



福 2013

Entanglement Polytopes

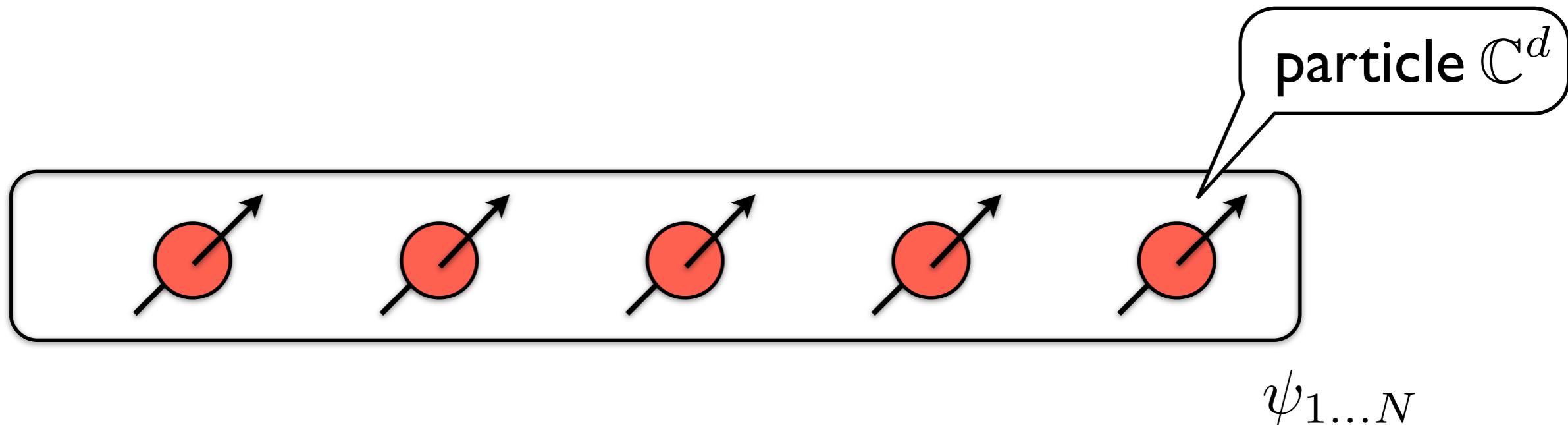
Detecting Genuine Multi-Particle Entanglement
by Single-Particle Measurements

Michael Walter

joint work with Matthias Christandl, Brent Doran
(ETH Zürich), and David Gross (Univ. Freiburg)

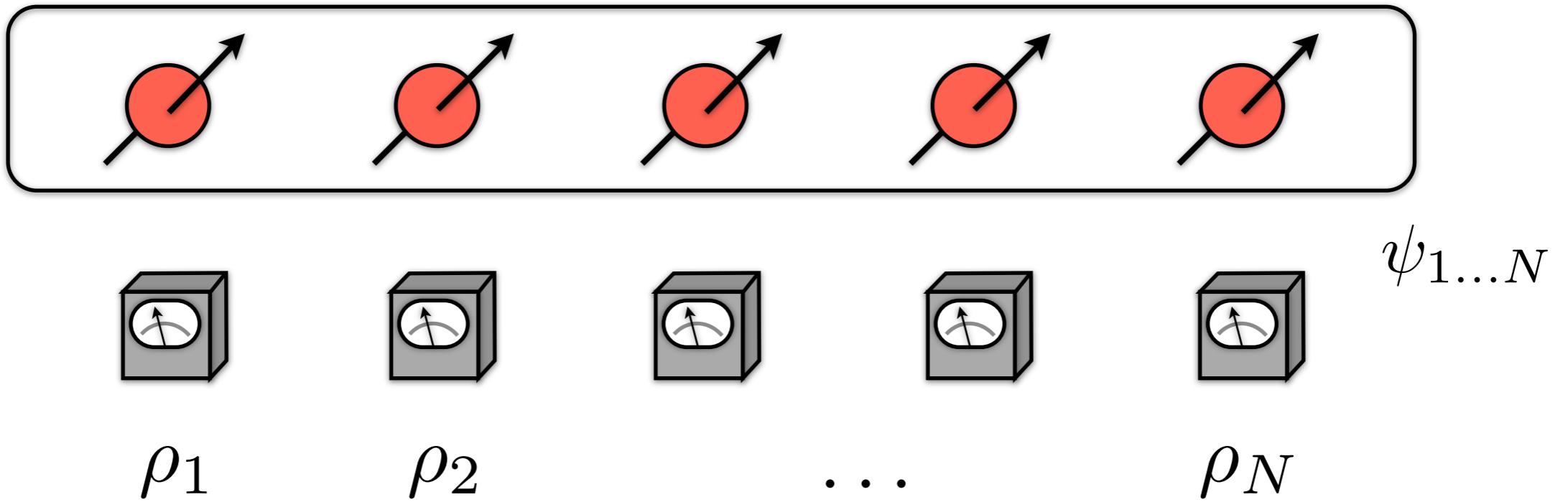


Multi-Particle Entanglement



How entangled is a given multi-particle quantum state $\psi_{1\dots N}$ (prepared in the laboratory)?

Multi-Particle Entanglement via Single-Particle Measurements



*What can be said about the multi-particle entanglement
of a pure state given its one-body density matrices only?*

Multi-Particle Entanglement

Operational approach: equivalent entanglement if related by operations which cannot “create entanglement”



Multi-Particle Entanglement

Operational approach: equivalent entanglement if related by operations which cannot “create entanglement”



Stochastic Local Operations and Classical Comm.:

- non-zero success probability
- convenient mathematical characterization:

$$\psi'_{1\dots N} = (A_1 \otimes \dots \otimes A_N) \psi_{1\dots N}$$

Dür, Vidal & Cirac (2000)

invertible local operators

Multi-Particle Entanglement

Entanglement class of a pure state $\psi_{1\dots N}$:

$$\mathcal{C}_\psi = \left\{ \psi' \xleftrightarrow{\text{SLOCC}} \psi \right\} = G \cdot |\psi\rangle\langle\psi|$$

$$G = SL(d) \times \dots \times SL(d)$$

$$\subseteq \mathbb{P}(\mathcal{H})$$

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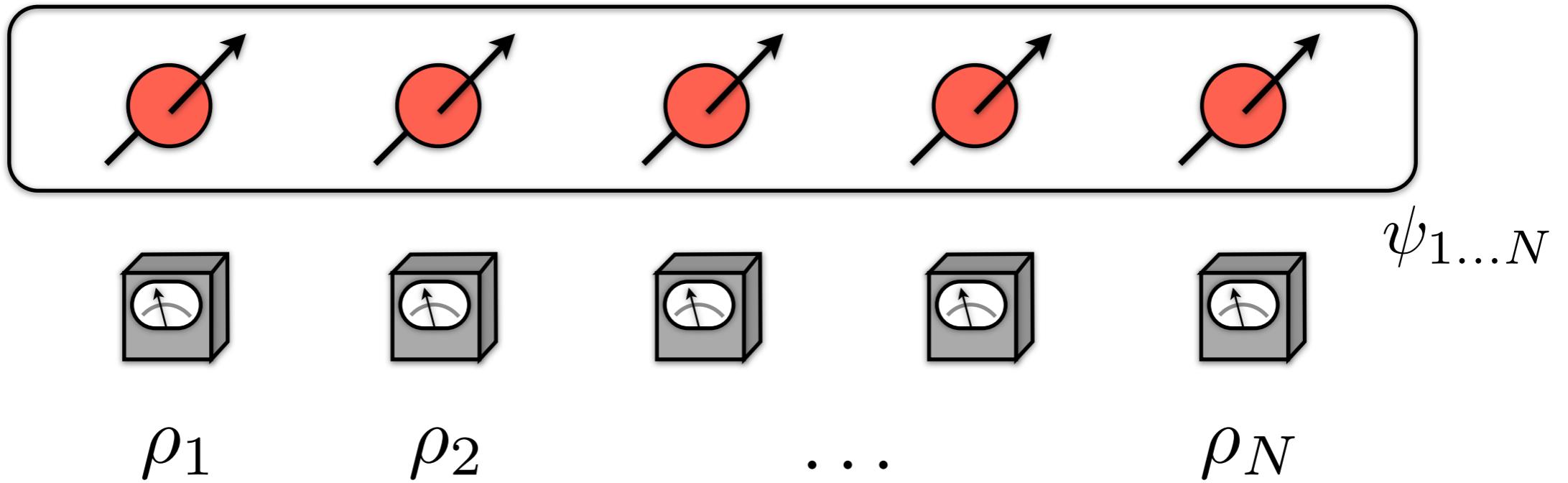
$$\mathcal{C}_\psi = \left\{ \psi' \xleftrightarrow{\text{SLOCC}} \psi \right\} = G \cdot |\psi\rangle\langle\psi| \subseteq \mathbb{P}(\mathcal{H})$$

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Three Qubits: six classes (GHZ, W, bi-separable, product)

Generically: infinitely many classes, labeled by $\exp(N)$
many continuous parameters

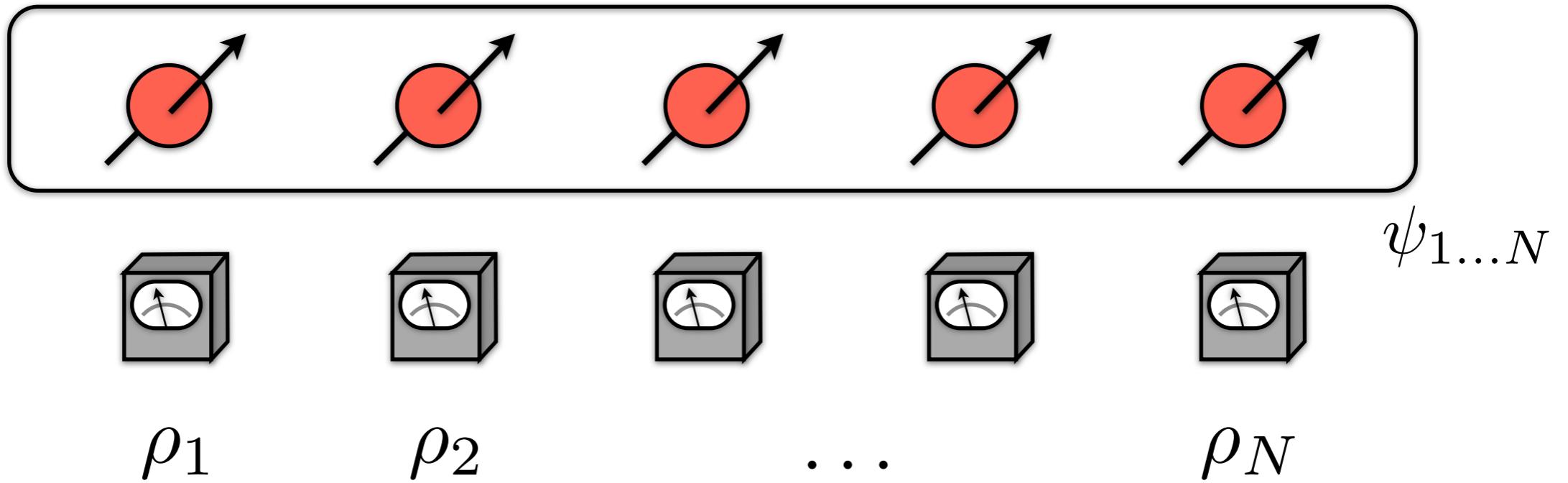
Single-Particle View of Multi-Particle Entanglement



What can be said about the entanglement class of a pure state given its one-body density matrices only?

What are the possible ρ_1, \dots, ρ_N for states in a given entanglement class?

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only depends on local eigenvalues

Entanglement Polytopes



$$\Delta_{\mathcal{C}} = \{ \vec{\lambda} = (\vec{\lambda}_1, \dots, \vec{\lambda}_N) \text{ for } \psi \in \overline{\mathcal{C}} \}$$

eigenvalues of
 ρ_1, \dots, ρ_N

Entanglement Polytopes



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entanglement
polytope

Our Main Result:

- convex polytope!
 - finite hierarchy
 - algorithm to compute using computational invariant theory (difficult)
- using results from Brion (1987),
Kempf & Ness (1979)
algebraic geometry / GIT

Entanglement Polytopes



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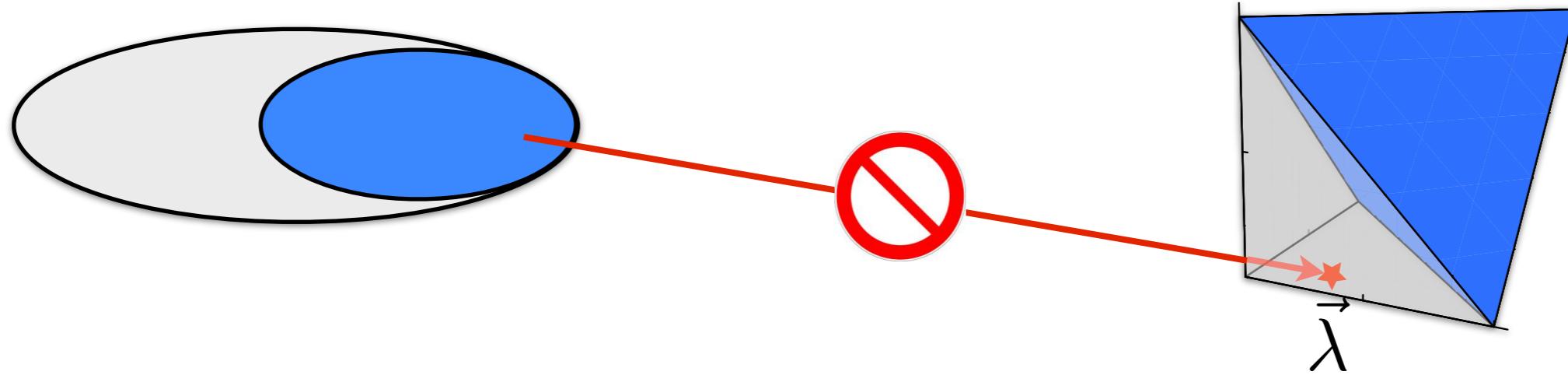
- convex polytope!
- finite hierarchy
- algorithm to compute using computational invariant theory

cf. Quantum Marginal Problem

using results from Brion (1987),
Kempf & Ness (1979)
algebraic geometry / GIT

Christandl-Mitchison (2004)
Klyachko (2004)
Daftuar-Hayden (2004)

Entanglement Criterion

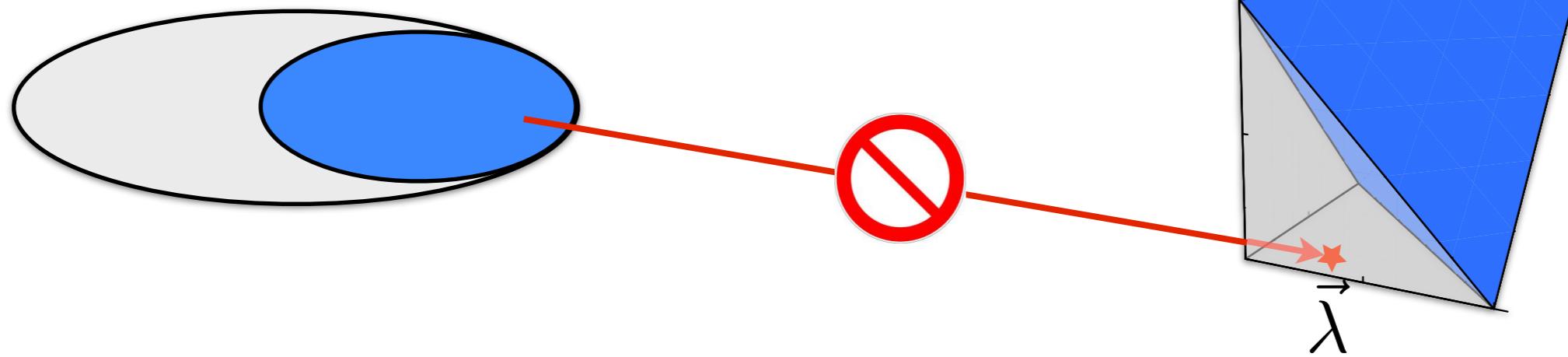


$$\vec{\lambda} \notin \Delta_{\mathcal{C}} \implies \psi \notin \mathcal{C}$$

- efficient, requires only linearly many measurements
- robust against experimental noise ($\psi \approx$ pure)

Entanglement Criterion

“Bell inequalities”



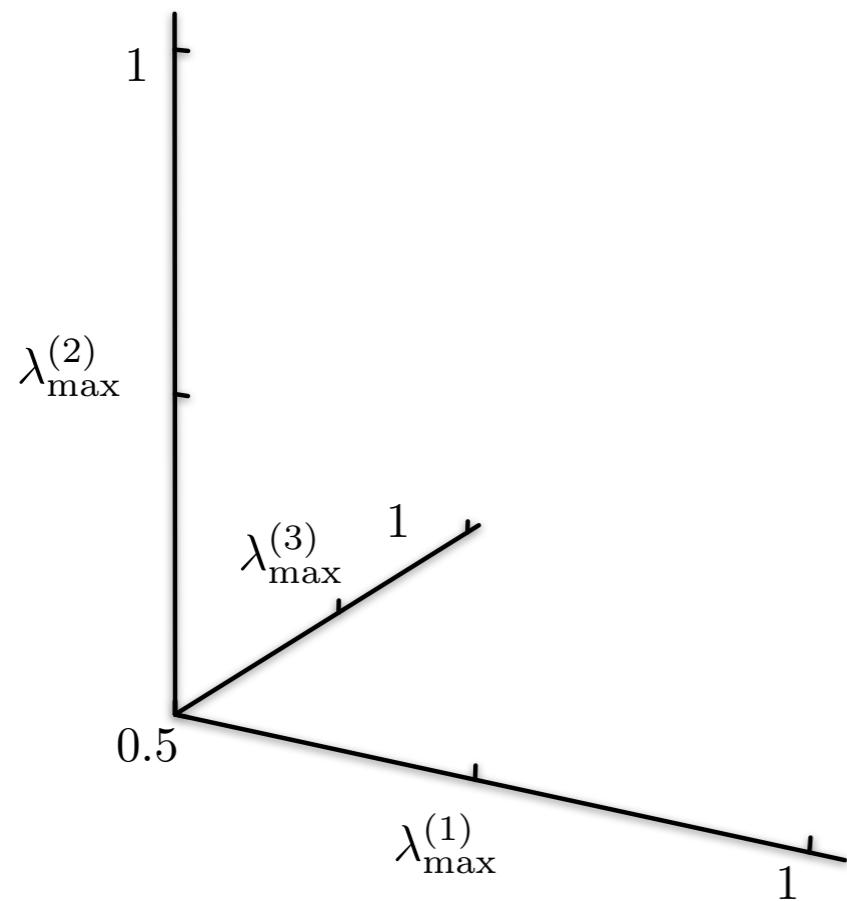
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violation of
“Bell inequality”

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Three Qubits

$$N = 3$$
$$\mathcal{H} = \mathbb{C}^2$$



Entanglement classes:

$$|GHZ\rangle = |000\rangle + |111\rangle$$

$$|W\rangle = |100\rangle + |010\rangle + |001\rangle$$

$$|B1\rangle = |0\rangle \otimes (|00\rangle + |11\rangle),$$

$$|B2\rangle, |B3\rangle$$

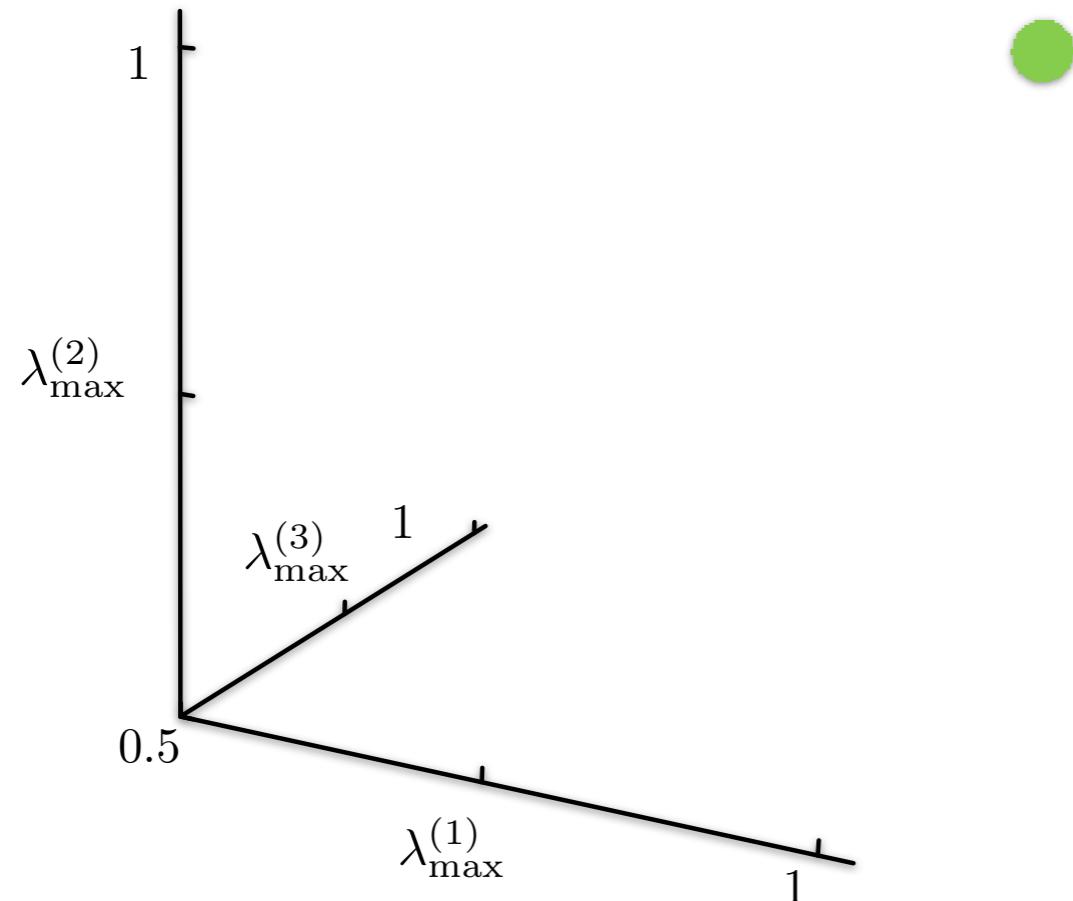
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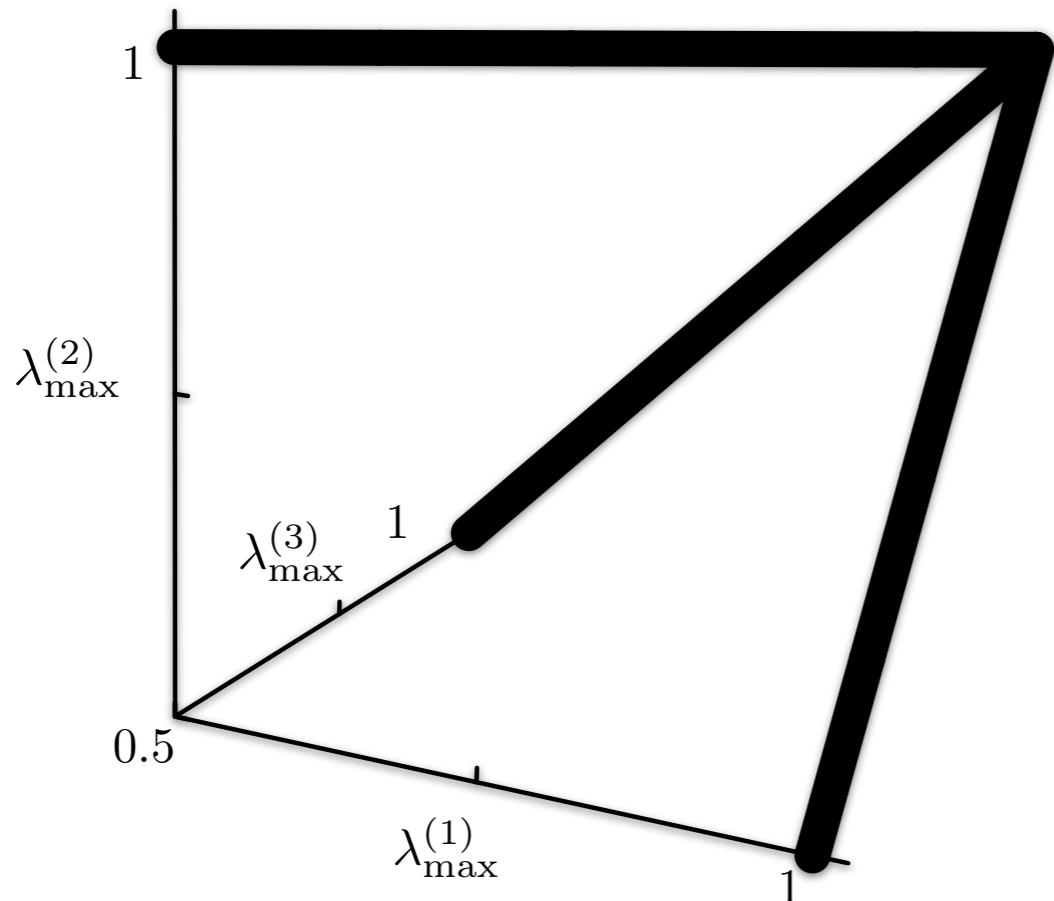
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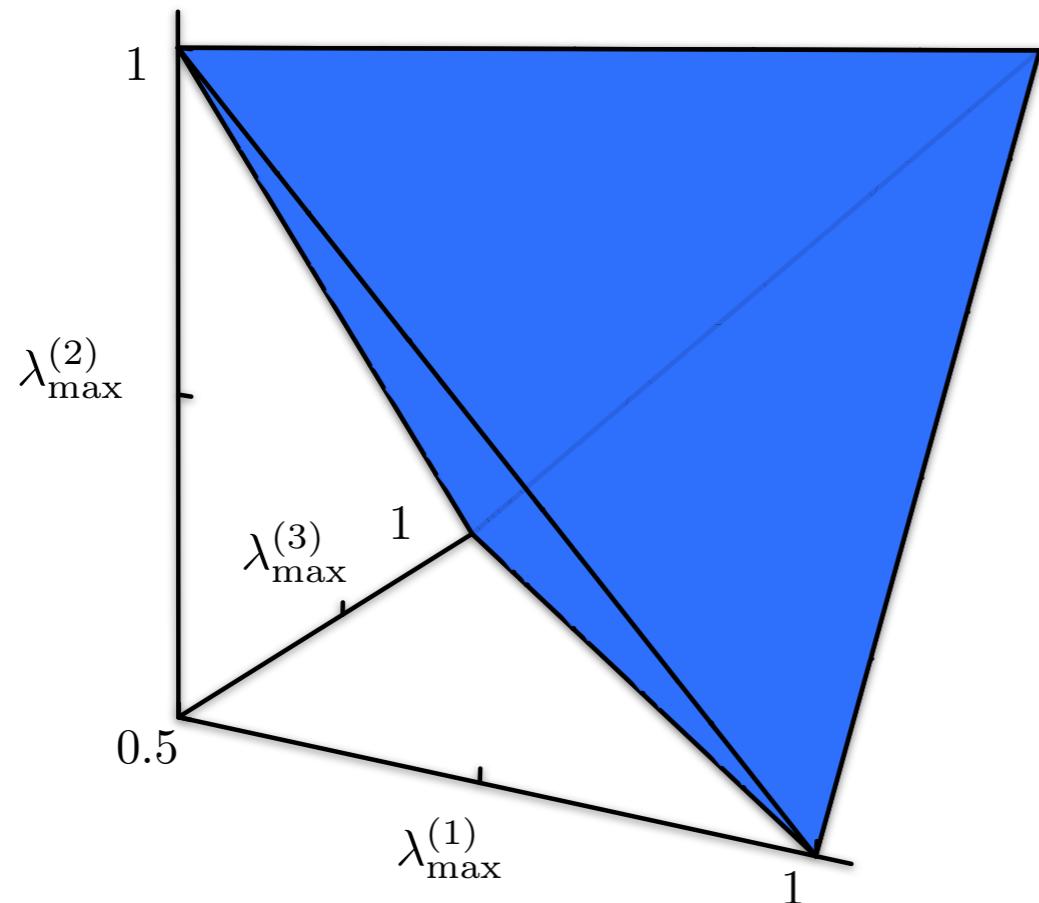
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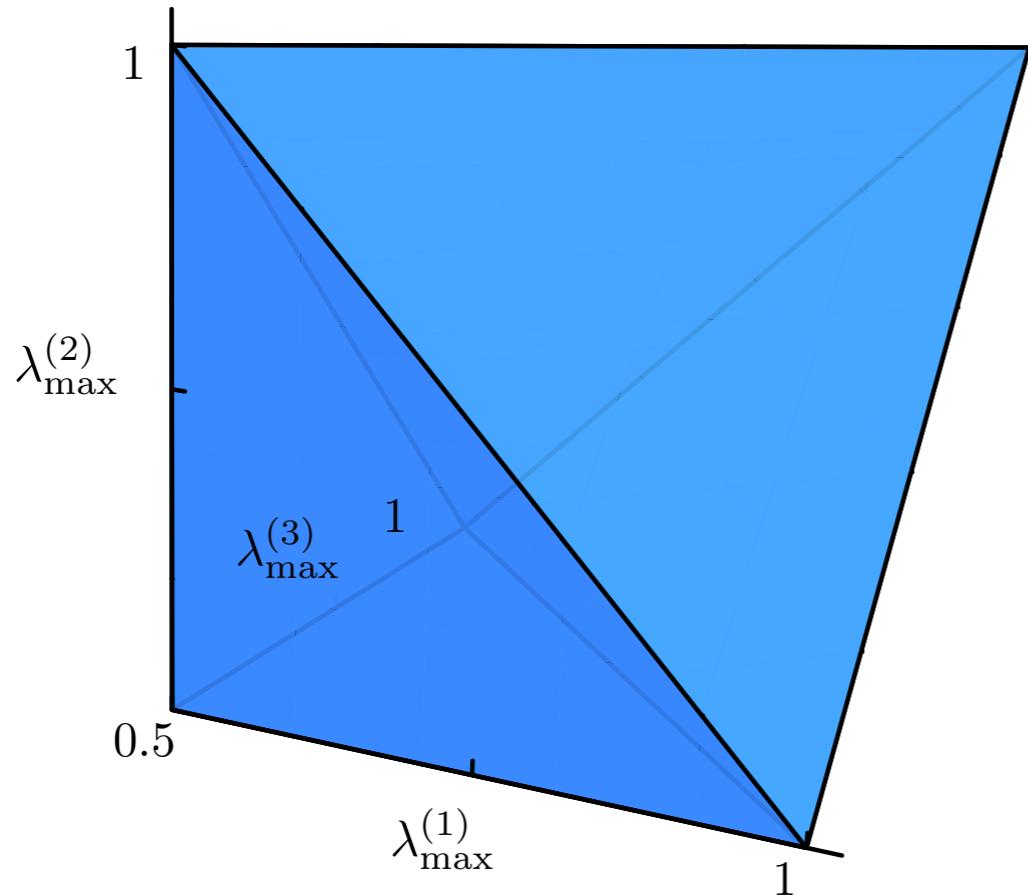
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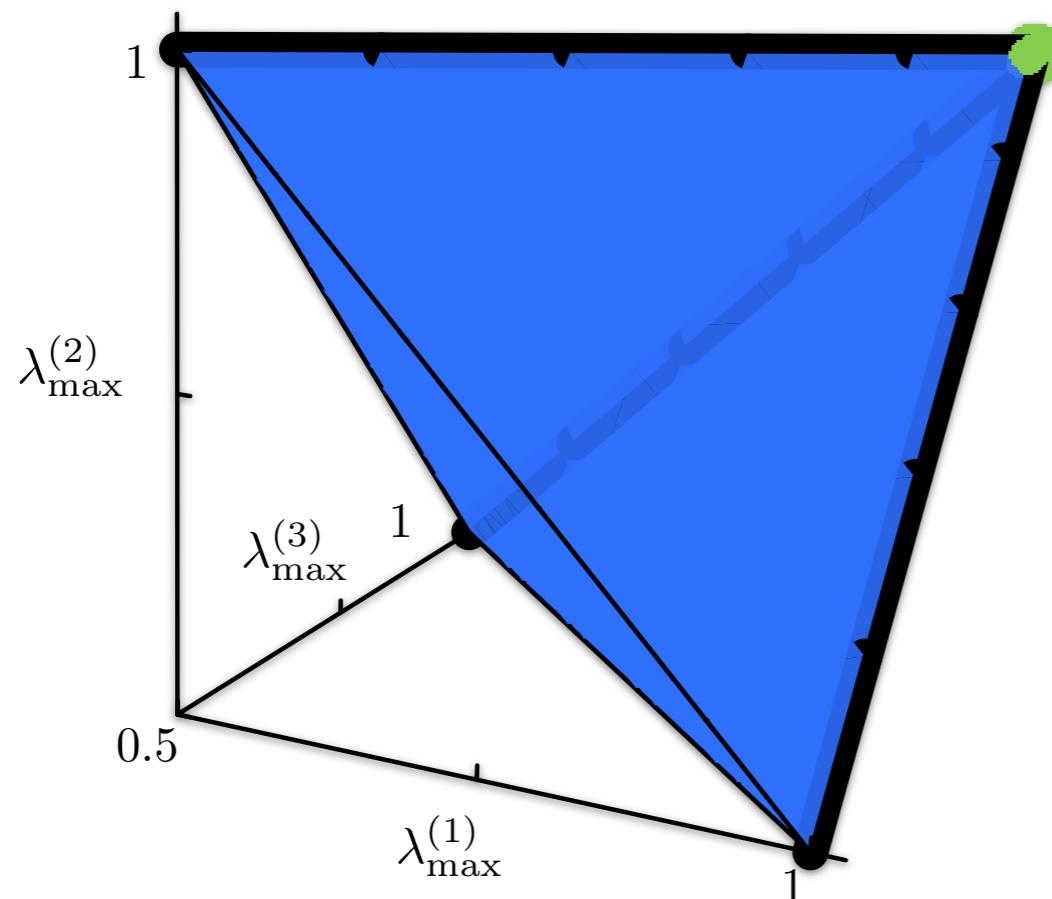
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Lower pyramid is
witness for GHZ class!

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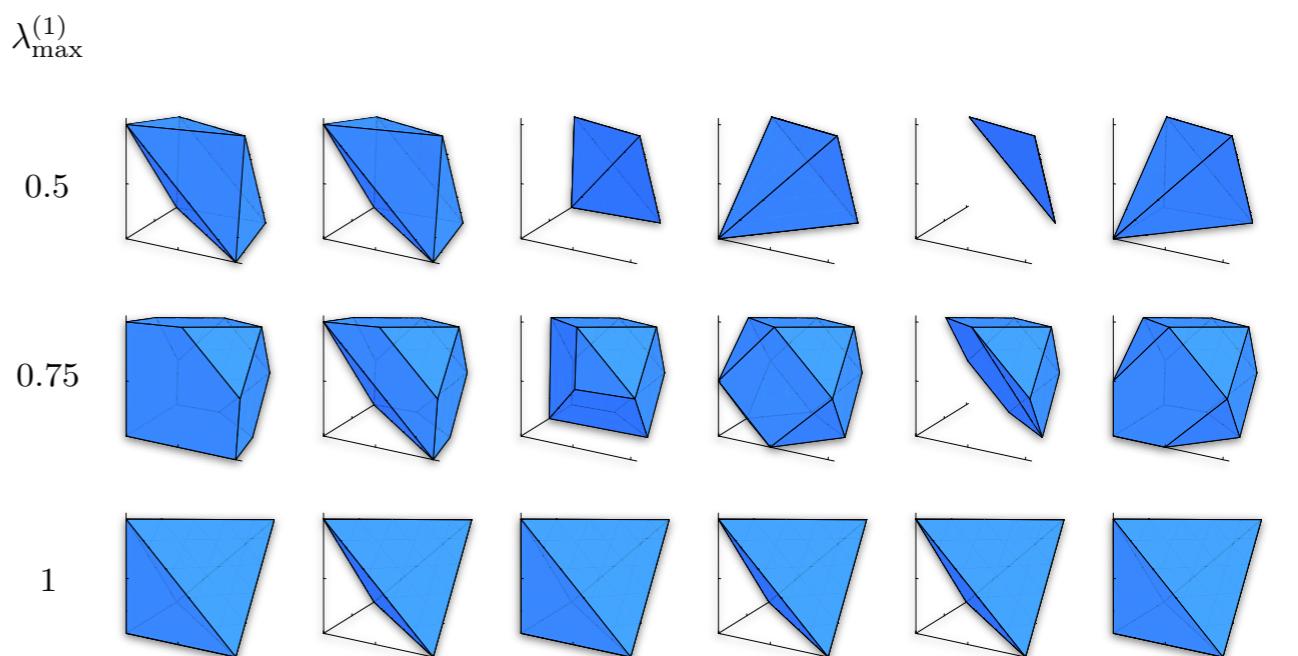
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Further Examples

- Four Qubits: six non-trivial entanglement polytopes (3d sections)



- Bosonic qubit systems, fermionic system, ...

$$\begin{matrix} \text{Sym}^N \mathbb{C}^2 \\ \wedge^3 \mathbb{C}^6 \end{matrix}$$

<http://www.itp.phys.ethz.ch/people/waltemic/polytopes>

- Genuine multi-particle entanglement:

$$\Delta \supseteq \bigcup_{\text{bipartition } S:S^c} \Delta_S \times \Delta_{S^c}$$

Invariant Theory

$$\begin{aligned}\mathcal{H} &= \mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d \\ G &= SL(d) \times \dots \times SL(d)\end{aligned}$$

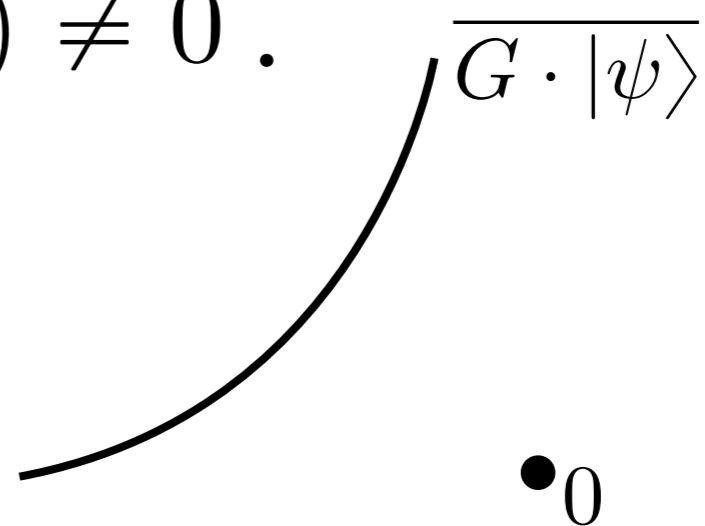
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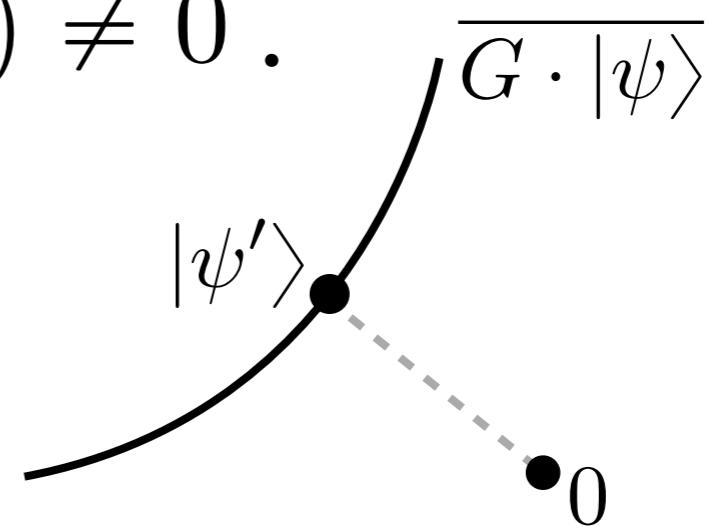
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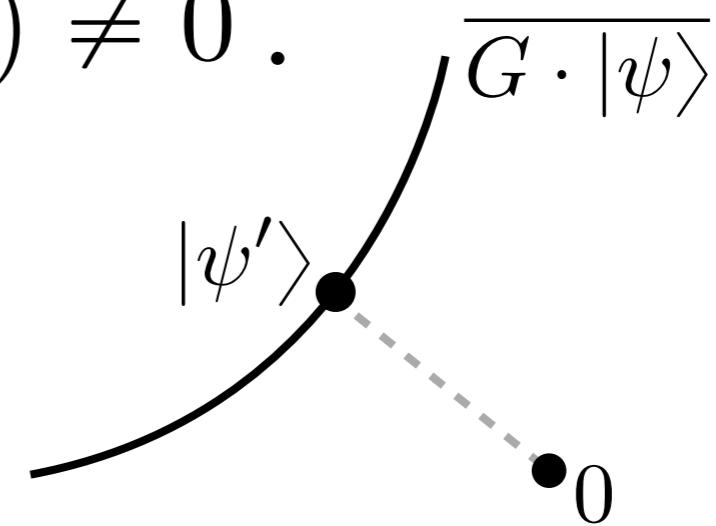


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$$0 = \frac{d}{dt} \Big|_{t=0} \|e^{tX} \cdot |\psi'\rangle\|^2 = 2\langle \psi' | X | \psi' \rangle$$

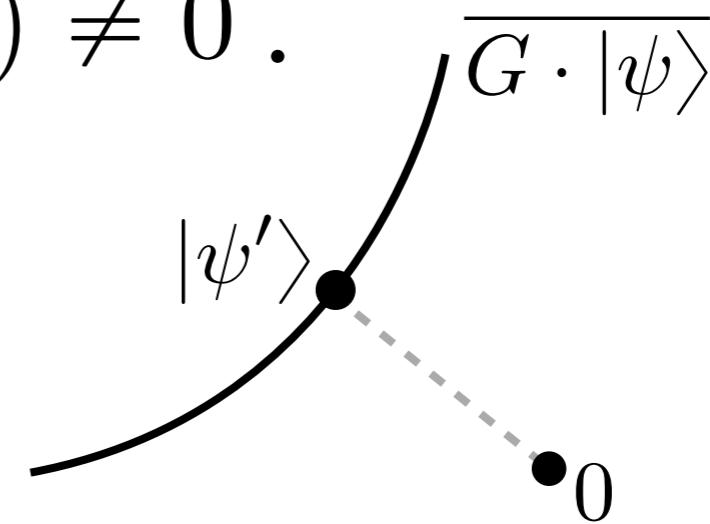
for all traceless local observables X .

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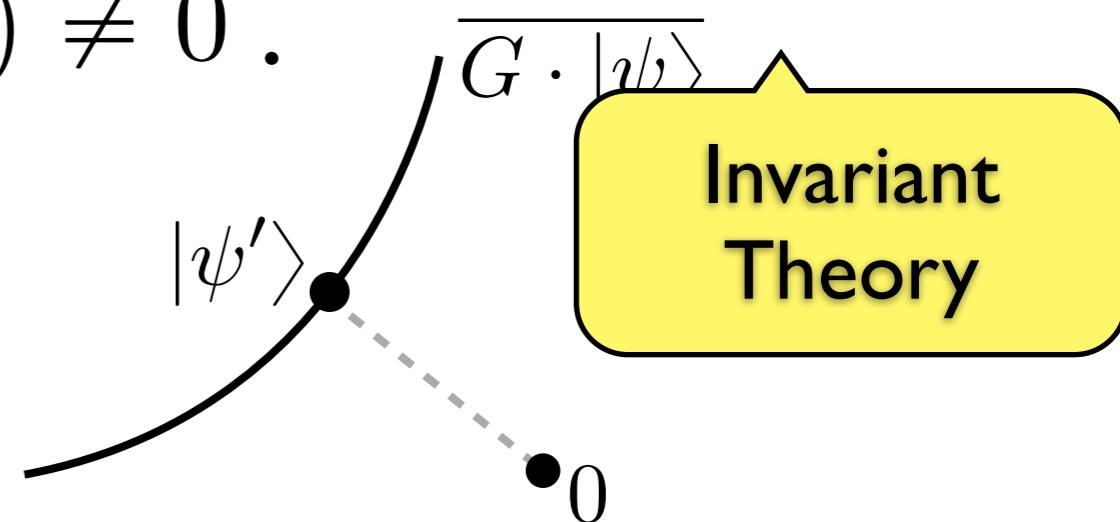
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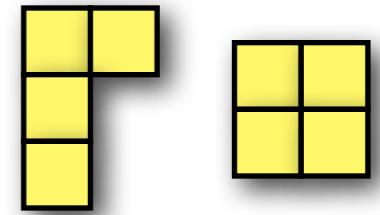
Kempf & Ness (1979)
Klyachko (2004)

Local eigenvalues

Entanglement Polytopes and Covariants

Covariant: G-equivariant homogeneous polynomial function

$$\Phi: \mathcal{H} \rightarrow V_\lambda$$

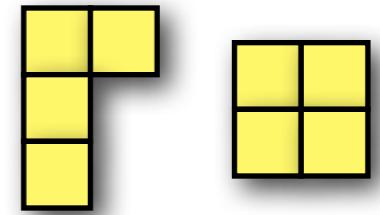


Irreducible G-representation
with highest weight λ

Entanglement Polytopes and Covariants

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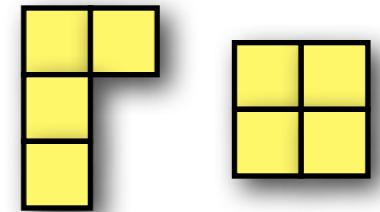


Choose a finite set of generators Φ_k with highest weight λ_k and degree d_k .

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Theorem: The entanglement polytope of an entanglement class \mathcal{C}_ψ is given by

proof via
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computational invariant theory

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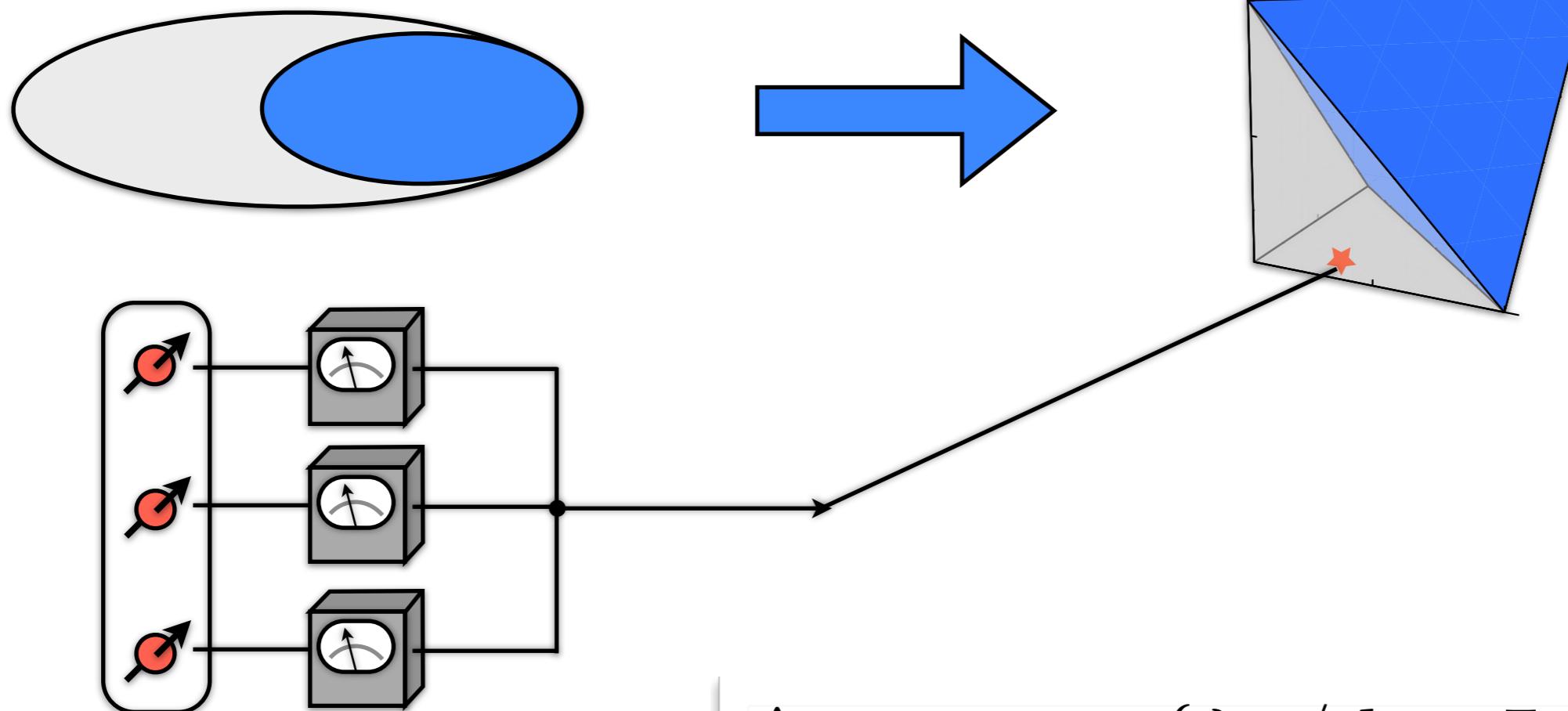
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Summary

arXiv:1208.0365

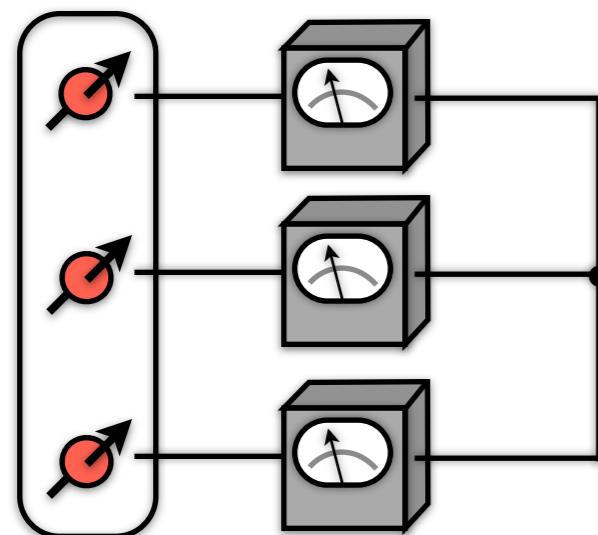
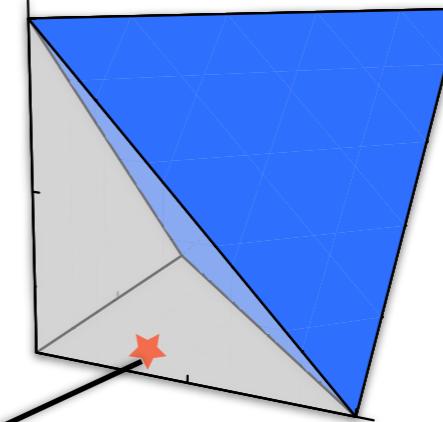
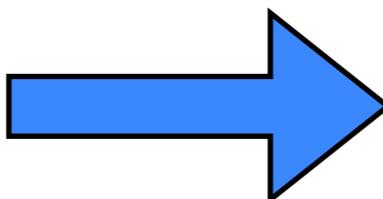
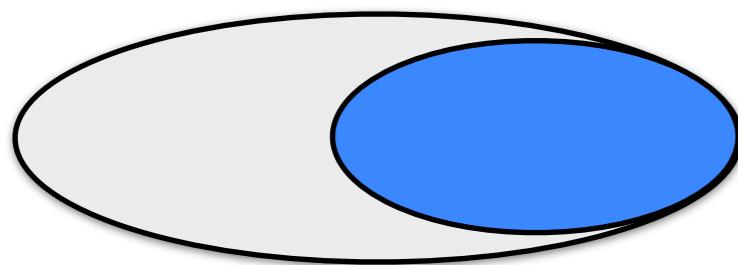


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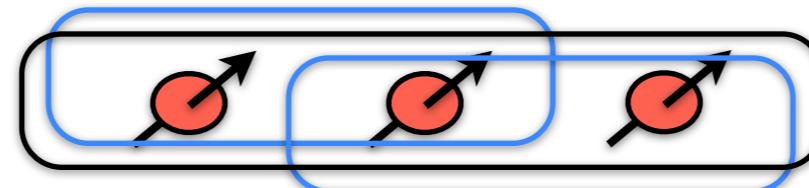
Thank you!

arXiv:1208.0365



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See also Burak's talk on “Recoupling Coefficients and Quantum Entropies”!



Thursday 9:55