Universidad de la Fuerzas Armadas ESPE - L

Nombre: Ismael Simbaña
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Derivada Numérica Orden Superior

Marco teórico

La derivación numérica es una técnica de análisis numérico para calcular una aproximación a la derivada de una función en un punto dado, utilizando los valores y propiedades de esta.

v(t) se puede expandir en una serie de Taylor del siguiente modo:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + \frac{v''(t_i)}{2!}(t_{i+1} - t_i)^2 + \cdots + R_n$$

Se trunca la serie después del término de la primera derivada

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(x_{i+1} - x_i)$$

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + R_1$$

$$v'(t_i) = \underbrace{\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}}_{\substack{\text{Aproximación} \\ \text{de primer orden}}} - \underbrace{\frac{R_1}{t_{i+1} - t_1}}_{\substack{\text{Error de} \\ \text{truncamiento}}}$$

Con el método de la serie de Taylor se ha obtenido una estimación del error de truncamiento asociado con esta aproximación de la derivada.

- Demostración de las fórmulas o expresiones

Diferencia Hacia adelante y h el tamaño del incremento

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(x_{i+1} - x_i)$$

Aproximación a la primera derivada con diferencia hacia atrás.

La serie de Taylor se expande hacia atrás para calcular un valor anterior sobre la base del valor actual

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \cdots$$

Truncando la ecuación después de la primera derivada y reordenando los términos se obtiene

$$f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{h} = \frac{\nabla f_1}{h}$$

Aproximación a la primera derivada con diferencias centradas.

Se resta de la serie de Taylor

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(x_{i+1} - x_i) \quad - \quad f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \cdots$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \cdots$$

para obtener

$$f(x_{i+1}) = f(x_{i-1}) + 2f'(x_i)h + \frac{2f^{(3)}(x_i)}{3!}h^3 + \cdots$$

de donde se despeja

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - \frac{f^{(3)}(x_i)}{6}h^2 - \cdots$$

o

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - O(h^2)$$

Ejemplos

Estimar la primera derivada en x = 0.5 con h = 0.5

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$f'(x) = -0.4x^3 - 0.45x^2 - 1.0x - 0.25$$

h =0.5

$x_{i-1} = 0$	$f(x_{i-1}) = 1.2$
$x_i = 0.5$	$f(x_i) = 0.925$
$x_{i+1} = 1.0$	$f(x_{i+1}) = 0.2$

Calculan las diferencias dividas

$$f'(0.5) \cong \frac{0.2 - 0.925}{0.5} = -1.45 \qquad |\varepsilon_t| = 58.9\%$$

$$f'(0.5) \cong \frac{0.925 - 1.2}{0.5} = -0.55 \qquad |\varepsilon_t| = 39.7\%$$

$$f'(0.5) \cong \frac{0.2-1.2}{1.0} = -1.0 \qquad |\varepsilon_t| = 9.6\%$$

5 ejercicios (desarrollados manualmente y en python)

- **Ejercicio 1**

$h = 0{,}01$

$$f(x) = \frac{2\tan(1{,}5+0{,}01) - e^{-(1{,}5+0{,}01)} - \left(2\tan(1{,}5) - e^{-3(1{,}5)}\right)}{0{,}01}$$

$$f'(x) = 463.145384$$

$$e = \left| \frac{399.73341 - 463.145384}{394.73341} \right| \cdot 100 = 15.863572\,\%$$

$h = 0{,}001$

$$f(x) = \frac{2\tan(1{,}5+0{,}001) - e^{-3(1{,}5+0{,}001)} - \left[2\tan(1{,}5) - e^{-3(1.5)}\right]}{0{,}001}$$

$$f'(1{,}5) = 383.232466$$

$$e = \left| \frac{394.73341 - 383.232466}{394.7341} \right| 100\% = 4.127987\,\%$$

$h = 0{,}0001$

$$f(x) = \frac{2\tan(1{,}5+0{,}0001) - e^{-3(1{,}5+0{,}0001)} - (2\tan(1{,}5) - e^{-3(1{,}5)})}{0{,}0001}$$

$f(1{,}5) = 178.1179114$

$$e = \left| \frac{399.73341 - 178.1179114}{399.73341} \right| \cdot 100 = 55.44\%$$

hacia atrás

$$f'(0{,}01) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5-0{,}01) - e^{-3(1{,}5-0{,}01)})}{0{,}01}$$

$f'(0{,}01) = 350{,}346533$ $\qquad e = \left| \frac{399.73341 - 350.346533}{399.73341} \right| = 12.35\%$

$$f'(0{,}001) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5-0{,}01) - e^{-3(1{,}5-0{,}01)})}{0{,}001}$$

$f(0{,}001) = 394.1756$

$$e = \left| \frac{399.73341 - 394.1756}{399.73341} \right| \cdot 100\% = 1{,}40\%$$

$$f'(1{,}5) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5-0{,}001) - e^{-3(1{,}5-0{,}0001)})}{0{,}0001}$$

$f'(1{,}5) = 399{,}1705822$

$$e = \left| \frac{399.7341 - 399.170582}{399.7341} \right| \cdot 100\% = 0{,}1499\%$$

Centrales $h = 0{,}01$

$$\frac{2\tan(1{,}5+0{,}01) - e^{-3(1{,}5+0{,}01)} - \left[2\tan(1{,}5-0{,}01) - e^{-3(1{,}5-0{,}01)}\right]}{2\cdot 0{,}01} = f'(1{,}5)$$

$$f'(1{,}5) = 407.856858$$

$$e = \left|\frac{394.73341 - 402.856858}{394.73341}\right| \cdot 100 = 2.032242019\ \%$$

$h = 0{,}001$

$$f'(1{,}5) = \frac{2\tan(1{,}5+0{,}001) - e^{-3(1{,}5-0{,}01)} - \left(2\tan(1{,}5-0{,}001) - e^{-3(1{,}5-0{,}001)}\right)}{2\cdot 0{,}001}$$

$$f'(1{,}5) = 394.813045\ \%$$

$$e = \left|\frac{394.73341 - 394.8130456}{394.73341}\right| \cdot 100 = 0{,}019\ \%$$

$h = 0{,}0001$

$$f'(1{,}5) = \frac{2(\tan(1{,}5+0{,}001) - e^{-3(1{,}5+0{,}0001)} - \left(2\tan(1{,}5-0{,}0001) - e^{-3(1{,}5-0{,}0001)}\right)}{2\cdot 0{,}0001}$$

$$e = \left|\frac{394.73341 - 394.7342111}{394.73341}\right| \cdot 100\% = 2.000 \cdot 10^{-4}\ \%$$

- **Ejercicio 2**

$h = 0,01$

$$f(x) = \frac{2\tan(1,5+0,01) - e^{-(1,5+0,01)} - \left(2\tan(1,5) - e^{-3(1,5)}\right)}{0,01}$$

$$f'(x) = 463.145384$$

$$e = \left| \frac{394.73341 - 463.145384}{394.73341} \right| \cdot 100 = 15.863572\,\%$$

$h = 0,001$

$$f(x) = \frac{2\tan(1,5+0,001) - e^{-3(1,5+0,001)} - \left[2\tan(1,5) - e^{-3(1,5)}\right]}{0,001}$$

$$f'(1,5) = 383.232466$$

$$e = \left| \frac{394.73341 - 383.232466}{394.73341} \right| 100\,\% = 4.127987\,\%$$

$h = 0{,}0001$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}0001) - e^{-3(1{,}5+0{,}0001)} - (2\tan(1{,}5) - e^{-3(1{,}5)})}{0{,}0001}$$

$$f(1{,}5) = 178{,}1179114$$

$$e = \left| \frac{399.73341 - 178.1179114}{399.73341} \right| \cdot 100 = 55.44\%$$

hacia atras

$$f'(0{,}01) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5-0{,}01) - e^{-3(1{,}5-0{,}01)})}{0{,}01}$$

$$f'(0{,}01) = 350{,}346533 \qquad e = \left| \frac{399.73341 - 350.346533}{399.73341} \right| = 12.35\%$$

$$f'(0{,}001) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5-0{,}01) - e^{-3(1{,}5-0{,}01)})}{0{,}001}$$

$$f(0{,}001) = 394{,}1756$$

$$e = \left| \frac{399.73341 - 394.1756}{399.73341} \right| \cdot 100\% = 1{,}40\%$$

$$f'(1{,}5) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5-0{,}001) - e^{-3(1{,}5-0{,}0001)})}{0{,}0001}$$

$$f'(1{,}5) = 399{,}1705822$$

$$e = \left| \frac{399.7341 - 399.170582}{399.7341} \right| \cdot 100\% = 0{,}1499\%$$

Centrales $h = 0{,}01$

$$\frac{2\tan(1{,}5 + 0{,}01) - e^{-3(1{,}5+0{,}01)} - \left[2\tan(1{,}5 - 0{,}01) - e^{-3(1{,}5-0{,}01)}\right]}{2 \cdot 0{,}01} = f'(1{,}5)$$

$$f'(1{,}5) = 407.856858$$

$$e = \left| \frac{399.73341 - 407.856858}{399.73341} \right| \cdot 100 = 2{,}032242019\ \%$$

$h = 0{,}001$

$$f'(1{,}5) = \frac{2\tan(1{,}5 + 0{,}001) - e^{-3(1{,}5-0{,}01)} - \left(2\tan(1{,}5 - 0{,}001) - e^{-3(1{,}5-0{,}001)}\right)}{2 \cdot 0{,}001}$$

$$f'(1{,}5) = 399.818045\ \%$$

$$e = \left| \frac{399.73341 - 399.8130456}{399.73341} \right| \cdot 100 = 0{,}019\ \%$$

$h = 0{,}0001$

$$f'(1{,}5) = \frac{2(\tan(1{,}5 + 0{,}001) - e^{-3(1{,}5+0{,}0001)} - \left(2\tan(1{,}5 - 0{,}0001) - e^{-3(1{,}5-0{,}0001)}\right)}{2 \cdot 0{,}0001}$$

$$e = \left| \frac{399{,}73341 - 399.7342121}{399.73341} \right| \cdot 100\% = 2.000 \cdot 10^{-4}\ \%$$

Ejercicio 3

$h = 0{,}01$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}01) - e^{-(1{,}5+0{,}01)} - \left(2\tan(1{,}5) - e^{-3(1{,}5)}\right)}{0{,}01}$$

$$f'(x) = 463.145384$$

$$e = \left| \frac{394.73341 - 463.145384}{394.73341} \right| \cdot 100 = 15.863572\,\%$$

$h = 0{,}001$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}001) - e^{-3(1{,}5+0{,}001)} - \left[2\tan(1{,}5) - e^{-3(1{,}5)}\right]}{0{,}001}$$

$$\dot f(1{,}5) = 383.232466$$

$$e = \left| \frac{394.73341 - 383.232466}{394.7341} \right| 100\,\% = 4.127987\,\%$$

$h = 0{,}0001$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}0001) - e^{-3(1{,}5+0{,}0001)} - \left(2\tan(1{,}5) - e^{-3(1{,}5)}\right)}{0{,}0001}$$

$f(1{,}5) = 178.1179114$

$$e = \left| \frac{399.73341 - 178.1179114}{399.73341} \right| \cdot 100 = 55.44\%$$

hacia atrás

$$f'(0{,}01) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - \left(2\tan(1{,}5 - 0{,}01) - e^{-3(1{,}5-0{,}01)}\right)}{0{,}01}$$

$f'(0{,}01) = 350{,}346333$

$$e = \left| \frac{399.73341 - 350.346333}{399.73341} \right| = 12.35\%$$

$$f'(0{,}001) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - \left(2\tan(1{,}5 - 0{,}01) - e^{-3(1{,}5-0{,}01)}\right)}{0{,}001}$$

$f(0{,}001) = 394.1756$

$$e = \left| \frac{399.73341 - 394.1756}{399.73341} \right| \cdot 100\% = 1{,}40\%$$

$$f'(1{,}5) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - \left(2\tan(1{,}5 - 0{,}001) - e^{-3(1{,}5-0{,}0001)}\right)}{0{,}0001}$$

$f'(1{,}5) = 399{,}1703822$

$$e = \left| \frac{399.7341 - 399.170382}{399.7341} \right| \cdot 100\% = 0{,}1499\%$$

Centrales $h = 0{,}01$

$$\frac{2\tan(1{,}5+0{,}01) - e^{-3(1{,}5+0{,}01)} - \left(2\tan(1{,}5-0{,}01) - e^{-3(1{,}5-0{,}01)}\right)}{2\cdot 0{,}01} = f'(1{,}5)$$

$$f'(1{,}5) = 407.856858$$

$$e = \left| \frac{394.73341 - 407.856858}{394.73341} \right| \cdot 100 = 2.032292019\ \%$$

$h = 0{,}001$

$$f'(1{,}5) = \frac{2\tan(1{,}5+0{,}001) - e^{-3(1{,}5-0{,}01)} - \left(2\tan(1{,}5-0{,}001) - e^{-3(1{,}5-0{,}001)}\right)}{2\cdot 0{,}001}$$

$$f'(1{,}5) = 394.813045\ \%$$

$$e = \left| \frac{394.73341 - 394.8130456}{394.73341} \right| \cdot 100 = 0{,}019\ \%$$

$h = 0{,}0001$

$$f'(1{,}5) = \frac{2(\tan(1{,}5+0{,}001) - e^{-3(1{,}5+0{,}0001)} - \left(2\tan(1{,}5-0{,}0001) - e^{-3(1{,}5-0{,}0001)}\right)}{2\cdot 0{,}0001}$$

$$e = \left| \frac{394{,}73341 - 394{,}7342111}{394{,}73341} \right| \cdot 100\% = 2.000\cdot 10^{-4}\ \%$$

Ejercicio 4

$h = 0{,}01$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}01) - e^{-(1{,}5+0{,}01)} - \left(2\tan(1{,}5) - e^{-3(1{,}5)}\right)}{0{,}01}$$

$f'(x) = 463.145384$

$$e = \left| \frac{394.73341 - 463.145384}{394.73341} \right| \cdot 100 = 15.863572\,\%$$

$h = 0{,}001$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}001) - e^{-3(1{,}5+0{,}001)} - \left[2\tan(1{,}5) - e^{-3(1{,}5)}\right]}{0{,}001}$$

$f'(1{,}5) = 383.232466$

$$e = \left| \frac{394.73341 - 383.232466}{394.7341} \right| 100\,\% = 4.127987\,\%$$

$h = 0{,}0001$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}0001) - e^{-3(1{,}5+0{,}0001)} - (2\tan(1{,}5) - e^{-3(1{,}5)})}{0{,}0001}$$

$$f(1{,}5) = 178.1179114$$

$$e = \left| \frac{399.73341 - 178.1179114}{399.73341} \right| \cdot 100 = 55.44\%$$

hacia atras

$$f'(0{,}01) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5 - 0{,}01) - e^{-3(1{,}5-0{,}01)})}{0{,}01}$$

$$f'(0{,}01) = 350{,}346533 \qquad e = \left| \frac{399.73341 - 350.346533}{399.73341} \right| = 12.35\%$$

$$f'(0{,}001) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5 - 0{,}01) - e^{-3(1{,}5-0{,}01)})}{0{,}001}$$

$$f(0{,}001) = 394.1756$$

$$e = \left| \frac{399.73341 - 394.1756}{399.73341} \right| \cdot 100\% = 1{,}40\%$$

$$f'(1{,}5) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5 - 0{,}001) - e^{-3(1{,}5-0{,}0001)})}{0{,}0001}$$

$$f'(1{,}5) = 399{,}1703822$$

$$e = \left| \frac{399.7341 - 399.170382}{399.7341} \right| \cdot 100\% = 0{,}1499\%$$

Centrales $h = 0{,}01$

$$\frac{2\tan(1{,}5+0{,}01) - e^{-3(1{,}5+0{,}01)} - \left(2\tan(1{,}5-0{,}01) - e^{-3(1{,}5-0{,}01)}\right)}{2 \cdot 0{,}01} = f'(1{,}5)$$

$$f'(1{,}5) = 407.856858$$

$$e = \left| \frac{394.73341 - 407.856858}{394.73341} \right| \cdot 100 = 2.032242019 \ \%$$

$h = 0{,}001$

$$f'(1{,}5) = \frac{2\tan(1{,}5+0{,}001) - e^{-3(1{,}5-0{,}01)} - \left(2\tan(1{,}5-0{,}001) - e^{-3(1{,}5-0{,}001)}\right)}{2 \cdot 0{,}001}$$

$$f'(1{,}5) = 394.813045 \ \%$$

$$e = \left| \frac{394.73341 - 394.8130456}{394.73341} \right| \cdot 100 = 0{,}019 \ \%$$

$h = 0{,}0001$

$$f'(1{,}5) = \frac{2(\tan(1{,}5+0{,}001) - e^{-3(1{,}5+0{,}0001)} - \left(2\tan(1{,}5-0{,}0001) - e^{-3(1{,}5-0{,}0001)}\right)}{2 \cdot 0{,}0001}$$

$$e = \left| \frac{394{,}73341 - 394{,}7342121}{394{,}73341} \right| \cdot 100\% = 2.000 \cdot 10^{-4} \ \%$$

- **Ejercicio 5**

$h = 0{,}01$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}01) - e^{-(1{,}5+0{,}01)} - \left(2\tan(1{,}5) - e^{-3(1{,}5)}\right)}{0{,}01}$$

$f'(x) = 463.145384$

$$e = \left| \frac{394.73341 - 463.145384}{394.73341} \right| \cdot 100 = 15.863572\,\%$$

$h = 0{,}001$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}001) - e^{-3(1{,}5+0{,}001)} - \left[2\tan(1{,}5) - e^{-3(1{,}5)}\right]}{0{,}001}$$

$f'(1{,}5) = 383.232466$

$$e = \left| \frac{394.73341 - 383.232466}{394.7341} \right| 100\,\% = 4.127987\,\%$$

$h = 0{,}0001$

$$f(x) = \frac{2\tan(1{,}5 + 0{,}0001) - e^{-3(1{,}5+0{,}0001)} - (2\tan(1{,}5) - e^{-3(1{,}5)})}{0{,}0001}$$

$f(1{,}5) = 178.1179114$

$$e = \left| \frac{399.73341 - 178.1179114}{394.73341} \right| \cdot 100 = 55.44\%$$

hacia atras

$$f'(0{,}01) = \frac{2\tan(1{,}5) - e^{-(3(1{,}5)} - (2\tan(1{,}5 - 0{,}01) - e^{-3(1{,}5-0{,}01)}}{0{,}01}$$

$f'(0{,}01) = 350{,}346533$

$$e = \left| \frac{399.73341 - 350.346533}{3.99{,}73341} \right| = 12.35\%$$

$$f'(0{,}001) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5 - 0{,}01) - e^{-3(1{,}5-0{,}01)})}{0{,}001}$$

$f(0{,}001) = 394.1756$

$$e = \left| \frac{399.73341 - 394.1756}{399.73341} \right| \cdot 100\% = 1{,}40\%$$

$$f'(1{,}5) = \frac{2\tan(1{,}5) - e^{-3(1{,}5)} - (2\tan(1{,}5 - 0{,}001) - e^{-3(1{,}5-0{,}0001)})}{0{,}0001}$$

$f'(1{,}5) = 399{,}1703822$

$$e = \left| \frac{399.7341 - 399.170382}{394.7341} \right| \cdot 100\% = 0{,}1499\%$$

Centrales $h = 0{,}01$

$$\frac{2\tan(1{,}5+0{,}01) - e^{-3(1{,}5+0{,}01)} - \left(2\tan(1{,}5-0{,}01) - e^{-3(1{,}5-0{,}01)}\right)}{2 \cdot 0{,}01} = f'(1{,}5)$$

$$f'(1{,}5) = 407.856858$$

$$e = \left| \frac{399.73341 - 407.856858}{399.73341} \right| \cdot 100 = 2.032292019\ \%$$

$h = 0{,}001$

$$f'(1{,}5) = \frac{2\tan(1{,}5+0{,}001) - e^{-3(1{,}5-0{,}001)} - \left(2\tan(1{,}5-0{,}001) - e^{-3(1{,}5-0{,}001)}\right)}{2 \cdot 0{,}001}$$

$$f'(1{,}5) = 399.813045\ \%$$

$$e = \left| \frac{399.73341 - 399.8130456}{399.73341} \right| \cdot 100 = 0{,}019\ \%$$

$h = 0{,}0001$

$$f'(1{,}5) = \frac{2(\tan(1{,}5+0{,}0001) - e^{-3(1{,}5+0{,}0001)} - \left(2\tan(1{,}5-0{,}0001) - e^{-3(1{,}5-0{,}0001)}\right)}{2 \cdot 0{,}0001}$$

$$e = \left| \frac{399{,}73341 - 399{,}7342121}{399{,}73341} \right| \cdot 100\% = 2{,}006 \cdot 10^{-4}\ \%$$

Conclusiones

- Se calcula una aproximación a la derivada de una función gracias al punto y los valores de esta función.
- El error según cada método va disminuyendo, el más exacto o mejor descrito el que contiene mejor precisión de error es el método central.
- La aproximación de la derivada entrega resultados aceptables con un determinado error, además con el tamaño mientras mas cercano a cero sea, el error seria sumamente pequeño.

Bibliografía

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- Chapra, S. C. & Canale R. P. (1998). Numerical methods for engineers (2da. ed.). New York: McGraw-Hill