Exercise n°1 :

1. The standard formulation of (P) is : $\begin{cases} \min_x \ c^T x \\ \text{s.t.} \ \ Ax - b = 0 \\ \quad\ \ -x \leqslant 0 \end{cases}$

- The Lagrangian is given by :

$$\mathcal{L}(x, \lambda, \nu) = c^T x - \lambda^T x + \nu^T (Ax - b)$$
$$= (c^T + \nu^T A - \lambda^T) x - \nu^T b$$
$$= (A^T \nu + c - \lambda)^T x - b^T \nu$$

Then, $g(\lambda, \nu) = \inf_{x \in \mathbb{R}^d} \mathcal{L}(x, \lambda, \nu) = \begin{cases} -b^T \nu & \text{if } A^T \nu + c - \lambda = 0 \\ -\infty & \text{otherwise.} \end{cases}$

Thus, the dual of (P) is : $\begin{cases} \max_{(\lambda,\nu) \in \mathbb{R}^d \times \mathbb{R}^n} g(\lambda, \nu) \\ \text{st. } \lambda \geqslant 0 \end{cases} \iff \begin{cases} \max_{\nu \in \mathbb{R}^n} -b^T \nu \\ \text{s.t. } \lambda \geqslant 0 \text{ and } A^T \nu + c = \lambda \end{cases}$

I replaced ν with $(-\nu)$

$$\iff \boxed{\begin{cases} \max_\nu \ b^T \nu \\ \text{s.t } \ A^T \nu \leqslant c \end{cases}}$$

We can notice that $\boxed{\text{the dual of (P) is (D)}}$.

2. The standard form of (D) is : $\begin{cases} \min_y \ -b^T y \\ \text{s.t } \ A^T y - c \leqslant 0 \end{cases}$

- The Lagrangian is given by :

$$\mathcal{L}(y, \lambda) = -b^T y + \lambda^T (A^T y - c)$$
$$= (A\lambda - b)^T y - \lambda^T c = (A\lambda - b)^T y - c^T \lambda$$

Then, $g(\lambda) = \inf_{y \in \mathbb{R}^n} \mathcal{L}(y, \lambda) = \begin{cases} -c^T \lambda & \text{if } A\lambda = b \\ -\infty & \text{otherwise.} \end{cases}$

- Thus, the dual of (D) is : $\begin{cases} \max_\lambda \ g(\lambda) \\ \text{s.t } \ \lambda \geqslant 0 \end{cases} \iff \boxed{\begin{cases} \min_\lambda \ c^T \lambda \\ \text{s.t. } \ A\lambda = b \\ \quad\quad \lambda \geqslant 0 \end{cases}}$

Therefore, $\boxed{\text{the dual of (D) is (P)}}$

3. We can notice that the "Self-Dual" problem is $(P) \cap (D)$, (P) and (D) are independant. So, $\text{Dual}((P) \cap (D)) = \text{Dual}(P) \cap \text{Dual}(D) = (P) \cap (D)$.

More rigously, let us compute the lagrangian :

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \nu) = c^T x - b^T y - \lambda_1^T x + \lambda_2^T (A^T y - c) + \nu^T (Ax - b)$$
$$= (c^T - \lambda_1^T + \nu^T A) x + (\lambda_2^T A^T - b^T) y - c^T \lambda_2 - b^T \nu$$
$$= (A^T \nu + c - \lambda_1)^T x + (A\lambda_2 - b)^T y - c^T \lambda_2 - b^T \nu$$

Then $g(\lambda_1, \lambda_2, \nu) = \inf_{x,y} \mathcal{L}(x, y, \lambda_1, \lambda_2, \nu)$

$$= \left(\inf_x \left(A^T\nu + c - \lambda_1\right)^T x\right) + \left(\inf_y \left(A\lambda_2 - b\right)^T y\right) - c^T\lambda_2 - b^T\nu.$$

$$= \begin{cases} -c^T\lambda_2 - b^T\nu & \text{if } A^T\nu + c = \lambda_1 \text{ and } A\lambda_2 = b \\ -\infty & \text{otherwise.} \end{cases}$$

Then, the dual of "Self-Dual" is given by:
$$\begin{cases} \max_{(\lambda_2,\nu)} & -c^T\lambda_2 - b^T\nu \\ \text{s.t.} & A^T\nu + c \geq 0 \\ & A\lambda_2 = b \\ & \lambda_2 \geq 0 \end{cases}$$

I just substituted ν with $(-\nu)$

, which is equivalent to
$$\begin{cases} \max_{(\lambda,\nu)} & -c^T\lambda + b^T\nu \\ \text{s.t.} & -A^T\nu + c \geq 0 \\ & A\lambda = b \\ & \lambda \geq 0 \end{cases}$$

which is equivalent to
$$\begin{cases} \min_{(\lambda,\nu)} & c^T\lambda - b^T\nu \\ & A^T\nu \leq c \\ & A\lambda = b \\ & \lambda \geq 0 \end{cases}$$

Thus, the dual of "Self-Dual" is "Self-Dual". i.e $\boxed{\text{the problem is self-dual}}$

4 a) Since (x^*, y^*) is optimal for "Self-Dual", then, $\begin{cases} Ax^* = b \\ x^* \geq 0 \end{cases}$ and $A^T y^* \leq c$.

So, (P) and (D) are <u>feasible</u>.

- Let $\bar{x}$ a feasible solution for (P). Suppose with absurd that $c^T\bar{x} < c^T x^*$.

Then, $c^T\bar{x} - b^T y^* < c^T x^* - b^T y^*$: impossible because (x^*, y^*) is optimal for "Self-Dual".

Then, $\boxed{x^* \text{ is optimal for (P)}}$.

- Similarly, let $\bar{y}$ a feasible solution for (D). Suppose with absurd that $b^T\bar{y} > b^T y^*$.

Then, $c^T x^* - b^T\bar{y} < c^T x^* - b^T y^*$: impossible because (x^*, y^*) is optimal for "Self-Dual".

Then, $\boxed{y^* \text{ is optimal for (D)}}$

b) - If we assume that strong duality is applicable for LPs: since (D) is the dual of (P) and both are feasible LPs, so, $d^* = p^*$ i.e $c^T x^* = b^T y^*$ i.e $c^T x^* - b^T y^* = 0$.

- Actually, we don't need strong duality. In fact, weak duality is sufficient. In fact, since (D) is the dual of (P), with weak duality, we get $b^T y^* \leq c^T x^*$. Besides, (P) is the dual of (D), so, with weak duality, we have $c^T x^* \leq b^T y^*$. Hence, $c^T x^* = b^T y^*$.

Therefore, $\boxed{\text{the optimal value of "Self-Dual" is exactly } 0}$.

Exercise n°2 :

① Let $F : x \in \mathbb{R}^d \longmapsto y^T x - \|x\|_1$, for $y \in \mathbb{R}^d$. Let $(\vec{e}_k)_{1 \leq k \leq d}$ the canonical basis of $\mathbb{R}^d$

- Suppose that $\exists k \in [\![1, d]\!]$ such that $y_k > 1$:

  Then, $\forall t > 0$, $F(t\vec{e}_k) = t(y_k - 1) \xrightarrow[t \to +\infty]{} +\infty$ $\Rightarrow$ F is unbounded.

- Suppose that $\exists k \in [\![1, d]\!]$ such that $y_k < -1$:

  Then, $\forall t > 0$, $F(-t\vec{e}_k) = -t(y_k + 1) \xrightarrow[t \to +\infty]{} +\infty$ $\Rightarrow$ F is unbounded.

Thus, F is bounded $\Rightarrow$ $\forall k \in [\![1, d]\!]$, $|y_k| \leq 1$ ie $\|y\|_\infty \leq 1$.

- Now, let $y \in \mathbb{R}^d$ such that $\|y\|_\infty \leq 1$:

  $\forall i \in [\![1, d]\!]$; $x_i y_i \leq |x_i y_i| \leq \|y\|_\infty |x_i|$

  Then, $\sum_{i=1}^{d} x_i y_i \leq \sum_{i=1}^{d} \|y\|_\infty |x_i| = \|y\|_\infty \|x\|_1 \leq \|x_1\|$

  i.e $F(x) \leq 0$.

Moreover, $F(0) = 0$. Then, $\sup_{x \in \mathbb{R}^d} F(x) = 0$

Thus, the conjugate of $\|x\|_1$ is given by : $\boxed{f^* : \mathbb{R}^d \longrightarrow \mathbb{R}, \quad y \longmapsto \begin{cases} 0 & \text{if } \|y\|_\infty \leq 1 \\ +\infty & \text{otherwise.} \end{cases}}$

② (RLS): $\min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2 + \|x\|_1 \iff \begin{cases} \min_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^n} \|y\|_2^2 + \|x\|_1 \\ \text{subject to } y = Ax - b. \end{cases}$

* The Lagrangian is given by :

$$\mathcal{L}(x, y, \nu) := \|y\|_2^2 + \|x\|_1 + \nu^T(y - (Ax - b))$$
$$= \|y\|_2^2 + \nu^T y + \|x\|_1 - (A^T\nu)^T x + \nu^T b$$

$$g(\nu) := \inf_{x,y} \mathcal{L}(x, y, \nu)$$
$$= \inf_y \left(\|y\|_2^2 + \nu^T y \right) + \inf_x \left(\|x\|_1 - (A^T\nu)^T x \right) + \nu^T b$$

- We notice that : $\inf_x \left(\|x\|_1 - (A^T\nu)^T x \right) = - \sup_x \left((A^T\nu)^T x - \|x\|_1 \right) = -f^*(A^T\nu)$

$$= \begin{cases} 0 & \text{if } \|A^T\nu\|_\infty \leq 1 \\ -\infty & \text{otherwise.} \end{cases}$$

- Let $\varphi : y \in \mathbb{R}^n \longmapsto \|y\|_2^2 + \nu^T y = y^T y + \nu^T y$.

  φ is differentiable ; $\nabla\varphi(y) = 2y + \nu$. Then, $\nabla\varphi(y) = 0 \iff y = \frac{-1}{2}\nu$.

  φ is a convex function , then, φ reaches its global minimum at $y = -\frac{1}{2}\nu$

  and $\inf_y \|y\|_2^2 + \nu^T y = \frac{-1}{4}\nu^T\nu$.

Thus, $g(\nu) = \begin{cases} -\frac{1}{4}\nu^T\nu + b^T\nu & \text{if } \|A^T\nu\|_\infty \leqslant 1 \\ -\infty & \text{otherwise} \end{cases}$

Then, the dual of (RLS) is: $\begin{cases} \max\limits_{\nu} & -\frac{1}{4}\nu^T\nu + b^T\nu \\ \text{s.t.} & \|A^T\nu\|_\infty \leqslant 1 \end{cases}$

. This is equivalent to: $\boxed{\begin{cases} \min\limits_{\nu} & \frac{1}{4}\nu^T\nu + b^T\nu \\ \text{s.t.} & \|A^T\nu\|_\infty \leqslant 1 \end{cases}}$

Exercise n°3:

(1) We notice that the problem (Sep2) is equivalent to $\begin{cases} \min\limits_{(\omega,z)} & \frac{1}{m}\mathbb{1}^T z + \frac{\tau}{2}\|\omega\|_2^2 \\ \text{s.t.} & z_i \geqslant \mathcal{L}(\omega, x_i, y_i) \quad \forall i \in [\![1,m]\!]. \end{cases}$

Let (w^*, z^*) is an optimal solution for (Sep2).

Suppose that $\exists \bar{\omega} \in \mathbb{R}^d$ such that: $\frac{1}{m}\sum_{i=1}^m \mathcal{L}(\bar{\omega}, x_i, y_i) + \frac{\tau}{2}\|\bar{\omega}\|_2^2 < \frac{1}{m}\sum_{i=1}^m \mathcal{L}(\omega^*, x_i, y_i) + \frac{\tau}{2}\|\omega^*\|_2^2$ (*)
(by absurd)

Let $\bar{z} \in \mathbb{R}^m$ such that $\bar{z}_i := \mathcal{L}(\bar{\omega}, x_i, y_i) \quad \forall i \in [\![1,m]\!]$.

Then, $(\bar{\omega}, \bar{z})$ is a feasible solution for (Sep2).

Moreover, we have: $\frac{1}{m}\mathbb{1}^T\bar{z} + \frac{\tau}{2}\|\bar{\omega}\|_2^2 = \frac{1}{m}\sum_{i=1}^m \mathcal{L}(\bar{\omega}, x_i, y_i) + \frac{\tau}{2}\|\bar{\omega}\|_2^2$ I used (*)

$$< \frac{1}{m}\sum_{i=1}^m \underbrace{\mathcal{L}(\omega^*, x_i, y_i)}_{\leqslant z_i^*} + \frac{\tau}{2}\|\omega^*\|_2^2$$

$$< \frac{1}{m}\mathbb{1}^T z^* + \frac{\tau}{2}\|\omega^*\|_2^2$$

$\Rightarrow$ Absurd because (ω^*, z^*) is optimal for (Sep2).

Therefore, $\forall \omega \in \mathbb{R}^d$, $\frac{1}{m}\sum_{i=1}^m \mathcal{L}(\omega, x_i, y_i) + \frac{\tau}{2}\|\omega\|_2^2 \geqslant \frac{1}{m}\sum_{i=1}^m \mathcal{L}(\omega^*, x_i, y_i) + \frac{\tau}{2}\|\omega^*\|_2^2$.

ie ω^* is an optimal solution for (Sep1).

Thus, $\boxed{\text{(Sep2) solves (Sep1)}}$.

(2) Let $\boxed{L := \operatorname{diag}(y) X}$ where $\operatorname{diag}(y) = \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_m \end{pmatrix} \in \mathbb{R}^{m\times m}$ and $X = (x_1, \dots, x_m)^T \in \mathbb{R}^{m\times d}$

which means that $[L\omega]_i = y_i \omega^T x_i \quad \forall i \in [\![1,m]\!]$.

- The Lagrangian is given by:

$$\mathcal{L}(\omega, z, \lambda, \pi) = \frac{1}{m\tau}\mathbb{1}^T z + \frac{1}{2}\|\omega\|_2^2 - \sum_{i=1}^m \lambda_i (z_i - 1 + y_i \omega^T x_i) - \pi^T z$$

$$= \frac{1}{m\tau}\mathbb{1}^T z + \frac{1}{2}\|\omega\|_2^2 - \lambda^T z + \mathbb{1}^T\lambda - \lambda^T L\omega - \pi^T z$$

$$\mathcal{L}(w,z,\lambda,\pi) = \left(\frac{1}{m\tau}\mathbb{1}^T - \lambda^T - \pi^T\right) z + \frac{1}{2}\|w\|_2^2 - \lambda^T L w + \mathbb{1}^T \lambda$$

$$= \left(\frac{1}{m\tau}\mathbb{1} - \lambda - \pi\right)^T z + \frac{1}{2}\|w\|_2^2 - \lambda^T L w + \mathbb{1}^T \lambda .$$

$$g(\lambda,\pi) = \inf_{w,z} \mathcal{L}(w,z,\lambda,\pi)$$

$$= \left(\inf_z \left(\frac{1}{m\tau}\mathbb{1} - \lambda - \pi\right)^T z\right) + \inf_w \left(\frac{1}{2}\|w\|_2^2 - \lambda^T L w\right) + \mathbb{1}^T \lambda .$$

- $\inf_z \left(\frac{1}{m\tau}\mathbb{1} - \lambda - \pi\right)^T z = \begin{cases} 0 & \text{if } \frac{1}{m\tau}\mathbb{1} - \pi = \lambda \\ -\infty & \text{otherwise.} \end{cases}$

- Let $f : w \mapsto \frac{1}{2}\|w\|_2^2 - \lambda^T L w$. Then, $\inf_w \frac{1}{2}\|w\|_2^2 - \lambda^T L w = -\frac{1}{2}\lambda^T L L^T \lambda$.
  $\nabla f(w) = w - L^T\lambda$; f is convex.

Thus, $g(\lambda,\pi) = \begin{cases} -\frac{1}{2}\lambda^T L L^T \lambda + \mathbb{1}^T\lambda & \text{if } \lambda = \frac{1}{m\tau}\mathbb{1} - \pi \\ -\infty & \text{otherwise} \end{cases}$

Therefore the dual is given by $\begin{cases} \max_{(\lambda,\pi)} g(\lambda,\pi) \\ \text{s.t. } \lambda \geq 0 \\ \quad\;\; \pi \geq 0 \end{cases} \iff \begin{cases} \max_{\lambda} -\frac{1}{2}\lambda^T L L^T \lambda + \mathbb{1}^T \lambda \\ \text{s.t. } \lambda \geq 0 \\ \quad\;\; \lambda \leq \frac{1}{m\tau}\mathbb{1}. \end{cases}$

$$\iff \boxed{\begin{cases} \min_{\lambda} \frac{1}{2}\lambda^T L L^T \lambda - \mathbb{1}^T \lambda \\ \text{s.t. } \lambda \geq 0 \\ \quad\;\; \lambda \leq \frac{1}{m\tau}\mathbb{1}. \end{cases}}$$

Exercise n°4:

$\boxed{\text{NB: } C \text{ is a matrix ; } C\in\mathbb{R}^{p\times n}\text{; } d\in\mathbb{R}^p\text{; } \mathcal{P}=\{a\in\mathbb{R}^n \mid Ca\leqslant d\}}$

We notice that $\sup_{a\in\mathcal{P}} a^T x$ is the optimal value of the following LP: $(*)\begin{cases}\max_a x^T a\\ \text{s.t. } Ca\leqslant d.\end{cases}$

(where the variable is a).

The associated Lagrangian is:
$$\mathcal{L}(a,\lambda) = -x^T a + \lambda^T(Ca-d) = (C^T\lambda - x)^T a - d^T\lambda$$

$$\Rightarrow g(\lambda) = \inf_a \mathcal{L}(a,\lambda) = \begin{cases} -d^T\lambda & \text{if } C^T\lambda = x\\ -\infty & \text{otherwise}\end{cases}$$

Then, the dual of $(*)$ is $\begin{cases}\max_{\lambda\in\mathbb{R}^p} -d^T\lambda\\ \text{s.t. } C^T\lambda = x\\ \quad \lambda\geqslant 0\end{cases} \iff (**)\begin{cases}\min_{\lambda\in\mathbb{R}^p} d^T\lambda\\ \text{s.t. } C^T\lambda = x\\ \quad\lambda\geqslant 0\end{cases}$

Since we are dealing with LPs here, we can apply the strong duality theorem.

Thus, $\sup_{a\in\mathcal{P}} a^T x$ is the optimal value of $(**)$.

Therefore, $\sup_{a\in\mathcal{P}} a^T x\leqslant b \iff \exists \lambda^*\in\mathbb{R}^p$ such that $\begin{cases} d^T\lambda^*\leqslant b\\ C^T\lambda^* = x\\ \lambda^*\geqslant 0\end{cases}$

Thus,
$$\boxed{\begin{cases}\min_{x\in\mathbb{R}^n} c^T x\\ \text{s.t. } \sup_{a\in\mathcal{P}} a^T x\leqslant b\end{cases} \iff \begin{cases}\min_{x\in\mathbb{R}^n} c^T x\\ \text{s.t. } d^T z\leqslant b\\ \quad C^T z = x\\ \quad z\geqslant 0\end{cases}}$$

Exercise n°5:

① $\mathcal{L}(x,\lambda,\nu) = c^T x + \lambda^T(Ax-b) - \sum_{i=1}^n \nu_i x_i(1-x_i)$

$$= c^T x + \lambda^T(Ax-b) - \nu^T x + x^T \operatorname{diag}(\nu)\, x$$

$$= x^T\operatorname{diag}(\nu)\, x + (A^T\lambda + c - \nu)^T x - b^T\lambda$$

- If $\exists i\in[\![1,n]\!]$ such that $\nu_i<0$:
  $\forall t>0$, $\mathcal{L}(t\vec{e_i},\lambda,\nu) = -\nu_i t^2 + ([A^T\lambda]_i + \nu_i + c_i)t - b^T\lambda \xrightarrow[t\to+\infty]{} -\infty$: unbounded

Then, to search $\inf_x \mathcal{L}(x,\lambda,\nu)$, we must have $\boxed{\nu\geqslant 0}$.

- Let $J\subseteq[\![1,n]\!]$ such that $\forall i\notin J,\ \nu_i = 0$ and $\forall i\in J,\ \nu_i>0$: (J can be $\emptyset$).
  $\nu_I = (\nu_{i_1},\dots,\nu_{i_k})^T$.
  We denote $J = \{i_1<\dots<i_k\}$ and
  Similarly, we denote $u_J = (u_{i_1},\dots,u_{i_k})^T$ for a given vector $u\in\mathbb{R}^d$.
  and $u_{\bar J} = (u_j,\ j\notin J)^T$.

So, $\mathcal{L}(x,\lambda,\nu) = x_J^T\operatorname{diag}(\nu_J)\, x_J + \left((A^T\lambda)_J + c_J - \nu_J\right)^T x_J + \left((A^T\lambda)_{\bar J} + c_{\bar J}\right) x_{\bar J} - b^T\lambda$

Suppose that $\exists j \notin J$ such that $(A^T\lambda)_j + c_j \neq 0$:

$\forall t \in \mathbb{R}$, $\mathcal{L}(t\vec{e}_j, \lambda, \nu) = \left((A^T\lambda)_j + c_j\right) t$

- If $(A^T\lambda)_j + c_j > 0$: then $\lim_{t \to -\infty} \mathcal{L}(t\vec{e}_j, \lambda, \nu) = -\infty$
- If $(A^T\lambda)_j + c_j < 0$: then $\lim_{t \to +\infty} \mathcal{L}(t\vec{e}_j, \lambda, \nu) = -\infty$

Thus, to define $\inf_x \mathcal{L}(x, \lambda, \nu)$, we must have $(A^T\lambda)_j + c_j = 0$ $\forall j \notin J$ (ie $\forall j$ such that $\nu_j = 0$)

- Now, we assume that $(A^T\lambda)_{\bar{J}} + c_{\bar{J}} = 0$ (ie $\forall j \notin J$, $(A^T\lambda)_j + c_j = 0$):

$$\mathcal{L}(x, \lambda, \nu) = x_J^T \operatorname{diag}(\nu_J)\, x_J + \left((A^T\lambda)_J + c_J - \nu_J\right)^T x_J - b^T\lambda.$$

By definition, $\nu_J > 0$ so $\operatorname{diag}(\nu_J) \in S^{++}_{|J|}$ (in particular, it is invertible).

$$g(\lambda, \nu) = \inf_x \mathcal{L}(x, \lambda, \nu) = \inf_{x_J} \underbrace{x_J^T \operatorname{diag}(\nu_J)\, x_J + \left((A^T\lambda)_J + c_J - \nu_J\right) x_J - b^T\lambda}_{\text{we note it } f(x_J)}.$$

We want to minimize a quadratic function. Thanks to the calculus done in the former exercises, we get: $\operatorname{argmin}_{x_J} f(x_J) = x_J^*$ where $[x_J^*]_j = -\dfrac{(A^T\lambda)_j + c_j - \nu_j}{\nu_j}$, $\forall j \in J$.

Thus, $\inf_{x_J} f(x_J) = f(x_J^*) = -\dfrac{1}{4} \sum_{j \in J} \dfrac{\left((A^T\lambda)_j + c_j - \nu_j\right)^2}{\nu_j} - b^T\lambda$

(*) For $\nu \geq 0$: to have simple notations, we will define, for $i \in [\![1, n]\!]$, $\dfrac{\left((A^T\lambda)_i + c_i - \nu_i\right)^2}{\nu_i}$ even if ν_i. In fact, if $\nu_i = 0$ and $(A^T\lambda)_i + c_i \neq 0$, then $\dfrac{\left((A^T\lambda)_i + c_i - \nu_i\right)^2}{\nu_i} = +\infty$ which is compatible to the fact that if $(A^T\lambda)_{\bar{J}} + c_{\bar{J}} \neq 0$, $\inf_x \mathcal{L}(x, \lambda, \nu) = -\infty$.
The other "pathologic" case for the formula $\dfrac{\left((A^T\lambda)_i + c_i - \nu_i\right)^2}{\nu_i}$ is when $\nu_i = 0$ and $(A^T\lambda)_i + c_i = 0$, but, in this case, $\dfrac{\left((A^T\lambda)_i + c_i - \nu_i\right)^2}{\nu_i} = \dfrac{0^2}{0}$, which can be interpreted as 0.

$\Rightarrow$ Thus, when $\nu \geq 0$, $\displaystyle\sum_{j \in J} \frac{\left((A^T\lambda)_j + c_j - \nu_j\right)^2}{\nu_j} = \sum_{i=1}^{n} \frac{\left((A^T\lambda)_i + c_i - \nu_i\right)^2}{\nu_i}$

Therefore,
$$g(\lambda, \nu) = \begin{cases} -\dfrac{1}{4} \displaystyle\sum_{i=1}^{n} \dfrac{\left((A^T\lambda)_i + c_i - \nu_i\right)^2}{\nu_i} - b^T\lambda & \text{if } \nu \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Then, the dual is given by: $\begin{cases} \max_{\lambda, \nu} g(\lambda, \nu) \\ \text{s.t. } \lambda \geq 0 \end{cases} \Leftrightarrow \begin{cases} \max_{\lambda, \nu} -\dfrac{1}{4} \displaystyle\sum_{i=1}^{n} \dfrac{\left((A^T\lambda)_i + c_i - \nu_i\right)^2}{\nu_i} - b^T\lambda \\ \text{s.t. } \lambda \geq 0 \\ \quad\;\; \nu \geq 0 \end{cases}$

Let A_i^T denote the ith row of A. So, $(A^T\lambda)_i = A_i^T \lambda_i$

Our purpose is to simplify $\max_{\substack{\lambda, \nu \\ \nu \geq 0}} -\frac{1}{4}\sum_{i=1}^{n} \frac{(A_i^T\lambda + c_i - \nu_i)^2}{\nu_i} - b^T\lambda$ (for $\nu \geq 0$).

We notice that:
$$\max_{\substack{\lambda, \nu \\ \nu \geq 0}} -\frac{1}{4}\sum_{i=1}^{n} \frac{(A_i^T\lambda + c_i - \nu_i)^2}{\nu_i} - b^T\lambda = \max_{\lambda}\left(\max_{\nu \geq 0} -\frac{1}{4}\sum_{i=1}^{n} \frac{(A_i^T\lambda + c_i - \nu_i)^2}{\nu_i} - b^T\lambda\right)$$
$$= \max_{\lambda}\left(\frac{1}{4}\sum_{i=1}^{n}\left(\max_{\nu_i \geq 0} -\frac{(A_i^T\lambda + c_i - \nu_i)^2}{\nu_i}\right) - b^T\lambda\right).$$

Using the hint, $\max_{\nu_i \geq 0} -\frac{(A_i^T\lambda + c_i - \nu_i)^2}{\nu_i} = 4\min\{0, c_i + A_i^T\lambda\}$

Therefore, the dual is given by:
$$\boxed{\begin{cases} \max_{\lambda} & \sum_{i=1}^{n} \min\{0, c_i + A_i^T\nu\} - b^T\lambda \\ \text{s.t.} & \lambda \geq 0 \end{cases}}$$ Ⓐ

② • We start by computing the dual of (2): $\begin{cases} \min c^Tx \\ \text{s.t. } Ax \leq b \\ \quad 0 \leq x_i \leq 1 \quad \forall i \in [\![1,n]\!] \end{cases}$

- The Lagrangian is:

$$\mathcal{L}(x, \lambda, \nu, \omega) = c^Tx + \lambda^T(Ax - b) - \nu^Tx + \omega^T(x - \mathbb{1})$$
$$= (A^T\lambda + c - \nu + \omega)^T - b^T\lambda - \mathbb{1}^T\omega.$$

So, $g(\lambda, \nu, \omega) = \inf_x \mathcal{L}(x, \lambda, \nu, \omega) = \begin{cases} -b^T\lambda - \mathbb{1}^T\omega & \text{if } A^T\lambda + c + \omega = \nu \\ -\infty & \text{otherwise.} \end{cases}$

Then, the dual of (2) is: $\begin{cases} \max_{\lambda, \nu, \omega} & g(\lambda, \nu, \omega) \\ \text{s.t} & \lambda \geq 0 \\ & \nu \geq 0 \\ & \omega \geq 0 \end{cases} \iff \begin{cases} \max_{\lambda, \omega} & -\mathbb{1}^T\omega - b^T\lambda \\ \text{s.t} & \lambda \geq 0 \\ & \omega \geq 0 \\ & A^T\lambda + c + \omega \geq 0 \end{cases}$

$$\iff \boxed{\begin{cases} \max_{\lambda, \omega} & \mathbb{1}^T\omega - b^T\lambda \\ \text{s.t.} & \lambda \geq 0 \\ & \omega \leq \min\{0, A^T\lambda + c\} \end{cases}}$$ Ⓑ

We notice that Ⓐ and Ⓑ are equivalent. Thus, they lead to the same optimal value.
Recall that Ⓐ is the dual of $(3)\begin{cases} \min c^Tx \\ \text{st. } Ax \leq b \\ x_i(1 - x_i) = 0 \quad \forall i \in [\![1,n]\!] \end{cases}$ and Ⓑ is the dual of $(2)\begin{cases} \min c^Tx \\ \text{st. } Ax \leq b \\ 0 \leq x \leq 1 \end{cases}$.

Therefore, $\boxed{\text{the lower bound obtained via Lagrangian relaxation (3) and via the LP relaxation (2) are the same.}}$