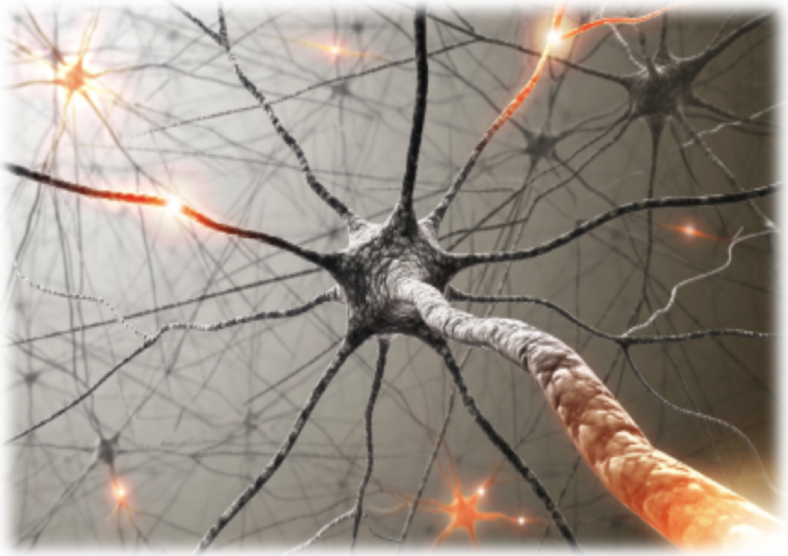
Machine learning: linear classification



Linear classification framework

training data

	x_1	x_2	y
example	0	2	1
example	-2	0	1
example	1	-1	-1

learning algorithm →

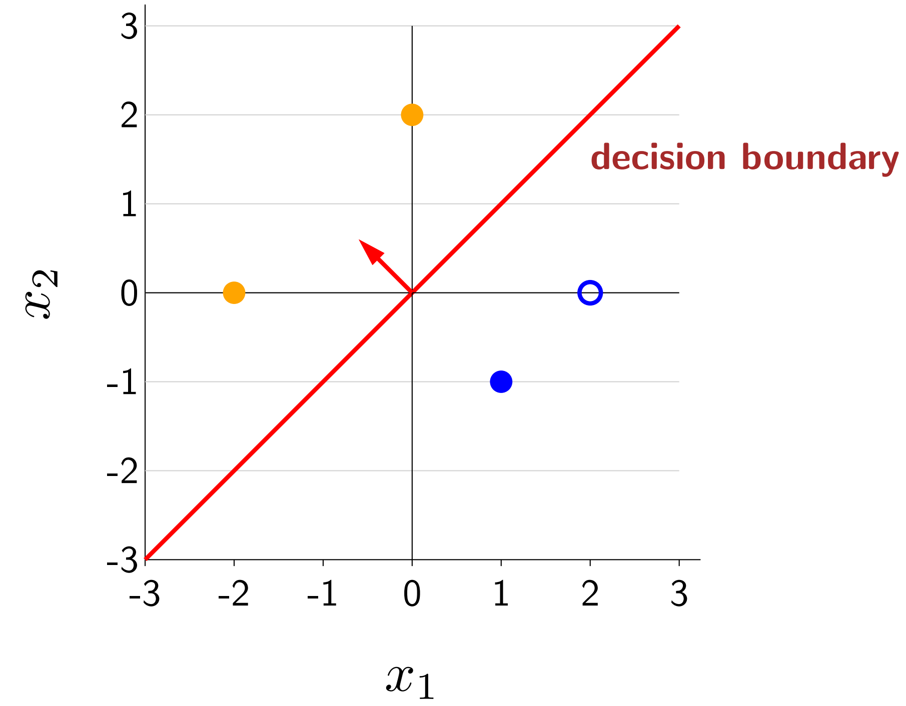
[2, 0] **input**

↓

f **classifier**

↓

-1 **label**



Design decisions:

Which classifiers are possible? **hypothesis class**

How good is a classifier? **loss function**

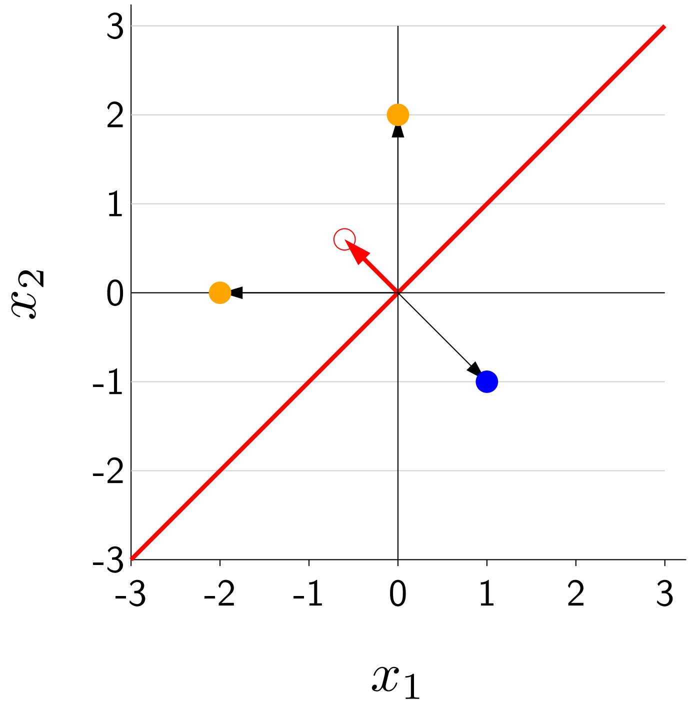
How do we compute the best classifier? **optimization algorithm**

An example linear classifier

$$f(x) = \text{sign}(\overbrace{[-0.6, 0.6]}^{\mathbf{w}} \cdot \overbrace{[x_1, x_2]}^{\phi(x)})$$

$$\text{sign}(z) = \begin{cases} +1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \\ 0 & \text{if } z = 0 \end{cases}$$

x_1	x_2	$f(x)$
0	2	1
-2	0	1
1	-1	-1



$$f([0, 2]) = \text{sign}([-0.6, 0.6] \cdot [0, 2]) = \text{sign}(1.2) = 1$$

$$f([-2, 0]) = \text{sign}([-0.6, 0.6] \cdot [-2, 0]) = \text{sign}(1.2) = 1$$

$$f([1, -1]) = \text{sign}([-0.6, 0.6] \cdot [1, -1]) = \text{sign}(-1.2) = -1$$

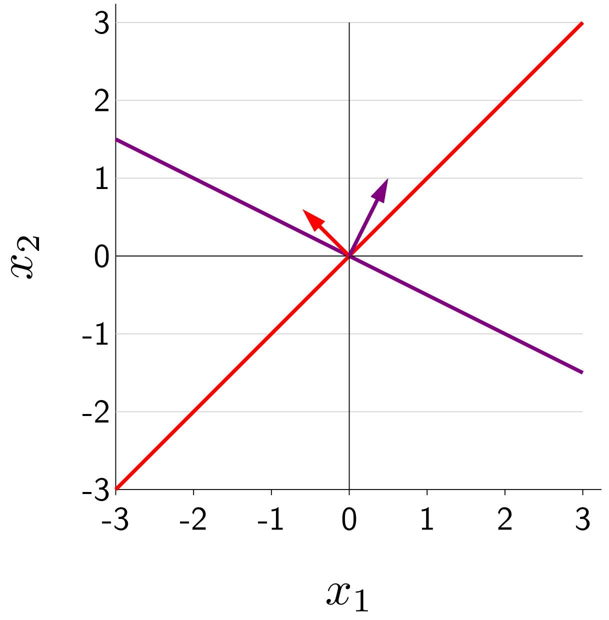
Decision boundary: x such that $\mathbf{w} \cdot \phi(x) = 0$

Hypothesis class: which classifiers?

$\phi(x) = [x_1, x_2]$

$f(x) = \text{sign}([-0.6, 0.6] \cdot \phi(x))$

$f(x) = \text{sign}([0.5, 1] \cdot \phi(x))$



General binary classifier:

$$f_{\mathbf{w}}(x) = \text{sign}(\mathbf{w} \cdot \phi(x))$$

Hypothesis class:

$$\mathcal{F} = \{f_{\mathbf{w}} : \mathbf{w} \in \mathbb{R}^2\}$$

Loss function: how good is a classifier?

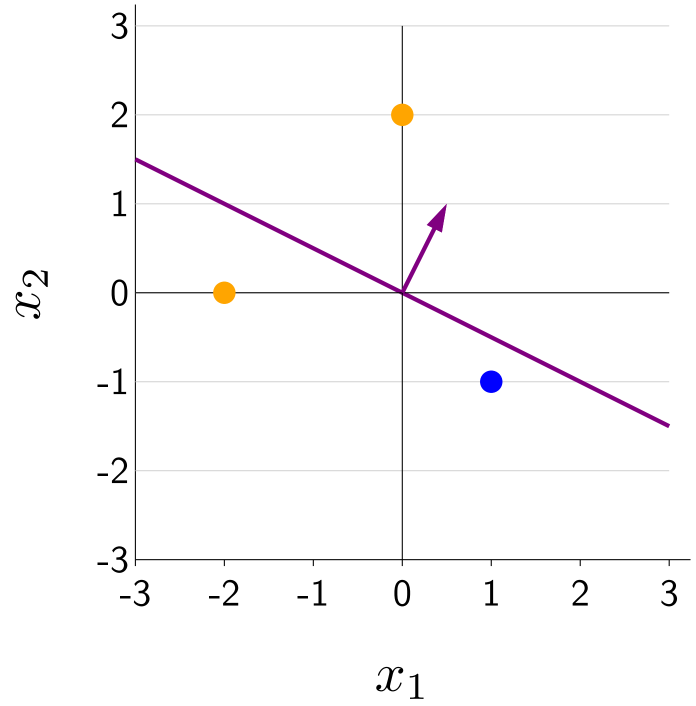
$$f_{\mathbf{w}}(x) = \mathbf{w} \cdot \phi(x)$$

$$\mathbf{w} = [0.5, 1]$$

$$\phi(x) = [x_1, x_2]$$

training data $\mathcal{D}_{\text{train}}$

x_1	x_2	y
0	2	1
-2	0	1
1	-1	-1



$$\text{Loss}_{\text{0-1}}(x, y, \mathbf{w}) = \mathbf{1}[f_{\mathbf{w}}(x) \neq y]$$ **zero-one loss**

$$\text{Loss}([0, 2], 1, [0.5, 1]) = \mathbf{1}[\text{sign}([0.5, 1] \cdot [0, 2]) \neq 1] = 0$$

$$\text{Loss}([-2, 0], 1, [0.5, 1]) = \mathbf{1}[\text{sign}([0.5, 1] \cdot [-2, 0]) \neq 1] = 1$$

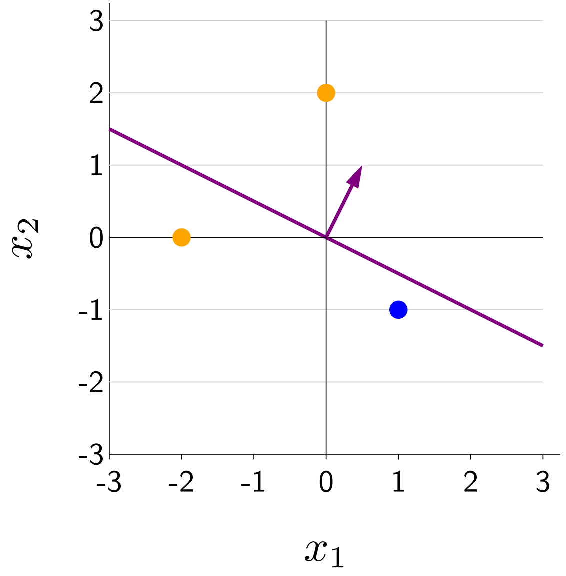
$$\text{Loss}([1, -1], -1, [0.5, 1]) = \mathbf{1}[\text{sign}([0.5, 1] \cdot [1, -1]) \neq -1] = 0$$

$$\text{TrainLoss}([0.5, 1]) = 0.33$$

Score and margin

Predicted label: $f_{\mathbf{w}}(x) = \text{sign}(\mathbf{w} \cdot \phi(x))$

Target label: y



Definition: score

The score on an example (x, y) is $\mathbf{w} \cdot \phi(x)$, how **confident** we are in predicting $+1$.

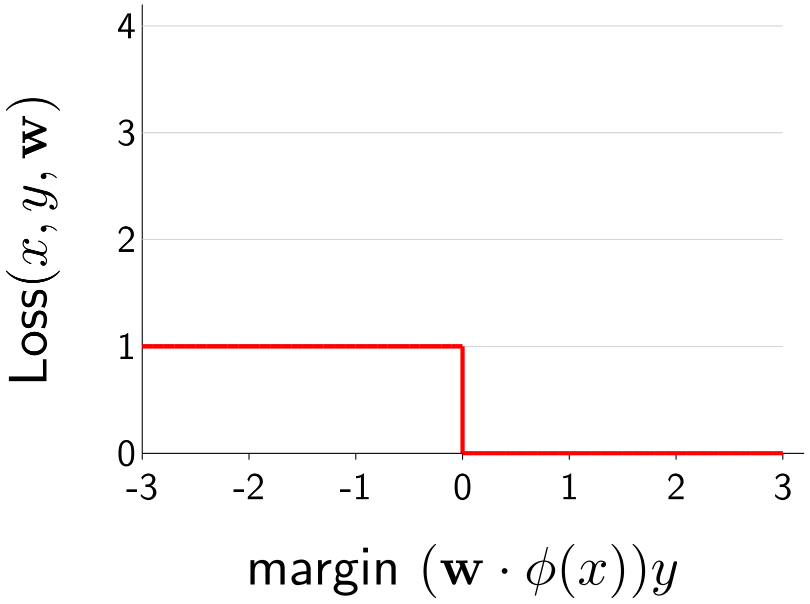
Definition: margin

The margin on an example (x, y) is $(\mathbf{w} \cdot \phi(x))y$, how **correct** we are.

Zero-one loss rewritten

Definition: zero-one loss

$$\text{Loss}_{\text{0-1}}(x, y, \mathbf{w}) = \mathbf{1}[f_{\mathbf{w}}(x) \neq y]$$
$$= \mathbf{1}[\underbrace{(\mathbf{w} \cdot \phi(x))y}_{\text{margin}} \leq 0]$$



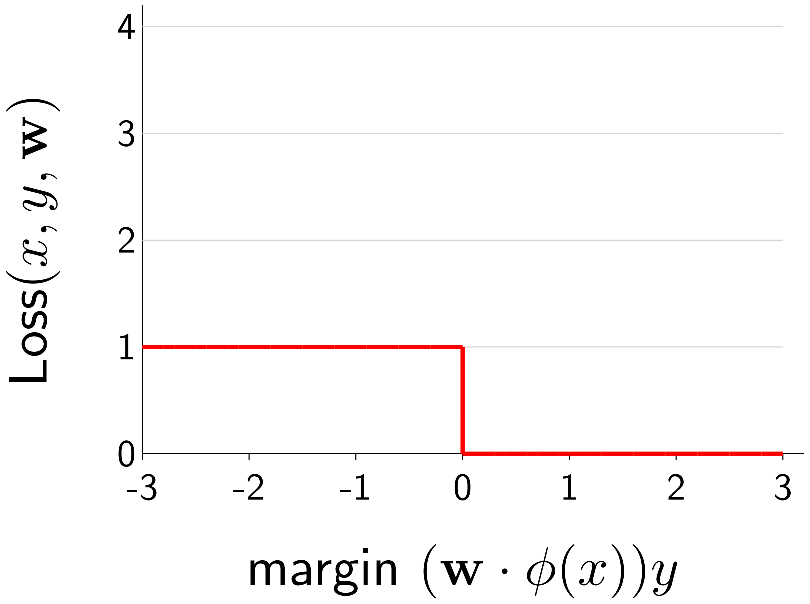
Optimization algorithm: how to compute best?

Goal: $\min_{\mathbf{w}} \text{TrainLoss}(\mathbf{w})$

To run gradient descent, compute the gradient:

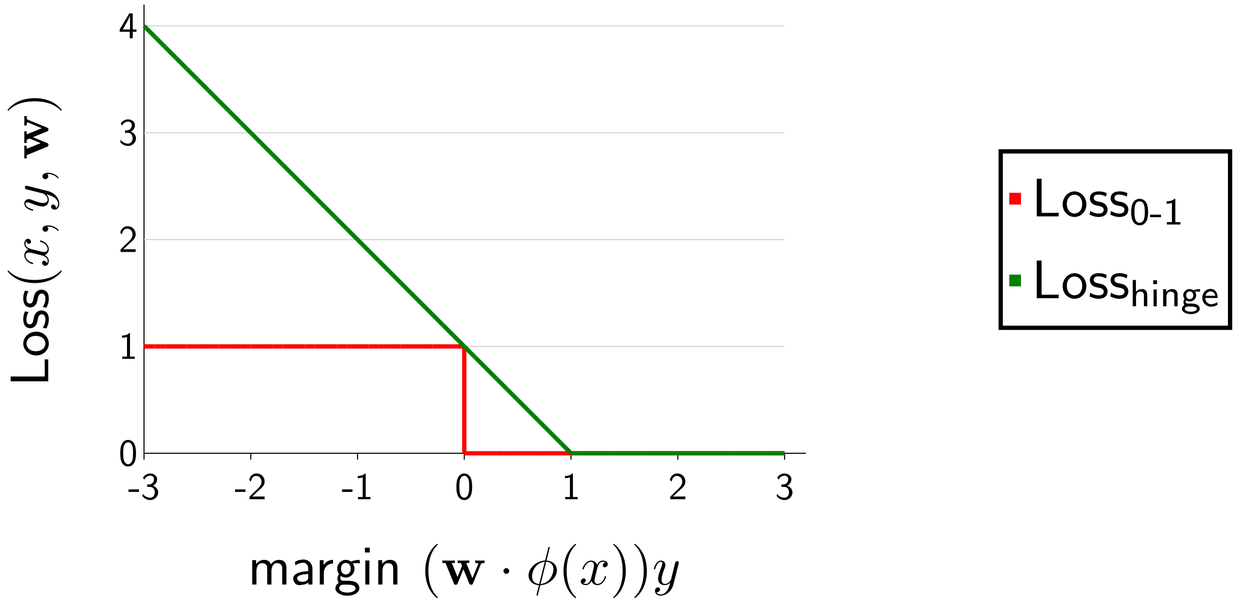
$$\nabla_{\mathbf{w}} \text{TrainLoss}(\mathbf{w}) = \frac{1}{|\mathcal{D}_{\text{train}}|} \sum_{(x,y)\in\mathcal{D}_{\text{train}}} \nabla \text{Loss}_{0\text{-}1}(x, y, \mathbf{w})$$

$$\nabla_{\mathbf{w}} \text{Loss}_{0\text{-}1}(x, y, \mathbf{w}) = \nabla \mathbf{1}[(\mathbf{w} \cdot \phi(x))y \leq 0]$$



Gradient is zero almost everywhere!

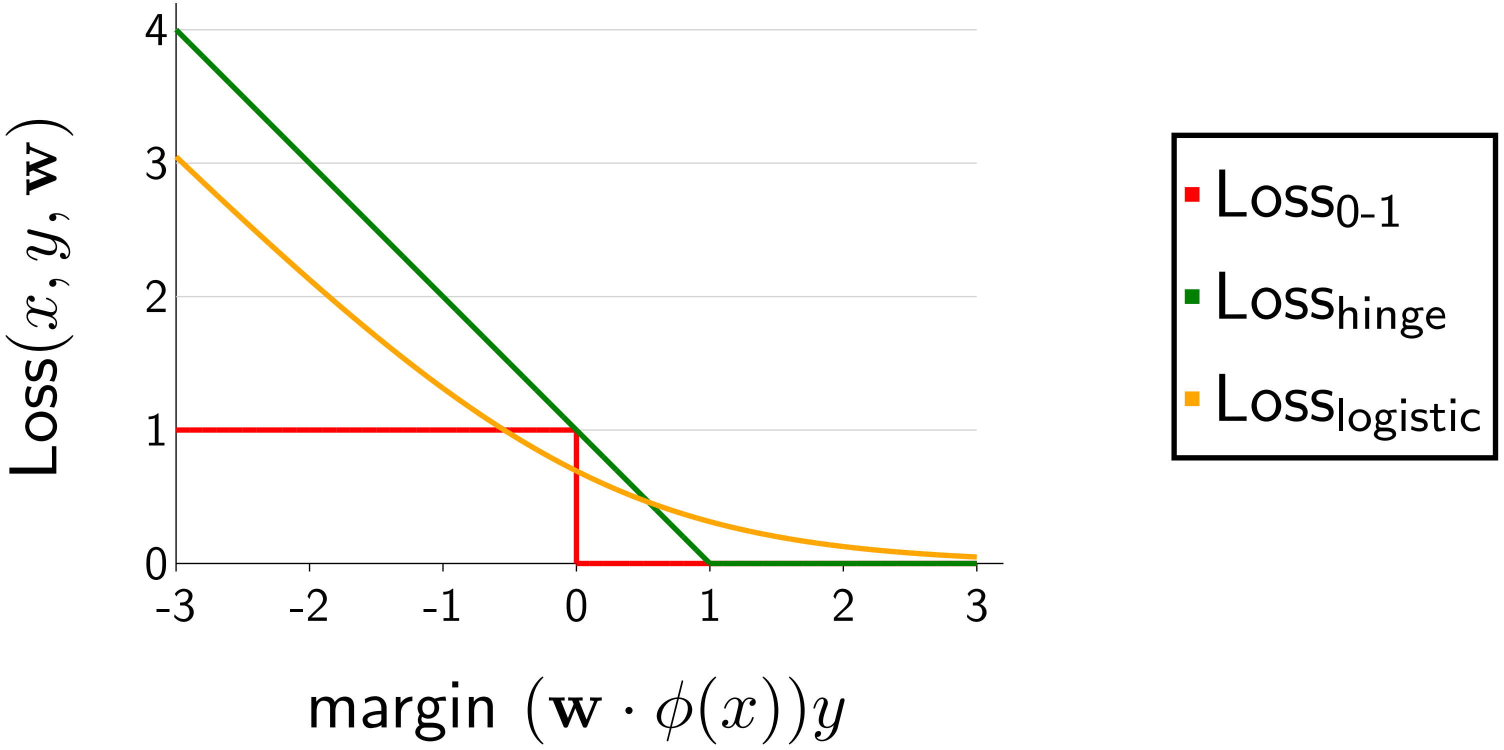
Hinge loss



$$\text{Loss}_{\text{hinge}}(x, y, \mathbf{w}) = \max\{1 - (\mathbf{w} \cdot \phi(x))y, 0\}$$

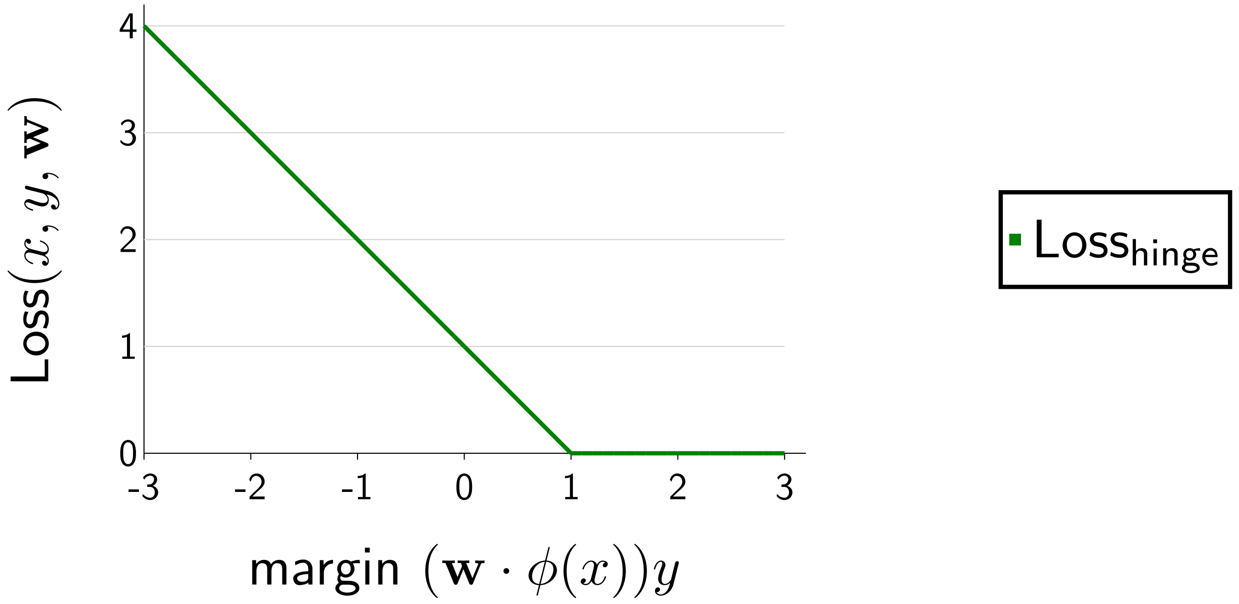
Digression: logistic regression

$$\text{Loss}_{\text{logistic}}(x, y, \mathbf{w}) = \log(1 + e^{-(\mathbf{w} \cdot \phi(x))y})$$



Intuition: Try to increase margin even when it already exceeds 1

Gradient of the hinge loss



$$\text{Loss}_{\text{hinge}}(x, y, \mathbf{w}) = \max\{1 - (\mathbf{w} \cdot \phi(x))y, 0\}$$

$$\nabla\text{Loss}_{\text{hinge}}(x, y, \mathbf{w}) = \begin{cases} -\phi(x)y & \text{if } \{1 - (\mathbf{w} \cdot \phi(x))y\} > \{0\} \\ 0 & \text{otherwise} \end{cases}$$

Hinge loss on training data

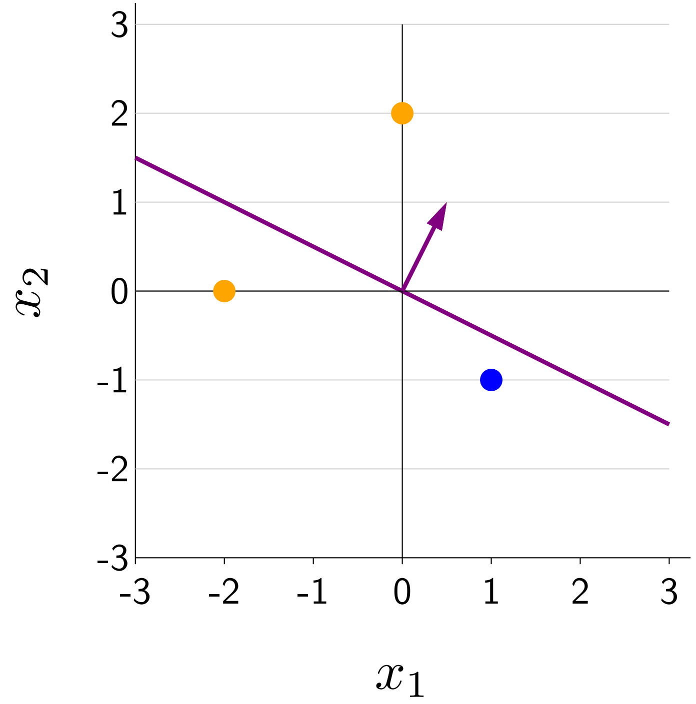
$$f_{\mathbf{w}}(x) = \mathbf{w} \cdot \phi(x)$$

$\mathbf{w} = [0.5, 1]$

$\phi(x) = [x_1, x_2]$

training data $\mathcal{D}_{\text{train}}$

x_1	x_2	y
0	2	1
-2	0	1
1	-1	-1



$$\text{Loss}_{\text{hinge}}(x, y, \mathbf{w}) = \max\{1 - (\mathbf{w} \cdot \phi(x))y, 0\}$$

$\text{Loss}([0, 2], 1, [0.5, 1]) = \max\{1 - [0.5, 1] \cdot [0, 2](1), 0\} = 0$

$\text{Loss}([-2, 0], 1, [0.5, 1]) = \max\{1 - [0.5, 1] \cdot [-2, 0](1), 0\} = 2$

$\text{Loss}([1, -1], -1, [0.5, 1]) = \max\{1 - [0.5, 1] \cdot [1, -1](-1), 0\} = 0.5$

$\text{TrainLoss}([0.5, 1]) = 0.83$

$\nabla\text{Loss}([0, 2], 1, [0.5, 1]) = [0, 0]$

$\nabla\text{Loss}([-2, 0], 1, [0.5, 1]) = [2, 0]$

$\nabla\text{Loss}([1, -1], -1, [0.5, 1]) = [1, -1]$

$\nabla\text{TrainLoss}([0.5, 1]) = [1, -0.33]$

Gradient descent (hinge loss) in Python

[code]

Summary so far

$$\underbrace{\mathbf{w} \cdot \phi(x)}_{\text{score}}$$

	Regression	Classification
Prediction $f_{\mathbf{w}}(x)$	score	sign(score)
Relate to target y	residual (score $- y$)	margin (score y)
Loss functions	squared absolute deviation	zero-one hinge logistic
Algorithm	gradient descent	gradient descent