

Visual Data Analytics - Formulas

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1 General Knowledge

Visualization Pipeline:

- Data Acquisition
- Filtering/Enhancement (obtain useful data/cleaning)
- Visualization/Mapping (how to represent data)
- Rendering (generate 2D images/video)

Independent Variables: 2D/3D space, time

Dependent Variables: temperature, velocity

Characteristics of data values:

- Attribute types
- Domain
- Value range
- Dimension
- Error & Uncertainty
- Physical Interpretation

Qualitative data: categorical, non-measurable, discrete

Nominal data: no natural ordering, membership

Ordinal data: natural ordering

Categorical data: values from fixed number or categories

Scalar data: if we can map from higher dimensional data to a lower dimensional data ($f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$)

Tensor data: multi-dimensional data

2 Interpolation

ISOContours: all points lying on the same line with the same values

Goal of interpolation: construct continuous function f which approximates given values

2.1 Radial Basis Functions

Each point (p_i, f_i) influences $f(x)$ based on distance:

$$r = \|p_i - x\|$$
$$f(x) = \sum_{i=1}^N f_i \cdot \varphi(\|p_i - x\|)$$
$$\varphi(r) = e^{-r^2}$$

Weighted radial basis functions

$f(p_j)$ interpolates the value f_j

$$f(p_j) = \sum_{i=1}^N w_i \cdot \varphi(\|p_i - p_j\|) = f_j$$

Yields a system of linear equations to be solved for w_i :

$$A = \begin{pmatrix} \varphi(\|p_1 - p_1\|) & \cdots & \varphi(\|p_N - p_1\|) \\ \vdots & \ddots & \vdots \\ \varphi(\|p_1 - p_N\|) & \cdots & \varphi(\|p_N - p_N\|) \end{pmatrix}$$

$$W = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \quad F = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$

$$W = A^{-1} \cdot F$$

Drawbacks: global influence of every sample, adding a new point requires solving the equation system, computationally expensive

Inverse Distance Weighting

Assumption: Nearby points are more similar than those further away.

$$f(x) = \sum_{i=1}^N f_i \varphi(\|p_i - x\|)$$

$$d_i = \|p_i - x\| \quad \varphi(r) = \frac{\frac{1}{r^2}}{\sum_{i=1}^N \frac{1}{d_i^2}}$$

Adjusted formula:

$$\sum_{i=1}^N \frac{f_i}{\|p_i - x\|^2} / \sum_{i=1}^N \frac{1}{\|p_i - x\|^2}$$

Drawbacks: still costly, still global influence of every sample

2.2 Triangulation

Giving up smooth, precise reconstruction in favor of speed

Generating a Triangulation

Goals: avoid long and thin triangles, maximize minimum angle in the triangulation, maximize $\frac{\text{radius of circle}}{\text{radius of circumcircle}}$

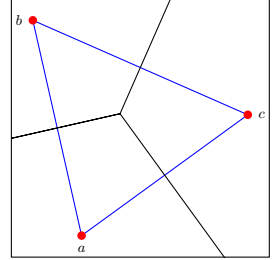
Delaunay Triangulation

1. Circumcircle does not contain another point of the set

2. Maximizes minimum angle of triangulation

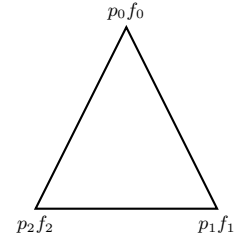
3. Triangulation is unique for all but trivial cases

Voronoi Diagram



Every **Voronoi Sample** (a, b, c) is a vertex of a **Delaunay** triangulation

2.3 Interpolation Inside a Triangle



Find a function f that interpolates f_i at the point p_i such that:

$$f(p_i) = f_i \quad i = 0, \dots, N$$

Linear function:

$$f(x) = a + bx + cy$$

Where a, b, c can be obtained with:

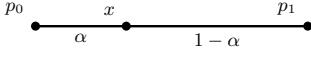
$$X = \begin{pmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{pmatrix} \quad A = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad F = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

$$A = X^{-1} \cdot F$$

2.4 Baricentric Interpolation

We want to have a smooth, continuous interpolation $\forall \alpha \in [0, 1]$

$$f(\alpha) = \alpha \cdot p_0 + (1 - \alpha) \cdot p_1$$



$$\alpha_0 = A_0/A, \alpha_1 = A_1/A, \alpha_2 = A_2/A$$

Where A = area of the triangle

$$x = \alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2$$

$$\alpha_0 + \alpha_1 + \alpha_2 = 1$$

If α_i are known, then $f(x)$ can be interpolated from values f_i at the vertices via:

$$f(x) = \alpha_0 f_0 + \alpha_1 f_1 + (1 - \alpha_0 - \alpha_1) \cdot f_2$$

Given a point Q we can find α by solving the linear system:

$$P = \begin{pmatrix} p_{0x} & p_{1x} & p_{2x} \\ p_{0y} & p_{1y} & p_{2y} \\ 1 & 1 & 1 \end{pmatrix}$$

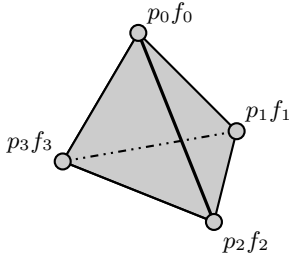
$$A = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \quad X = \begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix}$$

$$A = P^{-1} \cdot X$$

2.5 Scalar Interpolation

Formula of tetrahedron:

$$f(x) = a + bx + cy + dz$$



Can get values a, b, c, d by solving linear system:

$$X = \begin{pmatrix} 1 & x_0 & y_0 & z_0 \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{pmatrix}$$

$$A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad F = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$A = X^{-1} \cdot F$$

Computing Gradient of Scalar Field

The gradient in a scalar field $f(x)$:

$$\nabla f(x) = \begin{pmatrix} \frac{\delta f(x)}{\delta x} \\ \frac{\delta f(x)}{\delta y} \\ \frac{\delta f(x)}{\delta z} \end{pmatrix}$$

In the case of a tetrahedron with function the gradient is always constant:

$$f(x) = a + bx + cy + dz$$

$$\nabla f(x) = \begin{pmatrix} \frac{\delta f(x)}{\delta x} \\ \frac{\delta f(x)}{\delta y} \\ \frac{\delta f(x)}{\delta z} \end{pmatrix} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

2.6 Piece-Wise Linear Interpolation

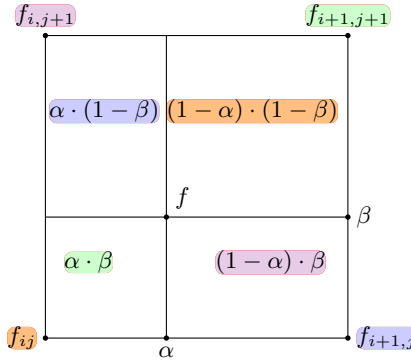
For data points $(x_0, y_0), \dots, (x_N, y_N)$
Evaluate

$$f(x) = (1 - \alpha)y_i + \alpha y_{i+1}$$

$$\text{where: } \alpha = \frac{x - x_i}{x_{i+1} - x_i}$$

Bilinear Interpolation

$$f(\alpha, \beta) = f_{ij} \cdot (1 - \alpha)(1 - \beta) + f_{i+1,j} \cdot \alpha(1 - \beta) + f_{i,j+1} \cdot (1 - \alpha)\beta + f_{i+1,j+1} \cdot \alpha\beta$$

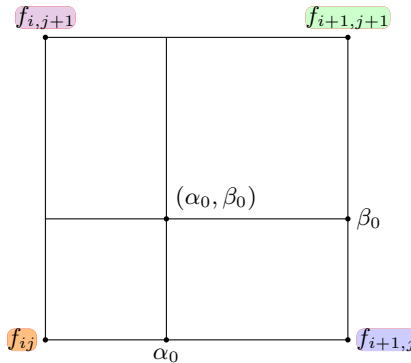


Asymptotic Decider

Hyperbola form:

$$f(\alpha, \beta) = \gamma(\alpha - \alpha_0)(\beta - \beta_0) + \delta$$

where δ is the value at point (α_0, β_0)



Transform to **hyperbola form**:

$$f(\alpha, \beta) = A\alpha + B\beta + C\alpha\beta + D$$

$$A = f_{i+1,j} - f_{ij}$$

$$B = f_{i,j+1} - f_{ij}$$

$$C = f_{ij} - f_{i,j+1} - f_{i+1,j} + f_{i+1,j+1}$$

$$D = f_{ij}$$

$$\delta = \frac{(f_{ij} \cdot f_{i+1,j+1} - f_{i+1,j} \cdot f_{i,j+1})}{C}$$

$$\alpha_0 = -B/C$$

$$\beta_0 = -A/C$$

$$\gamma = C$$

2.7 Marching Squares

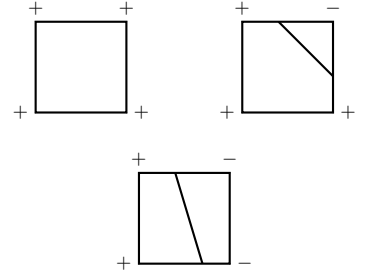
Algorithm used to compute isolines on a 2D surface, given a grid of values.

Step 1: mark all the data with:

value \geq isovalue: +

value $<$ isovalue: -

Step 2: 16 different combinations, but we are interested in 4 base cases:



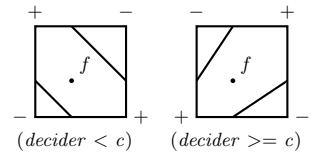
In the last case, we have ambiguity, which can be resolved in two ways:

• midpoint decider:

$$f_{center} = \frac{1}{4}(f_{ij} + f_{i+1,j} + f_{i,j+1} + f_{i+1,j+1})$$

• if midpoint decider is right in the middle, hence the ambiguity is still there, we can use the **asymptotic decider** previously defined as:

$$f(\alpha, \beta) = \gamma(\alpha - \alpha_0)(\beta - \beta_0) + \delta$$



3 Phong Illumination Model

Components:

- **Ambient Light:** background light, constant everywhere
- **Diffuse Reflector:** reflects equally into all directions
- **Specular Reflector:** reflects mostly into the mirror direction

3.1 Lighting

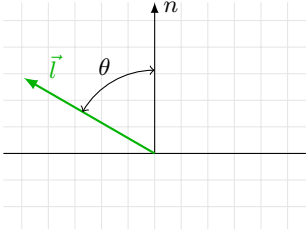
$$C = K_a C_a O_d$$

K_a : ambient reflection coefficient $\in [0, 1]$

C_a : color of the ambient light

O_a : object color

3.2 Diffuse Reflection



$$C = K_d C_p O_d \cdot \cos \theta$$

$$\cos \theta = \frac{\vec{n} \cdot \vec{l}}{|\vec{n}| \cdot |\vec{l}|}$$

K_d : diffuse reflection coefficient $\in [0, 1]$

C_p : color of point of light

O_d : object color

$\cos \theta$: angle between light vector \vec{l} and \vec{n}

Reminder:

$$|\vec{v}| = \sqrt{(v_0)^2 + \dots + (v_n)^2}$$

3.3 Specular Reflection

$$C = K_s C_p O_d \cdot \cos^n \varphi$$

$$\cos \varphi = \frac{\vec{r} \cdot \vec{v}}{|\vec{r}| \cdot |\vec{v}|}$$

$$\vec{r} = 2 \cdot \left(\frac{\vec{n} \cdot \vec{l}}{|\vec{n}| \cdot |\vec{l}|} \right) \cdot \frac{\vec{n}}{|\vec{n}|} - \frac{\vec{l}}{|\vec{l}|}$$

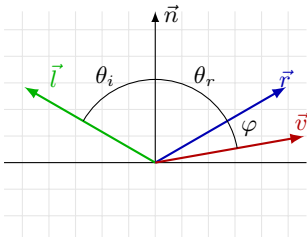
K_s : specular reflection coefficient $\in [0, 1]$

C_p : color of point of light

O_d : object color

$\cos^n \varphi$: angle between reflected light ray \vec{r} and the vector to the viewer \vec{v}

$(\dots)^n$: shininess factor (extent of highlight)



Perfect Mirror: shininess factor n has to be very large, going to infinity.

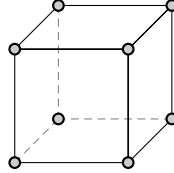
4 Volume Visualization

• **Indirect** Volume Rendering (Marching Cubes, data is reduced to intermediate representation, which can then be rendered)

• **Direct** Volume Rendering (Ray Casting, data is considered semi-transparent with physical properties, and directly rendered)

• **Voxel:** point sample in 3D

• **Transfer Function:** maps data values to color & opacity



4.1 Transfer Function

Associates distinct materials (value range) to distinct properties (color and opacity). Maps a different color to each scalar value.

$$T : \text{scalar value} \rightarrow \text{color} + \alpha$$

4.2 Direct Volume Rendering

Each **voxel** emits light of the color assigned to it, and absorbs light according to its opacity.

4.3 Light Emission and Attenuation

Volume rendering integral:

$$C(s) = \int_{s_0}^s C(s') \cdot e^{-\int_{s'}^s \alpha(t) dt} ds'$$

We are calculating the accumulated color until $C(s')$, with the absorption coefficient between the last color s' and the point of view s .

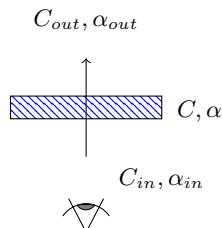
4.4 Ray Casting - Compositing

Front-to-back Strategy

$$C_{in} = (0, 0, 0), \alpha_{in} = 0$$

$$C_{out} = C_{in} + (a - \alpha_{in}) \cdot \alpha \cdot C$$

$$\alpha_{out} = \alpha_{in} + (a - \alpha_{in}) \cdot \alpha$$



4.5 Direct Volume Rendering: Phong Shading

We have to evaluate Phong's illumination model based on:

- **position** of current sample and light source
- sample's color emission assigned by transfer function
- sample's **normal/gradient** (central difference)

4.6 Gradient Estimation

We can estimate the gradient using finite differencing. However, this technique is not feasible with large amounts of data.

$$\nabla f(x, y, z) \approx$$

$$\approx \frac{1}{2h} \begin{pmatrix} f(x+h, y, z) - f(x-h, y, z) \\ f(x, y+h, z) - f(x, y-h, z) \\ f(x, y, z+h) - f(x, y, z-h) \end{pmatrix}$$

4.7 Compositing Schemes

- **Surface Rendering/First-Hit:** stop ray traversal if an ISOSurface is hit
- **Average:** simply accumulate colors, but ignore opacity
- **Maximum:** take maximum color along axis and display it

5 Flow Visualization

- Motion of fluids (gases, liquids)
- Geometric boundaries
- Conservation of mass, energy and momentum
- Velocity/Flow field $v(x, t)$

5.1 Vector Field Visualization

- Vector data representing **Direction** and **Magnitude**
- **Steady** (time-independent) flow:

$$v(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

- **Unsteady** (time dependent) flow:

$$v(x, t) : \mathbb{R}^n \times \mathbb{R}^1 \rightarrow \mathbb{R}^n$$

5.2 Vector and Scalar Functions

Scalar function:

$$\rho(x, t)$$

The gradient points to the direction of maximum change of $\rho(x, t)$:

$$\nabla \rho(x, t) = \begin{pmatrix} \frac{\partial \rho(x, t)}{\partial x} \\ \frac{\partial \rho(x, t)}{\partial y} \\ \frac{\partial \rho(x, t)}{\partial z} \end{pmatrix}$$

Jacobian matrix applied to vector field $v(x, t)$:

$$J = \nabla v(x, t) = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

Divergence: Diagonal of Jacobian, describes flow into/out of a region:

$$\text{div } v(x, t) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$\text{div } v(x_0) > 0$: v has **source** in x_0

$\text{div } v(x_0) < 0$: v has **sink** in x_0

$\text{div } v(x_0) = 0$: v is **source-free** in x_0

Curl/vorticity: tells how fast the flow is rotating, and the axis around which it is rotating.

$$\text{curl } v(x, t) = \nabla v(x, t) = \begin{pmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{pmatrix}$$

5.3 Computing Characteristic Lines

Characteristic lines are **tangential** to the flow.

$$\frac{\delta x(t)}{\delta t} = v(x(t), t)$$

- do not intersect
- they are the solutions to the initial value problem

5.4 Euler Method

- First order method
- higher accuracy with smaller step size (Δt)

$$x(t + \Delta t) \approx x(t) + \Delta t \cdot v(x(t), t)$$

5.5 Midpoint Method

- Also known as **Runge-Kutta 2nd order** method
- Better, faster than Euler method

$$\Delta x = \Delta t \cdot v(x, t)$$

$$v_{mid} = v\left(x + \frac{\Delta x}{2}\right)$$

$$x(t + \Delta t) = x(t) + \Delta t \cdot v_{mid}$$

5.6 Runge-Kutta 4th Order

- The most accurate

$$k_1 = \Delta t \cdot v(x, t)$$

$$k_2 = \Delta t \cdot v\left(x + \frac{k_1}{2}\right)$$

$$k_3 = \Delta t \cdot v\left(x + \frac{k_2}{2}\right)$$

$$k_4 = \Delta t \cdot v(x + k_3)$$

$$x(t + \Delta t) = x + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}$$

5.7 Vector Field Topology

- We want to draw along the most important lines
- Draw topological skeleton of flow

5.8 Critical Points

- Singularities in vector fields such that:

$$v(x^*) = 0$$

- Points where magnitude of vectors goes to zero and direction of vectors is undefined

To find critical points: points where:

$$v(x^*) = 0$$

Classification First, we build the Jacobian matrix for $v(x^*)$

$$J = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix}$$

Then, for each x^* calculate the **eigenvalues** λ_1, λ_2 of J .

To find the eigen values:

$$\det(J - \lambda I) = \det \begin{pmatrix} \frac{\partial v_x}{\partial x} - \lambda & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} - \lambda \end{pmatrix}$$

Circulating Source

$$\text{Im}(\lambda_1, \lambda_2) \neq 0 \text{ and } \text{Re}(\lambda_1, \lambda_2) > 0$$

Non-Circulating Source

$$\text{Im}(\lambda_1, \lambda_2) = 0 \text{ and } \text{Re}(\lambda_1, \lambda_2) > 0$$

Center

$$\text{Im}(\lambda_1, \lambda_2) \neq 0 \text{ and } \text{Re}(\lambda_1, \lambda_2) = 0$$

Circulating Sink

$$\text{Im}(\lambda_1, \lambda_2) \neq 0 \text{ and } \text{Re}(\lambda_1, \lambda_2) < 0$$

Non-Circulating Sink

$$\text{Im}(\lambda_1, \lambda_2) = 0 \text{ and } \text{Re}(\lambda_1, \lambda_2) < 0$$

Saddle Point

$$\text{Im}(\lambda_1, \lambda_2) = 0 \text{ and } \lambda_1 \cdot \lambda_2 < 0$$

5.9 Dense Texture-Based Flow Visualization

- Better information coverage
- Critical point detection and classification

5.10 Line-Integral Convolution (LIC)

- **Global** visualization technique
- Starts with random noise (white noise)
- *Smear out* the texture along trajectories of vector field
- Results in **low correlation** between neighbouring lines and **high correlation** along flow direction
- The larger the kernel, the more the values of the noise texture are blurred

Steps:

- Vector Field \rightarrow (integration) Stream Line
- Stream Line \rightarrow (convolution) Noise Texture
- Noise Texture \rightarrow (result) Output Image

5.11 Filtering By Convolution

- Sliding a function $g(x)$ along a function $f(x)$:

$$s(x') = [f * g](x') = \int_{-\infty}^{\infty} f(x)g(x' - x) dx$$

- Function f is averaged with a weight function g
- $(x' - x)$ centers g around x'
- We use the single stream lines that pass through a single pixel to smear out the noise
- $\Phi_0(u)$ is the stream line containing the point (x_0, y_0)
- $T(x, y)$ is the randomly generated noise texture
- Compute the intensity at (x_0, y_0) as:

$$I(x_0, y_0) = \int_{-L}^L k(u) \cdot T(\Phi(u)) du$$

6 Visualization Mapping

Decide:

- Which part of the data is shown
- How to represent the data

6.1 Mapping Techniques

Properties:

- Accuracy
- Pop-Out (emphasize important information)
- Discriminability
- Separability (judge each channel independently)
- Relative vs absolute judgement

Graphical Primitives:

- Points

- Lines
- Areas
- Surfaces

Visual Channels:

- Position
- Color
- Tilt
- Shape
- Size

Pop-Out:

- Pre-Attentive Processing: automatic and parallel detection of basic features in

visual information

- Speed independent of distractor count
- Works on many individual channels

From Most Accurate/Effective to Less Accurate/Effective:

- Position
- Length
- Slope/Tilt/Angle
- 2D Area
- Saturation/Luminance
- Curvature
- 3D Volume