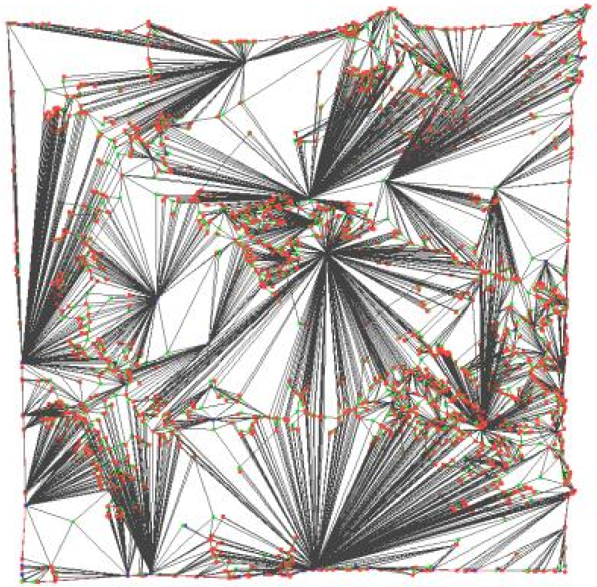
Results

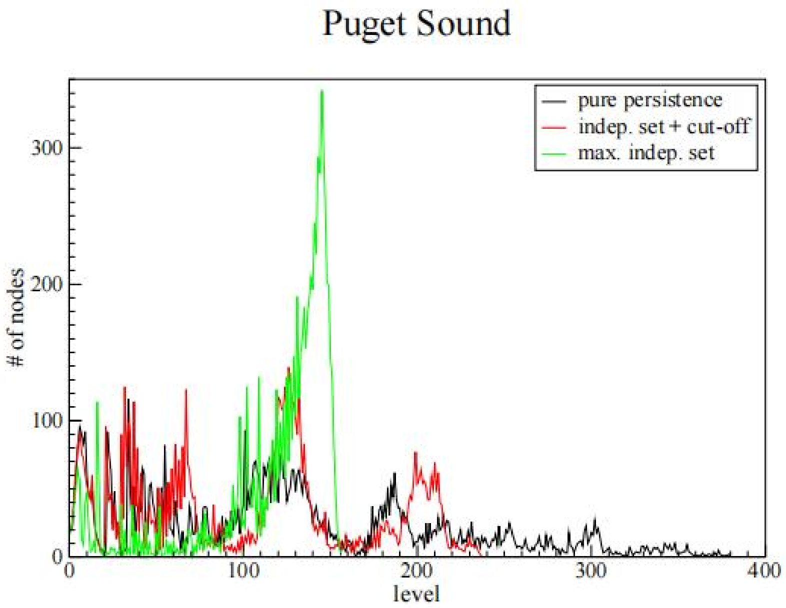
Three different hierarchy construction strategy:
1. pure persistence:performing cancellations in order of increasing persistence.
2. performing simplification in "batches" of maximal independent sets of cancellations.
3. constructing maximal independent sets with a restricted range of persistence values.

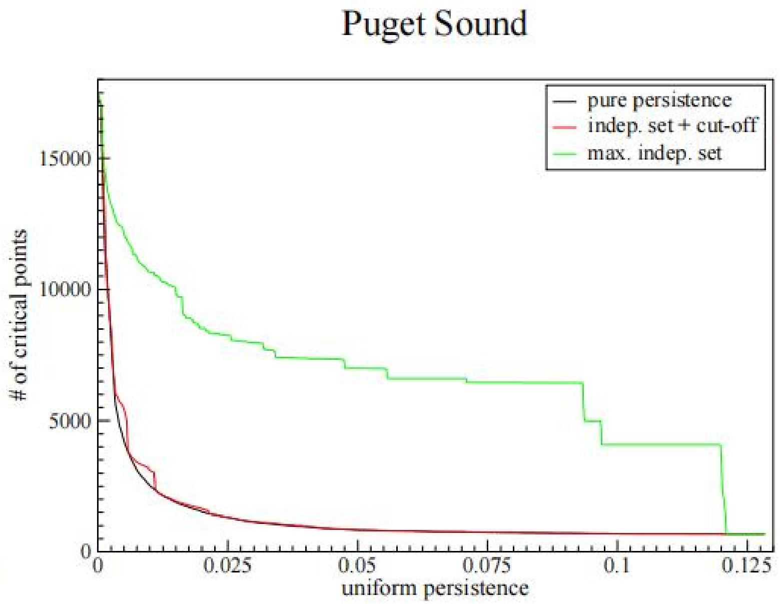
In all three methods, topology with persistence larger than 20% of the initial height range is not removed.

	depth	avg dep	max #p	max #c	avg deg
Puget Sound	org. no. of CPs 49185, no. of significant CPs 17470				
pure persistence	381	128	148	110	3.80
max. indep. set	157	118	131	112	4.28
indep. set-cut-off	238	105	147	106	3.94
Yakima	org. no. of CPs 21275, no. of significant CPs 6082				
pure persistence	119	56	73	32	3.60
max. indep. set	57	35	38	34	4.24
indep. set-cut-off	96	52	74	37	3.82
Dalles	org. no. of CPs 24617, no. of significant CPs 2144				
pure persistence	80	34	75	39	3.40
max. indep. set	54	31	82	43	3.88
indep. set-cut-off	73	33	63	57	3.52
Needles	org. no. of CPs 17375, no. of significant CPs 3772				
pure persistence	177	68	111	87	3.60
max. indep. set	113	70	87	87	3.88
indep. set-cut-off	149	62	124	101	3.68

Results

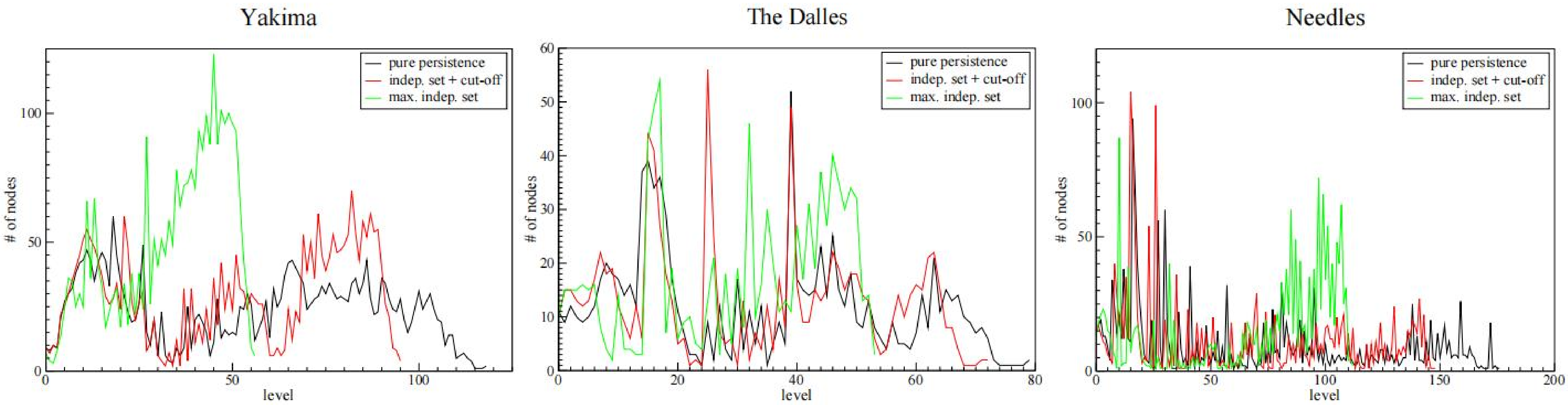






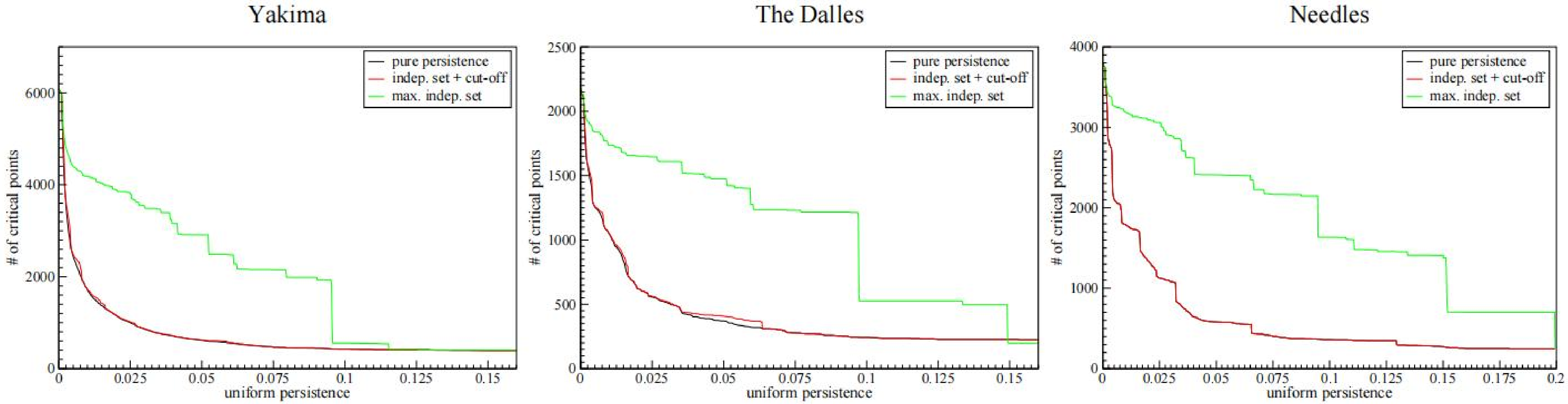
1. (left) Highest resolution MS complex of Needles data set.
2. (middle) Node distribution over the levels for different cancellation strategies for Puget Sound data set.
3. (right) Number of critical points in MS complex during uniform refinement of Puget Sound data set.

Results



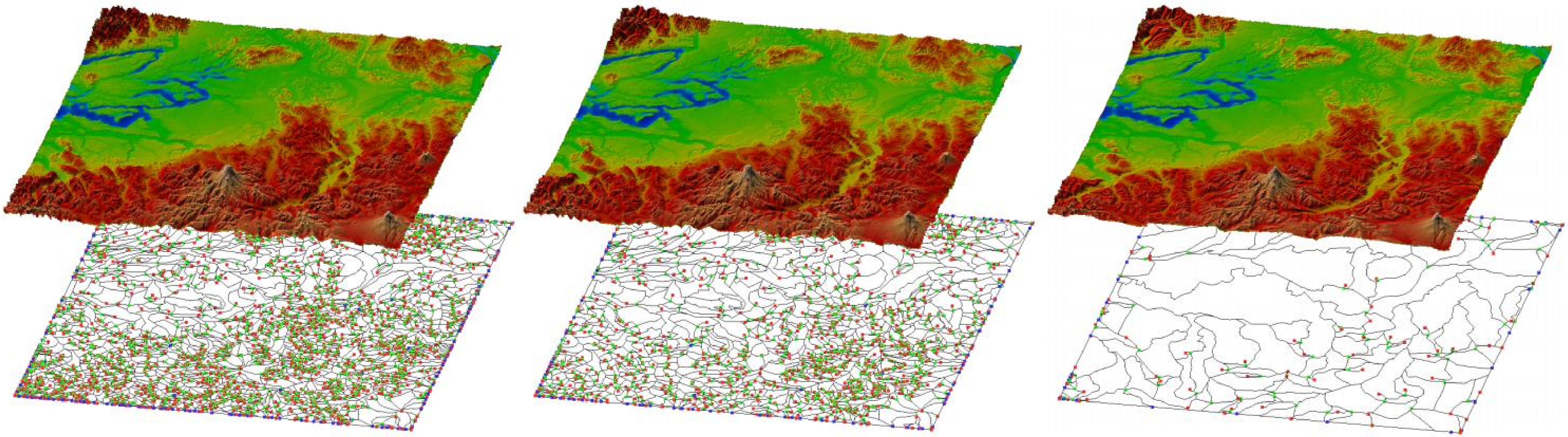
Nodes distribution over the levels for different cancellation strategies for the Yakima (left) , The Dalles (middle) , and the Needles (right) data sets.

Results



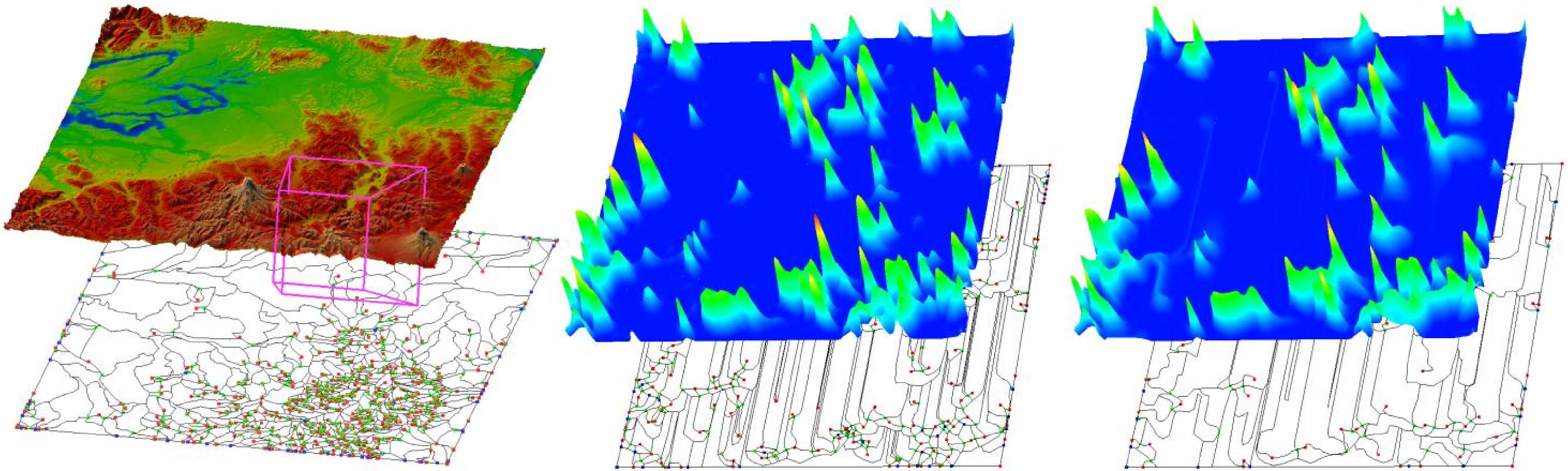
Number of critical points in MS complex for the Yakima (left), The Dalles (middle), and the Needles (right) data sets.

Results



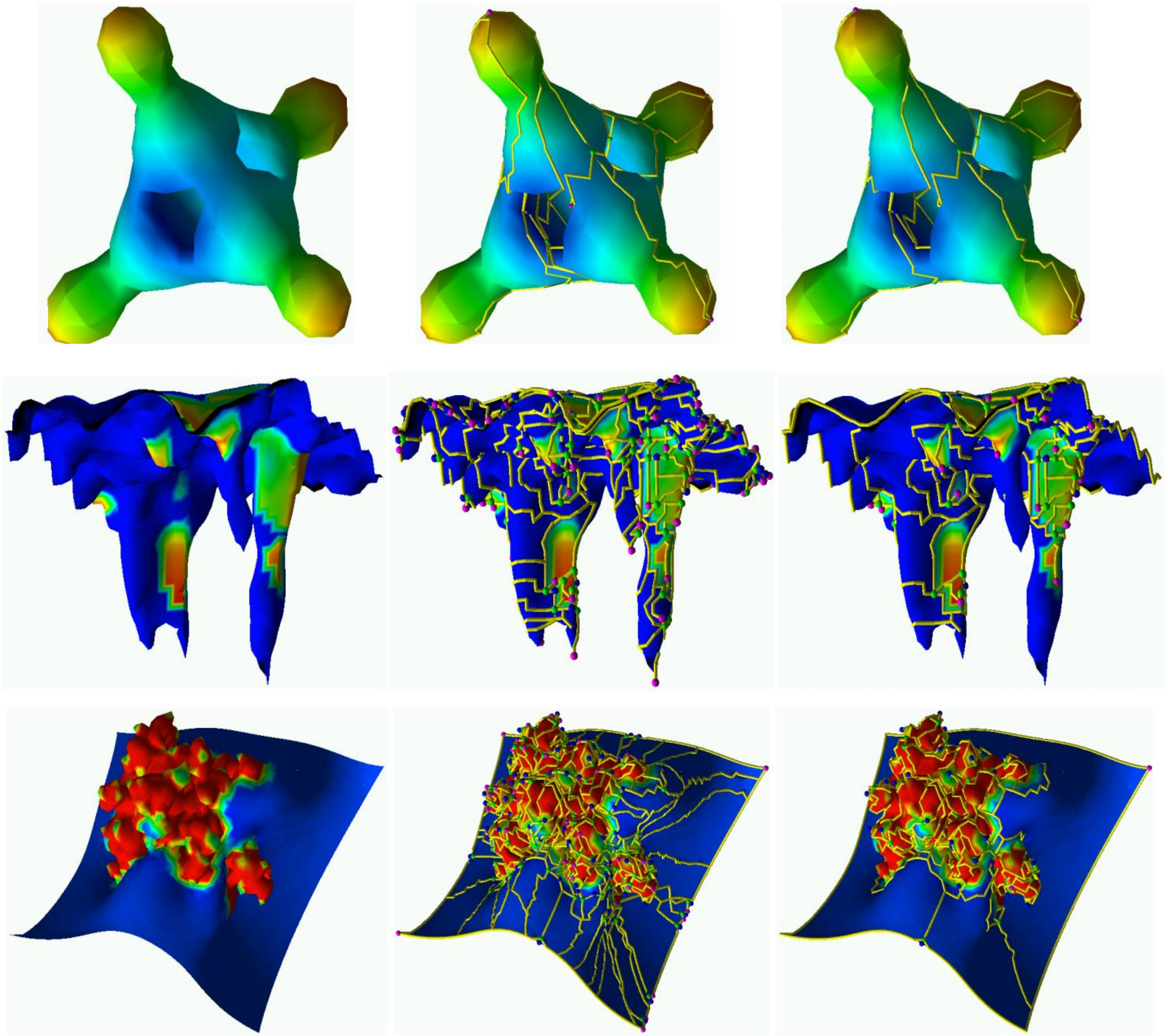
(Left) Puget Sound data after topological noise removal.
(Middle) Data at persistence of 1 2% of the maximum height.
(Right) Data at persistence 20% of maximum height.

Results



(Left) View-dependent refinement of Puget Sound data (purple: view frustum).
(Middle) Combustion data after topological noise removal.
(Right) Adaptive refinement of combustion data based on function value. All maxima above 90% of the maximal function value are preserved.

Results



Appendix

Source of image:http://www.offconvex.org/2016/03/22/saddlepoints/

Hessian

$L(\boldsymbol{\theta})$ around $\boldsymbol{\theta} = \boldsymbol{\theta}'$ can be approximated below

$$L(\boldsymbol{\theta}) \approx L(\boldsymbol{\theta}') + \boxed{\cancel{(\boldsymbol{\theta} - \boldsymbol{\theta}')^T \boldsymbol{g}}} + \boxed{\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}')^T H (\boldsymbol{\theta} - \boldsymbol{\theta}')}$$

At critical point

telling the properties of critical points