Ex 2.12 :

a) The slab is an intersection of two halfspaces $\{x \in \mathbb{R}^n \mid a^T x \leqslant \beta\}$ and $\{x \in \mathbb{R}^n \mid (-a)^T x \leqslant -\alpha\}$ (for $a \neq \vec{0}$). Since halfspaces are convex and the intersection preserves convexity, then, $\boxed{\text{the slab is convex}}$.

Notice that if $a = \vec{0}$, the slab is either $\mathbb{R}^n$ or $\emptyset$, which remains convex.

b) Let R denote the set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \leqslant x_i \leqslant \beta_i \quad \forall i \in [\![1, n]\!]\}$

Let $(e_i)_{1 \leqslant i \leqslant n}$ be the canonical basis of $\mathbb{R}^n$.

Then, $R = \bigcap_{i=1}^{n} \left(\{x \in \mathbb{R}^n \mid e_i^T x \leqslant \beta_i\} \cap \{x \in \mathbb{R}^n \mid (-e_i)^T x \leqslant -\alpha_i\} \right)$

R is an intersection of halfspaces so it is convex. Then, $\boxed{\text{a rectangle is convex}}$

c) Let $W := \{x \in \mathbb{R}^n \mid a_1^T x \leqslant b_1,\ a_2^T x \leqslant b_2\} \Rightarrow W = \{x \in \mathbb{R}^n \mid a_1^T x \leqslant b_1\} \cap \{x \in \mathbb{R}^n \mid a_2^T x \leqslant b_2\}$.

- If $a_1 \neq 0$ and $a_2 \neq 0$: W is an intersection of two halfspaces, then, W is convex.
- If $a_1 = 0$ and $a_2 \neq 0$: $W = \begin{cases} \{a_2^T x \leqslant b_2\} & \text{if } b_1 \geqslant 0 \\ \emptyset & \text{otherwise} \end{cases}$. So, W is convex.
  (similar if $a_2 = 0$ and $a_1 \neq 0$)
- If $a_1 = 0$ and $a_2 = 0$: W is either $\mathbb{R}^n$ or $\emptyset$, so, it's convex.

Therefore, $\boxed{\text{a wedge is convex}}$.

d) Let $B := \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leqslant \|x - y\|_2 \quad \forall y \in S\}$.

Then, $B = \bigcap_{y \in S} B_y$ where $B_y := \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leqslant \|x - y\|_2\}$.

- $x \in B_y \Leftrightarrow \|x - x_0\|_2^2 - \|x - y\|_2^2 \leqslant 0$

$\Leftrightarrow ((x - x_0) - (x - y))^T ((x - x_0) + (x - y)) \leqslant 0$ } we use the formula: $\|u\|^2 - \|v\|^2 = \langle u - v, u + v \rangle$

$\Leftrightarrow (y - x_0)^T (2x - (x_0 + y)) \leqslant 0$

$\Leftrightarrow (y - x_0)^T x \leqslant \frac{1}{2} (y - x_0)^T (y + x_0) = \frac{\|y\|^2 - \|x_0\|^2}{2}$

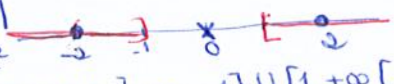
Thus, if $y \neq x_0$, B_y is a halfspace and if $y = x_0$, B_y is $\emptyset$ or $\mathbb{R}^n$.

In both cases, B_y is convex.

B is an intersection of convex sets. Then, $\boxed{B \text{ is convex}}$.

e) Counter example : Let us consider the case $n = 1$: $T = \{0\}$ and $S = \{-2, 2\}$

$\forall x \in \mathbb{R}$; $\operatorname{dist}(x, S) \leqslant \operatorname{dist}(x, T) \Leftrightarrow |x - 2| \leqslant |x|$ or $|x + 2| \leqslant |x|$



$\Leftrightarrow (x - 2)^2 \leqslant x^2$ or $(x + 2)^2 \leqslant x^2$

$\Leftrightarrow x \geqslant 1$ or $x \leqslant -1 \Leftrightarrow x \in]-\infty, -1] \cup [1, +\infty[$.

$]-\infty, -1] \cup [1, +\infty[$ is <u>not</u> convex. Then, $\boxed{\text{the set of points closer to one set than another is in general not convex}}$

f) Let $E := \{x \in \mathbb{R}^n \mid x + S_2 \subseteq S_1\}$

Then, $E = \{x \in \mathbb{R}^n \mid \forall y \in S_2,\ x + y \in S_1\} = \{x \in \mathbb{R}^n \mid \forall y \in S_2,\ x \in S_1 - y\}$

Thus, $E = \bigcap_{y \in S_2} (S_1 - y)$

For $y \in S_2$, $S_1 - y$ is obtained by a translation of S_1. Since S_1 is convex and, the translation preserves convexity, then, $S_1 - y$ is convex.

E is an intersection of convex sets. Then, $\boxed{E \text{ is convex}}$

g) Let $F := \{x \in \mathbb{R}^n \mid \|x-a\|_2 \leqslant \theta \|x-b\|_2\}$

$$x \in F \iff \|x-a\|_2^2 - \theta^2 \|x-b\|_2^2 \leqslant 0$$

$$\iff \left((x-a) - \theta(x-b)\right)^T \left((x-a) + \theta(x-b)\right) \leqslant 0$$

$$\iff \left((1-\theta)x - (a-\theta b)\right)^T \left((1+\theta)x - (a+\theta b)\right) \leqslant 0.$$

• If $\theta = 1$: $x \in F \iff -2(a-b)^T x + (a-b)^T(a+b) \leqslant 0$

$$\iff -2(a-b)^T x \leqslant \|b\|^2 - \|a\|^2$$

Since $a \neq b$, So, F is a halfspace then convex.

• If $\theta \neq 1$: $x \in F \iff (1-\theta^2) \times \left(x - \frac{a-\theta b}{1-\theta}\right)^T \left(x - \frac{a+\theta b}{1+\theta}\right) \leqslant 0$ since $1-\theta^2 > 0$

$$\iff \left(x - \frac{(1+\theta)(a-\theta b)}{1-\theta^2}\right)^T \left(x - \frac{(a+\theta b)(1-\theta)}{1-\theta^2}\right) \leqslant 0$$

$$\iff \left(x - \frac{a-\theta^2 b}{1-\theta^2} - \frac{\theta(a-b)}{1-\theta^2}\right)^T \left(x - \frac{a-\theta^2 b}{1-\theta^2} + \frac{\theta(a-b)}{1-\theta^2}\right) \leqslant 0$$

$$\iff \left\| x - \frac{a-\theta^2 b}{1-\theta^2} \right\|_2^2 \leqslant \frac{\theta^2 \|a-b\|_2^2}{(1-\theta^2)^2}$$

$$\iff \left\| x - \frac{a-\theta^2 b}{1-\theta^2} \right\|_2 \leqslant \frac{\theta \|a-b\|_2}{1-\theta^2}$$

Then, F is the ball $\overline{B}\left(\frac{a-\theta^2 b}{1-\theta^2}, \frac{\theta\|a-b\|_2}{1-\theta^2}\right)$, which is convex.

Therefore, $\boxed{F \text{ is convex}}$

Ex 3.21:

a) $\forall x \in \mathbb{R}^n$, $f(x) = \max_{1 \leq i \leq k} \|A^{(i)}x - b^{(i)}\| = \max_{1\leq i \leq k} f_i(x)$

where $f_i : x \in \mathbb{R}^n \longmapsto \|A^{(i)}x - b^{(i)}\|$.

f_i is the composition of a norm with an affine function and since a norm is convex, then, f_i is convex.

f is the pointwise maximum of k convex functions, then, $\boxed{f \text{ is convex}}$

b) $\forall x \in \mathbb{R}^n$, $f(x) = \max_{1 \leq i_1 < \dots < i_r \leq n} \sum_{k=1}^{r} |x_{i_k}|$.

For $i_k \in [\![1, n]\!]$, $g : x \longmapsto |x_{i_k}|$ is convex (it's easy to verify that $\forall \lambda \in [0,1]$, $\forall x, y \in \mathbb{R}^n$, $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y)$)

Then, $x \longmapsto \sum_{k=1}^{r} |x_{i_k}|$ is convex as a finite sum of convex functions.

f is the pointwise maximum of $\binom{n}{r}$ convex functions, then, $\boxed{f \text{ is convex}}$

Ex 3.32:

a) Suppose f and g are convex, non decreasing and positive over an interval I.

Let $\lambda \in [0,1]$ and $(x,y) \in I^2$:

$$(fg)(\lambda x + (1-\lambda)y) = f(\lambda x + (1-\lambda)y)\ g(\lambda x + (1-\lambda)y)$$
$$\leq [\lambda f(x) + (1-\lambda) f(y)]\,[\lambda g(x) + (1-\lambda) g(y)]$$

since f and g are convex and postive

$$\leq \lambda^2 f(x) g(x) + \lambda(1-\lambda)[f(x)g(y) + g(x)f(y)] + (1-\lambda)^2 f(y) g(y)$$

Then, $(fg)(\lambda x + (1-\lambda)y) - [\lambda (fg)(x) + (1-\lambda)(fg)(y)] \leq \underbrace{(\lambda^2 - \lambda)}_{\lambda(\lambda-1)} f(x)g(x) + \lambda(1-\lambda)[f(x)g(y) + g(x)f(y)] + \underbrace{[(1-\lambda)^2 - (1-\lambda)]}_{\lambda(\lambda-1)} f(y)g(y)$

$$= \lambda(1-\lambda)\,[f(x)g(y) + g(x)f(y) - f(x)g(x) - f(y)g(y)]$$
$$= \lambda(1-\lambda)\,(f(x)(g(y) - g(x)) - f(y)(g(y) - g(x)))$$
$$= \underbrace{\lambda(1-\lambda)}_{\text{positive since } 0\leq\lambda\leq 1}\ \underbrace{(f(x) - f(y))}_{\text{negative since } f \nearrow}\ \underbrace{(g(y) - g(x))}_{\text{positive since } g \nearrow}$$
$$\leq 0$$

Then, $(fg)(\lambda x + (1-\lambda)y) \leq \lambda (fg)(x) + (1-\lambda)(fg)(y)$. Thus, $\boxed{fg \text{ is convex over } I}$

b) Suppose that f and g are concave and positive over an interval I such that $f \nearrow$ and $g \searrow$. (Same if $f \searrow$ and $g \nearrow$)

Using the <u>reversed</u> inequalities in a), we get:

$\forall \lambda \in [0,1], \ \forall (x,y) \in I^2$:

$$(fg)(\lambda x + (1-\lambda)y) - [\lambda (fg)(x) + (1-\lambda)(fg)(y)] \geqslant \underbrace{\lambda(1-\lambda)}_{\geqslant 0} \underbrace{(f(x)-f(y))}_{\leqslant 0 \text{ because } f \nearrow} \underbrace{(g(y)-g(x))}_{\leqslant 0 \text{ because } g \searrow}$$

i.e. $(fg)(\lambda x + (1-\lambda)y) \geqslant \lambda (fg)(x) + (1-\lambda)(fg)(y)$

Then, $\boxed{fg \text{ is concave over } I}$.

c) g is concave and $g > 0$, then, $\frac{1}{g}$ is convexe and > 0.

In fact: $\forall \lambda \in [0,1], \ \forall (x,y) \in I^2$:

$$\lambda \frac{1}{g(x)} + (1-\lambda)\frac{1}{g(y)} = \lambda h(g(x)) + (1-\lambda) h(g(y)) \quad \text{where } h : x \in \mathbb{R}_+^* \longmapsto \frac{1}{x}.$$

$$\geqslant h(\lambda g(x) + (1-\lambda) g(y)) \quad \text{because } h \text{ is convex over } \mathbb{R}_+^* \ (h'' > 0)$$

$$= \frac{1}{\lambda g(x) + (1-\lambda) g(y)}$$

g is concave, then, $0 < \lambda g(x) + (1-\lambda) g(y) \leqslant g(\lambda x + (1-\lambda) y)$

so $\frac{1}{\lambda g(x) + (1-\lambda) g(y)} \geqslant \frac{1}{g(\lambda x + (1-\lambda) y)}$

Then, $\frac{1}{g}$ is concave.

g is decreasing and positive so $\frac{1}{g}$ is increasing.

We apply a) to f and $\frac{1}{g}$. Then, $\boxed{\frac{f}{g} = f \times \frac{1}{g} \text{ is convex}}$

Ex 3.36 :

Notation : Let $\boxed{(e_k)_{1\leq k\leq n}}$ the canonical basis of $\mathbb{R}^n$. Let $\boxed{u = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = \sum_{k=1}^{n} e_k}$

a) Let $F : x \longmapsto y^T x - f(x)$, $y \in \mathbb{R}^n$

- Compute the domain of f^*:

 * If there exists $k \in [\![1,n]\!]$ such that $y_k < 0$:

 $\forall t \geq 0$, $F(-t e_k) = -t y_k \xrightarrow[t\to+\infty]{} +\infty$. Then, F is unbounded.

 Then, if $y \in \mathrm{dom}(f^*) \Rightarrow y \succcurlyeq 0$.

 * Suppose $y \succcurlyeq 0$.

 $\forall t \in \mathbb{R}$, $F(tu) = t\left(\sum_{i=1}^{n} y_i - 1\right)$

 ▪ If $\sum_{i=1}^{n} y_i < 1$: $\lim_{t\to-\infty} F(tu) = +\infty \Rightarrow$ F is unbounded

 ▪ If $\sum_{i=1}^{n} y_i > 1$: $\lim_{t\to+\infty} F(tu) = +\infty \Rightarrow$ F is unbounded.

 Therefore, $y \in \mathrm{dom}(f^*) \Longrightarrow y \succcurlyeq 0$ and $\sum_{i=1}^{n} y_i = 1$.

- Let $y \in \mathbb{R}^n$ such that $y \succcurlyeq 0$ and $\sum_{i=1}^{n} y_i = 1$

$\forall x$, $F(x) = \sum_{i=1}^{n} \underbrace{x_i}_{\leq \max_j(x_j)} y_i - \max_j(x_j) \begin{cases} \leq \left(\sum_{i=1}^{n} y_i - 1\right) \max_j(x_j) = 0. & \text{if } \max_j(x_j) > 0 \\ \leq 0 & \text{if } \max_j(x_j) \leq 0 \end{cases}$

$\left.\begin{array}{l} \forall x \in \mathbb{R}^n,\ F(x) \leq 0. \\ \text{Moreover, } F(\vec{0}) = 0 \end{array}\right\}$ Then, $\sup_{x\in\mathbb{R}^n} F(x) = 0$.

Thus, $\boxed{f^* : y \in \mathbb{R}^n \longmapsto \begin{cases} 0 & \text{if } y \succcurlyeq 0 \text{ and } \sum_{i=1}^{n} y_i = 1 \\ +\infty & \text{otherwise} \end{cases}}$

b) $F(x) = y^T x - \sum_{i=1}^{r} x_{[i]}$, $y \in \mathbb{R}^n$

- Domain of f^*:

* If $\exists k \in [\![1,n]\!]$ such that $y_k < 0$:

 $\forall t \geq 0$, $F(-t e_k) = -t y_k \xrightarrow[t\to+\infty]{} +\infty \Rightarrow$ F is unbounded.

* Let $y \in \mathbb{R}^n$ such that $y \succcurlyeq 0$. Suppose that $\exists k \in [\![1,n]\!]$ such that $y_k > 1$.

 $\forall t \in \mathbb{R}_+$, $F(t e_k) = t y_k - t = t(y_k - 1) \xrightarrow[t\to+\infty]{} +\infty \Rightarrow$ F is unbounded

* Let $y \in \mathbb{R}^n$ such that $0 \preccurlyeq y \preccurlyeq u$.

 $\forall t \in \mathbb{R}$, $F(tu) = \sum_{i=1}^{n} y_i t - rt = t\left(\sum_{i=1}^{n} y_i - r\right)$

 ▪ If $\sum_{i=1}^{n} y_i > r$: $\lim_{t\to+\infty} F(tu) = +\infty \Rightarrow$ F is unbounded

 ▪ If $\sum_{i=1}^{n} y_i < r$: $\lim_{t\to-\infty} F(tu) = +\infty \Rightarrow$ F is unbounded.

Then, $y \in \mathrm{dom}(f^*) \Longrightarrow 0 \leqslant y \leqslant u$ and $\sum_{i=1}^{n} y_i = r$.

- Let $y \in \mathbb{R}^n$ such that $0 \leqslant y \leqslant u$ and $\sum_{i=1}^{n} y_i = r$:

Let $i_1 < i_2 < \dots < i_m \in [\![1, n]\!]$ such that $x_{[1]} = x_{i_1}, \dots, x_{[m]} = x_{i_m}$ (ie $x_{i_1} \leqslant \dots \leqslant x_{i_m}$)

$$F(x) = \sum_{i=1}^{n} x_i y_i - \sum_{k=1}^{r} x_{i_k}$$

$$= \sum_{k=1}^{n} x_{i_k} y_{i_k} - \sum_{k=1}^{r} x_{i_k}$$

$$= \sum_{k=1}^{r} x_{i_k}(y_{i_k} - 1) + \sum_{k=r+1}^{n} \underbrace{x_{i_k}}_{\leqslant x_{i_r}} y_{i_k}$$

$$\leqslant \sum_{k=1}^{r} x_{i_k}(y_{i_k} - 1) + x_{i_r} \times \underbrace{\sum_{k=r+1}^{n} y_{i_k}}_{= r - \sum_{k=1}^{r} y_{i_k} = \sum_{k=1}^{r}(1 - y_{i_k})}$$

$$\overset{=}{\leqslant} \sum_{k=1}^{r} \underbrace{(1 - y_{i_k})}_{\geqslant 0} \underbrace{(x_{i_r} - x_{i_k})}_{\leqslant 0}$$

$F(x) \leqslant 0$

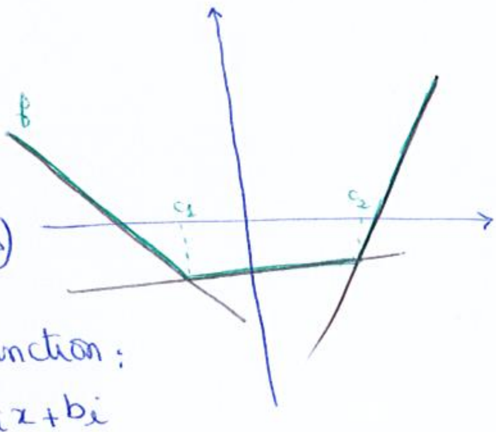
Notice that, $F(\vec{0}) = 0$. $\Big\} \Rightarrow \sup_{x \in \mathbb{R}^n} F(x) = 0$.

Therefore, $\boxed{f^* : y \in \mathbb{R}^n \longmapsto \begin{cases} 0 & \text{if } 0 \leqslant y \leqslant u \text{ and } \sum_{i=1}^{n} y_i = r \\ +\infty & \text{otherwise.} \end{cases}}$

c)

$$f(x) = \max_{1 \leqslant i \leqslant m} (a_i x + b_i)$$

$F : x \longmapsto xy - \max_{1 \leqslant i \leqslant m}(a_i x + b_i)$, $y \in \mathbb{R}$.



For $i \in [\![1, m-1]\!]$, let $c_{i+1} = \dfrac{b_i - b_{i+1}}{a_{i+1} - a_i}$ (abscisses of intersections)

By definition, f is a piecewise-affine function:

$$\begin{cases} \bullet\ \forall i \in [\![2, m-1]\!],\ \forall x \in [c_i, c_{i+1}]\ f(x) = a_i x + b_i \\ \bullet\ \forall x \in]-\infty, c_1],\ f(x) = a_1 x + b_i \\ \bullet\ \forall x \in [c_m, +\infty[,\ f(x) = a_m x + b_m \end{cases}$$

- If $y > a_m$, then $\forall x \in [c_m, +\infty[$, $F(x) = x(y - a_m) + b_m \xrightarrow[x \to +\infty]{} +\infty$
- If $y < a_1$: then $\forall x \in]-\infty, c_2]$, $F(x) = x(y - a_1) + b_1 \xrightarrow[x \to -\infty]{} +\infty$

Then, $y \in \mathrm{dom}(f^*) \Longrightarrow y \in [a_1, a_m]$.

- Let $y \in [a_1, a_m]$. Then, $\exists i \in [\![1, m-1]\!]$ such that $y \in [a_i; a_{i+1}]$.

  * Over $]-\infty, c_2]$, $F : x \mapsto x(y-a_1) - b_1$ is increasing (since $y - a_1 \geqslant 0$)

    and reaches its maximum at c_2.

    This max $= c_2(y-a_1) - b_1$

  * Over $[c_j, c_{j+1}]$ (when $j \leqslant i$); $F : x \mapsto x(y-a_j) - b_j$

    F is increasing (since $y - a_j \geqslant 0$) and reaches its maximum at c_{j+1}.

    This max $= c_{j+1}(y-a_j) - b_j$.

  * Over $[c_j, c_{j+1}]$ (where $j > i$): $F : x \mapsto x(y-a_j) + b_j$

    F is decreasing (because $y - a_j < 0$) and reaches its maximum at c_j

    This max $= c_j\,(y-a_j) - b_j$

  * Over $[c_m, +\infty[$: $F : x \mapsto x(y-a_m) + b_m$ is decreasing. It reaches its maximum at c_m

    This max $= c_m(y-a_m) - b_m$

Then, for $y \in [a_i; a_{i+1}]$,

$$\sup_{\mathbb{R}} F = \max\left(\max_{1\leqslant j \leqslant i}\left(c_{j+1}(y-a_j) - b_j\right),\ \max_{i<j\leqslant m}\left(c_j(y-a_j)-b_j\right)\right)$$

$$= c_{i+1}(y-a_i) - b_i$$

$$= (y-a_i)\,\frac{b_{i+1}-b_i}{a_i - a_{i+1}} - b_i$$

Therefore

$$\boxed{f^* : y \mapsto \begin{cases} \displaystyle\sum_{i=1}^{m-1}\left[\frac{b_{i+1}-b_i}{a_i - a_{i+1}}(y-a_i) - b_i\right] \mathbb{1}_{[a_i; a_{i+1}]}(y) & \text{if } y \in [a_1, a_m] \\ +\infty & \text{if } y < a_1 \text{ or } y > a_m \end{cases}}$$

d) Case 1 : if $p > 1$:

$F : x \in \mathbb{R}_+^* \longmapsto yx - x^p$, $y \in \mathbb{R}$.

F is differentiable over $\mathbb{R}_+^*$; $\forall x > 0$, $F'(x) = y - px^{p-1}$

- If $y \leq 0$: Since $-px^{p-1} < 0$ then $F'(x) < 0$ so $F \searrow$ $\mathbb{R}_+^*$
Since F is continuous and $\searrow$ then F realizes a bijection of $]0,+\infty[$ on $F(]0,+\infty[) =]\lim\limits_{+\infty} F, \lim\limits_{0^+} F[=]-\infty, 0[$. Then, $\sup\limits_{\mathbb{R}_+^*} F = 0$.

- If $y > 0$:

$F'(x) = 0 \iff y = px^{p-1} \iff x = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

$F''(x) = -p(p-1)x^{p-2} < 0 \quad \forall x > 0$

Then, F reaches its maximum (global over $\mathbb{R}_+^*$) at $\left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

and $F\left(\left(\frac{y}{p}\right)^{\frac{1}{p-1}}\right) = y\left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}} = p \times \left(\frac{y}{p}\right)^{\frac{p}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}} = (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}}$

Then, $\boxed{f^* : y \in \mathbb{R} \longmapsto \begin{cases} (p-1)\left(\frac{y}{p}\right)^{\frac{1}{p-1}} & \text{if } y > 0 \\ 0 & \text{otherwise.} \end{cases}}$

Case 2 : if $p < 0$

- If $y > 0$: $\lim\limits_{x \to +\infty} F(x) = +\infty \Rightarrow F$ is unbounded.

- If $y < 0$: $F'(x) = 0 \iff y = px^{p-1} \iff \left(\frac{y}{p}\right)^{\frac{1}{p-1}} = x$

$F''(x) = -\underbrace{p}_{<0}\underbrace{(p-1)}_{<0}x^{p-2} < 0 \quad \forall x > 0$

Then, F reaches its maximum (global over $\mathbb{R}_+^*$) at $\left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

and $F\left(\left(\frac{y}{p}\right)^{\frac{1}{p-1}}\right) = y\left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}} = (p-1)\left(\frac{y}{p}\right)^{\frac{1}{p-1}}$.

- If $y = 0$: $F(x) = -x^p$. F is continuous and $\nearrow$ then $F(]0,+\infty[) =]\lim\limits_{0^+} F, \lim\limits_{+\infty} F[=]-\infty, 0[$

Then, $\sup\limits_{\mathbb{R}_+^*} F = 0$. $\left(0 = (p-1)\left(\frac{0}{p}\right)^{\frac{p}{p-1}}\right)$

$0^\alpha = 0$
$\forall \alpha > 0$

Therefore, $\boxed{f^* : y \in \mathbb{R} \longmapsto \begin{cases} (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & \text{if } y \geq 0 \\ +\infty & \text{otherwise} \end{cases}}$

e) $F : x \in (\mathbb{R}_+^*)^n \longmapsto y^T x + \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$, $y \in \mathbb{R}^n$

- Domain of f^*:

(*) If $\exists k \in [\![1, n]\!]$ such that $y_k \geqslant 0$:

$$\forall t \geqslant 0, \quad F(u + (t-1)e_k) = F\left(\begin{pmatrix} 1 \\ \vdots \\ t \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right) = y_k t + \sum_{\substack{i=1 \\ i \neq k}}^{n} y_i + t^{\frac{1}{n}} \xrightarrow[t \to +\infty]{} +\infty$$

k^{th} position

Then, F is unbounded.

(*) Let $y \in \mathbb{R}^n$ such that $y < 0$: Let $z = -y$ (ie $\forall i \in [\![1, n]\!]$ $z_i = -y_i$).

$$\forall t > 0: \quad F\left(\frac{t}{z}\right) = F\left(\begin{pmatrix} \frac{t}{z_1} \\ \vdots \\ \frac{t}{z_n} \end{pmatrix}\right) = \left(\prod_{i=1}^{n} \left(\frac{t}{z_i}\right)\right)^{\frac{1}{n}} - nt = t\left(\underbrace{\left(\prod_{i=1}^{n} \frac{1}{z_i}\right)^{\frac{1}{n}}}_{= \left(\prod_{i=1}^n z_i\right)^{-\frac{1}{n}}} - n\right)$$

- If $\left(\prod_{i=1}^{n} \frac{1}{z_i}\right)^{\frac{1}{n}} > n$: $F\left(\frac{t}{z}\right) \xrightarrow[t \to +\infty]{} +\infty \Rightarrow F$ is unbounded.

Then, $y \in \mathrm{dom}(f^*) \Longrightarrow y \leqslant 0$ and $\left(\prod_{i=1}^{n} -y_i\right)^{\frac{1}{n}} \geqslant \frac{1}{n}$.

- Let $y \in \mathbb{R}^n$ such that $y < 0$ and $\left(\prod_{i=1}^{n} -y_i\right)^{\frac{1}{n}} \geqslant \frac{1}{n}$:

Let $z = -y$:

$$\forall x \in \mathbb{R}_{++}^n: \quad F(x) = n\left(\frac{1}{n}\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} - \frac{1}{n}\sum_{i=1}^{n} x_i z_i\right)$$

$$\leqslant n\left(\left(\prod_{i=1}^{n} z_i\right)^{\frac{1}{n}}\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} - \frac{1}{n}\sum_{i=1}^{n} x_i z_i\right)$$

$$F(x) \leqslant n\left(\left(\prod_{i=1}^{n} x_i z_i\right)^{\frac{1}{n}} - \frac{1}{n}\sum_{i=1}^{n} x_i z_i\right)$$

We know that $\ln$ is concave over $\mathbb{R}_+^*$, then:

$$\ln\left(\sum_{i=1}^{n} \frac{x_i z_i}{n}\right) \geqslant \sum_{i=1}^{n} \frac{1}{n} \ln(x_i z_i)$$

we compose by exp, which increasing

$$\sum_{i=1}^{n} \frac{x_i z_i}{n} \geqslant \underbrace{\exp\left(\frac{1}{n}\sum_{i=1}^{n} \ln(x_i z_i)\right)}_{= \left(\prod_{i=1}^{n} x_i z_i\right)^{\frac{1}{n}}}$$

Then $F(x) \leqslant 0 \quad \forall x \in \mathbb{R}_{++}^n$.

$\forall t > 0, \quad F\left(\frac{1}{tz}\right) = \frac{1}{t}\left(\left(\prod_{i=1}^{n} \frac{1}{z_i}\right) - n\right) \xrightarrow[t \to +\infty]{} 0$

$\Big\} \Rightarrow \sup_{\mathbb{R}_{++}^n} F = 0$

Therefore, $\boxed{f^* : y \in \mathbb{R}^n \longmapsto \begin{cases} 0 & \text{if } y < 0 \text{ and } \left(\prod_{i=1}^{n} (-y_i)\right)^{\frac{1}{n}} \geqslant \frac{1}{n} \\ +\infty & \text{otherwise} \end{cases}}$

f) Let $\mathcal{E} := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 < t\}$ (10)

$F : (x,t) \in \mathcal{E} \longmapsto y^T x + st + \ln(t^2 - x^T x)$. for $(y,s) \in \mathbb{R}^n \times \mathbb{R}$.

- Domain of f^*:

* If $s > 0$: $\forall t > 0$, $F(0,t) = st + 2\ln(t) = t\left(s + 2\frac{\ln(t)}{t}\right) \xrightarrow[t \to +\infty]{} +\infty$ $\Rightarrow$ F is unbounded.

* Let $y \in \mathbb{R}^n$ et $s \leq 0$:

Let $\alpha > 0$: $(\alpha y, \alpha\|y\|_2 + 1) \in \mathcal{E}$.

$$F(\alpha y, \alpha\|y\|_2 + 1) = \alpha\|y\|_2^2 + s(\alpha\|y\|_2 + 1) + \ln\left((\alpha\|y\|_2 + 1)^2 - \alpha^2\|y\|_2^2\right)$$

$$= \alpha\|y\|_2(\|y\|_2 + s) + s + \ln(2\alpha\|y\|_2 + 1)$$

- if $\|y\|_2 + s > 0$: $\lim_{\alpha \to +\infty} F(\alpha y, \alpha\|y\|_2 + 1) = +\infty$
- if $\|y\|_2 + s = 0$ and $y \neq 0$: $F(\alpha\|y\|_2, \alpha\|y\|_2 + 1) = s + \ln(2\alpha\underbrace{\|y\|_2}_{>0} + 1) \xrightarrow[\alpha \to +\infty]{} +\infty$
- if $y = 0$ and $s = 0$: $F(0,t) = \ln(t^2) = 2\ln(t) \xrightarrow[t \to +\infty]{} +\infty$.

Then $(y,s) \in \mathrm{dom}(f^*) \Longrightarrow \|y\|_2 < -s$.

- Now, let $(y,s) \in \mathbb{R}^n \times \mathbb{R}_-$ such that $\|y\|_2 < -s$: (ie $s^2 - \|y\|_2^2 > 0$)

F is differentiable.

$$\frac{\partial F}{\partial t}(x,t) = s + \frac{2t}{t^2 - \|x\|_2^2} \quad ; \quad \frac{\partial F}{\partial x}(x,t) = y + \frac{-2x}{t^2 - x^T x}$$ (it is a vector)

↳ a better notation: $\frac{\partial F}{\partial x_i} = y_i - \frac{2x_i}{t^2 - \|x\|^2}$

Critical Points: $\nabla F(x,t) = 0 \iff \begin{cases} s = -\dfrac{2t}{t^2 - \|x\|_2^2} & (1) \\ y = \dfrac{2x}{t^2 - \|x\|_2^2} & (2) \end{cases}$.

$$(1)^2 - (2)^2 \Rightarrow s^2 - y^T y = \frac{4t^2}{(t^2 - \|x\|_2^2)^2} - \frac{4\|x\|_2^2}{(t^2 - \|x\|_2^2)^2} = \frac{4}{t^2 - \|x\|_2^2}$$

Then, $\nabla F(x^*, t^*) = 0 \iff \begin{cases} t^* = -\frac{1}{2}(t^2 - \|x\|_2^2)s \\ x^* = \frac{1}{2}(t^2 - \|x\|_2^2)y \\ s^2 - \|y\|_2^2 = \dfrac{4}{t^{*2} - \|x^*\|_2^2} \end{cases} \iff \begin{cases} t^* = \dfrac{-2s}{s^2 - \|y\|_2^2} \\ x^* = \dfrac{2y}{s^2 - \|y\|_2^2} \end{cases}$

We can show(*) that:
$$\nabla^2 F(x,t) = \frac{-4}{(t^2 - \|x\|^2)^2} \underbrace{\left(\begin{array}{c|c} xx^T + \frac{1}{2}(t^2 - \|x\|^2) I_n & -tx \\ \hline -tx^T & t^2 - \frac{1}{2}(t^2 - \|x\|^2) \end{array}\right)}_{H}$$

$\boxed{\text{(*) : calculus in page (12)}}$

Let us show that $H \geqslant 0$:

Let $z \in \mathbb{R}^n$, $u \in \mathbb{R}$ such that $\|z\|_2 < u$, then:

$$(z^T \; u)\, H \begin{pmatrix} z \\ u \end{pmatrix} = z^T\left(xx^T + \frac{1}{2}(t^2 - \|x\|^2) I_n\right) z - ut\, x^T z - ut\, z^T x + u^2\left(t^2 - \frac{1}{2}(t^2 - \|x\|^2)\right)$$

$$= (z^Tx)^2 + \frac{1}{2}(t^2 - \|x\|^2)\|z\|^2 - 2ut\, z^T x + u^2t^2 - \frac{1}{2}u^2(t^2 - \|x\|^2)$$

$$= (z^Tx)^2 - 2ut\, z^Tx + u^2t^2 - \frac{1}{2}(t^2 - \|x\|^2)(\|z\|^2 - u^2)$$

$$= (z^Tx - ut)^2 - \underbrace{\frac{1}{2}(t^2 - \|x\|^2)(u^2 - \|z\|^2)}$$

$$= (t - \|x\|)(t + \|x\|)(u - \|z\|)(\|z\| + u)$$

$$= (ut - \|x\|\|z\| + t\|z\| - u\|x\|)(ut - \|x\|\|z\| - t\|z\| + u\|x\|)$$

$$= (ut - \|x\|\|z\|)^2 - (t\|z\| - u\|x\|)^2$$

$$= (z^Tx - ut)^2 - \frac{1}{2}(ut \;\; \|x\|\|z\|)^2 + \frac{1}{2}(t\|z\| - u\|x\|)^2$$

$$= \frac{1}{2}\left[(z^Tx - ut)^2 - (ut - \|x\|\|z\|)^2\right] + \frac{1}{2}(z^Tx - ut)^2 + \frac{1}{2}(t\|z\| - u\|x\|)^2$$

$$= \frac{1}{2}\underbrace{(z^Tx - \|x\|\|z\|)}_{\leqslant 0 \text{ with Cauchy-Schwartz}}\underbrace{(z^Tx + \|x\|\|z\| - 2ut)}_{\substack{\leqslant 2\|x\|\|z\| - 2ut \text{ (with Cauchy-Schwartz)} \\ \leqslant 0 \text{ (because } \|x\| < t \text{ and } \|z\| < u)}} + \frac{1}{2}(z^Tx - ut)^2 + \frac{1}{2}(t\|z\| - u\|x\|)^2$$

Then, $(z^T \; u)\, H \begin{pmatrix} z \\ u \end{pmatrix} \geqslant 0 \quad \forall (z, u) \in \mathbb{R}^n \times \mathbb{R}$ such that $\|z\|_2 < u$.

Then, $H \succeq_{\mathcal{E}} 0$. Therefore, $\boxed{\nabla^2 F(x,t) \preceq_{\mathcal{E}} 0} \quad \forall (x,t) \in \mathcal{E}$

(Recall that $\mathcal{E} := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 < t\}$.
norm-cone, which is convex set.)

Then, $\sup_{(x,t) \in \mathcal{E}} F(x,t) = F(x^*, t^*)$ we use that $t^{*2} - \|x^*\|_2^2 = \frac{4}{s^2 - \|y\|^2}$

$$= y^T\left(\frac{2y}{s^2 - \|y\|_2^2}\right) + s\left(\frac{-2s}{s^2 - \|y\|_2^2}\right) + \ln\left(\frac{4}{s^2 - \|y\|_2^2}\right)$$

$$= \frac{2\|y\|_2^2}{s^2 - \|y\|_2^2} - \frac{2s^2}{s^2 - \|y\|_2^2} + \ln(4) - \ln(s^2 - \|y\|_2^2)$$

$$= -2 + \ln(4) + \ln(s^2 - \|y\|_2^2).$$

Therefore, $\boxed{f^*: (y,s) \in \mathbb{R}^n \times \mathbb{R} \longmapsto \begin{cases} -2 + \ln(4) + \ln(s^2 - \|y\|_2^2) & \text{if } \|y\|_2 < s \\ +\infty & \text{otherwise} \end{cases}}$

Details of calculus of $\nabla^2 F$:

we have : $\begin{cases} \forall i \in [\![1, n]\!] \quad \dfrac{\partial F(x,t)}{\partial x_i} = y_i - \dfrac{2x_i}{t^2 - \|x\|^2} \\ \dfrac{\partial F}{\partial t}(x,t) = s + \dfrac{2t}{t^2 - \|x\|^2} \end{cases}$

- $$\frac{\partial^2 F}{\partial x_i^2}(x,t) = -2 \frac{(t^2 - \|x\|^2) - x_i(-2x_i)}{(t^2 - \|x\|^2)^2} = -4 \frac{x_i^2 + \frac{1}{2}(t^2 - \|x\|^2)}{(t^2 - \|x\|^2)^2}$$

- $$\frac{\partial^2 F}{\partial t^2}(x,t) = 2 \frac{(t^2 - \|x\|^2) - t \times 2t}{(t^2 - \|x\|^2)^2} = -4 \frac{t^2 - \frac{1}{2}(t^2 - \|x\|^2)}{(t^2 - \|x\|^2)^2}$$

$i \neq j$
- $$\frac{\partial^2 F}{\partial x_i \partial x_j}(x,t) = 2x_i \frac{-2x_j}{(t^2 - \|x\|^2)^2} = -4 \frac{x_i x_j}{(t^2 - \|x\|^2)^2}$$

- $$\frac{\partial^2 F}{\partial x_i \partial t}(x,t) = -2x_i \frac{-2t}{(t^2 - \|x\|^2)^2} = 4 \frac{x_i t}{(t^2 - \|x\|^2)^2}$$

Notice that $x_i^2 = [x\, x^T]_{ii}$, $x_i x_j = [x\, x^T]_{ij}$

$x_i t = [t\, x]_i$

So,
$$\nabla^2 F(x,t) = \frac{-4}{(t^2 - \|x\|^2)^2} \left(\begin{array}{c|c} x\, x^T + \frac{1}{2}(t^2 - \|x\|^2) I_n & -t x \\ \hline -t x^T & t^2 - \frac{1}{2}(t^2 - \|x\|^2) \end{array} \right)$$