Visual Data Analytics - Formulas

Adrian Castro

February 2021

1 General Knowledge

Visualization Pipeline:
- Data Acquisition
- Filtering/Enhancement (obtain useful data/cleaning)
- Visualization/Mapping (how to represent data)
- Rendering (generate 2D images/video)

Indipendent Variables: 2D/3D space, time
Dependent Variables: temperature, velocity
Characteristics of data values:
- Attribute types
- Domain
- Value range
- Dimension
- Error & Uncertainty
- Physical Interpretration

Qualitative data: categorical, non-measurable, discrete
Nominal data: no natural ordering, membership
Ordinal data: natural ordering
Categorical data: values from fixed number or categories
Scalar data: if we can map from higher dimensional data to a lower dimensional data ($f(x) : \mathbb{R}^n \to \mathbb{R}$)
Tensor data: multi-dimensional data

2 Interpolation

ISOContours: all points lying on the same line with the same values
Goal of interpolation: construct continuous function f which approximates given values

2.1 Radial Basis Functions

Each point (p_i, f_i) influences $f(x)$ based on distance:

$$r = \|p_i - x\|$$

$$f(x) = \sum_{i=1}^{N} f_i \cdot \varphi(\|p_i - x\|)$$

$$\varphi(r) = e^{-r^2}$$

Weighted radial basis functions
$f(p_j)$ interpolates the value f_j

$$f(p_j) = \sum_{i=1}^{N} w_i \cdot \varphi(\|p_i - p_j\|) = f_j$$

Yields a system of linear equations to be solved for w_i:

$$A = \begin{pmatrix} \varphi(\|p_1 - p_1\|) & \cdots & \varphi(\|p_N - p_1\|) \\ \vdots & \ddots & \vdots \\ \varphi(\|p_1 - p_N\|) & \cdots & \varphi(\|p_N - p_N\|) \end{pmatrix}$$

$$W = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \quad F = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$

$$W = A^{-1} \cdot F$$

Drawbacks: global influence of every sample, adding a new point requires solving the equation system, computationally expensive
Inverse Distance Weighting
Assumption: Nearby points are more similar than those further away.

$$f(x) = \sum_{i=1}^{N} f_i \varphi(\|p_i - x\|)$$

$$d_i = \|p_i - x\| \qquad \varphi(r) = \frac{\frac{1}{r^2}}{\sum_{i=1}^{N} \frac{1}{d_i^2}}$$

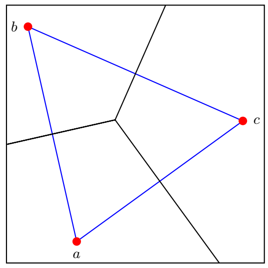
Adjusted formula:

$$\sum_{i=1}^{N} \frac{f_i}{\|p_i - x\|^2} \Big/ \sum_{i=1}^{N} \frac{1}{\|p_i - x\|^2}$$

Drawbacks: still costly, still global influence of every sample

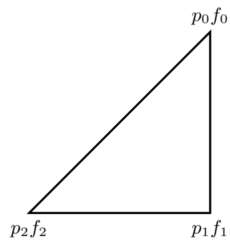
2.2 Triangulation

Giving up smooth, precise reconstruction in favor of speed
Generating a Triangulation
Goals: avoid long and thin triangles, maximize minimum angle in the triangulation, maximize $\frac{\text{radius of circle}}{\text{radius of circumcircle}}$
Delaunay Triangulation
1. Circumcircle does not contain another point of the set
2. Maximizes minimum angle of triangulation
3. Triangulation is unique for all but trivial cases
Voronoi Diagram



Every **Voronoi Sample** (a, b, c) is a vertex of a **Delaunay** triangulation

2.3 Interpolation Inside a Triangle



Find a function f that interpolates f_i at the point p_i such that:

$$f(p_i) = f_i \qquad i = 0, \cdots, N$$

Linear function:

$$f(x) = a + bx + cy$$

Where a, b, c can be obtained with:

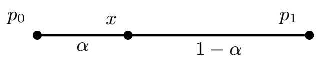
$$X = \begin{pmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{pmatrix} \; A = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \; F = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

$$A = X^{-1} \cdot F$$

2.4 Baricentric Interpolation

We want to have a smooth, continuous interpolation $\forall \alpha \in [0, 1]$

$$f(\alpha) = \alpha \cdot p_0 + (1 - \alpha) \cdot p_0$$



$$\alpha_0 = A_0/A, \ \alpha_1 = A_1/A, \ \alpha_2 = A_2/A$$

Where A = area of the triangle

$$x = \alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2$$

$$\alpha_0 + \alpha_1 + \alpha_2 = 1$$

If α_i are known, then $f(x)$ can be interpolated from values f_i at the vertices via:

$$f(x) = \alpha_0 f_0 + \alpha_1 f_1 + (1 - \alpha_0 - \alpha 1) \cdot f_2$$

Given a point Q we can find α by solving the linear system:

$$P = \begin{pmatrix} p_{0x} & p_{1x} & p_{2x} \\ p_{0y} & p_{1y} & p_{2y} \\ 1 & 1 & 1 \end{pmatrix}$$

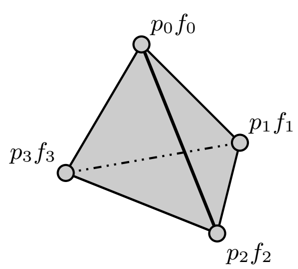
$$A = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} X = \begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix}$$

$$A = P^{-1} \cdot X$$

2.5 Scalar Interpolation

Same concepts for 2D applied in 3D.

$$f(x) = a + bx + cy + dz$$



Can get values a, b, c, d by solving linear system:

$$X = \begin{pmatrix} 1 & x_0 & y_0 & z_0 \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{pmatrix}$$

$$A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} F = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$A = X^{-1} \cdot F$$

Computing Gradient of Scalar Field

The gradient in a scalar field $f(x)$:

$$\nabla f(x) = \begin{pmatrix} \frac{\delta f(x)}{\delta x} \\ \frac{\delta f(x)}{\delta y} \\ \frac{\delta f(x)}{\delta z} \end{pmatrix}$$

In the case of a tetrahedron with function the gradient is always constant:

$$f(x) = a + bx + cy + dz$$

$$\nabla f(x) = \begin{pmatrix} \frac{\delta f(x)}{\delta x} \\ \frac{\delta f(x)}{\delta y} \\ \frac{\delta f(x)}{\delta z} \end{pmatrix} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

2.6 Piece-Wise Linear Interpolation

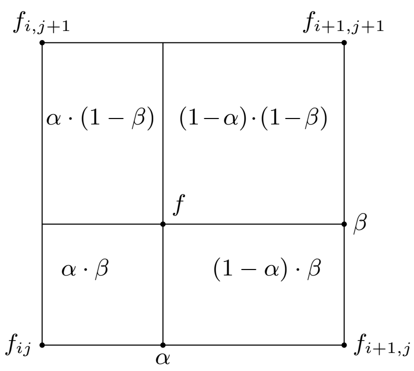
For data points $(x_0, y_0), \cdots, (x_N, y_N)$ Evaluate

$$f(x) = (1 - \alpha) y_i + \alpha y_i$$

$$\text{where: } \alpha = \frac{x - x_1}{x_{i+1} - x_i}$$

Bilinear Interpolation

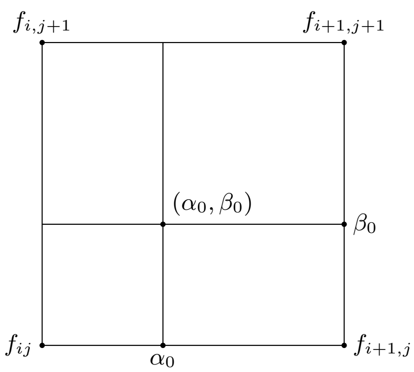
$$\begin{aligned} f(\alpha, \beta) = & f_{ij} \cdot (1 - \alpha)(1 - \beta) + \\ & f_{i+i,j} \cdot \alpha(1 - \beta) + \\ & f_{i,j+1} \cdot (1 - \alpha)\beta + \\ & f_{i+1,j+1} \cdot \alpha\beta \end{aligned}$$



Asymptotic Decider
Hyperbola form:

$$f(\alpha, \beta) = \gamma(\alpha - \alpha_0)(\beta - \beta_0) + \delta$$

where δ is the value at point (α_0, β_0)



Transform to **hyperbola form**:

$$f(\alpha, \beta) = A\alpha + B\beta + C\alpha\beta + D$$

$$\begin{aligned} A &= f_{i+1,j} - f_{ij} \\ B &= f_{i,j+1} - f_{ij} \\ C &= f_{ij} - f_{i,j+1} - f_{i+1,j} + f_{i+1,j+1} \\ D &= f_{ij} \\ \delta &= (f_{ij} \cdot f_{i+1,j+1} - f_{i+1,j} \cdot f_{i,j+1}) \\ \alpha_0 &= -B/C \\ \beta_0 &= -A/C \\ \gamma &= C \end{aligned}$$

3 Phong Illumination Model

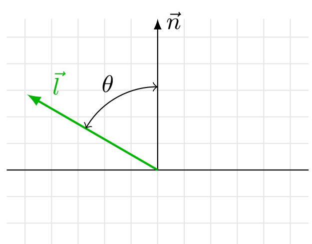
Ambient Light: background light, constant everywhere
Diffuse Reflector: reflects equally into all directions
Specular Reflector: reflects mostly into the mirror direction

3.1 Lighting

$$C = K_a C_a O_d$$

K_a : ambient reflection coefficient $\in [0, 1]$
C_a : color of the ambient light
O_a : object color

3.2 Diffuse Reflection



$$C = K_d C_p O_d \cdot \cos\theta$$

$$\cos\theta = \frac{\vec{n} \cdot \vec{l}}{|\vec{n}| \cdot |\vec{l}|}$$

K_d : diffuse reflection coefficient $\in [0, 1]$
C_p : color of point of light
O_d : object color
$\cos\theta$: angle between light vector $\vec{l}$ and $\vec{n}$

3.3 Specular Reflection

$$C = K_s C_p O_d \cdot \cos^n \varphi$$

$$\cos^n \varphi = \frac{\vec{r} \cdot \vec{v}}{|\vec{r}| \cdot |\vec{v}|}$$

K_s : specular reflection coefficient $\in [0, 1]$

C_p : color of point of light

O_d : object color

$\cos^n \varphi$: angle between reflected light ray $\vec{r}$ and the vector to the viewer $\vec{v}$

$(\cdots)^n$: shininess factor (extent of highlight)