Problem Set 10

1. $A(z) = a[0] + a[1]z^{-1} + a[2]z^{-2} + \dots + a[N]z^{-N}$

$B(z) = b[0] + b[1]z^{-1} + b[2]z^{-2} + \dots + b[M]z^{-M}$,

$A(z)B(z) = c[0] + c[1]z^{-1} + c[2]z^{-2} + \dots + c[N+M]z^{-(N+M)}$

Q: Derive expression for $c[n]$ as a function of coefficients $\{a[k]\}_{k=0}^{N}$ and $\{b[k]\}_{k=0}^{M}$. Describe a procedure uses FFTs to compute $\{c[n]\}$ quickly.

Solution: clearly $c[n] = \sum_{k=0}^{n} a[k]b[n-k]$ $\quad n=0,1,\cdots N+M$.

clearly, this is a linear convolution.

To use FFT. we first add zero to both sequence $\{a[n]\}$, $\{b[n]\}$ and let their length to be $N+M$. we have $a[n]'$, $b[n]'$

$\therefore$ $c[n] = \text{ifft}(\ \text{FFT}(a[n]') \ .*\ \text{FFT}(b[n]'))$

2. $\delta_s = 2\times10^{-3}$

$\delta_p = 2\times10^{-2}$

$\omega_s = 3\pi/8$

$\omega_p = \pi/4$

$$\varepsilon^2 = \frac{1}{(1-\delta_p)^2} - 1 = 0.0412$$

$$A^2 - 1 \approx \frac{1}{\delta_s^2} = 2.5\times10^5$$

$$N = \frac{1}{2}\,\frac{\log(2.5\times10^5/0.0412)}{0.4782} \approx 17$$

$$H_a(s) = \frac{\Omega_c^{17}}{\prod_{\ell=1}^{17}(s-p_\ell)} \;;\; p_\ell = \Omega_c e^{j\pi\left(\frac{16+2\ell}{34}\right)} = \Omega_c e^{j\pi\left(\frac{8+\ell}{17}\right)}$$

$$H(z)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{(0.4636)^{17}\left(\frac{2}{T}\right)^{17}}{\prod_{\ell=1}^{17}\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} - \Omega_c e^{j\pi\left(\frac{8+\ell}{17}\right)}\right)}$$

$$= \frac{(0.4636)^{17}(1+z^{-1})^{17}}{\prod_{\ell=1}^{17}\left(1-z^{-1} - 0.4636 e^{j\left(\frac{8+\ell}{17}\right)\pi}(1+z^{-1})\right)}$$

$$\{a_\ell[k]\} = \left\{1 - 0.4636 e^{j\left(\frac{8+\ell}{17}\right)\pi},\; -1 - 0.4636 e^{j\left(\frac{8+\ell}{17}\right)\pi}\right\} \quad \text{\& add 16 0's}$$

$$\text{coefficient under} = \mathrm{ifft}\left(\prod_{\ell=1}^{17} \mathrm{fft}(a_\ell)\right)$$

2

```
% solution2
a = zeros(17,18);
c = 0.4636;
for l = 1 : 17
    a(l,1:2) = [1-c*exp(1j*( (8+l)/17 )*pi),-1-c*exp(1j*( (8+l)/17 )*pi)];
end
multifft = ones(1,18);
for l = 1 : 17
    multifft = fft(a(l,:)) .* multifft;
end
coefficientDe = ifft(multifft);
% dominator
dominator = zeros(1,18);
for i = 0 : 17
    dominator(i+1) = nchoosek(17,i);
end
dominator = c^17 * dominator;
[h,w]=freqz(dominator,coefficientDe);
plot(w/pi,abs(h));
title('magnitude');
```

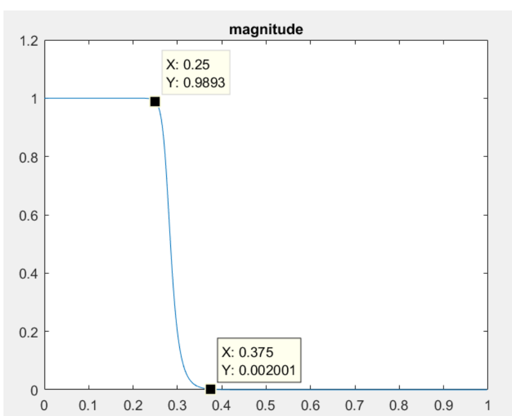

Figure 1: *magnitude responce*

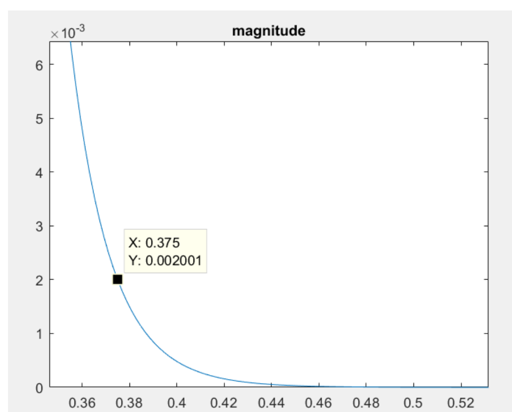

Figure 2: *ripple*

3(a)

```
b = firpm(14, [0,0.25,0.45,1], [1,1,0,0]);
[h,w]=freqz(b);
plot(w/pi,abs(h));
title('magnitude')

figure
b = firpm(14, [0,0.5,0.6,1], [1,1,0,0]);
```

```
[h,w]=freqz(b);
plot(w/pi,abs(h));
title('magnitude')
```

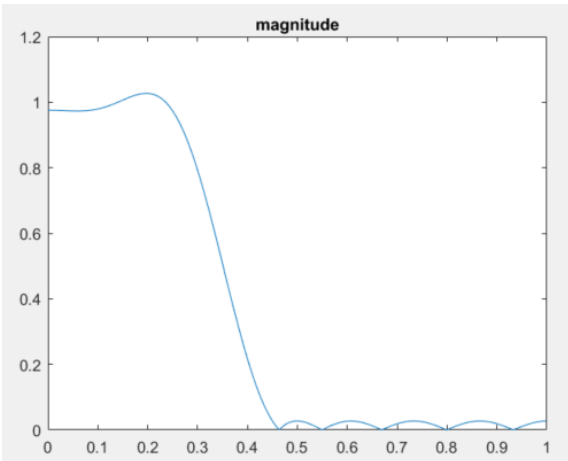

Figure 3: $\omega_c = 0.35\pi$

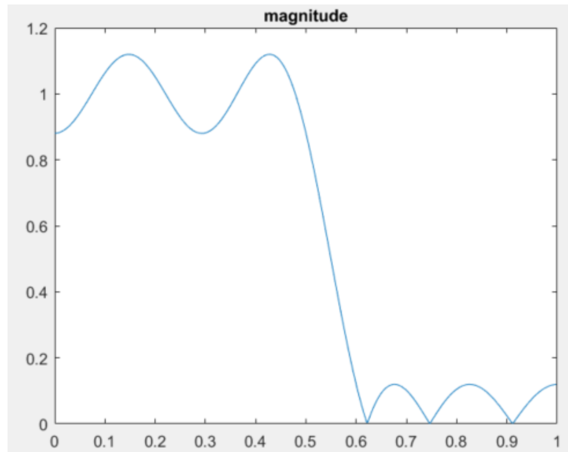

Figure 4: $\omega_c = 0.55\pi$

3(b)

```
b = firpm(14, [0,0.25,0.45,1], [1,1,0,0]);
[h,w]=freqz(b);
plot(w/pi,abs(h));
title('magnitude')


beta = -0.3129;
Mat = zeros(14,15,15);
for i = 1 : 14
    Mat(i,:,1) = [1,-beta,zeros(1,13)];
end
for i = 1 : 14
    Mat(i,:,15) = [-beta,1,zeros(1,13)];
end
for j = 2:13
    for i = 1 : j-1
        Mat(i,:,j) = [-beta,1,zeros(1,13)];
    end
    for i = j : 14
        Mat(i,:,j) = [1,-beta,zeros(1,13)];
    end
end

sum = zeros(1,15);
```

3(b): we get $\{b[n]\}$ of the filter.

$$\therefore H(z) = b[0] + b[1]z^{-1} + b[2]z^{-2} + \cdots + b[14]z^{-14}$$

$$\downarrow z^{-1} = \frac{z^{-1}-\beta}{1-\beta z^{-1}}$$

$$H'(z) = b[0] + b[1]\left(\frac{z^{-1}-\beta}{1-\beta z^{-1}}\right) + b[2]\left(\frac{z^{-1}-\beta}{1-\beta z^{-1}}\right)^2 + \cdots + b[14]\left(\frac{z^{-1}-\beta}{1-\beta z^{-1}}\right)^{14}$$

$$= \frac{b[0](1-\beta z^{-1})^{14} + b[1](z^{-1}-\beta)(1-\beta z^{-1})^{13} + \cdots + b[14](z^{-1}-\beta)^{14}}{(1-\beta z^{-1})^{14}}$$

Construct $M_\ell = \begin{array}{c} \text{13 0's} \\ \begin{bmatrix} -\beta, & 1, & 0, & 0, & \cdots & 0 \\ 1, & -\beta, & 0, & 0, & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \}\ \ell \\ \}\ 14-\ell \end{array} \quad \ell = 0 \sim 14$

$$\sum_{\ell=0}^{14}\left\{ \text{ifft}\left(\prod_{i=1}^{14} \text{FFT}(M_{\ell_i}) \right) \cdot b[\ell] \right\}$$ is the coefficient

$$\beta = \frac{\sin[(-0.55\pi + 0.35\pi)/2]}{\sin[(0.55\pi + 0.35\pi)/2]} = \frac{\sin(-0.1\pi)}{\sin(0.45\pi)} = -0.3129$$

```
for l = 1 : 15
    multiTemp = ones(1,15);
    for i = 1 : 14
        multiTemp = multiTemp .* fft(Mat(i,:,l));
    end
    sum = sum + ifft(multiTemp) * b(l);
end
dedominator = zeros(1,15);
for i = 0 : 14
    dedominator(i+1) = nchoosek(14,i)*((-beta)^i);
end
figure
[h,w]=freqz(sum,dedominator);
plot(w/pi,abs(h));
title('magnitude')
```

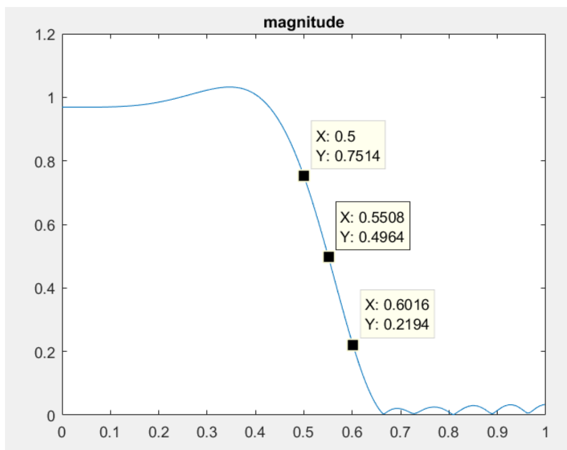

Figure 5: *Applying frequency transformation*

3(c) Using this way, we have an IIR filter with smaller ripple, and nearly equiripple.However, it has larger transitionband

4(a)

```
b = firpm(14, [0,0.25,0.45,1], [1,1,0,0]);
[h,w]=freqz(b);
plot(w/pi,abs(h));
title('magnitude')

figure
b = firpm(14, [0,0.5,0.6,1], [0,0,1,1]);
[h,w]=freqz(b);
plot(w/pi,abs(h));
title('magnitude')
```

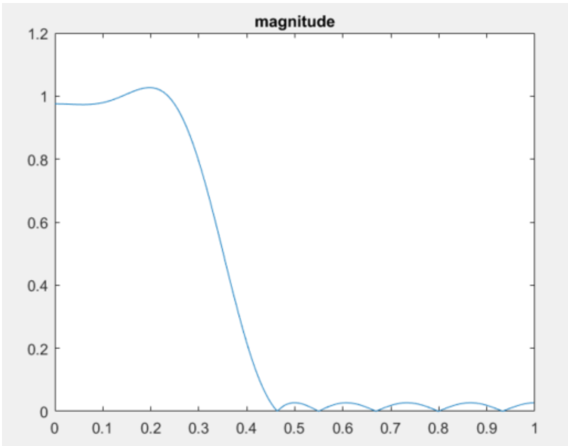

Figure 6: $\omega_c = 0.35\pi$

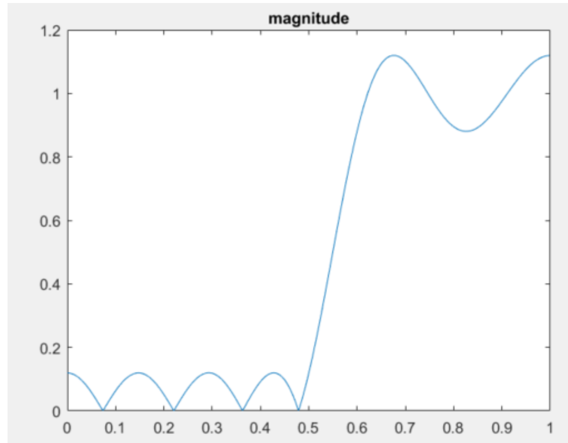

Figure 7: $\omega_c = 0.55\pi$

4b

```
b = firpm(14, [0,0.25,0.45,1], [1,1,0,0]);
[h,w]=freqz(b);
plot(w/pi,abs(h));
title('magnitude')


beta = cos(0.45*pi)/cos(0.1*pi);
Mat = zeros(14,15,15);
for i = 1 : 14
    Mat(i,:,1) = [1,-beta,zeros(1,13)];
end
for i = 1 : 14
    Mat(i,:,15) = [beta,-1,zeros(1,13)];
end
for j = 2:13
    for i = 1 : j-1
        Mat(i,:,j) = [beta,-1,zeros(1,13)];
    end
    for i = j : 14
        Mat(i,:,j) = [1,-beta,zeros(1,13)];
    end
end

sum = zeros(1,15);
for l = 1 : 15
    multiTemp = ones(1,15);
    for i = 1 : 14
        multiTemp = multiTemp .* fft(Mat(i,:,l));
    end
    sum = sum + ifft(multiTemp) * b(l);
end
```

```
dedominator = zeros(1,15);
for i = 0 : 14
    dedominator(i+1) = nchoosek(14,i)*((-beta)^i);
end
figure
[h,w]=freqz(sum,dedominator);
plot(w/pi,abs(h));
title('magnitude')
```

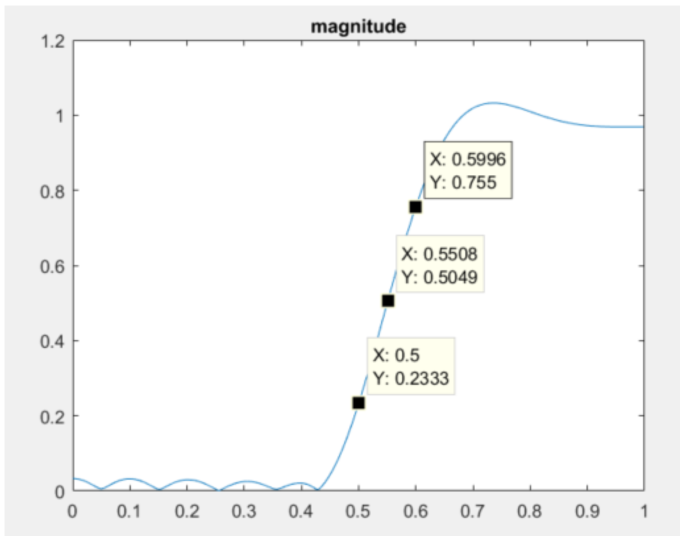

Figure 8: *Frequency transformation*