Universal operations in resource theories and local thermodynamics

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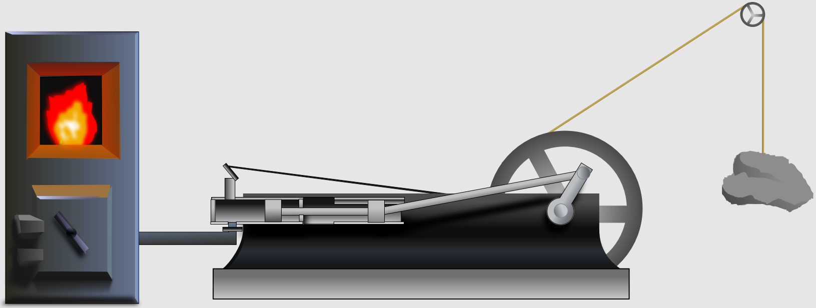
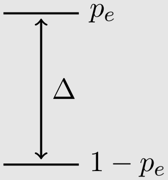
Thermodynamics



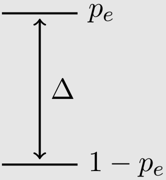
Quantum Thermodynamics



Quantum Thermodynamics

$$\begin{array}{c} \overline{}\ p_e \\ \updownarrow \Delta \\ \overline{}\ 1 - p_e \end{array}$$

Quantum Thermodynamics



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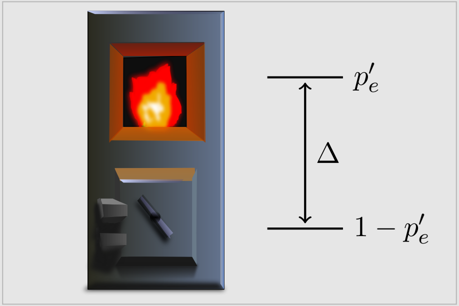

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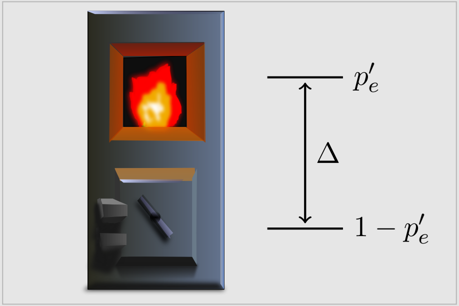
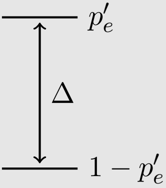


Thermalising map

Quantum Thermodynamics



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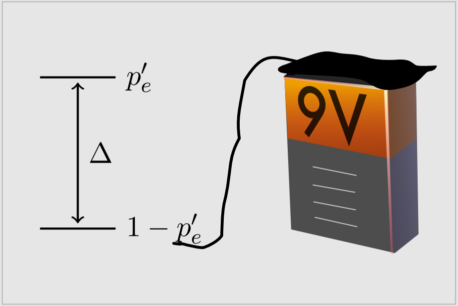

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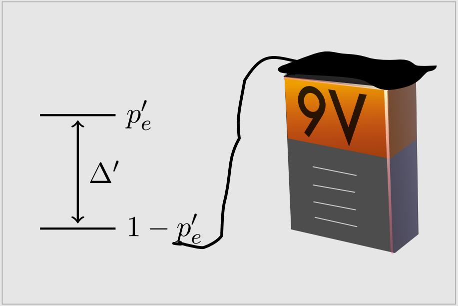

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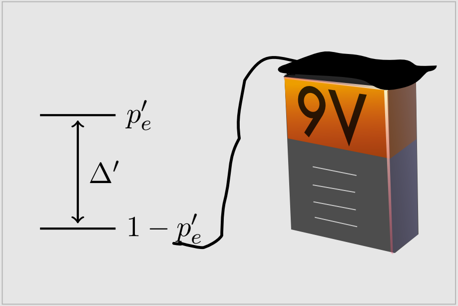


Charge battery

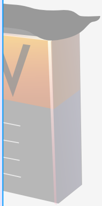
Quantum Thermodynamics

Unitary dynamics of the form

$$(\rho_0, H_0) \mapsto (U_t \rho_0 U_t^\dagger, H_t)$$

charges the battery by an amount of work

$$\langle W \rangle = \mathbf{Tr}\left(\rho_0 H_0 - U_t \rho_0 U_t^\dagger H_t\right).$$



Quantum Thermodynamics

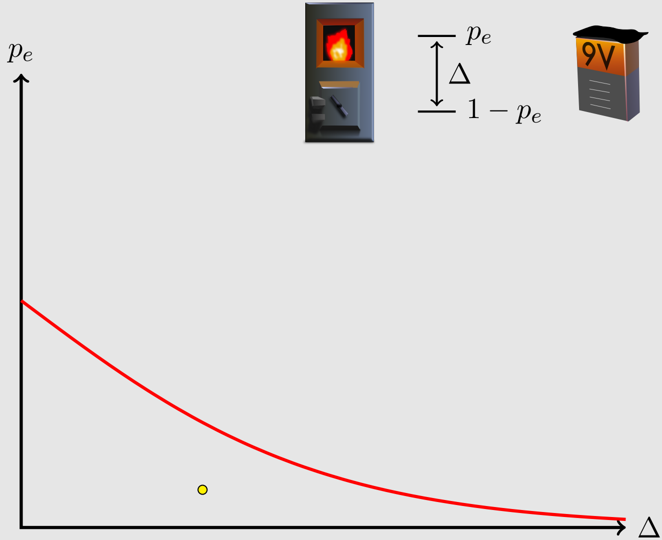
p'_e

Δ'

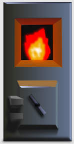
$1 - p'_e$

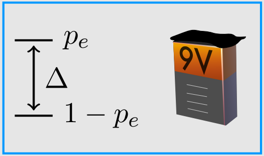
Goal: Charge the battery as much as possible

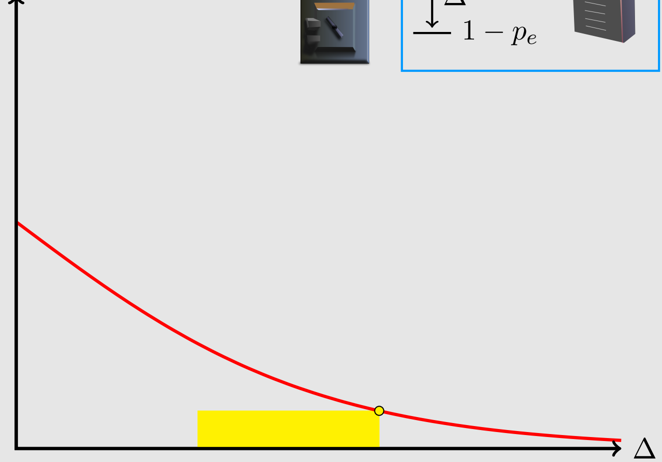
Thermodynamical Operations



Thermodynamical Operations







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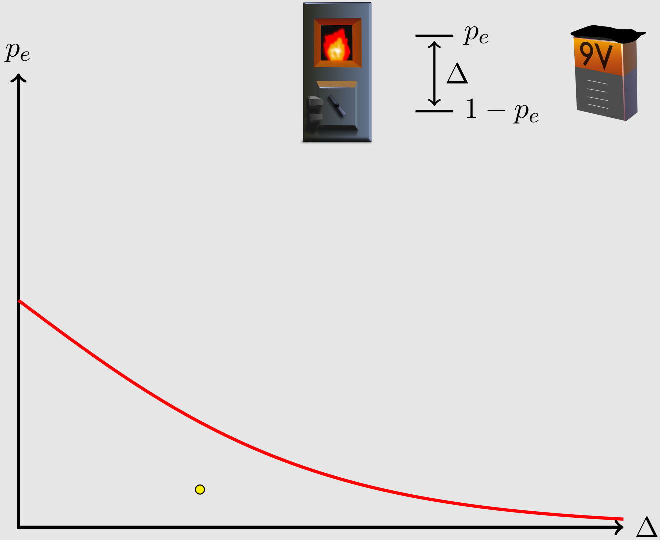

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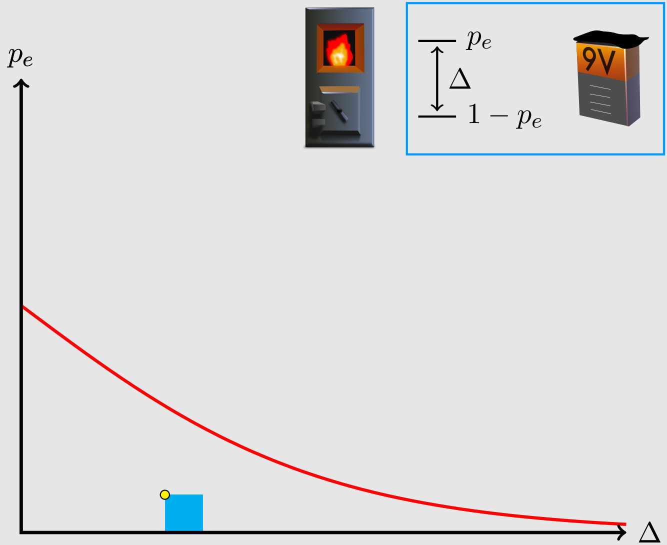

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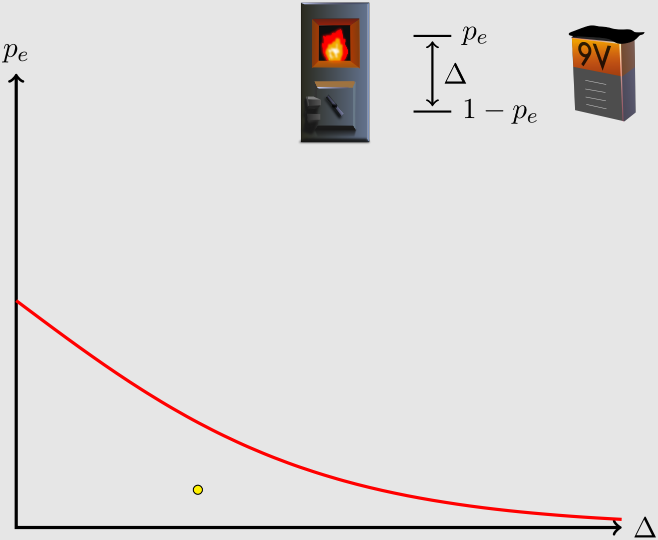

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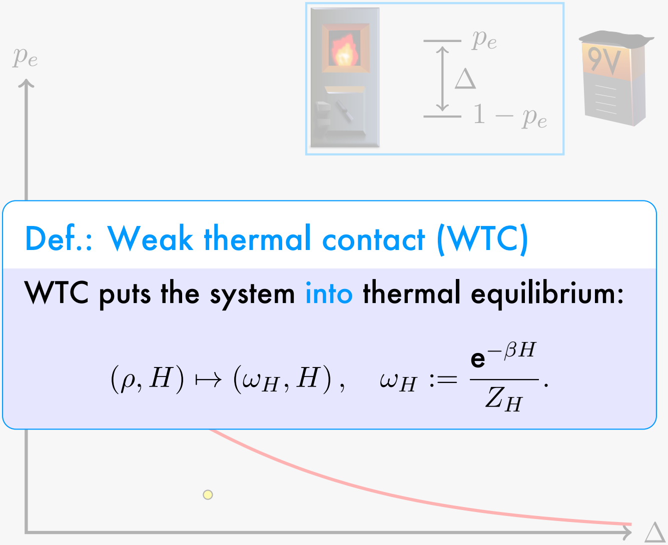
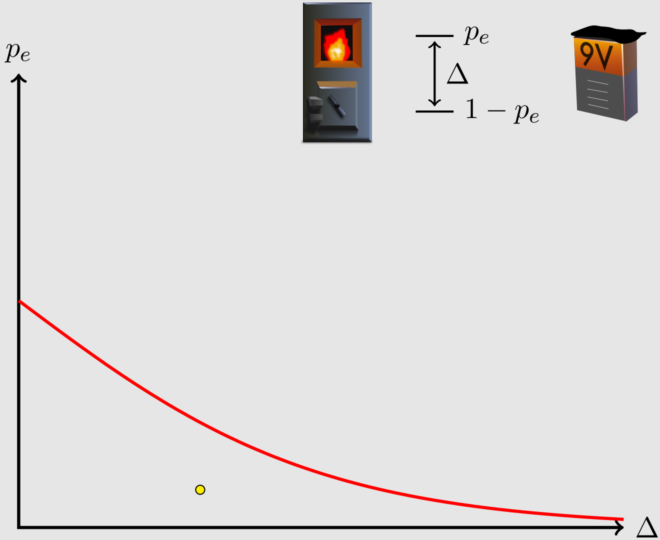


Def.: Weak thermal contact (WTC)

WTC puts the system into thermal equilibrium:

$$(\rho, H) \mapsto (\omega_H, H)\,, \quad \omega_H := \frac{\mathrm{e}^{-\beta H}}{Z_H}.$$

Thermodynamical Operations



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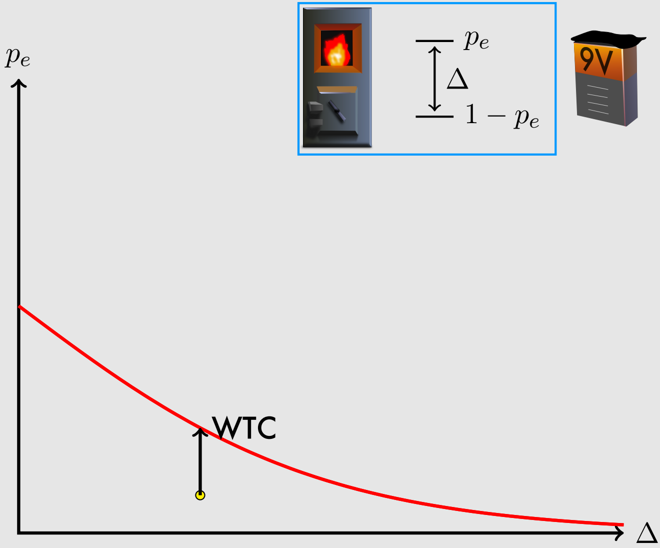

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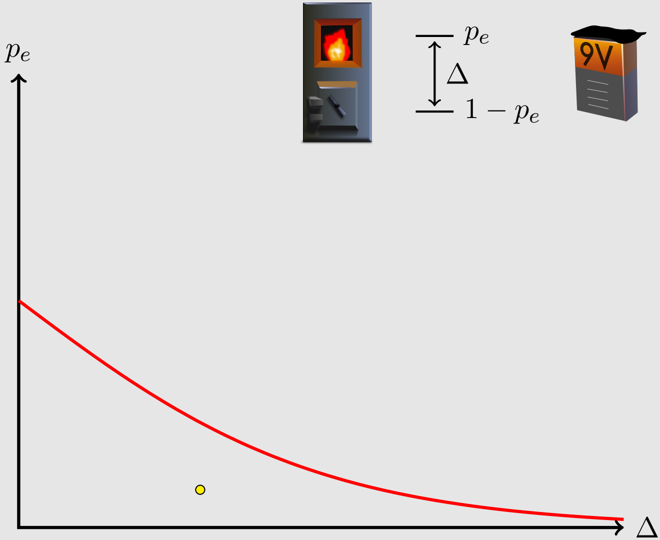

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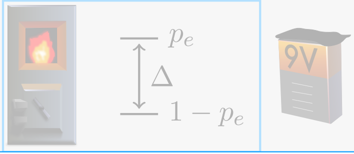


Def.: Thermal Operations (TO) [7]

Let H_B be a Hamiltonian (on a bath). A thermal operation is of the form

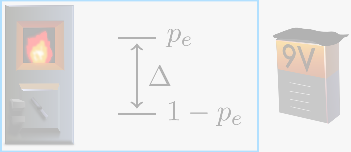
$$(\rho, H) \mapsto \left(\mathrm{Tr}_B\left(U\rho \otimes \omega_{H_B} U^\dagger\right), H\right),$$

where U is any unitary such that $[U, H \otimes \mathbb{1} + \mathbb{1} \otimes H_B] = 0$.

p_e

Δ

Thermodynamical Operations

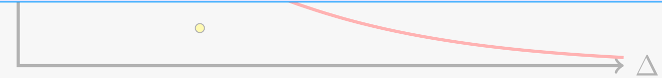


Def.: Gibbs-perserving map (GP-map)

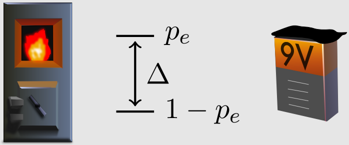
A Gibbs-preserving map is a transformation on pairs that cannot bring the system out of thermal equilibrium:

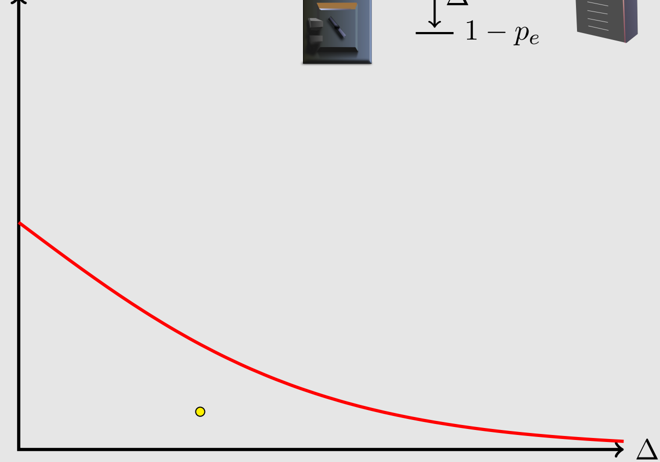
$$(\rho, H) \mapsto (\mathcal{G}_H(\rho), H)\,, \quad \mathcal{G}_H(\omega_H) = \omega_H$$

with a quantum channel $\mathcal{G}_H$.



Thermodynamical Operations





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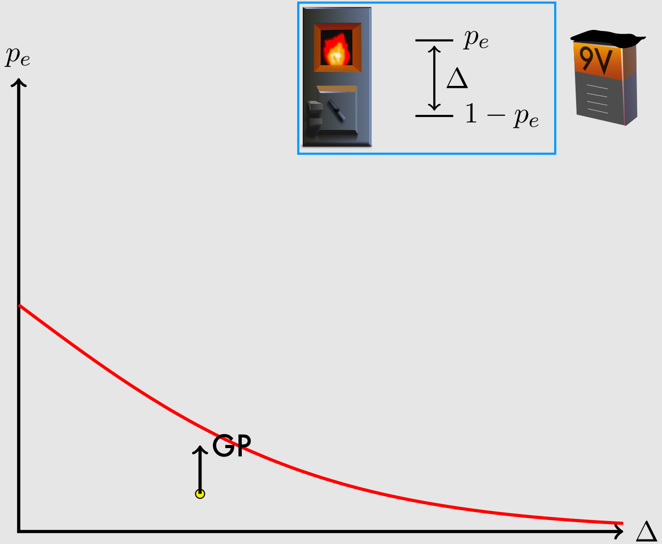

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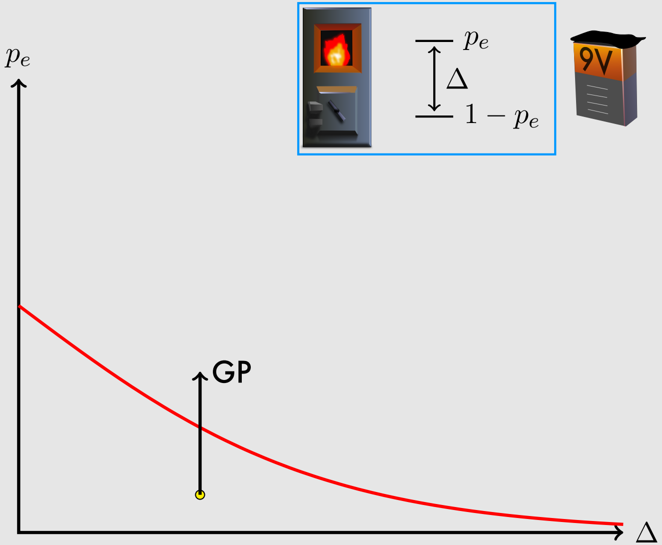

Thermalising maps

- **Weak thermal contact (WTC):**

$$(\rho, H) \mapsto (\omega_H, H).$$

- **Thermal operations (TO):**

$$(\rho, H) \mapsto \left(\mathrm{Tr}_B \left(U \rho \otimes \omega_{H_B} U^\dagger\right), H\right).$$

- **Gibbs-Preserving maps (GP):**

$$(\rho, H) \mapsto ((\mathcal{G}_H(\rho), H), \quad \mathcal{G}_H(\omega_H) = \omega_H.$$

The game of work-extraction

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You:

$$\boxed{(\rho_0, H_0)}$$

The game of work-extraction

You:

$$
\begin{array}{c}
\boxed{(\rho_0, H_0)} \\
\downarrow \\
\boxed{\text{Unitary}}
\end{array}
$$

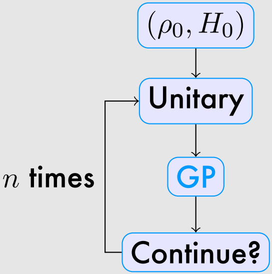
The game of work-extraction

You:

$$\begin{tikzcd} (\rho_0, H_0) \arrow[d] \\ \text{Unitary} \arrow[d] \\ \text{GP} \end{tikzcd}$$

The game of work-extraction

You:



The game of work-extraction

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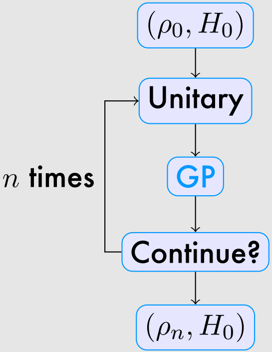

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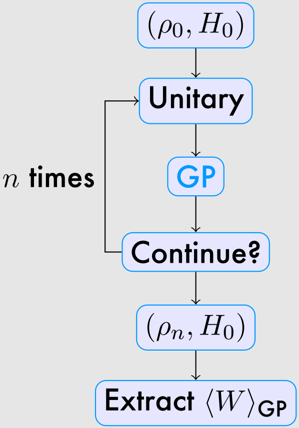

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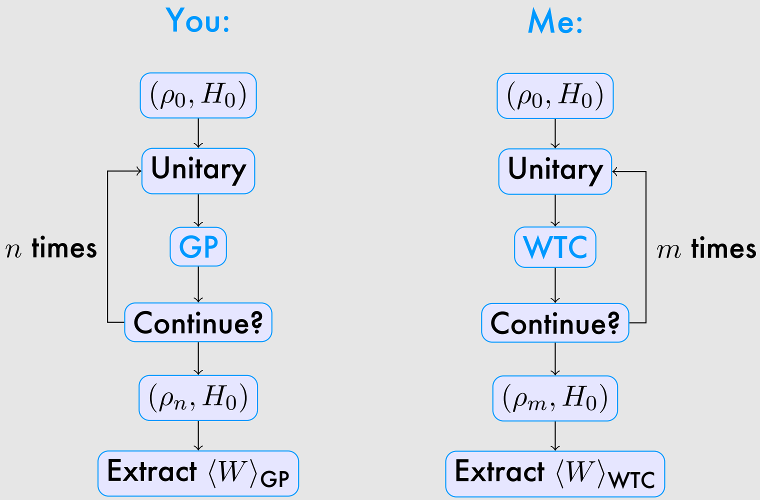
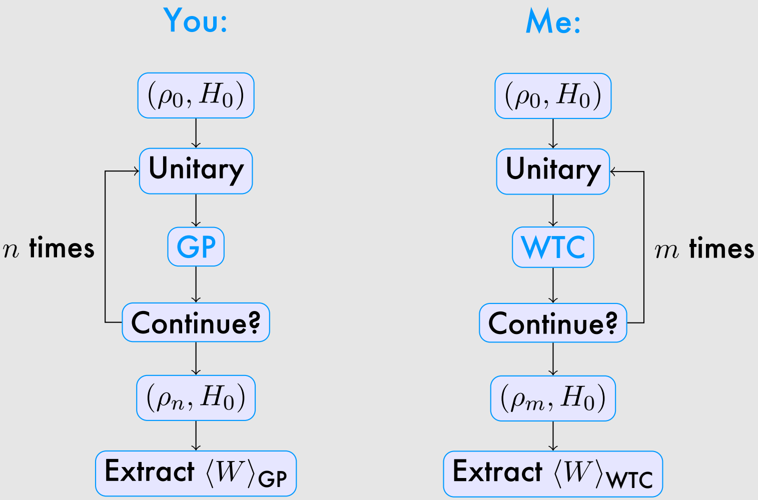
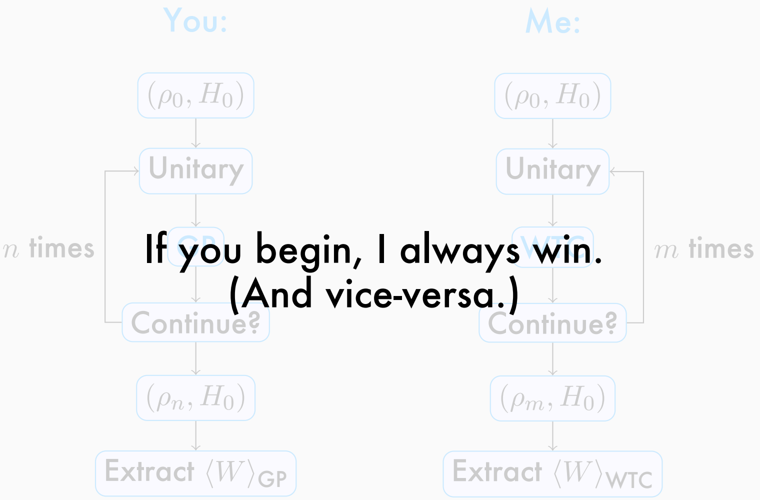


The game of work-extraction



I win if $\langle W \rangle_{\text{GP}} \leq \langle W \rangle_{\text{WTC}}$.

The game of work-extraction



If you begin, I always win.
(And vice-versa.)

I win if $\langle W \rangle_{\text{GP}} \leq \langle W \rangle_{\text{WTC}}$.

Universality of WTC

Theorem [2, 4]

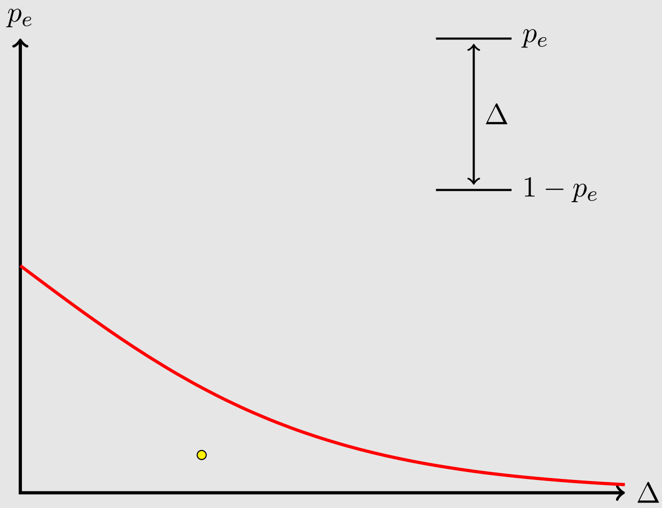
The work yield of a cyclic Hamiltonian process using GP-maps as thermalising maps is bounded as

$$\langle W \rangle_{\mathsf{GP}}(\rho_0, H_0) \leq \frac{1}{\beta} S(\rho_0 || \omega_{H_0}).$$

The bound can be saturated arbitrarily well already with WTC as thermalising maps.

[2] J. Aberg, Phys. Rev. Lett. 113, 150402 (2014).
[4] H. Wilming, R. Gallego, J. Eisert, arXiv:1411.3754 (2014).

Optimal Protocol



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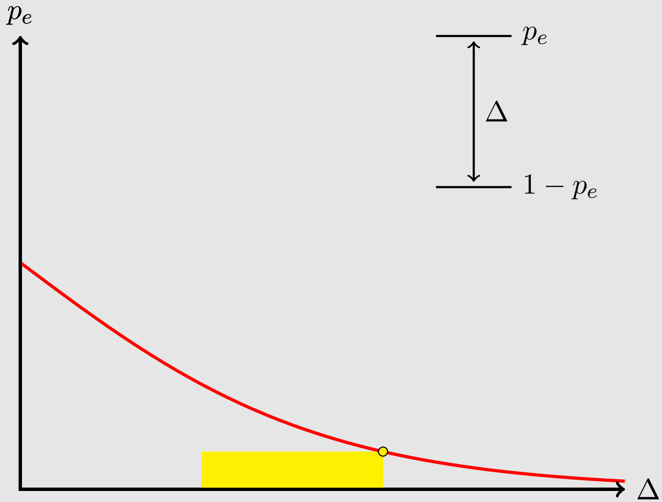

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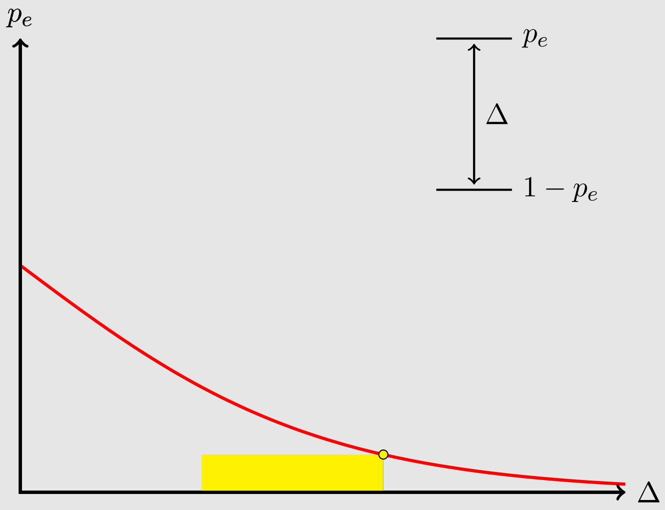

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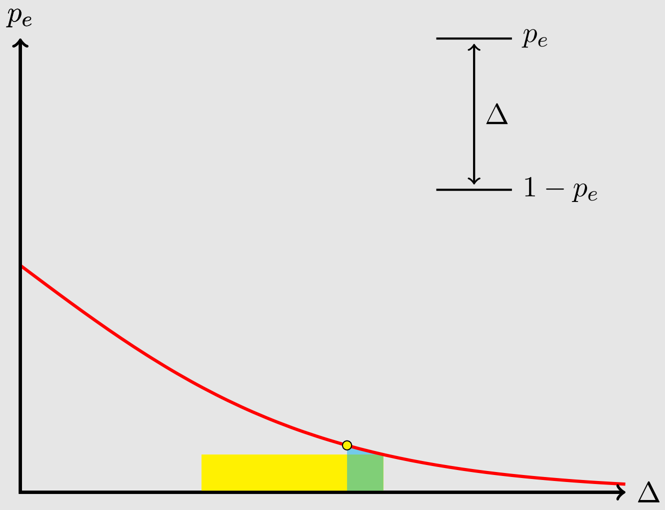

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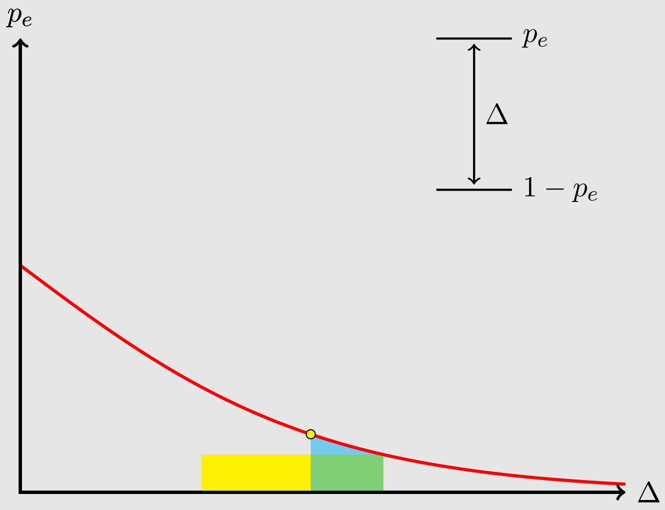

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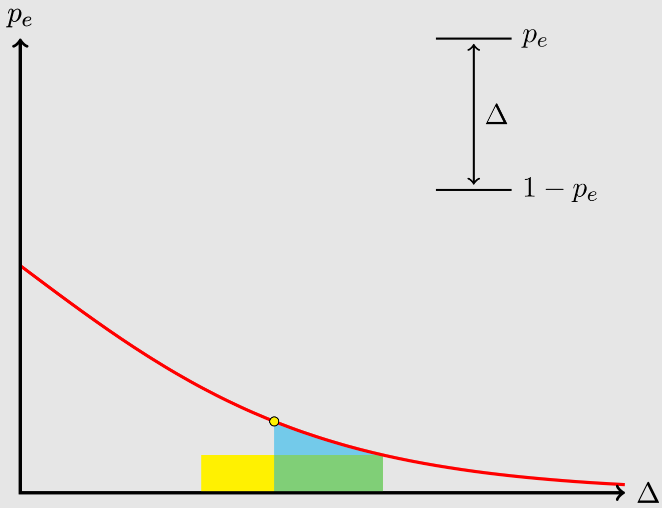

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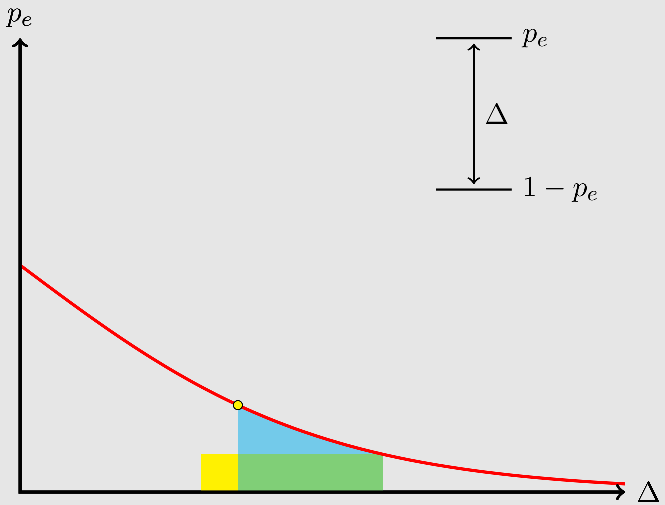

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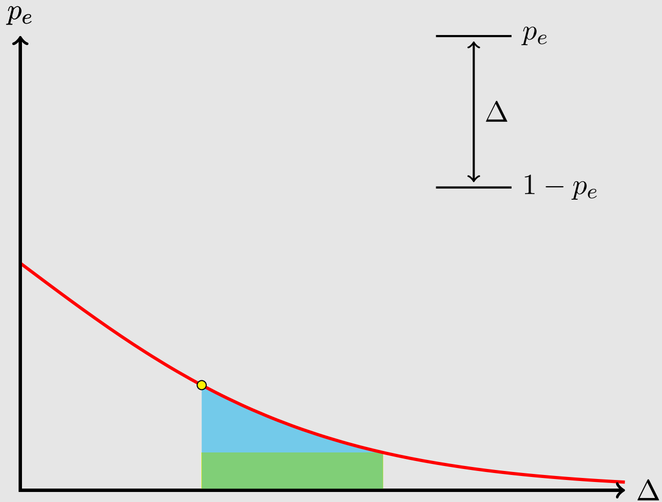

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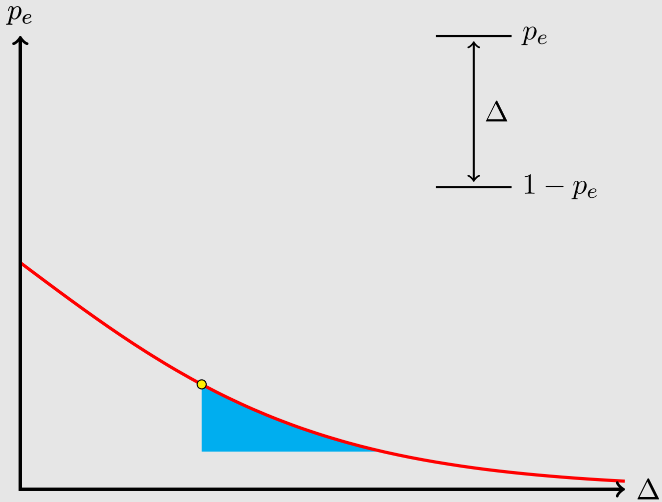

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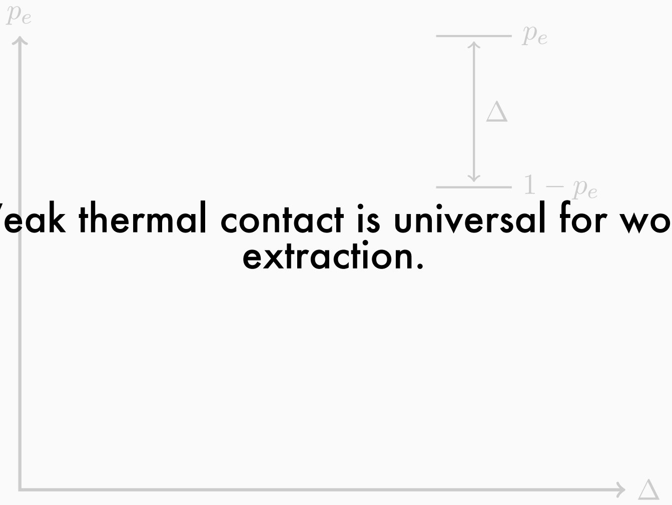
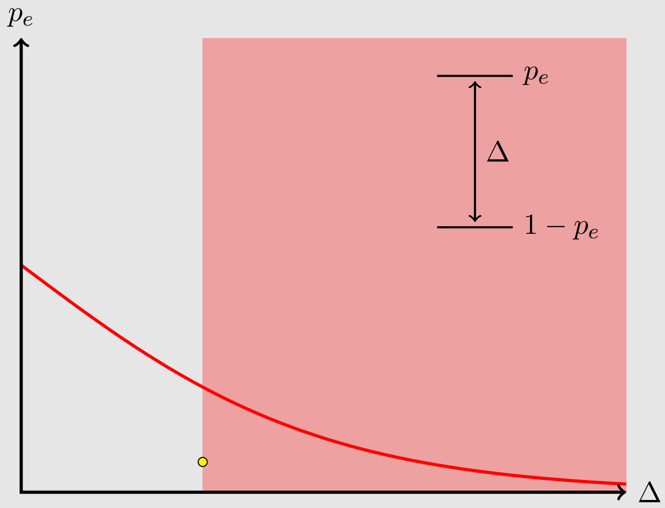


Weak thermal contact is universal for work extraction.

New rules for the game



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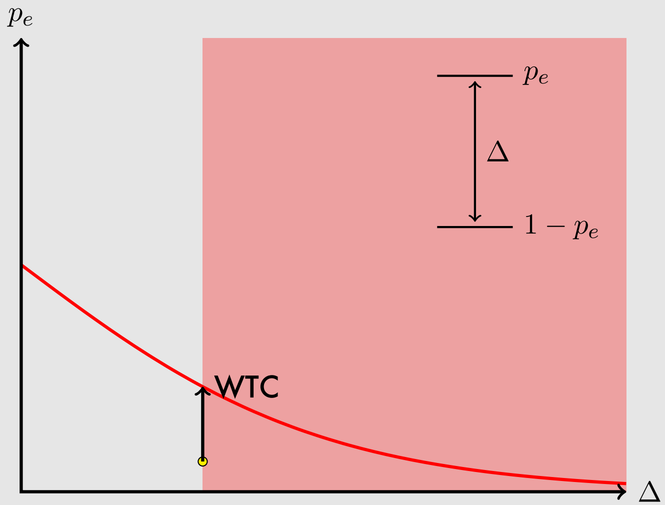

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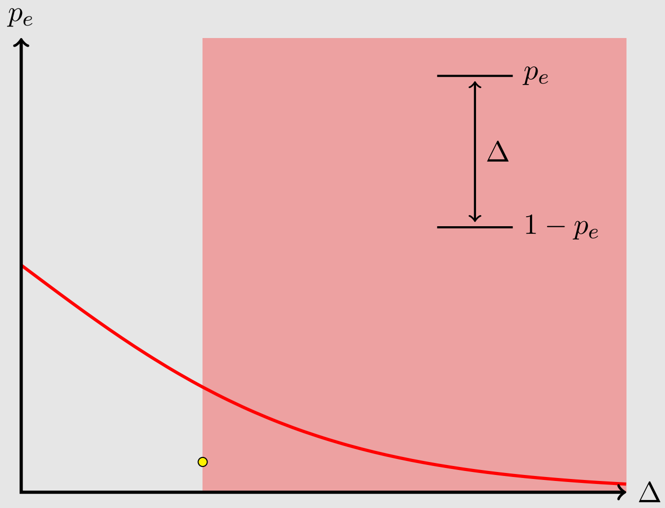

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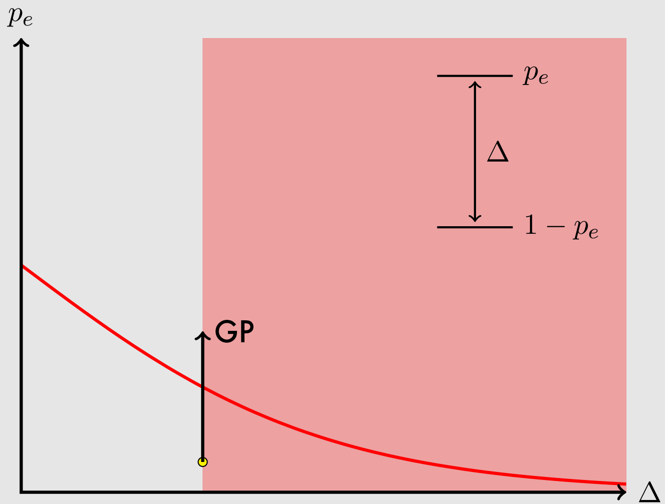

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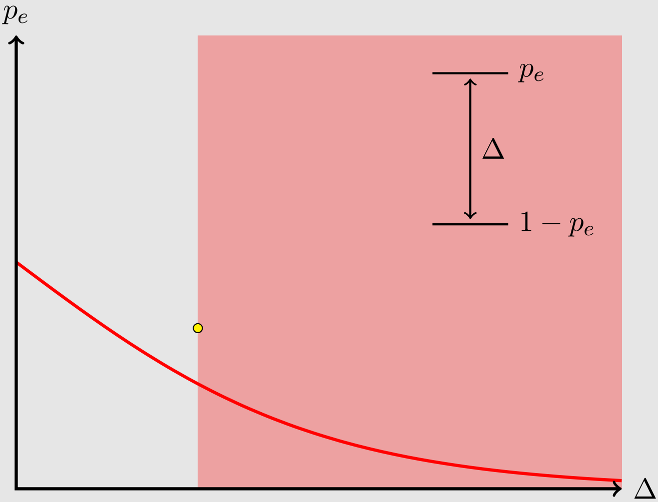

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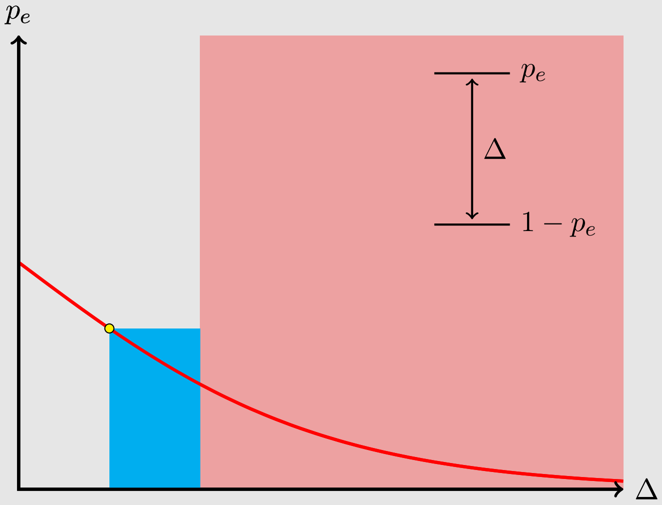

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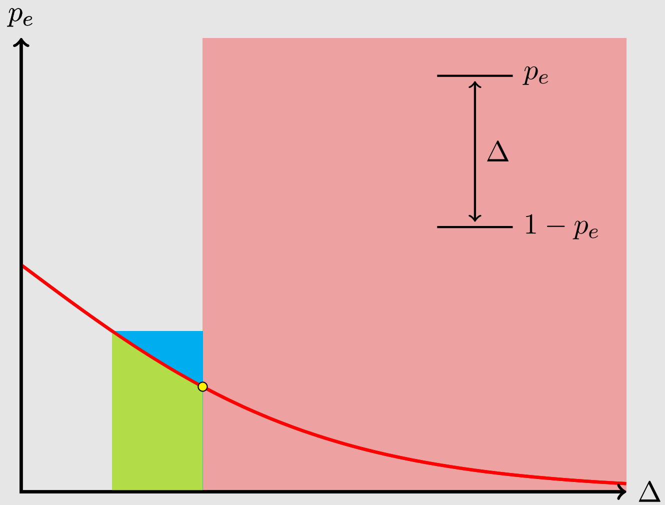

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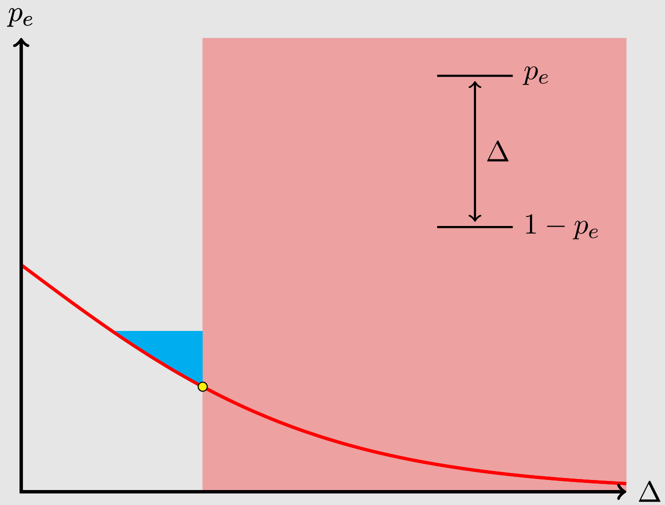

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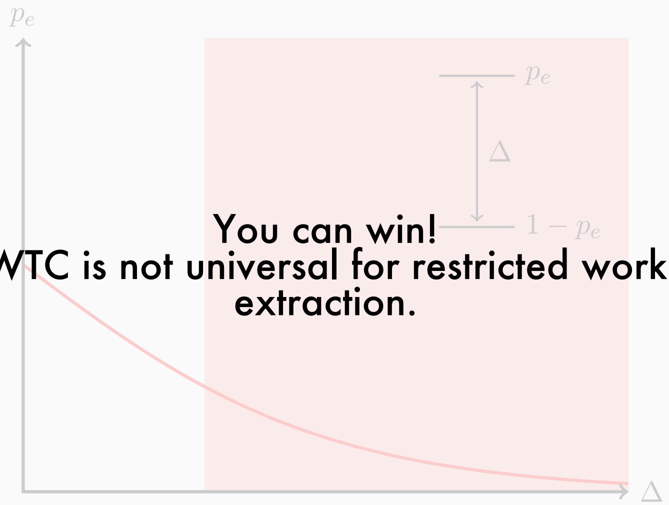

You can win!
WTC is not universal for restricted work extraction.

More general restrictions

Given an initial state (ρ_0, H_0) we restrict the possible Hamiltonians to a set $\mathcal{H}(H_0)$.

WTC under restrictions

Theorem (General work bound)

The work that can be extracted from a pair (ρ_0, H_0) using WTC is bounded as

$$\langle W \rangle^{\mathcal{H}}_{\mathsf{WTC}}(\rho_0, H_0) \leq \frac{1}{\beta} S(\rho_0 || \omega_{H_0}) - \inf_{\substack{H_t \in \mathcal{H}(H_0) \\ U \in \mathcal{U}[H_0]}} \frac{1}{\beta} S(U \rho_0 U^\dagger || \omega_{H_t}),$$

with $\mathcal{U}[H_0]$ being the unitary group generated by $\mathcal{H}(H_0)$. The bound can be saturated arbitrarily well.

Restriction on locality

$$\mathcal{H}_{\mathsf{loc}}(H_0) := \left\{ H_0 + \sum_i H_i \mid H_i \text{ local} \right\}$$

Models the situation of local control over an interacting Hamiltonian.

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Models the situation of local control over an interacting Hamiltonian.

Theorem (Non-universality of WTC)

Under the restriction to $\mathcal{H}_{\text{loc}}$, there exist initial states (ρ_0, H_0) such that no work can be extracted using WTC but work can be extracted using GP-maps.

Proof (Example)

- Consider two qubits with maximally mixed initial state Ω, Hamiltonian $Z \otimes Z$ and $\beta = 1$. W.l.o.g. assume all Hamiltonians are traceless.

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- General bound implies

$$\langle W \rangle(\Omega, Z \otimes Z) \leq \sup_{\substack{H_t \in \mathcal{H}_{\mathsf{loc}}(H_0) \\ U \in \mathcal{U}[H_0]}} \Big(S(\Omega || \omega_{Z \otimes Z}) - S(U \Omega U^\dagger || \omega_{H_t}) \Big) \, ,$$

Proof (Example)

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- General bound implies

$$
\begin{aligned}
\langle W \rangle(\Omega, Z \otimes Z) &\leq \sup_{\substack{H_t \in \mathcal{H}_{\mathsf{loc}}(H_0) \\ U \in \mathcal{U}[H_0]}} \Big(S(\Omega || \omega_{Z \otimes Z}) - S(U \Omega U^\dagger || \omega_{H_t}) \Big) , \\
&= \sup_{H_t \in \mathcal{H}_{\mathsf{loc}}(H_0)} \left(S(\Omega \, || \omega_{Z \otimes Z}) - S(\Omega \, || \omega_{H_t}) \right) ,
\end{aligned}
$$

Proof (Exa

$$\frac{1}{\beta}S(\rho||\omega_H) = F(\rho, H) - F(\omega_H, H)$$

with the **free energy**

$$F(\rho, H) = \mathrm{Tr}(\rho H) - \frac{1}{\beta}S(\rho).$$

- Cons … state Ω, Hami … all Hami
- Gene

$$\langle W \rangle (\ldots \Omega U^\dagger||\omega_{H_t}) \Big), \quad U \in \mathcal{U}[H_0]$$

$$= \sup_{H_t \in \mathcal{H}_{\mathrm{loc}}(H_0)} \left(S(\Omega\,||\omega_{Z\otimes Z}) - S(\Omega\,||\omega_{H_t}) \right),$$

$$\frac{1}{\beta} S(\rho || \omega_H) = F(\rho, H) - F(\omega_H, H)$$

with the **free energy**

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Proof (Exa

- Cons … state Ω, Hami … all Hami
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$$\langle W \rangle (\ldots \Omega U^\dagger || \omega_{H_t}) \Big),$$

$$\qquad {\scriptstyle H_t \in \mathcal{H}_{\mathsf{loc}}(H_0),\ U \in \mathcal{U}[H_0]}$$

$$= \sup_{H_t \in \mathcal{H}_{\mathsf{loc}}(H_0)} \left(S(\Omega \,||\omega_{Z\otimes Z}) - S(\Omega\,||\omega_{H_t}) \right),$$

$$= \sup_{H_t \in \mathcal{H}_{\mathsf{loc}}(H_0)} \left(F(\omega_{H_t}, H_t) - F(\omega_{Z\otimes Z}, Z \otimes Z) \right).$$

Proof (Exc

Peierls-Bogoliubov inequality:

$$F(\omega_{A+B}, A+B) \leq F(\omega_A, A) + \mathrm{tr}(\omega_A B)$$

Any traceless local Hamiltonian is orthogonal to $\omega_{Z\otimes Z}$. Hence $F(\omega_{H_t}, H_t) \leq F(\omega_{Z\otimes Z}, Z\otimes Z)$.

$$= \sup_{H_t \in \mathcal{H}_{\mathrm{loc}}(H_0)} \left(F(\omega_{H_t}, H_t) - F(\omega_{Z\otimes Z}, Z\otimes Z)\right).$$

Proof (Exa

- Cons … state Ω, Hami … all Hami
- Gene

$$\langle W\rangle(\ldots \Omega U^\dagger||\omega_{H_t})\Big)\,,$$

$$= \sup_{H_t\in\mathcal{H}_{\mathsf{loc}}(H_0)} \ldots ||\omega_{Z\otimes Z}) - S(\ldots||\omega_{H_t}))\,,$$

Peierls-Bogoliubov inequality:

$$F(\omega_{A+B}, A+B) \leq F(\omega_A, A) + \mathrm{tr}(\omega_A B)$$

Any traceless local Hamiltonian is orthogonal to $\omega_{Z\otimes Z}$. Hence $F(\omega_{H_t}, H_t) \leq F(\omega_{Z\otimes Z}, Z\otimes Z)$.

$$= \sup_{H_t\in\mathcal{H}_{\mathsf{loc}}(H_0)} \left(F(\omega_{H_t}, H_t) - F(\omega_{Z\otimes Z}, Z\otimes Z)\right).$$

$$\leq \left(F(\omega_{Z\otimes Z}, Z\otimes Z) - F(\omega_{Z\otimes Z}, Z\otimes Z)\right) \leq 0$$

Proof (Example)

Thermomajorization [5, 6, 7] implies that there exists a thermal operation $\mathcal{G}$ such that

$$\mathcal{G}(\Omega) = \omega_{Z\otimes Z + tZ\otimes \mathbb{1}},$$

for $|t| < 0.46$.

Since $Z \otimes Z + tZ \otimes \mathbb{1} \in \mathcal{H}_{\text{loc}}(Z \otimes Z)$, we can extract

$$\langle W \rangle = S\left(\omega_{Z\otimes Z+tZ\otimes \mathbb{1}} || \omega_{Z\otimes Z}\right) > 0.$$

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[5] E. Ruch, R. Schranner, and T. H. Seligman, J. Chem. Phys. 69, 386 (1978).
[6] H. D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, Int. J. Th. Phys. 39, 2717 (2000).
[7] M. Horodecki and J. Oppenheim, Nature Comm. 4, 2059 (2013).

Summarizing the effect of restrictions

Restrictions on the Hamiltonians

$\downarrow$

Passive states for WTC

$\downarrow$

GP-maps can activate passive states

$\downarrow$

WTC ceases to be universal

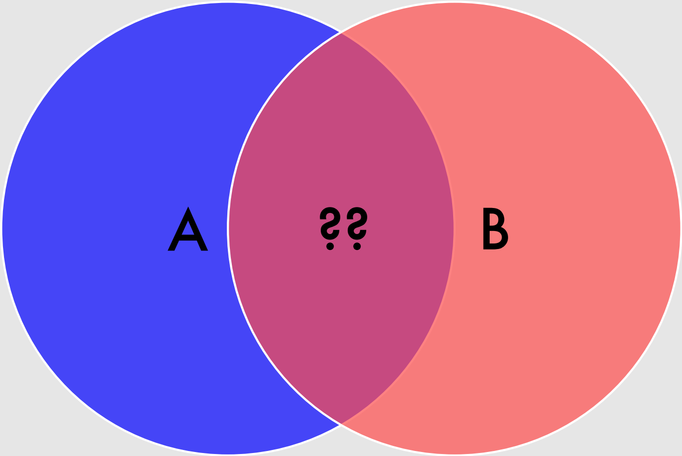
Outlook

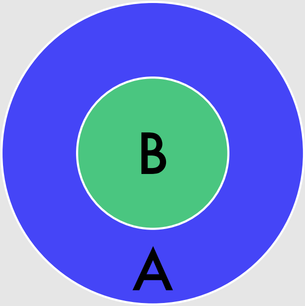
Resource theory (roughly)

A class of operations that is closed under composition and contains the identity together with a distinction between free and costly states and operations [9].

[9] B. Coecke, T. Fritz, R. W. Spekkens, arXiv:1409.5531 (2014).

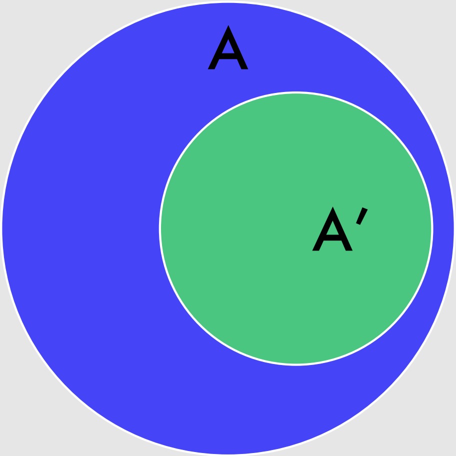
What happens if we “combine” two resource theories?



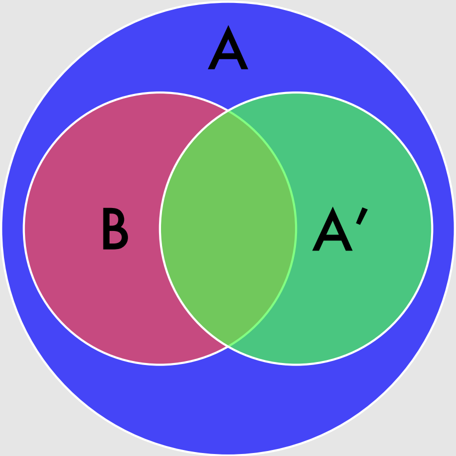


We call a sub-theory $B \subset A$ universal for a specific task if the task can already be achieved optimally only with operations from B.

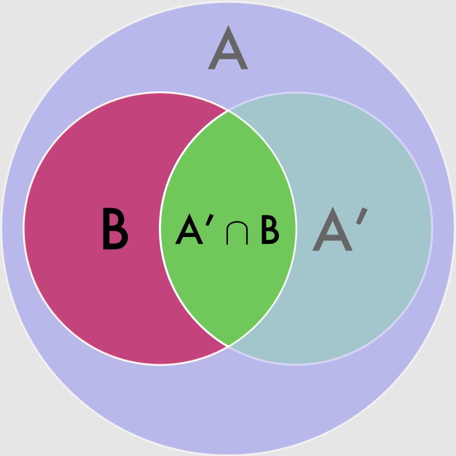
Suppose $A' \subset A$ is universal for some task.



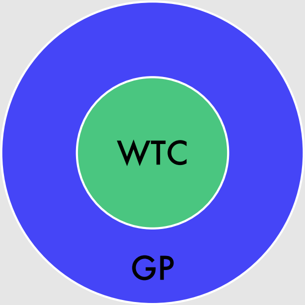
Suppose $A' \subset A$ is universal for some task.



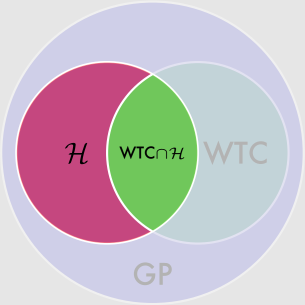
Suppose $A' \subset A$ is universal for some task.



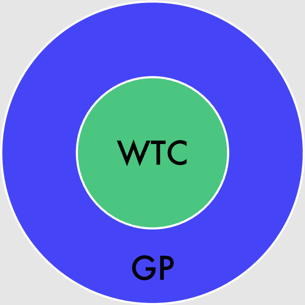
Is $A' \cap B$ still universal in B?



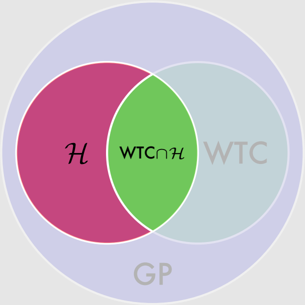
WTC is universal for unrestricted work extraction



WTC is not universal for restricted work extraction

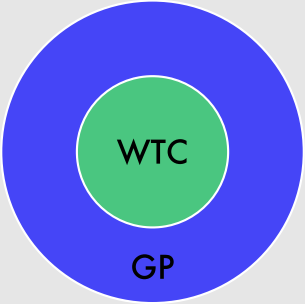


WTC is universal for unrestricted work extraction

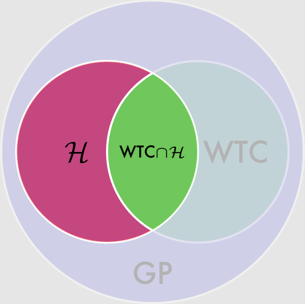


WTC is not universal for restricted work extraction

Open questions



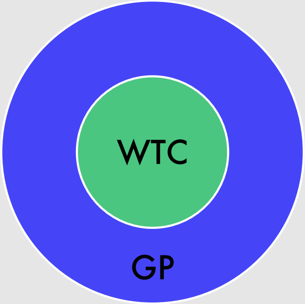
WTC is universal for unrestricted work extraction



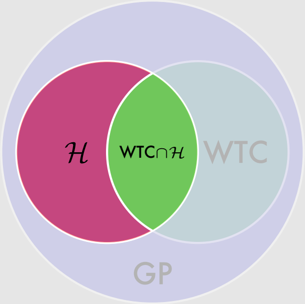
WTC is not universal for restricted work extraction

Open questions

- Can we find similar situations outside of thermodynamics?



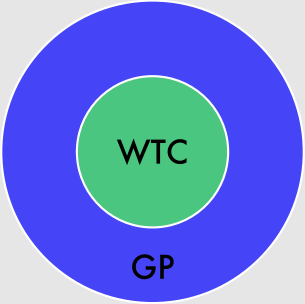
WTC is universal for unrestricted work extraction



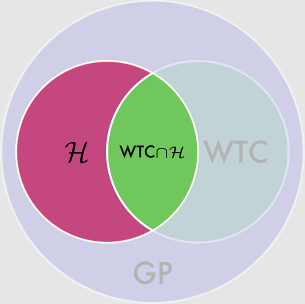
WTC is not universal for restricted work extraction

Open questions

- Can we find similar situations outside of thermodynamics?
- Other operationally meaningful types of restrictions?



WTC is universal for unrestricted work extraction



WTC is not universal for restricted work extraction

Open questions

- Can we find similar situations outside of thermodynamics?
- Other operationally meaningful types of restrictions?
- What about single-shot statistical mechanics / different notions of work?

Thanks for listening!
For details see arXiv:1411.3754.

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[8] P. Faist, J. Oppenheim, and R. Renner, arXiv:1406.3618 (2014).
[9] B. Coecke, T. Fritz, R. W. Spekkens, arXiv:1409.5531 (2014).