A generalized Grothendieck inequality and entanglement in XOR games

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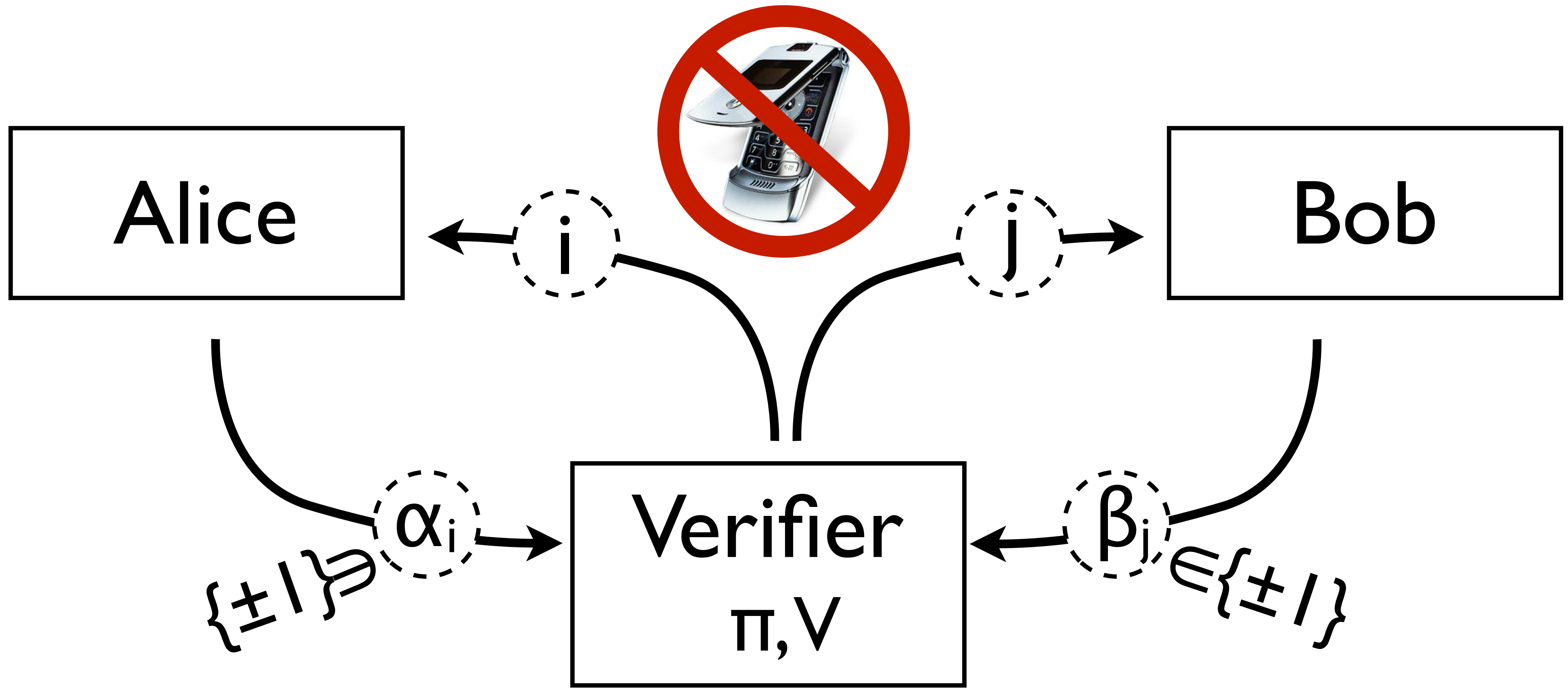
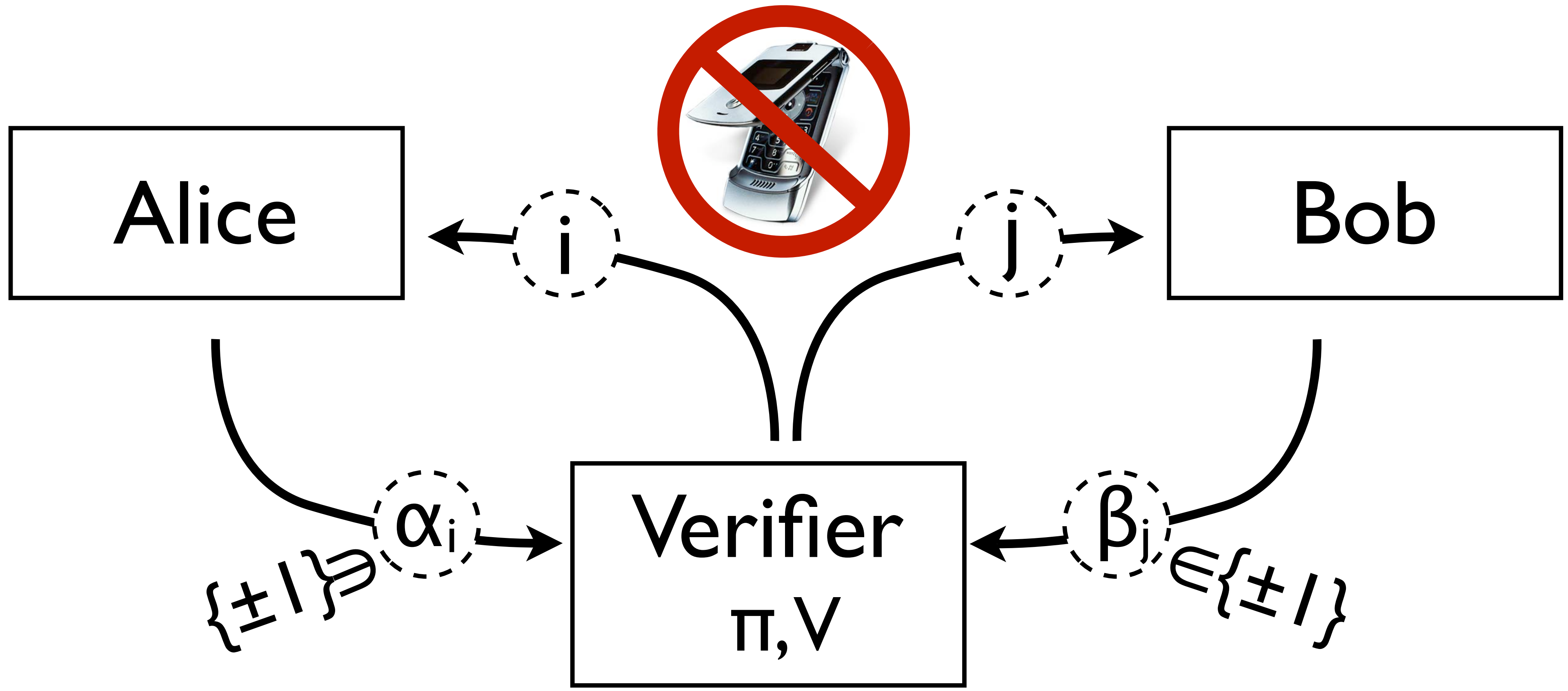
XOR games

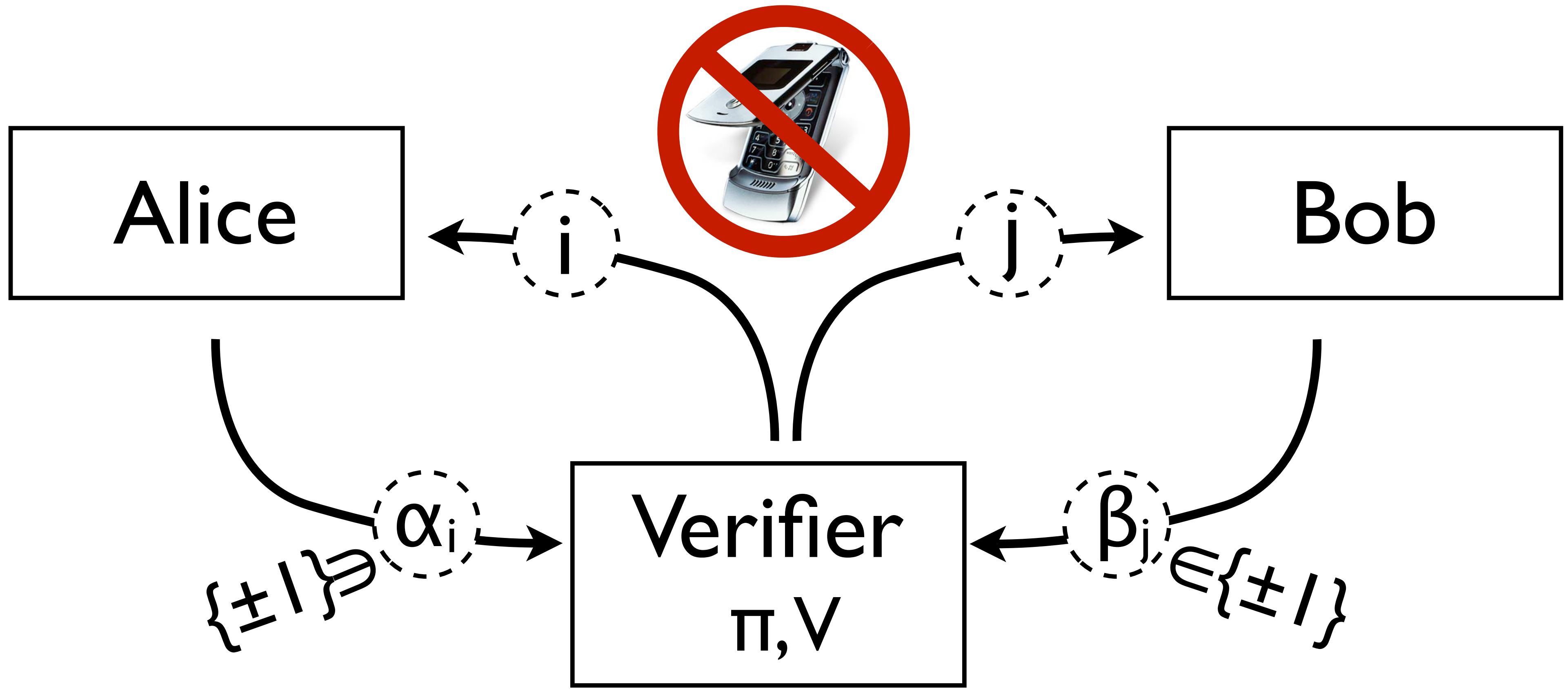


XOR games



- The verifier picks questions $i,j \in \{1,\ldots,r\}$ according to probability distribution $\pi(i,j)$

XOR games



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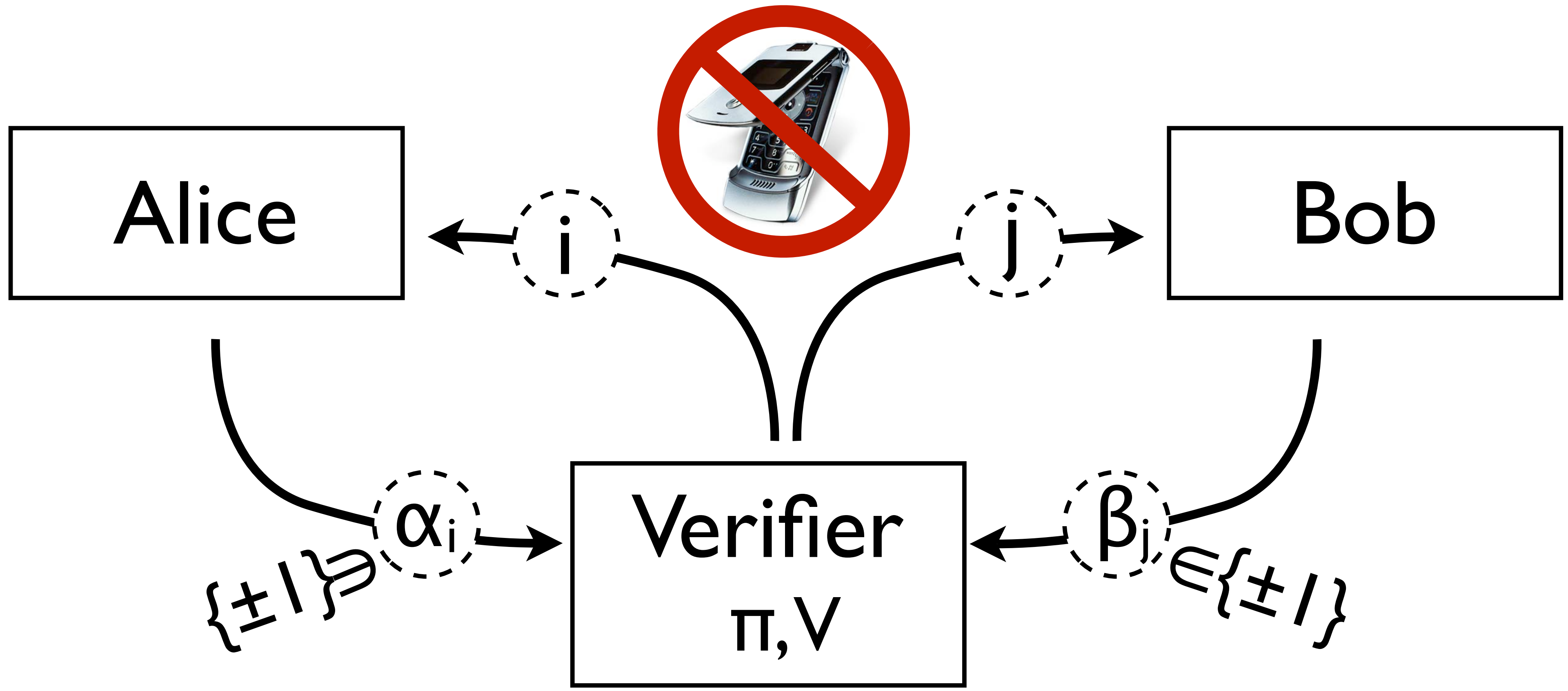

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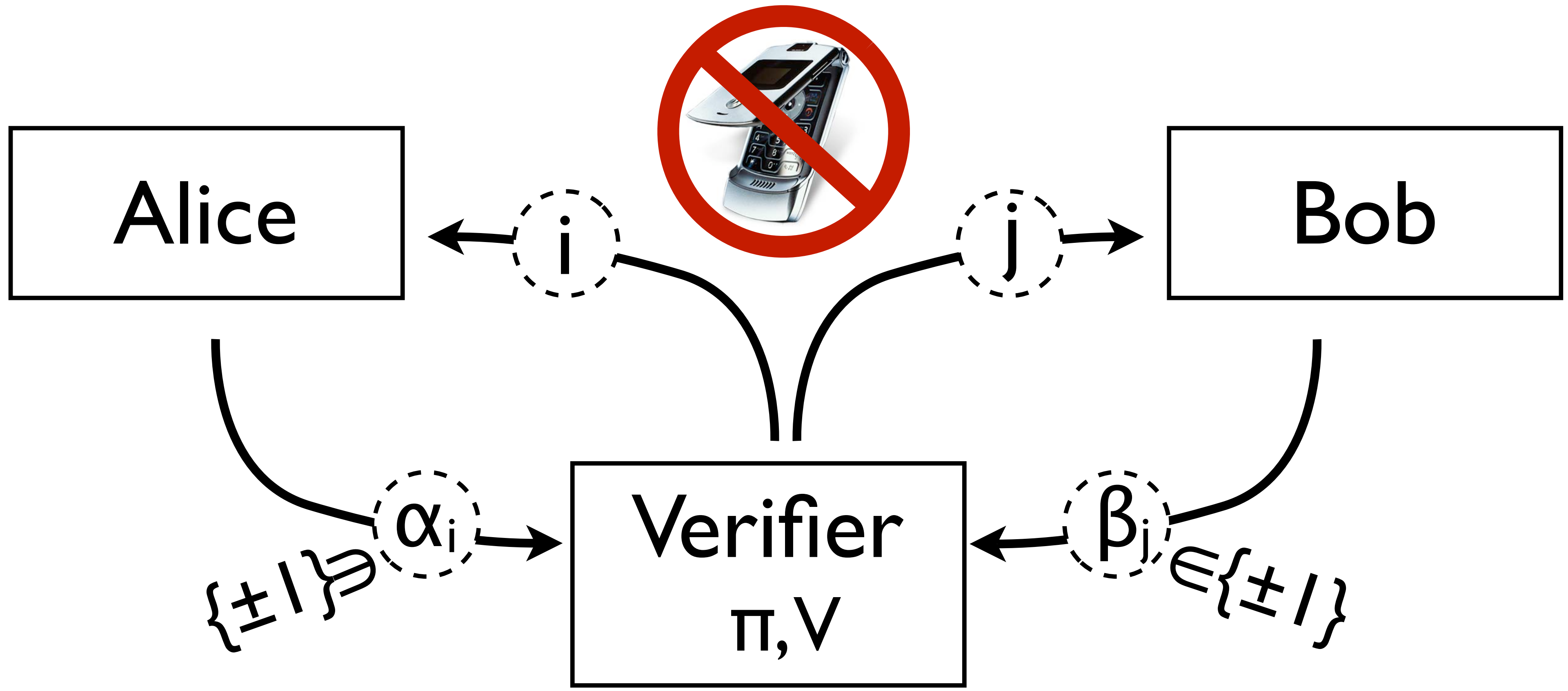
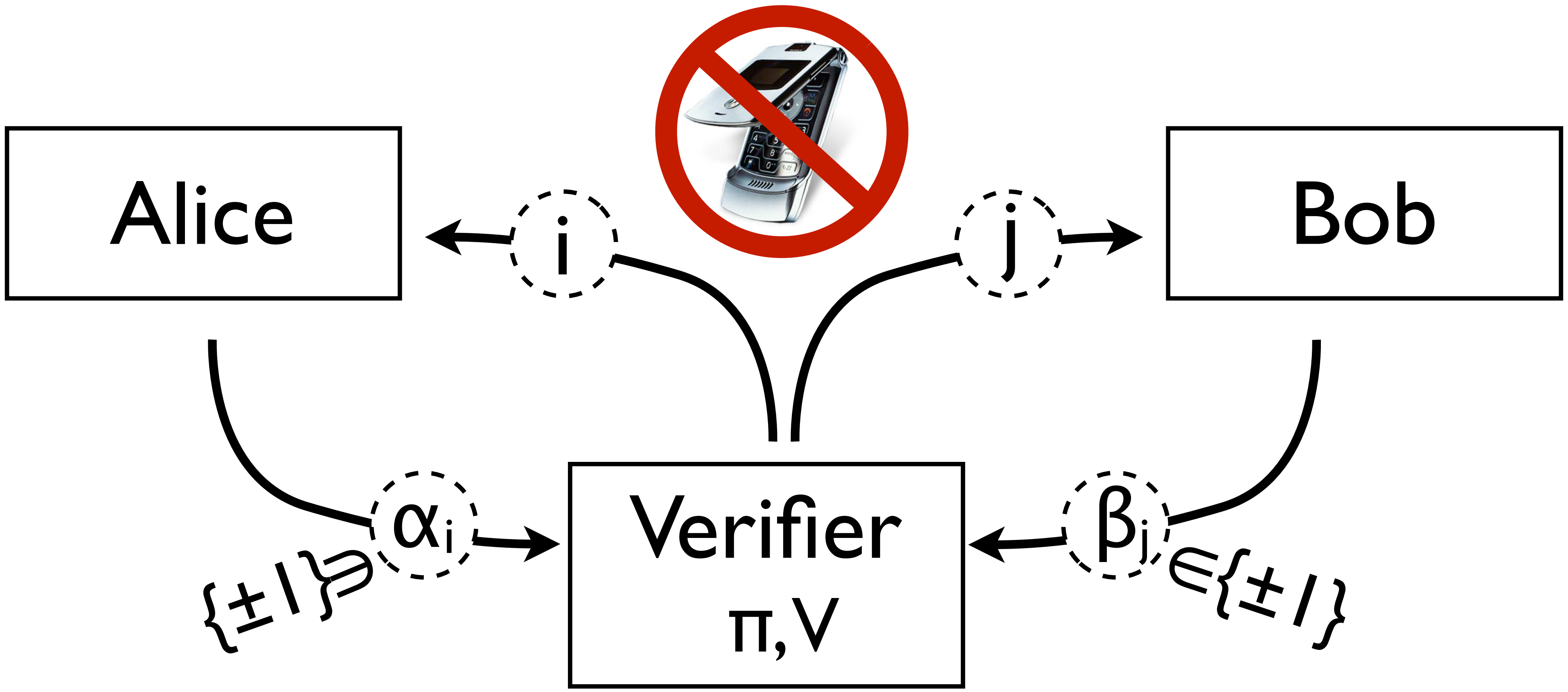


- The verifier picks questions $i,j \in \{1,\ldots,r\}$ according to probability distribution $\pi(i,j)$
- He computes function $V(i,j) \in \{\pm 1\}$, sends i to Alice and j to Bob
- Alice answers $\alpha_i \in \{\pm 1\}$ and Bob $\beta_j \in \{\pm 1\}$
- Alice and Bob win if $\alpha_i \beta_j = V(i,j) \quad \Leftrightarrow \quad V(i,j) \cdot (\alpha_i \beta_j) = +1$

XOR games



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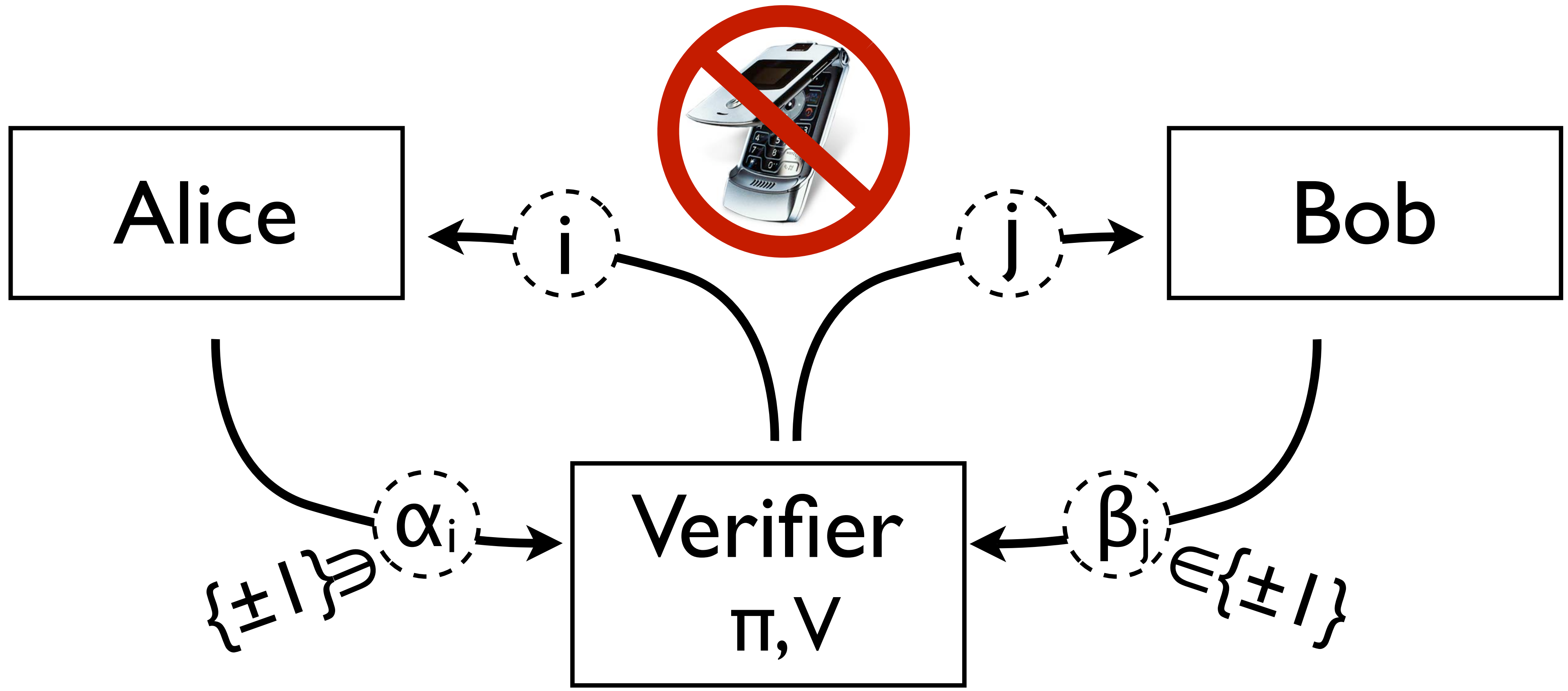
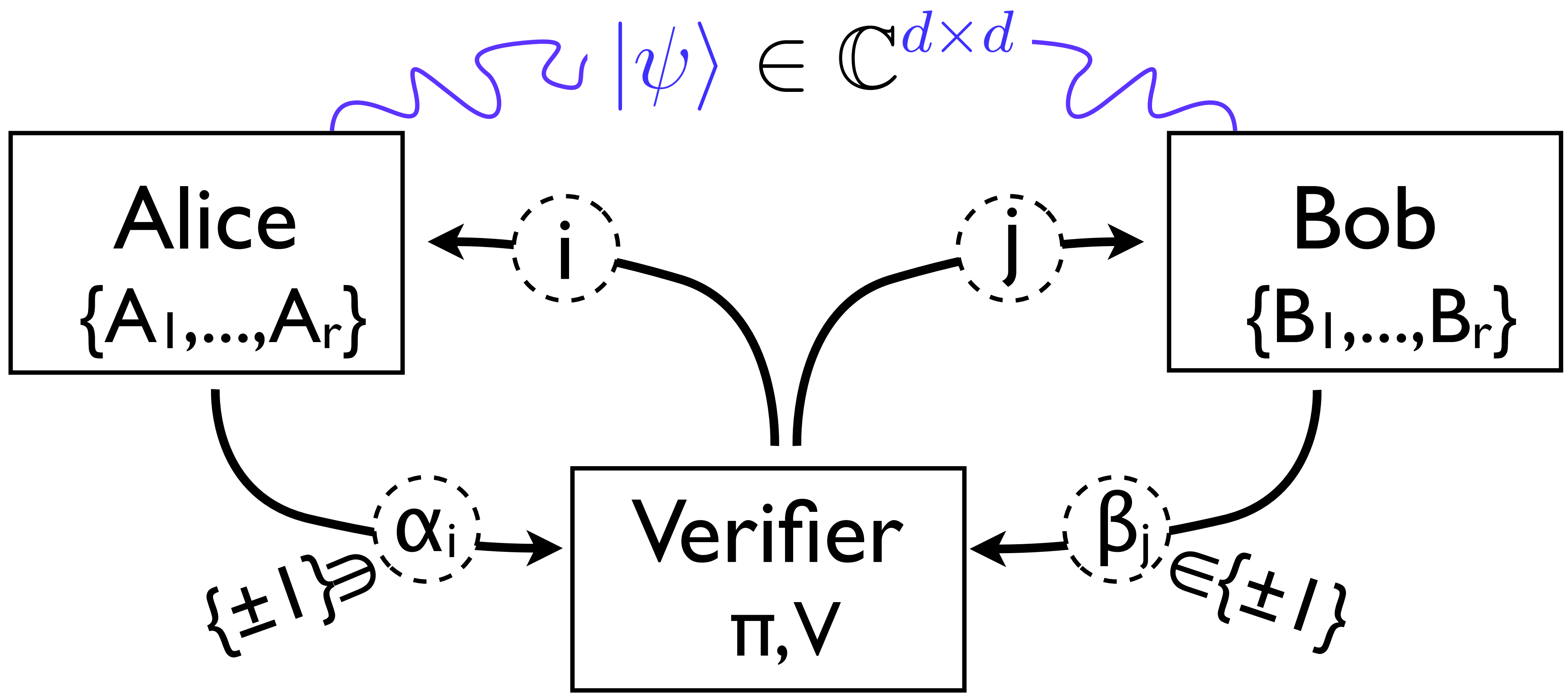


- We can quantify how well Alice and Bob can play game G(π,V) by the *correlation bias*

$$\omega(G) := \max_{\{\alpha_i\},\{\beta_j\}\in\{\pm 1\}} \sum_{i,j} \pi(i,j) V(i,j)\, \alpha_i \beta_j$$

(Deterministic strategies are optimal)

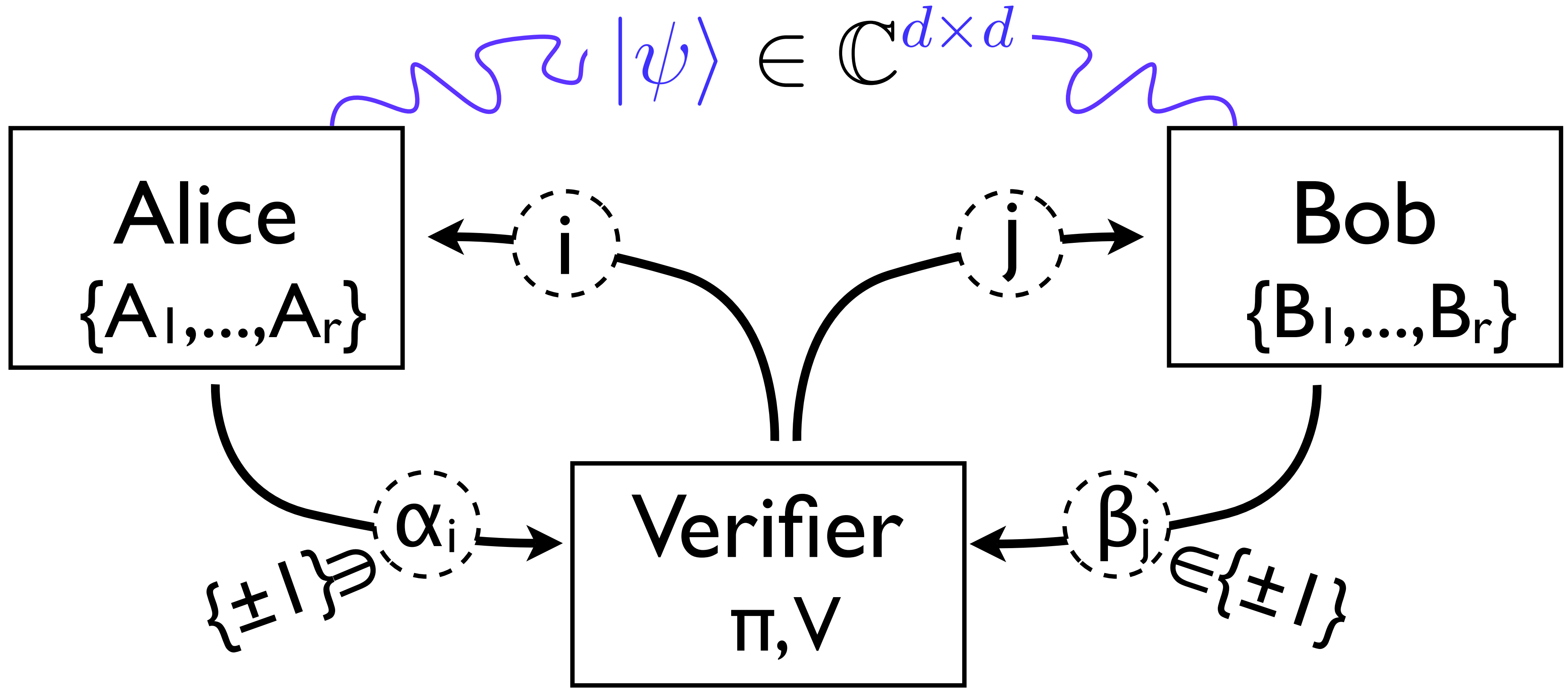
Non-local XOR games



- Alice and Bob share *entangled state* $\psi \in C^{dxd}$
- α_i is the outcome of a $\{\pm 1\}$-valued *measurement of observable* A_i on Alice's part of ψ
- β_j is the outcome of a $\{\pm 1\}$-valued *measurement of observable* B_j on Bob's part of ψ

(Projective measurements and pure states are optimal)

Non-local XOR games



- Measurements give *expectation* $E[\alpha_i\beta_j] = \langle\psi|A_i \otimes B_j|\psi\rangle$
- The d-*dimensional non-local correlation bias*:

$$\omega_d^*(G) := \max_{\{A_i\},\{B_j\},|\psi\rangle\in\mathbb{C}^{d\times d}} \sum_{i,j} \pi(i,j)V(i,j)\,\langle\psi|A_i \otimes B_j|\psi\rangle$$

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Example: CHSH game

- Questions: $i,j \in \{0,1\}$

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At most 3 equations can simultaneously be satisfied

- Classically, the players win with probability $\leq 3/4$ (correlation bias $\leq 1/2$)

CHSH with entanglement

- Players can do better if they share a Bell state

$$\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

CHSH with entanglement

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- Both carry out the same protocol:
  - Question 0: Perform R(-π/16) on qubit
  - Question 1: Perform R(3π/16) on qubit
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- Entangled players win with probability ≈ 0.85

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- Entangled players win with probability ≈ 0.85 (correlation bias ≈ 0.71)
- Tsirelson showed this is best possible

Observables and vectors

- Tsirelson (1987) discovered a correspondence between measurements sets of observables on pure states, and inner products of sets of unit vectors.

For pure state $\psi \in C^{dxd}$ and observables A_a, B_b, there exist unit vectors $a,b \in S_{2d^2-1}$ s.t.

$$a \cdot b = \langle \psi | A_a \otimes B_b | \psi \rangle$$

Conversely, there exist mappings A,B from S_{n-1} to observables on maximally entangled state $\psi \in C^{dxd}$ with $d=2^{n/2}$ s.t. for $a,b \in S_n$

$$\langle \psi | A(a) \otimes B(b) | \psi \rangle = a \cdot b$$

XOR games and Grothendieck's inequality

[Tsirelson87]:

- Associate a matrix $M \in [-1,1]^{r \times r}$ with game (π, V): $M_{ij} = \pi(i,j)V(i,j)$.

- Tsirelson's map lets us upper bound the d-dim. quantum correlation by correlations with $2d^2$-dim unit vectors:

$$\omega_d^*(M) = \max_{\{A_i\},\{B_j\},|\psi\rangle \in \mathbb{C}^{d\times d}} \sum_{i,j} M_{ij} \langle\psi| A_i \otimes B_j |\psi\rangle \leq \max_{\{a_i\},\{b_j\} \in S_{2d^2-1}} \sum_{i,j} M_{ij} a_i \cdot b_j$$

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- This gives an upper bound on the quantum correlation in terms of the classical correlation and Grothendieck's constant

$$\omega_d^*(M) \leq K_G \cdot \omega(M)$$

(Davie `84, Reeds `91) $1.676 \leq K_G \leq 1.782$ (Krivine `71)

Grothendieck constant of order n

- $K_G(n)$ is the smallest constant s.t. for every real rxr matrix M

$$\frac{\max_{\{a_i\},\{b_j\}\in S_{n-1}} \sum_{i,j\in[r]} M_{ij} a_i \cdot b_j}{\max_{\{\alpha_i\},\{\beta_j\}\in\{\pm 1\}} \sum_{i,j\in[r]} M_{ij}\alpha_i\beta_j} \leq K_G(n)$$

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- $K_G = \sup_n K_G(n)$

- Arose in study of norms on tensor products of Banach spaces
- Has also found applications in:
  - Quantum information
  - Communication complexity
  - Approximation algorithms

- $K_G(3) < K_G$ (Krivine)
- $K_G(n)$ is not known to be strictly increasing with n

Our results

Dimension witnesses [Brunner et al.`08]

Alice

$\{A_1,\ldots,A_r\}$

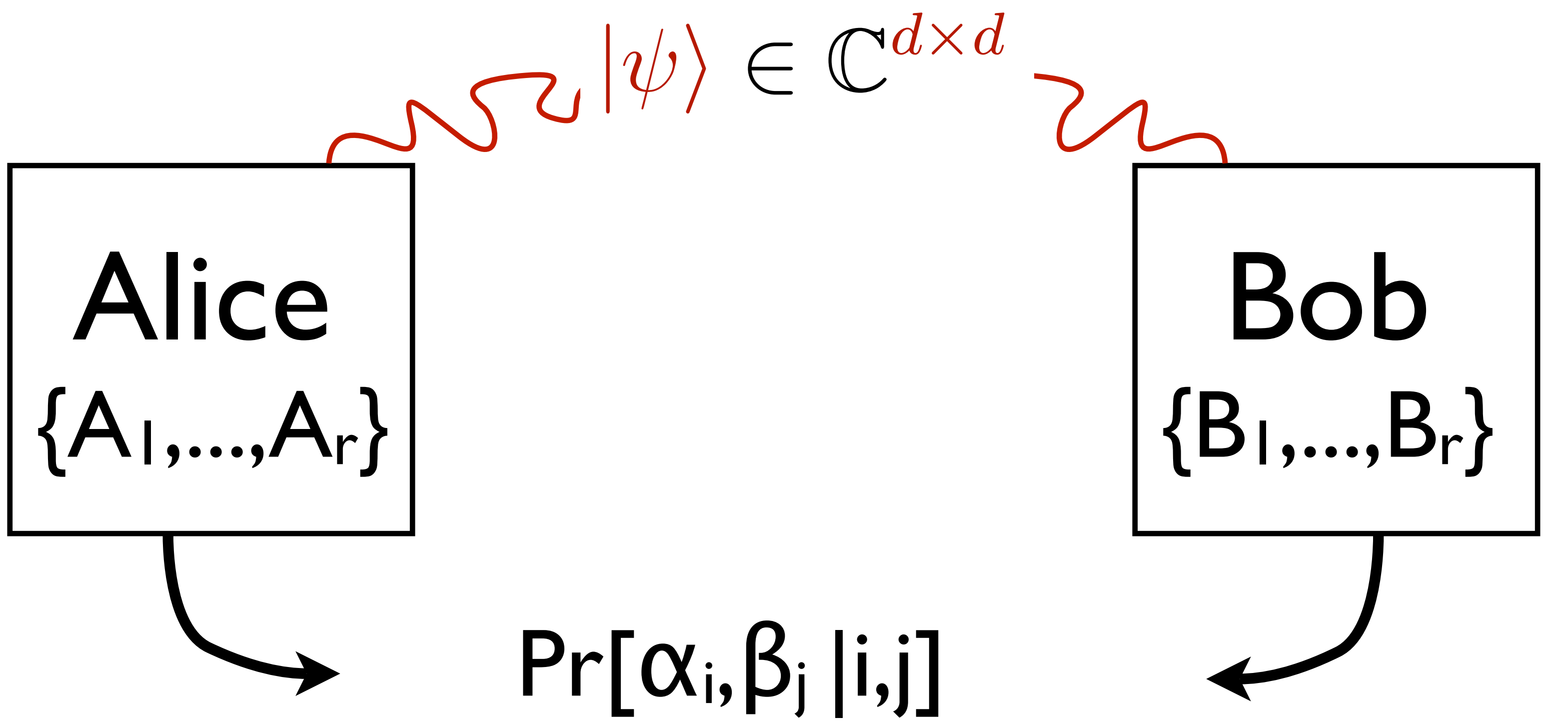
Bob

$\{B_1,\ldots,B_r\}$

$\Pr[\alpha_i,\beta_j \mid i,j]$

- Suppose we get a list of probabilities of Alice and Bob's measurements outcomes, conditioned on their measurements

Dimension witnesses [Brunner et al.`08]



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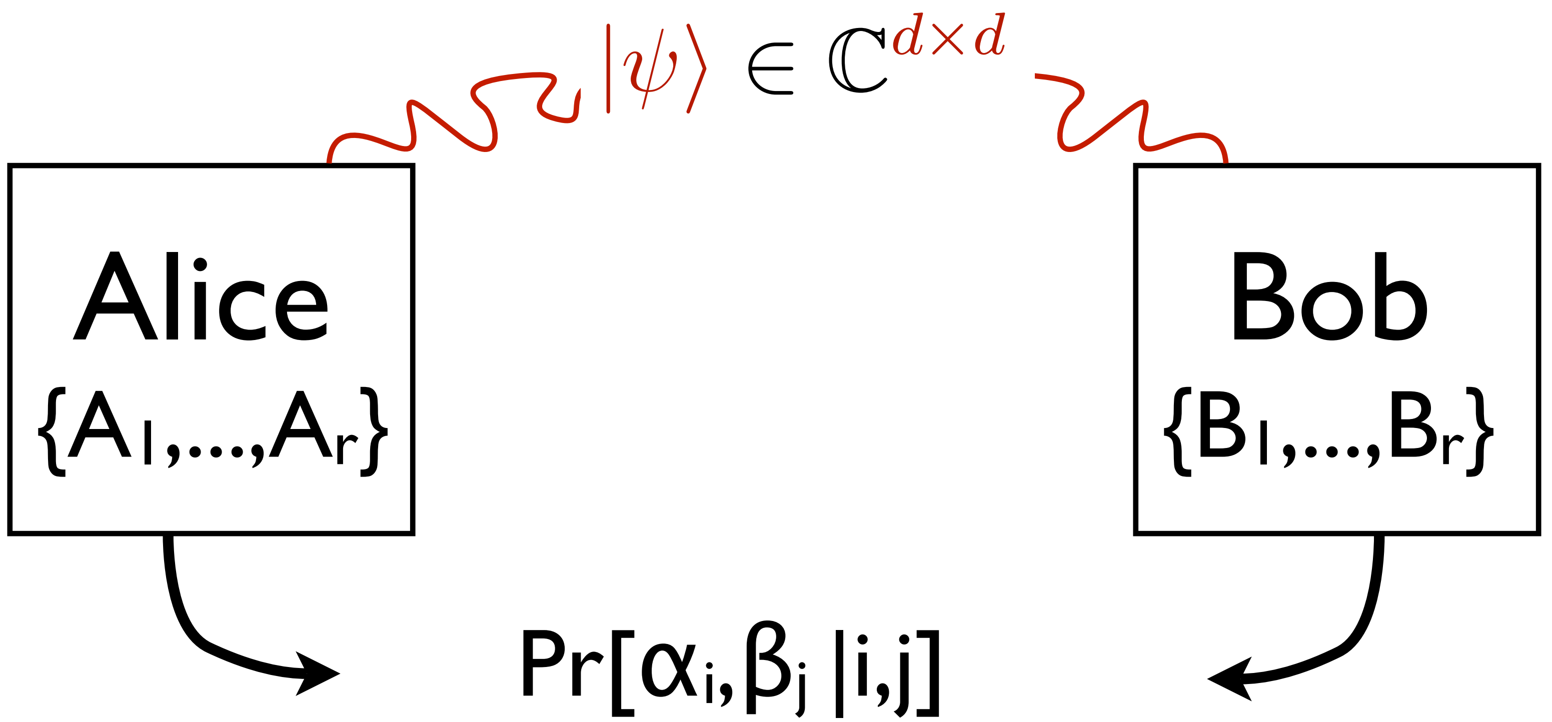

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- But can we also lower bound the dimension of the state?

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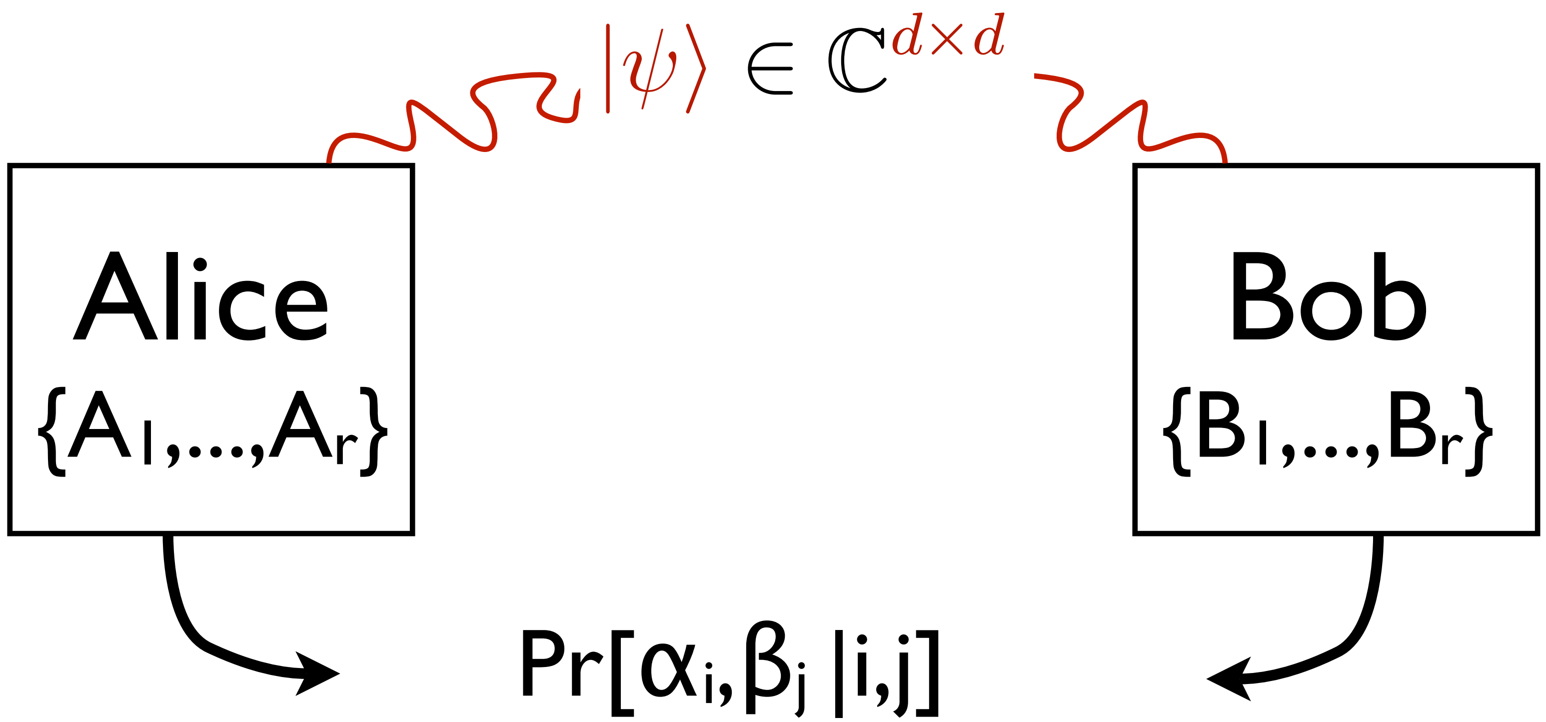

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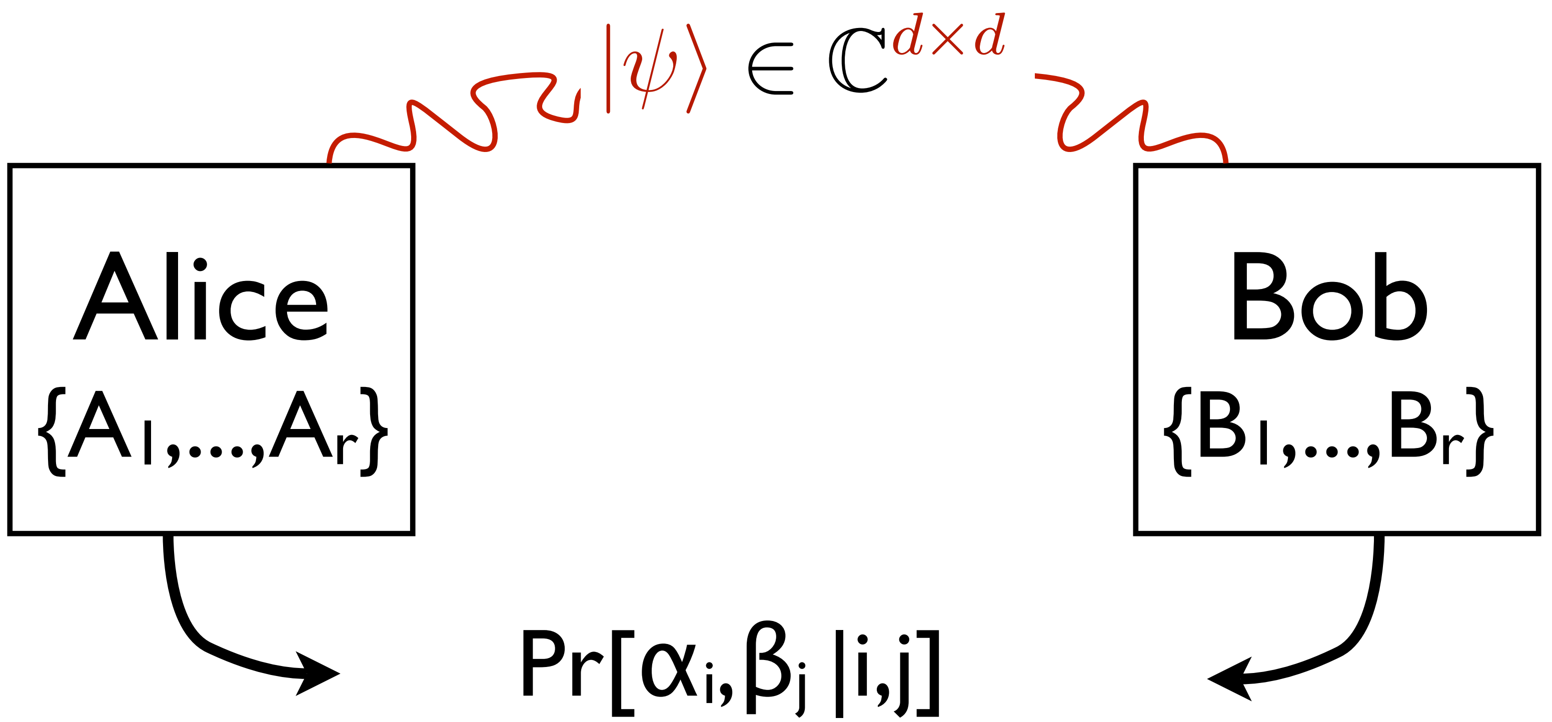

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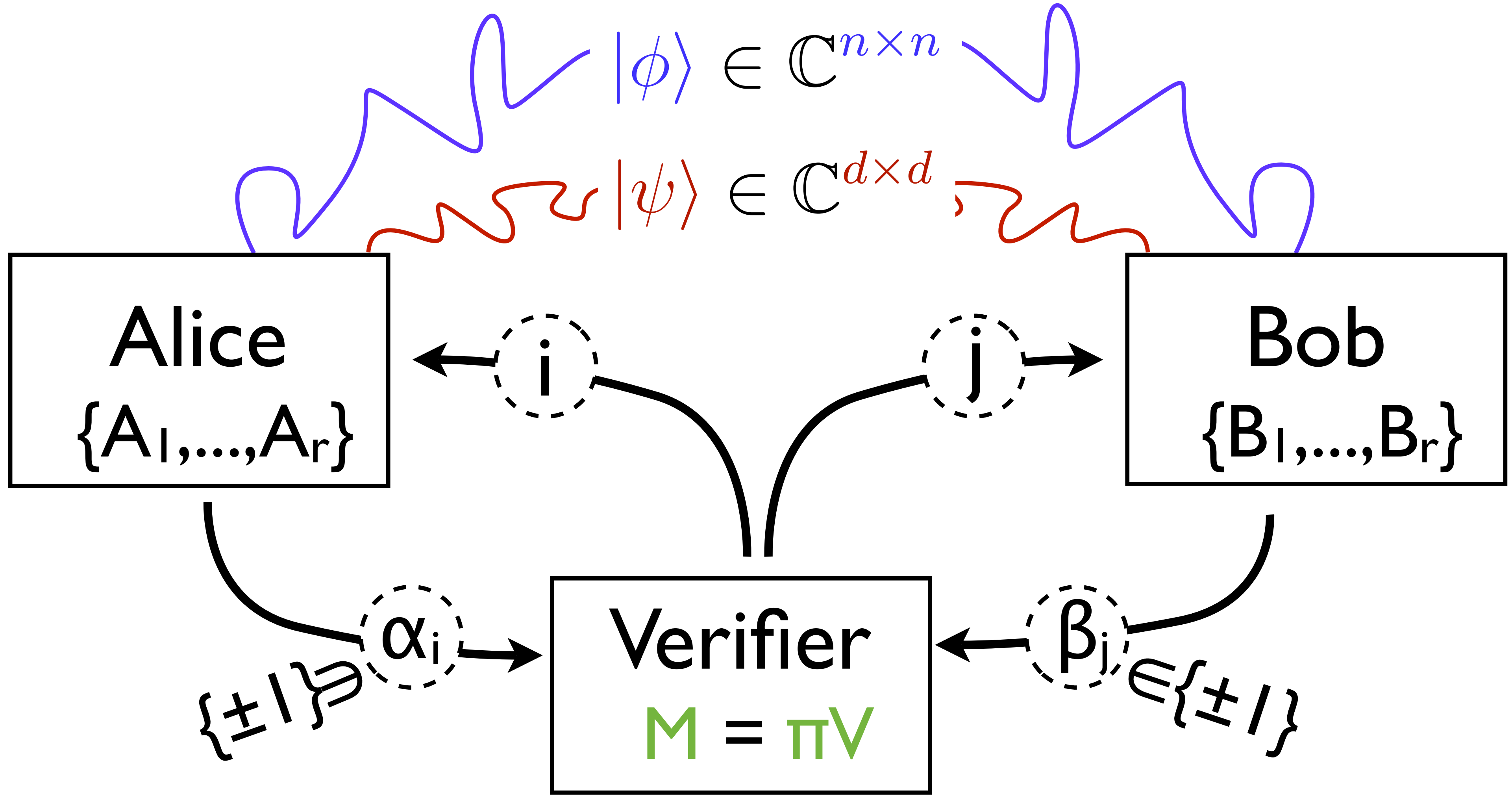
☑ This talk: Yes, for some XOR games!

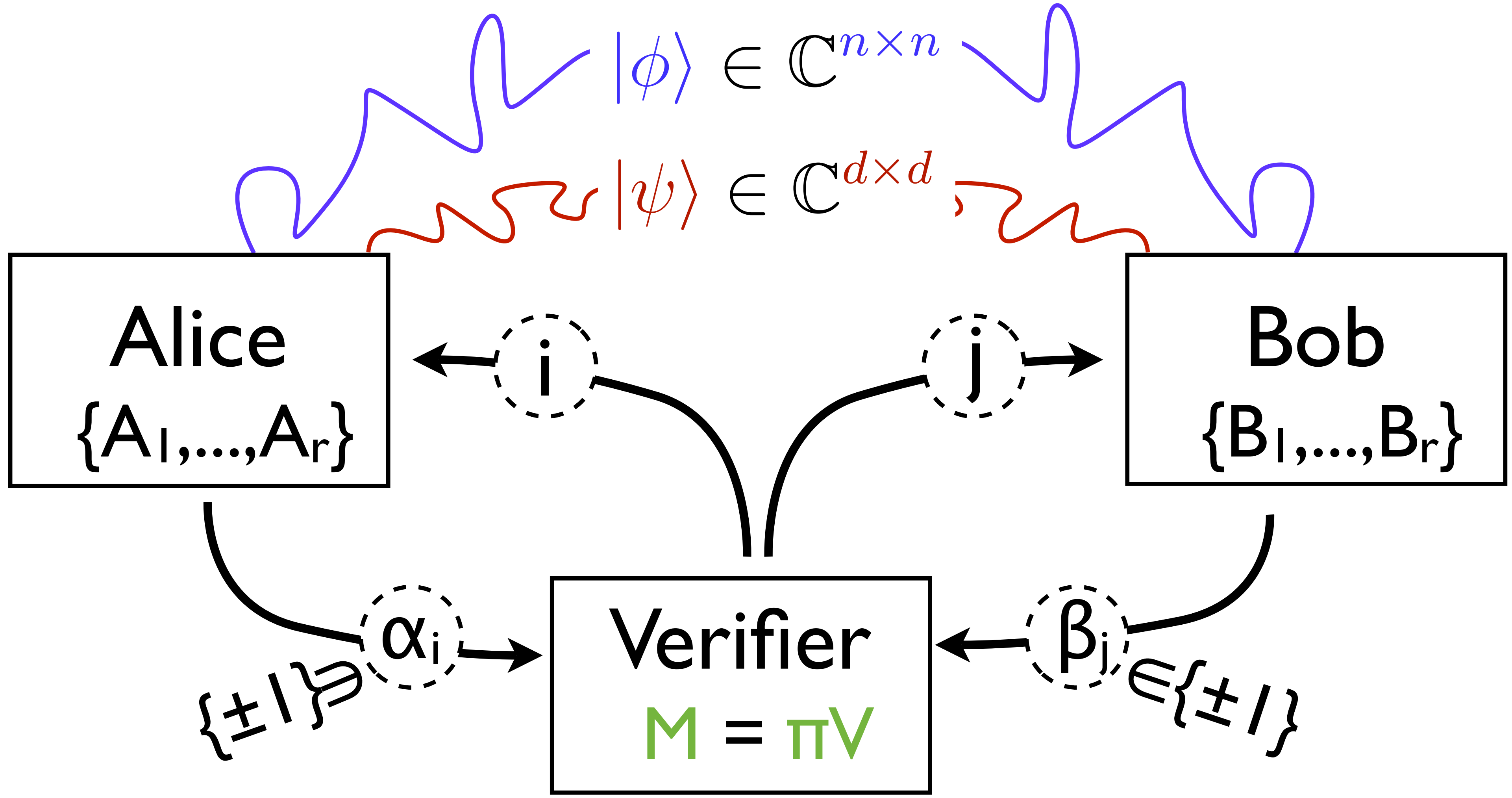
Dimension witnesses and XOR games



For every d, there is an XOR game M and finite n>d, s.t.

$$\omega_d^*(M) < \omega_n^*(M)$$

Dimension witnesses and XOR games



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$$\omega_d^*(M) < \omega_n^*(M)$$

Obtained independently by Vértesi and Pál

Other related work: [Wehner et al. `08]

Recall: $$\frac{\omega_d^*(M)}{\omega(M)} \leq \frac{\max_{\{a_i\},\{b_j\}\in S_{2d^2-1}} \sum_{i,j} M_{ij} a_i \cdot b_j}{\max_{\{\alpha_i\},\{\beta_j\}\in\{\pm 1\}} \sum_{i,j} M_{ij}\, \alpha_i \beta_j} \leq K_G(2d^2)$$

[Brunner et al. `08]

Since $K_G(3) < K_G$ there exists an XOR game M s.t. for some n>2, we have $\omega_2^*(M) < \omega_n^*(M)$
(there exist qubit witnesses)

If $K_G(n)$ is strictly increasing with n (not known), then there are XOR games (M) that can be played better with more entanglement (as conjectured)

We take a different approach: generalize $K_G(n)$

Generalization of Grothendieck's constant

Definition: For m<n, let $K_G(n \longmapsto m)$ be the smallest constant for, s.t. for every real rxr matrix M

$$K_G(n \mapsto m) \geq \frac{\max_{\{a_i\},\{b_j\} \in S_{n-1}} \sum_{i,j \in [r]} M_{ij} a_i \cdot b_j}{\max_{\{a'_i\},\{b'_j\} \in S_{m-1}} \sum_{i,j \in [r]} M_{ij} a'_i \cdot b'_j}$$

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- $K_G(n \mapsto 1) = K_G(n)$
- Trivial upper bound: $K_G(n \mapsto m) \leq K_G(n)$
- We prove first lower bound: $K_G(n \mapsto m) > 1+1/2m-1/2n - O(1/m^2)$

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Our lower bound is proved by choosing M: For $a,b \in S_{n-1}$,

let M be a function: $M(a,b) = a \cdot b$

$$K_G(n \mapsto m) \geq \frac{\max_{A,B:S_{n-1}\to S_{n-1}} \int_{a,b\in S_{n-1}} da\,db\,(a\cdot b)\big(A(a)\cdot B(b)\big)}{\max_{A',B':S_{n-1}\to S_{m-1}} \int_{a,b\in S_{n-1}} da\,db\,(a\cdot b)\big(A'(a)\cdot B'(b)\big)} > 1$$

Relation to dimension witness game

- Define a game (π,V) by infinitely many questions, labelled as unit vectors $a,b \in S_{n-1}$, and $\pi(a,b)V(a,b) = M(a,b) = a \cdot b$

- By Tsirelson, the numerator and denominator bound non-local correlations

$$\max_{A,B:S_{n-1}\to S_{n-1}} \int_{a,b\in S_{n-1}} dadb(a\cdot b)\big(A(a)\cdot B(b)\big) \leq \omega^*_{2^{n/2}}(M)$$

$$\omega^*_d(M) \leq \max_{A',B':S_{n-1}\to S_{2d^2-1}} \int_{a,b\in S_{n-1}} dadb(a\cdot b)\big(A'(a)\cdot B'(b)\big)$$

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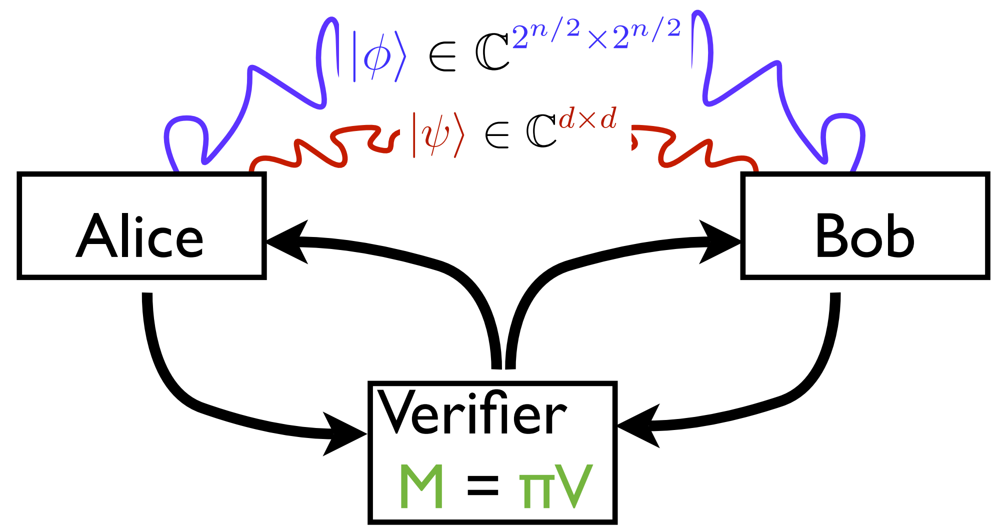

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Set n = $2d^2+1$ so that m>n

$$\frac{\omega^*_{2^{n/2}}(M)}{\omega^*_d(M)} \geq \frac{\max_{A,B:S_{n-1}\to S_{n-1}} \int_{a,b\in S_{n-1}} dadb(a\cdot b)\big(A(a)\cdot B(b)\big)}{\max_{A',B':S_{n-1}\to S_{2d^2-1}} \int_{a,b\in S_{n-1}} dadb(a\cdot b)\big(A'(a)\cdot B'(b)\big)} > 1$$

- This is a game that Alice and Bob can play better with more entanglement

Lower bound on $K_G(n \mapsto m)$

$$K_G(n \mapsto m) \geq \frac{\max_{A,B:S_{n-1}\rightarrow S_{n-1}} \int_{a,b\in S_{n-1}} da\,db\,(a\cdot b)\big(A(a)\cdot B(b)\big)}{\max_{A',B':S_{n-1}\rightarrow S_{m-1}} \int_{a,b\in S_{n-1}} da\,db\,(a\cdot b)\big(A'(a)\cdot B'(b)\big)} > 1$$

We need to find the maximizing maps A and B

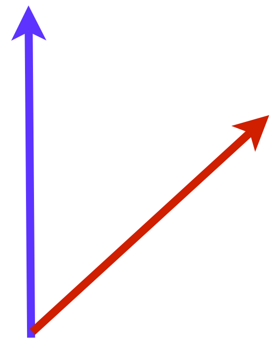
How to choose A and B

$$\text{Denominator} \quad = \quad \max_{A,B:S_{n-1}\to S_{m-1}} \int dadb(a\cdot b)\big((A(a)\cdot B(b)\big)$$

- The integrand can be written as an inner product

$$(a\cdot b)\big((A(a)\cdot B(b)\big) = \big(a\otimes A(a)\big)\cdot\big(b\otimes B(b)\big)$$

$$\left(\int da\big(a\otimes A(a)\big)\right)\cdot\left(\int db\big(b\otimes B(b)\big)\right)$$



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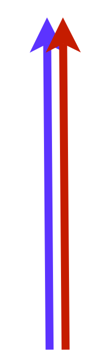
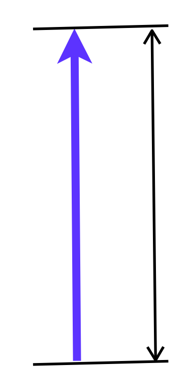


- This is maximal if these vectors are parallel. So set B=A.

- We need to calculate the maximal length of a vector

$$\left\|\int da\big(a\otimes A(a)\big)\right\|^2 = \|\chi v\|^2 = \chi^2$$

unit vector



How to choose A

- Take the Schmidt decomposition of v: $$v = \sum_{i=1}^{m} \lambda_i x_i \otimes y_i$$

({x_1,...,x_n} and {y_1,...,y_m} are bases for R^n and R^m respectively)

$$\begin{aligned}\chi = (\chi v) \cdot v &= \left(\int da\big(a \otimes A(a)\big)\right) \cdot \left(\sum_{i=1}^{m} \lambda_i x_i \otimes y_i\right) \\ &= \int da\Big(\sum_{i=1}^{m} \lambda_i (a \cdot x_i) y_i\Big) \cdot A(a)\end{aligned}$$

- Again we should set the vectors parallel. This gives an integral we can just evaluate

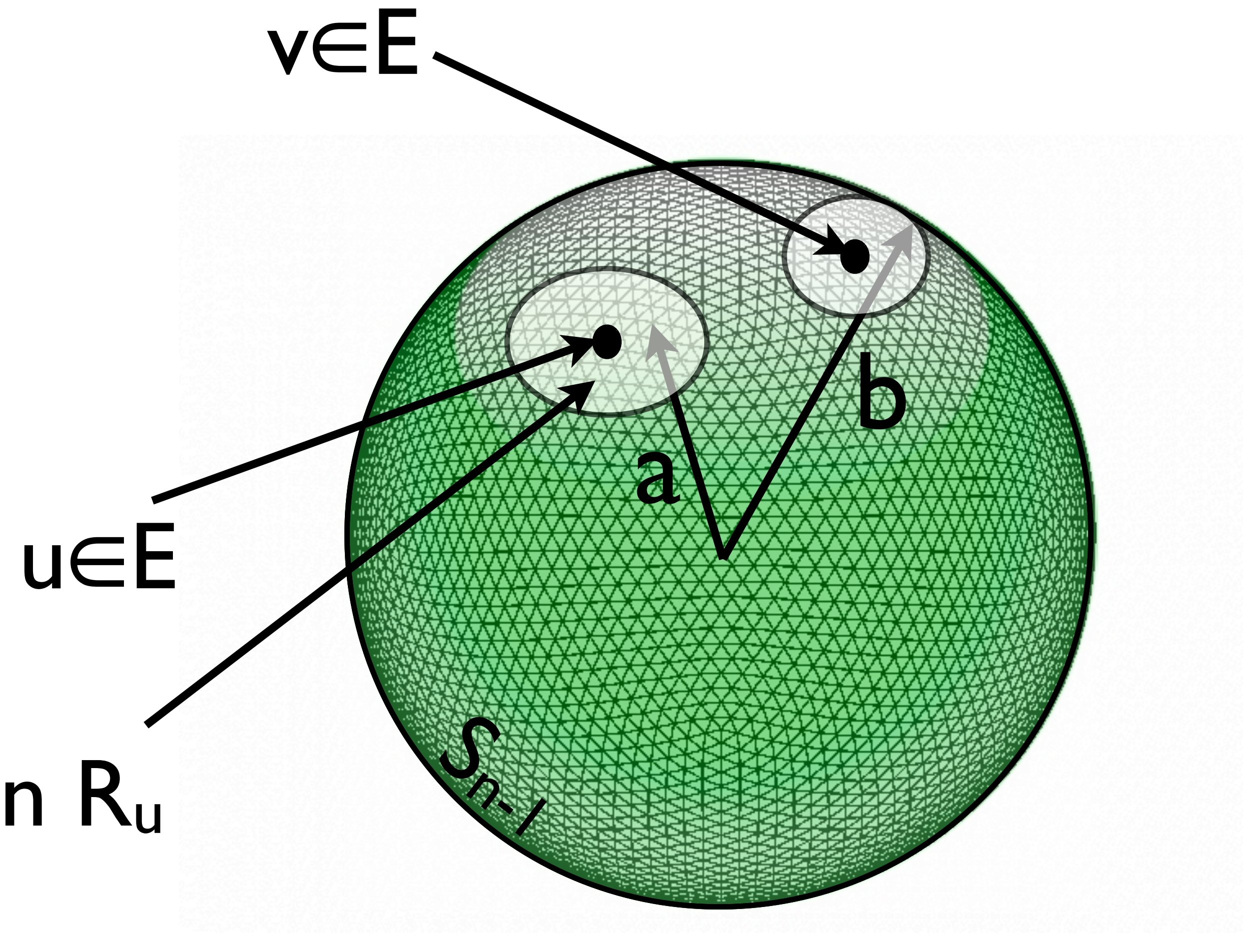
$$\chi = \int_{a \in S_{n-1}} da\Big(\sum_{i=1}^{m} \frac{1}{m} a_i^2\Big)^{1/2}$$

Now just evaluate the integrals

$$
\begin{aligned}
K_G(n \mapsto m) &\geq \left(\frac{\int_{a\in S_{n-1}} da \left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)^{1/2}}{\int_{a\in S_{n-1}} da \left(\frac{1}{m}\sum_{i=1}^{m} a_i^2\right)^{1/2}} \right)^2 \\
&= \left(\frac{\Gamma(\frac{n+1}{2})}{\sqrt{n}\Gamma(\frac{n}{2})} \right)^2 \Big/ \left(\frac{\Gamma(\frac{m+1}{2})}{\sqrt{m}\Gamma(\frac{m}{2})} \right)^2 \\
&\geq 1 + \frac{1}{2m} - \frac{1}{2n} - O(\frac{1}{m^2}) \\
&> 1
\end{aligned}
$$

Making the game finite

- Take questions from an **ϵ-net**: $E=\{u_1,\ldots,u_t\}\subseteq S_{n-1}$ s.t. for every $a \in S_{n-1}$, there exist a $u \in E$ close to a: $\|a - u\|_2 \leq \epsilon$



- The *region* R_u is the set $a \in S_{n-1}$ to which u is the closest
- Finite game: For questions $u,v \in E$ $\quad M_{\text{finite}}(u,v) := \int_{\substack{a\in R_u \\ b \in R_v}} da\,db\,(a \cdot b)$

$$\frac{\omega^*_{2^{n/2}}(M_{\text{finite}}) + 2\varepsilon}{\omega^*_d(M_{\text{finite}})} \geq \frac{\omega^*_{2^{n/2}}(M_{\text{infinite}})}{\omega^*_d(M_{\text{infinite}})} > 1$$

- Set n and ϵ appropriately

Summary

- Grothendieck's inequality: non-local vs classical games
- Generalization ($K_G(n \longmapsto m)$): non-local vs non-local
- For every d, there is an XOR game M and finite $n>d$, s.t. $\omega_d^*(M) < \omega_n^*(M)$.
  (Binary dimension witnesses exist.)
- We showed this by proving $K_G(n \longmapsto m)>1$