MDPs: policy evaluation

Evaluating a policy

Definition: utility

Following a policy yields a **random path**.

The **utility** of a policy is the (discounted) sum of the rewards on the path (this is a random variable).

Path	Utility
[in; stay, 4, end]	4
[in; stay, 4, in; stay, 4, in; stay, 4, end]	12
[in; stay, 4, in; stay, 4, end]	8
[in; stay, 4, in; stay, 4, in; stay, 4, in; stay, 4, end]	16
...	...

Definition: value (expected utility)

The **value** of a policy at a state is the **expected** utility.

Value: 12

Evaluating a policy: volcano crossing

Run (or press ctrl-enter)

2.4	-0.5	**-50**	**40**
3.7	5	**-50**	31
2	12.6	16.3	26.2

a	r	s
		(2,1)
E	-0.1	(2,2)
S	-0.1	(3,2)
E	-0.1	(3,3)
E	-0.1	(3,4)
N	-0.1	(2,4)
N	-50.1	(2,3)

Value: 3.73

Utility: -29.99

Discounting

Definition: utility

Path: $s_0, a_1 r_1 s_1, a_2 r_2 s_2, \ldots$ (action, reward, new state).

The **utility** with discount γ is

$$u_1 = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$$

Discount $\gamma = 1$ (save for the future):

[stay, stay, stay, stay]: $4 + 4 + 4 + 4 = 16$

Discount $\gamma = 0$ (live in the moment):

[stay, stay, stay, stay]: $4 + 0 \cdot (4 + \cdots) = 4$

Discount $\gamma = 0.5$ (balanced life):

[stay, stay, stay, stay]: $4 + \frac{1}{2} \cdot 4 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 4 = 7.5$

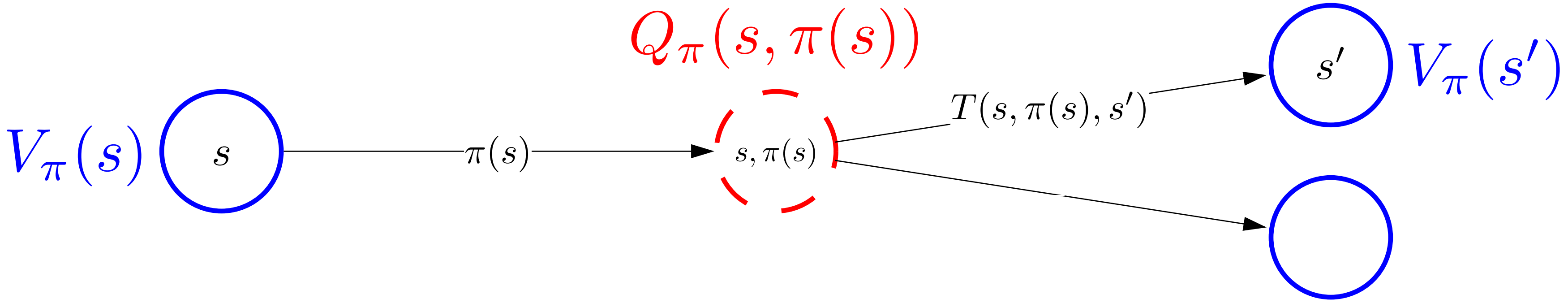
Policy evaluation

Definition: value of a policy

Let $V_\pi(s)$ be the expected utility received by following policy π from state s.

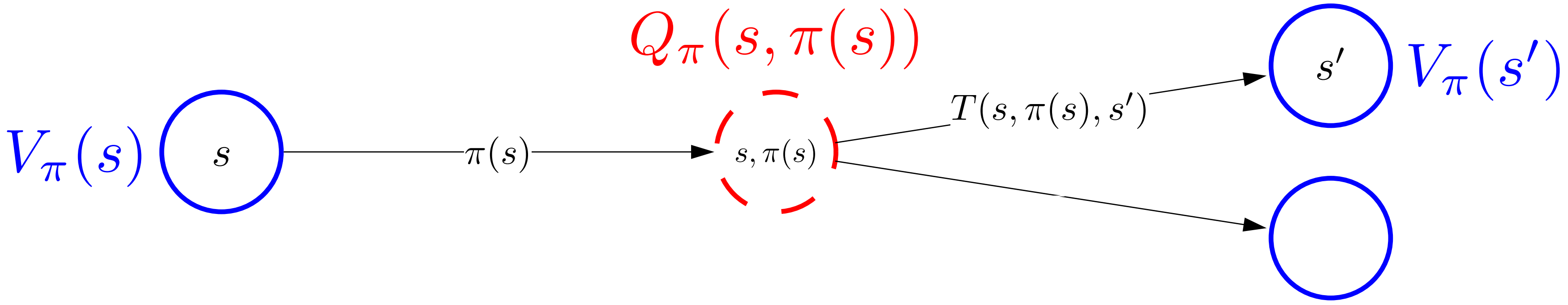
Definition: Q-value of a policy

Let $Q_\pi(s, a)$ be the expected utility of taking action a from state s, and then following policy π.



Policy evaluation

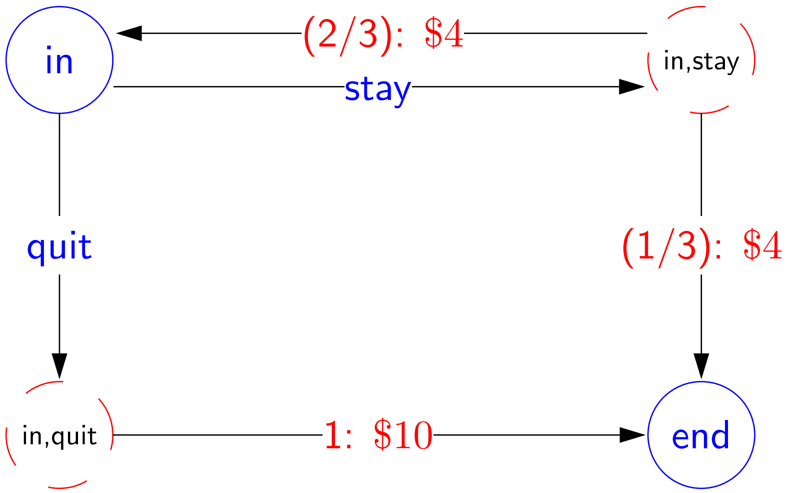
Plan: define recurrences relating value and Q-value



$$V_\pi(s) = \begin{cases} 0 & \text{if IsEnd}(s) \\ Q_\pi(s, \pi(s)) & \text{otherwise.} \end{cases}$$

$$Q_\pi(s, a) = \sum_{s'} T(s'|s, a)[\text{Reward}(s, a, s') + \gamma V_\pi(s')]$$

Dice game



(assume $\gamma = 1$)

Let π be the "stay" policy: $\pi(\text{in}) = \text{stay}$.

$$V_\pi(\text{end}) = 0$$

$$V_\pi(\text{in}) = \frac{1}{3}(4 + V_\pi(\text{end})) + \frac{2}{3}(4 + V_\pi(\text{in}))$$

In this case, can solve in closed form:

$$V_\pi(\text{in}) = 12$$

Policy evaluation

Key idea: iterative algorithm

Start with arbitrary policy values and repeatedly apply recurrences to converge to true values.

Algorithm: policy evaluation

Initialize $V_\pi^{(0)}(s) \leftarrow 0$ for all states s.

For iteration $t = 1, \ldots, t_{\text{PE}}$:

For each state s:

$$V_\pi^{(t)}(s) \leftarrow \underbrace{\sum_{s'} T(s'|s, \pi(s))[\text{Reward}(s, \pi(s), s') + \gamma V_\pi^{(t-1)}(s')]}_{Q^{(t-1)}(s,\pi(s))}$$

Policy evaluation implementation

How many iterations ($t_{\textsf{PE}}$)? Repeat until values don't change much:

$$\max_{s \in \textsf{States}} |V_\pi^{(t)}(s) - V_\pi^{(t-1)}(s)| \le \epsilon$$

Don't store $V_\pi^{(t)}$ for each iteration t, need only last two:

$$V_\pi^{(t)} \text{ and } V_\pi^{(t-1)}$$

Complexity

Algorithm: policy evaluation

Initialize $V_\pi^{(0)}(s) \leftarrow 0$ for all states s.

For iteration $t = 1, \ldots, t_{\text{PE}}$:

For each state s:

$$V_\pi^{(t)}(s) \leftarrow \underbrace{\sum_{s'} T(s'|s,\pi(s))[\text{Reward}(s,\pi(s),s') + \gamma V_\pi^{(t-1)}(s')]}_{Q^{(t-1)}(s,\pi(s))}$$

MDP complexity

S states

A actions per state

S' successors (number of s' with $T(s'|s,a) > 0$)

Time: $O(t_{\text{PE}} S S')$

Policy evaluation on dice game

Let π be the "stay" policy: $\pi(\text{in}) = \text{stay}$.

$$V_\pi^{(t)}(\text{end}) = 0$$

$$V_\pi^{(t)}(\text{in}) = \frac{1}{3}(4 + V_\pi^{(t-1)}(\text{end})) + \frac{2}{3}(4 + V_\pi^{(t-1)}(\text{in}))$$

s	end	in	
$V_\pi^{(t)}$	0.00	12.00	($t = 100$ iterations)

Converges to $V_\pi(\text{in}) = 12$.

Summary so far

- MDP: graph with states, chance nodes, transition probabilities, rewards

- Policy: mapping from state to action (solution to MDP)

- Value of policy: expected utility over random paths

- Policy evaluation: iterative algorithm to compute value of policy