# Introductory Applied Machine Learning

Naïve Bayes

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## Overview

- Naïve Bayes classifier
  - components and their function
  - independence assumption
  - dealing with missing data
- Continuous example
- Discrete example
- Pros and cons



## Bayesian classification

- Goal: learning function $f(x) \rightarrow y$
  - $y$ ... one of $k$ classes (e.g. spam/ham, digit 0-9)
  - $x = x_1 \ldots x_n$ – values of attributes (numeric or categorical)
- Probabilistic classification:
  - most probable class given observation: $\hat{y} = \underset{y}{arg\,max}\, P(y|x)$
- Bayesian probability of a class:

$$P(y|x) = \frac{\overbrace{P(x|y)}^{\text{class model}}\overbrace{P(y)}^{\text{prior}}}{\underbrace{\sum_{y'} P(x|y')P(y')}_{\text{normalizer } P(x)}}$$



## Bayesian classification: components

$$P(y|x) = \frac{P(x|y)P(y)}{\sum_{y'} P(x|y')P(y')}$$

Example:
y ... patient has Avian flu
x ... observed symptoms

- P($y$): prior probability of each class
  - encodes how which classes are common, which are rare
  - apriori much more likely to have common cold than Avian flu
- P($x|y$): class-conditional model
  - describes how likely to see observation $x$ for class y
  - assuming it's Avian flu, do the symptoms look plausible?
- P($x$): normalize probabilities across observations
  - does not affect which class is most likely (**arg max**)

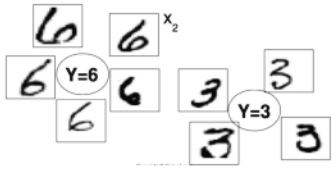


## Bayesian classification: normalization

Normalizer: $P(x) = \sum_{y'} P(x|y')P(y')$

- an "outlier" has a low probability under every class

$P(X=x_1 \mid Y=3) < P(X=x_2 \mid Y=3)$

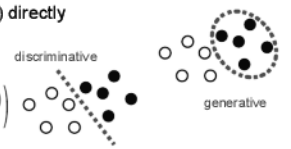

normalizer makes $P(Y=3|X=x_1)$ comparable to non-outliers



## Naïve Bayes: a generative model

- A complete probability distribution for each class
  - defines likelihood for any point $x$
  - P(class) via P(observation) $P(y|x) \propto P(x|y)P(y)$
  - can "generate" synthetic observations
    - will share many properties of the original data
- Not all probabilistic classifiers do this
  - possible to estimate P($y|x$) directly
  - e.g. logistic regression:

$$P(y|x) = \frac{1}{z_y}\exp\left(\sum_i \lambda_i g_i(y,x)\right)$$






## Independence assumption

- Compute $P(x_1 \ldots x_n|y)$ for every observation $x_1 \ldots x_n$
  - class-conditional "counts", based on training data
  - problem: may not have seen every $x_1 \ldots x_n$ for every $y$
    - digits: $2^{400}$ possible black/white patterns (20x20)
    - spam: every possible combination of words: $2^{10,000}$
  - often have observations for individual $x_i$ for every class

- idea: assume $x_1 \ldots x_n$ conditionally independent given $y$

$$P(x_1 \ldots x_d|y) = \underbrace{\prod_{i=1}^{d} P(x_i|x_1 \ldots x_{i-1}, y)}_{\text{chain rule (exact)}} = \underbrace{\prod_{i=1}^{d} P(x_i|y)}_{\text{independence}}$$

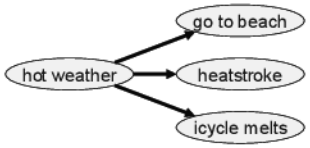


## Conditional independence

- Probabilities of going to the beach and getting a heat stroke are not independent: $P(B,S) > P(B)\,P(S)$
- May be independent if we know the weather is hot

$$P(B,S|H) = P(B|H)\,P(S|H)$$

- Hot weather "explains" all the dependence between beach and heatstroke
- In classification:
  - class value explains all the dependence between attributes






Example © Amos Storkey, 2007

## Overview

- Naïve Bayes classifier
- Continuous example
  - general concepts
  - working example
  - example of failure
- Discrete example
  - general concepts
  - problems with Naïve Bayes
- Pros and cons

$$P(y|x) = \frac{\overbrace{P(x|y)}^{\text{class model}}\overbrace{P(y)}^{\text{prior}}}{\underbrace{\sum_{y'} P(x|y')P(y')}_{\text{normalizer } P(x)}}$$

## Continuous example

- Distinguish children from adults based on size
  - classes: {*a*,*c*}, attributes: height [cm], weight [kg]
  - training examples: {$h_i$, $w_i$, $y_i$}, 4 adults, 12 children
- Class probabilities: $P(a)=\frac{4}{4+12}=0.25\ ;\ P(c)=0.75$
- Model for adults:
  - height ~ Gaussian with mean, variance
  - weight ~ Gaussian $\left(\mu_{w,a},\sigma^2_{w,a}\right)$
  - assume height and weight independent

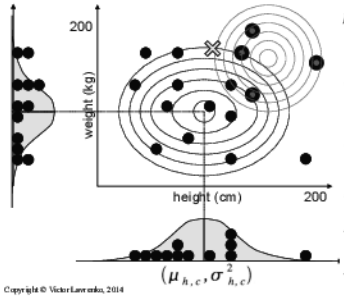
$$\begin{cases}\mu_{h,a}=\frac{1}{4}\sum_{i:\,y_i=a}h_i\\ \sigma^2_{h,a}=\frac{1}{4}\sum_{i:\,y_i=a}(h_i-\mu_{h,a})^2\end{cases}$$

- Model for children: same, using $(\mu_{h,c},\sigma^2_{h,c}),(\mu_{w,c},\sigma^2_{w,c})$



## Continuous example

$$P(a)=\frac{4}{4+12}=0.25\ ;\ P(c)=0.75$$




$$p(h_x|c)=\frac{1}{\sqrt{2\pi\,\sigma^2_{h,c}}}\exp-\frac{1}{2}\left(\frac{(h_x-\mu_{h,c})^2}{\sigma^2_{h,c}}\right)$$

$$p(w_x|c)=\frac{1}{\sqrt{2\pi\,\sigma^2_{w,c}}}\exp-\frac{1}{2}\left(\frac{(w_x-\mu_{w,c})^2}{\sigma^2_{w,c}}\right)$$

$$p(h_x|a)=\frac{1}{\sqrt{2\pi\,\sigma^2_{h,a}}}\exp-\frac{1}{2}\left(\frac{(h_x-\mu_{h,a})^2}{\sigma^2_{h,a}}\right)$$

$$p(w_x|a)=\frac{1}{\sqrt{2\pi\,\sigma^2_{w,a}}}\exp-\frac{1}{2}\left(\frac{(w_x-\mu_{w,a})^2}{\sigma^2_{w,a}}\right)$$

$$P(x|a)=p(h_x|a)\,p(w_x|a)$$

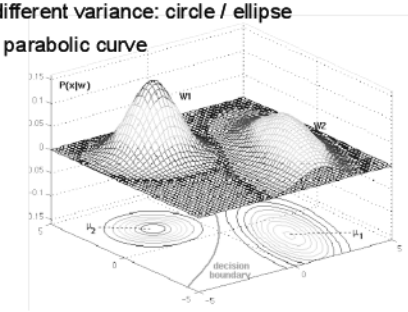
$$P(x|c)=p(h_x|c)\,p(w_x|c)$$

$$P(a|x)=\frac{P(x|a)P(a)}{P(x|a)P(a)+P(x|c)P(c)}$$



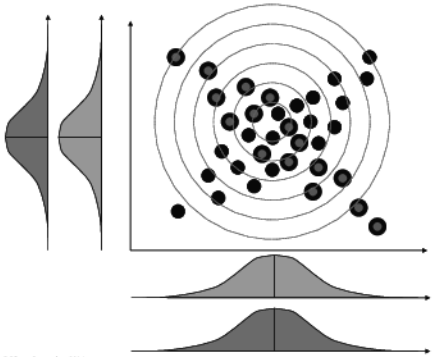
## Decision boundary

- Different means, same variance: straight line / plane
- Same mean, different variance: circle / ellipse
- General case: parabolic curve






## Problems with Naïve Bayes





## Discrete example: spam

- Separate spam from valid email, attributes = words

| | |
|---|---|
| D1: "send us your password" | spam |
| D2: "send us your review" | ham |
| D3: "review your password" | ham |
| D4: "review us" | spam |
| D5: "send your password" | spam |
| D6: "send us your account" | spam |

new email: "review us now"

P (spam) = 4/6  P ( ham) = 2/6

| spam | ham | |
|---|---|---|
| 2/4 | 1/2 | password |
| 1/4 | 2/4 | review |
| 3/4 | 1/2 | send |
| 3/4 | 1/2 | us |
| 3/4 | 1/2 | your |
| 1/4 | 0/2 | account |

$$P(\text{review us}\,|\text{spam})=P(0,1,0,1,0,0\,|\text{spam})=(1-\tfrac{2}{4})(\tfrac{1}{4})(1-\tfrac{3}{4})(\tfrac{3}{4})(1-\tfrac{3}{4})(1-\tfrac{1}{4})$$

$$P(\text{review us}\,|\text{ham})=P(0,1,0,1,0,0\,|\text{ham})=(1-\tfrac{1}{2})(\tfrac{2}{2})(1-\tfrac{1}{2})(\tfrac{1}{2})(1-\tfrac{1}{2})(1-\tfrac{0}{2})$$

$$P(\text{ham}\,|\text{review us})=\frac{0.0625\times 2/6}{0.0625\times 2/6+0.0044\times 4/6}=0.87$$ (note identical example)



## Problems with Naïve Bayes

- Zero-frequency problem
  - any mail containing "account" is spam: P(account|ham) = 0/2
  - solution: never allow zero probabilities
    - Laplace smoothing: add a small positive number to all counts:
    - may use global statistics in place of ε: *num(w) / num*

$$P(w|c)=\frac{num(w,c)+\epsilon}{num(c)+2\epsilon}$$

  - very common problem (Zipf's law: 50% words occur once)
- Assumes word independence
  - every word contributes independently to P(spam|email)
  - fooling NB: add lots of "hammy" words into spam email



## Overview

- Naïve Bayes classifier
- Continuous example
- Discrete example
- Pros and Cons
  - dealing with missing data
  - computational cost and incremental updates



## Missing data

- Suppose don't have value for some attribute $X_i$
  - applicant's credit history unknown
  - some medical test not performed on patient
  - how to compute $P(X_1=x_1 \ldots X_j=? \ldots X_d=x_d\,|\,y)$
- Easy with Naïve Bayes
  - ignore attribute in instance where its value is missing

$$P(x_1\ldots\boxed{X_j}\ldots x_d|y)=\prod_{i\neq j}^{d}P(x_i|y)$$

  - compute likelihood based on observed attribtues
  - no need to "fill in" or explicitly model missing values
  - based on conditional independence between attributes



## Missing data (2)

- Ex: three coin tosses: Event = {$X_1$=H, $X_2$=?, $X_3$=T}
  - event = head, unknown (either head or tail), tail
  - event = {H,H,T} + {H,T,T}
  - *P*(event) = P(H,H,T) + P(H,T,T)


- General case: $X_j$ has missing value

$$P(x_1\ldots\boxed{x_j}\ldots x_d|y)=P(x_1|y)\cdots\boxed{P(x_j|y)}\cdots P(x_d|y)$$

$$\sum_{\boxed{x_j}}P(x_1\ldots\boxed{x_j}\ldots x_d|y)=\sum_{x_j}P(x_1|y)\cdots\boxed{P(x_j|y)}\cdots P(x_d|y)$$

$$=P(x_1|y)\cdots\boxed{\sum_{x_j}P(x_j|y)}\cdots P(x_d|y)$$

$$=P(x_1|y)\cdots\boxed{1}\cdots P(x_d|y)$$

# Summary

- Naïve Bayes classifier $P(y|x) = \frac{P(x|y)P(y)}{\sum_{y'} P(x|y')P(y')}$
  - explicitly handles class priors
  - "normalizes" across observations: outliers comparable
  - assumption: all dependence is "explained" by class label
- Continuous example
  - unable to handle correlated data
- Discrete example
  - fooled by repetitions
  - must deal with zero-frequency problem
- Pros:
  - handles missing data
  - good computational complexity
  - incremental updates



# Computational complexity

- One of the fastest learning methods
- O(*nd*+*cd*) training time complexity
  - *c* … number of classes
  - *n* … number of instances
  - *d* … number of dimensions (attributes)
  - both learning and prediction
  - no hidden constants (number of iterations, etc.)
  - testing: O(ndc)
- O(*dc*) space complexity
  - only decision trees are more compact



# Incremental updates

- Allows incremental updates: O(*d*) insertion / deletion
- Bernoulli: store raw counts instead of probabilities
  - new example of class *c*:
    - $n_{cd}$ += $x_d$ for each *d* in example, $n_c$ += 1, *n* += 1
  - when need to classify:
    - $P(x_d=1 \mid c) = (n_{cd} + \varepsilon) / (n_c + 2\varepsilon)$
    - $P(c) = n_c / n$
- Gaussian: store partial sums instead of mean/variance
  - $s_{cd}$ += $x_d$  $s^2_{cd}$ += $x_d^2$
  - when need to classify:

    mean = $s_{cd} / n$  variance = $s^2_{cd} / n$ – mean$^2$



# General structure for Naïve Bayes

- **Task**
  - *c*-class classification ($c \geq 2$)
- **Model structure**
  - $c \times d$ independent distributions
  - continuous: Gaussian, discrete: Bernoulli
- **Score function**
  - class-conditional likelihood
- **Optimization / search method**
  - analytic solution
- Book: section 4.2