EE511

14 Feb 2017

Discrete Time Stochastic Processes:

Consider system in state $X(t)$ at time t. $X(t)$ is a random variable.

First limit $X(t)$ to outcomes from a countable sample space:

$$\Omega = \{0, 1, 2, \dots\}.$$

Also limit time index t to integer values.

Sometimes $X(t+1)$ depends on entire history of system, e.g. @ $t, t-1, t-2, \dots$

Defn: A Markov Chain is a system whose next state depends only on its current state:

$$\Pr[X(t+1)=j \mid X(t)=i_t, X(t-1)=i_{t-1}, \dots, X(0)=i_0] = \Pr[X(t+1)=j \mid X(t)=i_t].$$

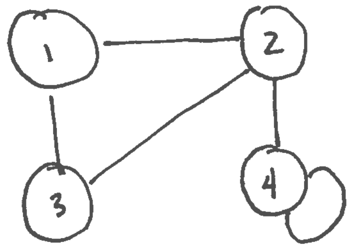
Defn: The transition matrix for a Markov chain is an arrangement of the individual $\Pr[X(t+1)=j \mid X(t)=i_t]$ values such that:

$$P(t) = \{P_{ij}(t)\} = \Pr[X(t+1)=j \mid X(t)=i_t].$$

Often P does not depend on time so $P(t)=P$.

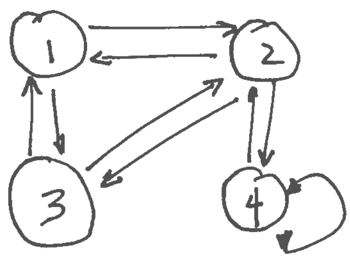
Ex: Cave system.

Consider 4 caves connected by tunnels.



Start in cave 1. Select (randomly) a tunnel and move to the next cave. Suppose each tunnel takes 1 hr to traverse.

Represent the caves as a state transition diagram:



then

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 \end{bmatrix}$$

Defn: A stochastic matrix is a matrix whose rows sum to 1.

Notice that P is a stochastic matrix.

Ex (cont)

Find $\Pr[\text{Cave } k \text{ after 3 transitions}]$

Define $\pi(t) = [\pi_1(t) \;\; \pi_2(t) \;\; \pi_3(t) \;\; \pi_4(t)]$.

$\pi_i(t) = \text{Prob}[\text{in cave } 1 \text{ at time } t]$.

Start at: $\pi(0) = [1 \;\; 0 \;\; 0 \;\; 0]$.

Then $\pi_i(t+1) = \sum_{j=1}^{4} \pi_j(t) \cdot P_{ji}$ [matrix form: $\pi(t+1) = \pi(t) P$].

that is: $\pi(1) = \pi(0) \cdot P = [0, \frac{1}{2} \;\; \frac{1}{2} \;\; 0]$.

$\pi(2) = \pi(1) \cdot P = [\frac{5}{12} \;\; \frac{3}{12} \;\; \frac{2}{12} \;\; \frac{2}{12}]$.

$\pi(3) = \pi(2) \cdot P = [\frac{12}{72} \;\; \frac{25}{72} \;\; \frac{21}{72} \;\; \frac{14}{72}]$

In this case: as $t \to \infty$, $\pi(t) = \underset{\text{constant.}}{\pi}$ a limiting (steady-state)

value that does not depend on $\pi(0)$.

Defn: A Markov chain is aperiodic iff $d = \gcd\{n : P(X_n = i \mid X_0 = i) > 0\} = 1$ for all states.

Notice a Markov chain with transition matrix P is

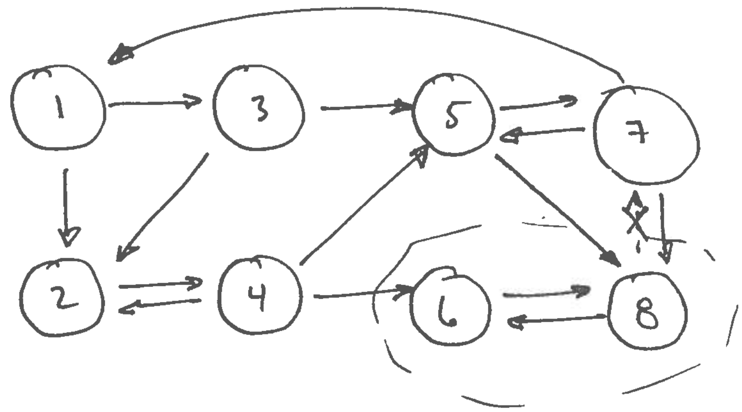
periodic if $|\lambda_1| = |\lambda_2| = 1$ for 2 distinct eigenvalues.

$(x \cdot P = \lambda \cdot x)$.

eigenvalue eigenvector

Defn: A Markov chain is irreducible iff $P_{ij}^{(n)} > 0$ for all i and j and some $n \geq 1$.

i.e. every state "communicates" with every other state.



not irreducible.
states 6 and 8 are attracting.

Thrm: (Perron Frobenius): Let P be a regular stochastic matrix. Suppose P is irreducible and aperiodic. Then P has a unique positive eigenvector π with $\pi_i > 0$ for $1 \leq i \leq N$,

$$\pi P = \pi.$$

Thrm: (Markov Chain Ergodic Theorem): For every non negative vector $x \in \mathbb{R}^N$ then if P satisfies the requirements for Perron-Frobenius

$$\lim_{n \to \infty} \underbrace{x \cdot P^n}_{} = \pi.$$

think $\left[\left[\left[x(0) \cdot P\right] \cdot P\right] \cdot P\right] \cdots \longrightarrow \pi.$

Analytic solution: (intuitive w/o proof):

If it exists then: $\pi P = \pi$. ← Linear equations.

it exists then "flow of probability" into each state must balance flow leaving.

i.e. $$\sum_j \pi_i P_{ij} = \sum_j \pi_j P_{ji}$$

leaving j ($i \to j$) entering j. ($j \leftarrow i$).

Also remember that $\sum_i \pi_i = 1$.

Solving gives: $\pi = \left[\frac{2}{10} \ \frac{3}{10} \ \frac{2}{10} \ \frac{3}{10}\right]$.

Ex: (In honor of Valentines Day) Picking hearts.

Pick card from a deck of 52 cards (4 suits, 13 each). If heart drawn keep it. Else randomly return it to the deck. How long to pick all 13 hearts?

Note: After picking h hearts prob[next card is heart] = $\frac{13-h}{52-h}$.

Using a Markov chain: $H(k)$ # of hearts picked up after k draws:

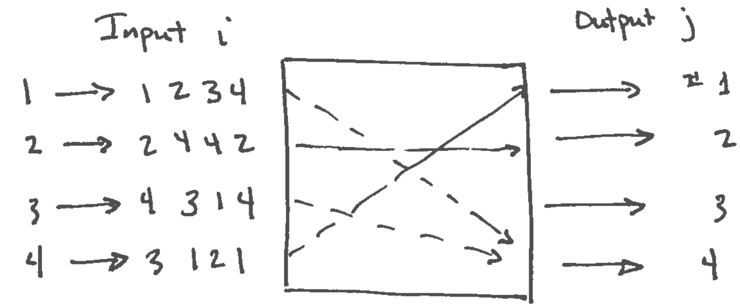
$$P_{ij}(k) = P_{ij} = \begin{cases} 0 & j \neq i, i+1 \\ 39/52-h & j = i \\ 13-h/52-h & j = i+1. \end{cases}$$

instead interested in the transient behavior.

Note: here limiting distribution not very interesting: $\lim_{k \to \infty} \{\pi_{13}(k)\} = 1$.

Ex: Head of Line (HOL) Blocking switch

Consider $N \times N$ blocking switch ... only one packet delivered at a time to a particular output.



exacerbated by non-uniform popularity.

conflict (multiple packets to same destination) cause blocking

Start w/ 2×2 HOL switch.

Consider heavily loaded switch. → every input always has a packet for transmission

Then state (X_1, X_2) w/ $X_i \in \{0,1\}$ specifies output port is requested by each input.

There are 4- possible states: $(0,0)$; $(0,1)$; $(1,0)$; $(1,1)$

↳ $(0,0)$ and $(1,1)$ allow only a single packet to transmit

$(0,1)$ and $(1,0)$ allow both packets to transmit.

So for instance: current state $(0,0)$. One packet delivered, one packet held.

replaced on next interval: equally likely 0 or 1. "uniform popularity".

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