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Dead Cell Analysis in Hex and the Shannon Game

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Abstract. In 1981 Claude Berge asked about combinatorial properties that might be used to solve Hex puzzles. In response, we establish properties of dead, or negligible, cells in Hex and the Shannon game.

A cell is dead if the colour of any stone placed there is irrelevant to the theoretical outcome of the game. We show that dead cell recognition is NP-complete for the Shannon game; we also introduce two broader classifications of ignorable cells and present a localized method for recognizing some such cells. We illustrate our methods on Hex puzzles created by Berge.

Keywords. Game theory, Hex, Shannon game, dead cell, negligible, induced path.

1. Introduction

Claude Berge, who loved to play Hex, commented in [4] that

> "[*it*] *would be nice to solve some Hex problem by using nontrivial theorems about combinatorial properties of sets* (*the sets considered are groups of critical* [*board cells*])."

In response, we investigate properties of cells that are dead, or ignorable in a certain sense, in Hex and the Shannon game and show how these properties simplify the solution of Hex puzzles.

In §2 we review Hex and the Shannon game. In §3 we define dead cells and show some basic properties; in particular, dead cell recognition is NP-complete in the Shannon game. In §4 we introduce captured and dominated sets, two broader classes of ignorable cells, and explain how some such sets can be defined in terms of local subgames. In §5 we explain the strategic implications of these results, while in §6 we describe how some such sets can be recognized. In §7 we illustrate our analysis on some Hex puzzles created by Berge.

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2. Hex and the Shannon game

Hex is played on a board containing a rhombic array of hexagonal cells with an equal number of hexagons on each side.[1] Commonly used board sizes are 11×11[2] and 14×14.[3] The two players, Black and White, take turns placing a stone on the board. White's goal is to connect the lower-left and upper-right sides of the board with a chain of white stones; Black's goal is to connect the upper-left and lower-right sides with a black chain. Lest the players forget their goals, each marks their two sides with a pair of extra stones off the board. Figure 1 shows a completed game which White has won.

FIGURE 1. An empty 6×6 Hex board (left) and a completed game (right).

For Hex on any board size, there exists a winning strategy for the first player; however, the only known proof is by *reductio ad absurdum* and no explicit general winning strategy is known [9]. Indeed, determining the existence of a winning strategy is PSPACE-complete for Hex [18] and so also for the Shannon game [6]. For more on Hex, see the website by Thomas Maarup [16], the survey by two of the authors of this paper [12], or the book by Cameron Browne [5].

The Shannon game is played on any graph G with two distinguished *terminal vertices*. Shannon originally formulated the game as played by colouring the edges of the graph; in this paper we consider only the vertex colouring game.[4] Further, we restrict our attention to finite graphs. We assume that the colours used are χ_s and χ_c.

The two players of a Shannon game are called *Short* and *Cut*. To start, the terminal vertices are coloured χ_s and all other vertices are uncoloured. Play proceeds with each player in turn colouring any previously uncoloured vertex. Short's goal is to form a monochrome path containing the terminal vertices; Cut's goal is to prevent this. Thus Short needs to create a χ_s-coloured terminal-connecting path while Cut needs to establish a χ_c-coloured terminal-separating cutset.

Hex is a special case of the Shannon game. Each Hex position, or board state, can be represented as a Shannon graph in two dual ways. Figure 2 shows

[1]If the side lengths are unequal, the game is trivial due to an easy pairing strategy [9].
[2]This is the size used by Piet Hein, the original inventor of Hex [14].
[3]This is the size preferred by Berge [3, 10].
[4]The edge colouring game, known as the *Shannon switching game*, is actually a special case of the Shannon game since it is equivalent to colouring vertices on the line graph of the original graph. Lehman found a polynomial-time algorithmic solution for the Shannon switching game [15].

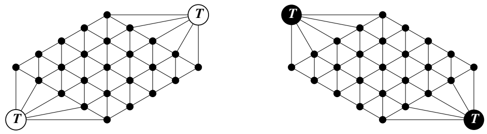

FIGURE 2. The two dual Shannon game representations of the 5×5 Hex board. Short is White on the left and Black on the right. Terminal vertices are labelled 'T'.

the two graphs that correspond to the empty 5×5 Hex board. Neither Hex nor the Shannon game can end in a draw; proofs which hold for Hex are in [1, 8].

For the Shannon game, note that when a vertex is coloured by Cut, it may equivalently be *cut*, or deleted, from the graph. On the other hand, when a vertex is coloured by Short, it may equivalently be *shorted* or contracted by adding edges between all pairs of its neighbors and then deleting the vertex. We refer to the graph that results by shorting and cutting all coloured nonterminal vertices as the *Shannon reduced graph*. For a Shannon board state represented by a graph G, the vertices of the Shannon reduced graph G' are the terminals and uncoloured vertices of G, with two vertices adjacent in G' if and only if they are adjacent or connected by a χ_s-coloured path in G. Short has a winning path in G if and only if the terminals are adjacent in the Shannon reduced graph G'; Cut has a winning cutset in G if and only if the terminals are disconnected in G'. Figure 3 shows a Hex position and the two Shannon reduced graphs that represent it.

In the remainder of this paper we use *node* to refer to both a vertex in a Shannon graph and a cell on a Hex board.

3. Dead nodes

The Shannon game is a special case of a more general class of games known as *division games*, which can be seen as set-colouring games with some function that assigns a winner to each completely coloured set. Following Yamasaki, who gave a theory of division games [21], an element in a particular state of a division game is *regular* if it is never disadvantageous to own it, *misère* if it is never advantageous to own it, and *negligible* if it does not matter who owns it. In the Shannon game,

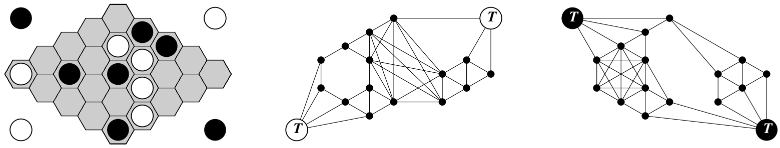

FIGURE 3. A Hex position and its two Shannon reduced graph representations.

all nodes are regular and some nodes may become negligible during the course of the game.

We shall refer to negligible nodes as *dead*. The notion of a dead cell in Hex or other related games was implicitly recognized by Schensted and Titus in their discussion of “worthless triangles” [19] and Beck et al. in the proofs of their opening move results [1, 2].

Formally, define a *board state* (G, T, Ψ) of a Shannon game as a graph G with a set of terminal nodes T and a colouring Ψ of some subset of the nonterminal nodes. For colourings Ψ_1 and Ψ_2, say that Ψ_2 *extends* Ψ_1 if every coloured node in Ψ_1 has the same colour in Ψ_2. A colouring is *complete* if every node is coloured. If a colouring is complete, then the game is over and the winner is known.

Definition 3.1. A nonterminal node v in a board state (G, T, Ψ) is *dead* if, for every complete colouring Ψ^* that extends Ψ, the winner of (G, T, Ψ^*) is independent of the colour of v; v is *live* otherwise.

Note that this definition applies to both coloured and uncoloured nodes.

The reader may enjoy working out which nodes are dead in Figure 3; the answer is shown in Figure 4. Each of the three representations contains exactly one dead node, and it is the same node in each case. As we shall see shortly, this is no coincidence.

Consider a board state (G, T, Ψ) and a set S of uncoloured nodes. When Ψ is extended to a complete colouring by assigning χ_π to all nodes in S and $\chi_{\overline{\pi}}$, the colour different from χ_π, to all other uncoloured nodes, the resulting colouring is denoted by $\Psi \oplus_\pi S$. With respect to a board state, we say that S is a *short set* if $\Psi \oplus_s S$ has a winning chain for Short. A short set is *minimal* if no proper subset is a short set.

Theorem 3.2. *For a board state of the Shannon game, an uncoloured node is live if and only if some minimal short set contains it.*

Proof. ($\Leftarrow$) Let S be a minimal short set that contains v, and let $\Psi^* = \Psi \oplus_s S$. Short is the winner in (G, T, Ψ^*). If the colour of v is subsequently changed then Cut is the winner, since otherwise $S - v$ would be a short set and S would not be minimal. Therefore v is live.

($\Rightarrow$) Suppose v is live. Then there is an extension of Ψ to a complete colouring Ψ^* in which the colour of v determines the winner for Ψ^*. Let Ψ^*_s and Ψ^*_c be Ψ^* with v repainted with χ_s and χ_c respectively, and let S be the uncoloured nodes of

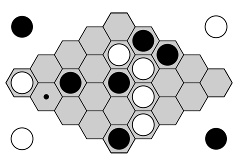

FIGURE 4. The only dead cell for this Hex position.

Ψ which are coloured χ_s in both Ψ_s^* and Ψ_c^*. Since Short wins (G, T, Ψ_s^*), $S \cup \{v\}$ contains a minimal short set S' for (G, T, Ψ). Since Cut wins (G, T, Ψ_c^*), S contains no short set for (G, T, Ψ), so S' is not a subset of S, so S' contains v. □

For a Shannon board state with all non-terminal nodes uncoloured, a set is a short set if and only if it contains a path between the terminals, and a set is a minimal short set if and only if it is an induced, or chordless, inter-terminal path. Thus we have the following.

Corollary 3.3. *In the Shannon game, an uncoloured node is live if and only if it is on some induced inter-terminal path in the Shannon reduced graph.*

A coloured node is live if and only if it is live when uncoloured, so the preceding corollary can also be used to tell whether a coloured node is live.

Recognizing dead nodes simplifies Shannon game analysis, since:

Observation 3.4. *A dead node can be assigned an arbitrary colour or removed from the game without affecting the outcome.*

Thus playing a move at a dead node is equivalent to skipping a move. In the Shannon game no move can be worse than skipping a move, since all nodes are regular, so every move wins if a dead cell represents a winning move. Also, if the game has not ended, then at least one move is to a live node. Hence there is a winning move if and only if there is a live winning move, and so:

Theorem 3.5. *In the Shannon game, a player with a winning strategy has a winning strategy in which every move is to a live node.*

In the interest of streamlining the search for a winning strategy, a player would thus like to be able to recognize dead nodes efficiently. So, how hard is it to recognize dead nodes?

For a vertex subset S of a graph G, the set of all vertices on some induced path between two vertices of S is known as the *monophonic interval* $J(S)$. By the preceding theorem, for a Shannon board state with reduced graph G', the set of live nodes is the monophonic interval $J(T)$ in G', where T consists of the two terminal nodes.

The *induced path pairs* problem is as follows: given a graph and vertex pairs $(a_1, b_1), \ldots, (a_k, b_k)$, is there a vertex induced subpath consisting of k disjoint induced paths joining a_j to b_j? Fellows showed the following [7].

Theorem 3.6 (Fellows). *For $k \geq 2$ the induced path pairs problem is NP-complete.*

For $k = 2$, Marcus Schaefer observed[5] that the induced path pairs problem reduces to the problem of finding a monophonic interval by adding a new vertex x adjacent to b_1 and b_2 and asking whether x is in $J(a_1, a_2)$ Thus we have the following.

Corollary 3.7. *Determining membership of a monophonic interval is NP-complete.*

[5]Private communication.

Corollary 3.8. *In the Shannon game, recognizing dead nodes is NP-complete.*

Dead cell recognition might be easier in Hex than in the Shannon game. In particular, in the Shannon reduced graph of a Hex position, dead nodes are often simplicial or, more generally, separated from both terminals by a clique cutset. Such nodes can be efficiently recognized using Whitesides's algorithm for finding clique cutsets [20].

Unfortunately, these observations do not simplify Hex analysis as much as one might hope, since dead cells arise infrequently in typical Hex positions. On the other hand, considering nodes that are at risk of becoming dead seems more useful, as we explain in the next section.

4. Beyond death: captured and dominated sets

We introduce in this section two concepts that generalize node death. Informally, we call a set of uncoloured nodes π-*captured* if player π can "own" the entire set no matter who plays first, and π-*dominated* if π can own it provided π has the first move. These notions evolved from [17, 11, 13].

By the *value* of a position, we mean the outcome assuming perfect play.

Consider for example the two Hex diagrams shown in Figure 5. On the left, the two marked nodes are effectively captured by Black, since a White stone played at either node becomes dead if Black then plays at the other. Thus adding Black stones to these two nodes will not change the value of the game. On the right, the three marked nodes form a dominating set for Black, since a Black move to the node with a large dot captures the two other nodes.

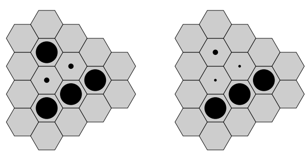

Figure 5. A Black-captured set (left) and a Black-dominated set (right).

Before formally defining capture and domination, we first introduce a generalization of the Shannon game.

The *multi-Shannon game* is a Shannon game played on a graph with two or more terminals. At the start of the game, all terminals are χ_s-coloured. Short's goal is to join each pair of connected terminals with some χ_s-coloured path; these paths may intersect. Cut's goal is to separate each pair of nonadjacent terminals with some χ_c-coloured cutset; these cutsets may intersect. This game can end in a draw.

Consider the two graphs of Figure 6. On the left graph, Cut has a second-player winning strategy, meaning that Cut wins even when Short goes first. On the right graph, Short has a second-player winning strategy.

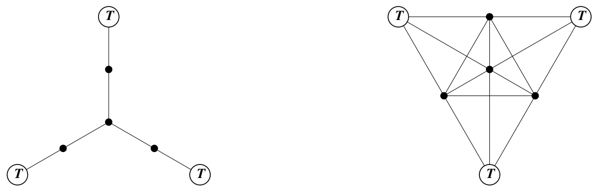

FIGURE 6. Multi-Shannon games with second-player wins for Cut (left) and Short (right).

When such a graph occurs as a particular subgraph in a regular Shannon game, we can simplify the analysis of the game. In a graph, the *neighbourhood* $N(S)$ of a set S of nodes is the set of all nodes not in S but adjacent to some node in S. Let G be a reduced Shannon graph with a set S of nonterminal nodes and let Γ_S be the multi-Shannon game played on the subgraph of G induced by $S \cup N(S)$ with terminals $N(S)$. Then:

Theorem 4.1. *If player π has a second-player winning strategy for Γ_S, then S can be filled in with χ_π stones without changing the Shannon value of Γ.*

Proof. Let $\Gamma = (G, T, \Psi)$ be a board state, and let $\Gamma' = (G, T, \Psi')$ where Ψ' is the extension of Ψ by filling in S with χ_π. The theorem is equivalent to stating that π wins Γ if and only if π wins Γ'.

($\Rightarrow$) If π wins Γ, then π trivially also wins on Γ', since Γ' is formed by giving π a number of "free" moves.

($\Leftarrow$) If π has a winning strategy for Γ' and a second-player winning strategy for Γ_S, then π can adopt the following strategy for Γ. Whenever $\overline{\pi}$ plays in S, π responds with a move in S according to the winning strategy for Γ_S. Otherwise, π plays a move in $G - S$ according to the winning strategy for Γ'. Let Γ^* be the final board state when all nodes are coloured. There are two cases to distinguish:

- π plays Cut: If Short forms a chordless winning path P in Γ then P cannot intersect S, for otherwise the two nodes $P \cap T$ would be nonadjacent and Short would have at least achieved a draw in Γ_S with the path $P \cap S$. But if P does not intersect S then Short has won Γ', contradicting Cut's winning strategy for Γ'.
- π plays Short: Short forms some winning path P' in Γ'. If this path does not intersect S then it is also a winning path for Short on Γ. If it does intersect S then there is a winning path on Γ that consists of $P' - S$ plus some Short path connecting the two nodes $P \cap S_T$; the latter is guaranteed to exist due to Short's win on Γ_S.

Therefore π wins Γ. □

For example, let S consist of the two marked nodes in the left diagram of Figure 5 and consider the resulting subgame Γ_S; thus $N(S)$ consists of the twelve

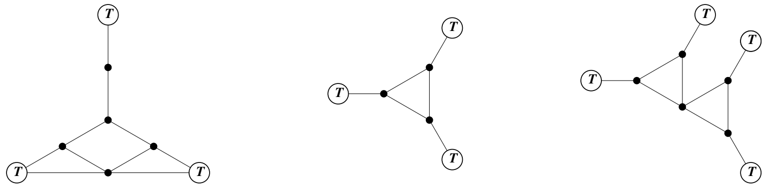

FIGURE 7. Multi-Shannon games with first-player wins for Cut.

nodes forming the diagram boundary. Short has a second-player winning strategy for Γ_S: play at whichever node of S that Cut does not play at. Thus the theorem tells us that for any Hex position containing this diagram the marked nodes can be filled in with black stones without changing value of the position.

In the graph on the left in Figure 7, the game ends in a draw when Short has the first move. If Cut has the first move, then Cut wins by colouring the bottom node. If this graph occurs as a subgraph in a regular Shannon graph, then a Cut move in the bottom node is equivalent to cutting all the nodes in the subgraph, since the remaining part of the subgraph is a second-player win for Cut and can therefore be filled in with χ_c. This leads to the following theorem.

Theorem 4.2. *If player π has a first-player winning strategy for Γ_S with winning move v, then in Γ v is at least as good for π as any other move in S.*

Proof. A π-move to v is equivalent to simultaneous π-moves to all nodes of S. □

We now formally define capture and domination with respect to a Shannon game Γ, a set of uncoloured nodes S, and the local multi-Shannon game Γ_S.

Definition 4.3. S is *π-captured* if π has a second-player winning strategy for Γ_S.

Definition 4.4. S is *π-dominated* if π has a first-player winning strategy for Γ_S. For any initial move m in such a strategy, we say that m *π-dominates* S.

Based on the usual recursive definitions of wins and losses in games, we have the following.

Observation 4.5. *S is π-captured if and only if S is the empty set or, for each $\overline{\pi}$-move m in S, both* (i) *$S - m$ is π-dominated and* (ii) *m is dead if $S - m$ is filled in with χ_π stones.*

Observation 4.6. *S is π-dominated if and only if S is the empty set or there is some π-move m in S such that $S - m$ is π-captured.*

In Observation 4.5, (i) guarantees that π can capture all cells of $S - m$ after an $\overline{\pi}$ move at m, while (ii) guarantees that the $\overline{\pi}$-stone at m would then also be dead; thus π captures all cells of S.

		Short moves first		
		Short wins	draw	Cut wins
Cut moves first	Short wins	Short-captured	–	–
	draw	Short-dominated	indifferent	–
	Cut wins	both dominated	Cut-dominated	Cut-captured

TABLE 1. Possible outcomes of a local position. E.g., for the event "Short wins when both Short and Cut go first", the outcome is "Short-captured".

Note that for any position of a Shannon game, each uncoloured node is dominated by both players and each dead node is captured by both players. Also, note that captured and dominated sets are defined only for uncoloured nodes, whereas dead cells are defined for coloured and uncoloured nodes.

5. Strategic advice

A local subgame Γ_S can be a win for Short or Cut or a draw, depending on whether Short or Cut has the first move. The possible outcomes are listed in Table 1. The boxes marked '–' represent impossible combinations, since having the first move cannot be a disadvantage in the multi-Shannon game.

Suppose player π is considering a move in Γ_S. According to the outcomes in Table 1, the following cases can be distinguished:

- One of the players has a second-player win. Then the set is captured and can be filled in, as per Theorem 4.1. Any π-move in S would be wasted.
- Set S is dominated by π. Then π has a locally winning move in Γ_S. By Theorem 4.2, π can safely play such a move.
- Set S is dominated by $\overline{\pi}$ but not by π. Then the best local move available for π is a local draw, while other moves may be local losses. Since a locally losing move can be followed by a $\overline{\pi}$-move that captures S and kills π's move, π should avoid locally losing moves.
- Set S is not dominated by either player. Any π-move leads to a local draw with locally optimal play. The choice of move depends on which pairs of terminals are favourable for π to connect or disconnect in the global game.

This information can be summarized by the following theorem.

Theorem 5.1. *Let Γ be a multi-Shannon game, and let v and w be moves in a subgame Γ_S defined by a set of nodes S. If v is at least as good as w for player π in Γ_S, then v is at least as good as w for π in Γ.*

Thus, if π is going to move in S, then π should make a move that is optimal in Γ_S. In short, "do not make any local mistakes". Here we consider the act of making no move at all to be optimal in a second-player win game, since it is better than wasting a move.

Once dominated and captured sets have been identified, the strategic advice for player π to move, is:

1. Fill in all captured sets, iterate until no new captured nodes are found.
2. For any π-dominated set, pick one dominating move and ignore the rest.
3. For any $\overline{\pi}$-dominated set, ignore the locally losing moves.

A special case of Step 3 is a $\overline{\pi}$-dominated set with two nodes $\{v, w\}$. There, the dominating move for $\overline{\pi}$, say v, is also the only locally drawing move for π. If π moves in w then $S - w$ is still dominated by $\overline{\pi}$, since it is a single uncoloured node, and v is dead after $\overline{\pi}$ moves in w, since S was $\overline{\pi}$-dominated by w.

Theorems 4.1–5.1 also hold in the more general case where a multi-Shannon game is a subgame of another multi-Shannon game.

6. Recognizing captured and dominated sets

In order for a player to benefit from our results so far, the player should be able to recognize captured and dominated sets efficiently. One way to achieve this is to build a library of local patterns, or subgames, that yield such sets.

Consider a set Q of nodes of a Hex position, where Q may contain Black or White stones. Let S be the uncoloured nodes of Q. Create a new board position by uncolouring all coloured nodes in $N(Q)$. Now let Γ be the reduced Shannon graph of this new position. Any move in S that is suboptimal in the multi-Shannon game Γ_S is also suboptimal in the original Hex position. This allows moves in Q to be analysed while ignoring the rest of the board. Moreover, whenever this same node pattern occurs anywhere else on the board, the local analysis results are the same. We consider such a pattern to be *irreducible* if none of the captured cells or suboptimal moves could have been detected by considering a smaller pattern.

Such a library need not be exhaustive, as there is a trade-off in effort saved by disregarding locally suboptimal moves and effort invested in detecting them. Experimental computer results suggest that in Hex games such sets are almost always reducible to a few base cases that can be derived from simplicial nodes.

Such a derivation is illustrated in Figure 8, which shows irreducible local Hex patterns with two empty cells that can be built up by starting from all cases in which a node is found to be simplicial by considering only its immediate neighborhood in the nonreduced graph. In the five starting patterns shown on the bottom of the figure, the empty cell is dead. Removing a White stone from each pattern yields a pattern with a two-cell White-dominated set; the dominating move is indicated with a dot. Now, pairs of these six patterns are combined to form larger patterns with two empty cells, such that each empty cell White-dominates the other; these two cells form a White-captured set. The eleven larger patterns that can be created in this way are shown as entries of the central array of the chart, with each entry indexed by the two smaller patterns that yield it.

There are no irreducible captured set patterns with three connected empty cells. Figure 9 shows examples of irreducible captured sets with four empty cells.

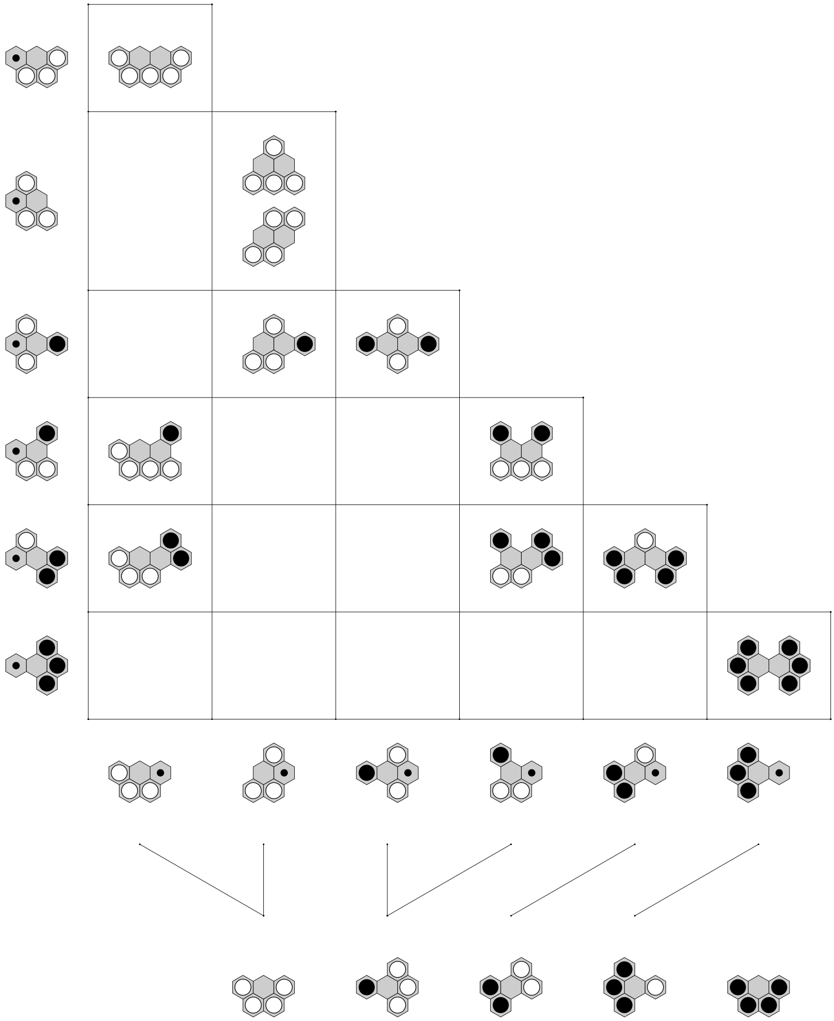

FIGURE 8. Constructing irreducible White-captured sets (empty hexagons of table entry patterns) and White-dominated sets (empty hexagons of row and column index patterns) from the five bottom dead-cell patterns.

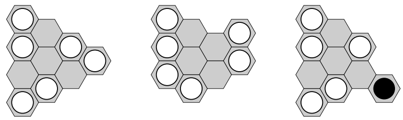

FIGURE 9. Some irreducible White-captured sets.

7. Hex examples

Claude Berge presented five Hex puzzles in his introduction to the game [3][6] (see also [10]). Figures 10–13 show the static analysis of these puzzles according to the notions of captured and dominated sets. Captured cells, including dead cells, have been marked by a large dot. Small dots represent suboptimal moves in locally dominated sets. Table 2 summarizes the statistics for these puzzles.

FIGURE 10. Berge's Puzzles 1 and 2. White to play can ignore all but the unmarked empty cells. Large dots are captured; small dots are locally inferior.

puzzle number	available moves	ignored moves	viable moves	percentage ignored
1	15	11	4	73%
2	19	13	6	68%
3	163	65	98	40%
4	145	75	70	52%
5	120	59	61	49%

TABLE 2. Statistics for dead cell analysis of Berge's puzzles.

Acknowledgment

We thank Cameron Browne for the use of his Hex drawing software and Marcus Schaefer for bringing Theorem 3.6 and Corollary 3.7 to our attention.

[6]The Puzzle 5 that we include is from an early version of Berge's manuscript; a later version has two additional white stones near the lower-right side.

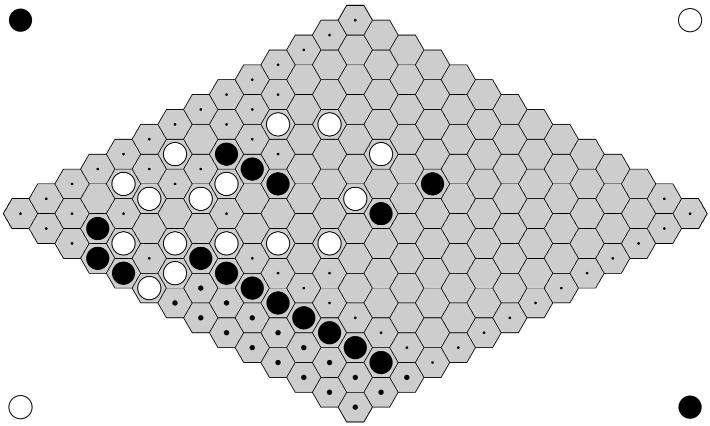

FIGURE 11. Berge's Puzzle 3: Black to play.

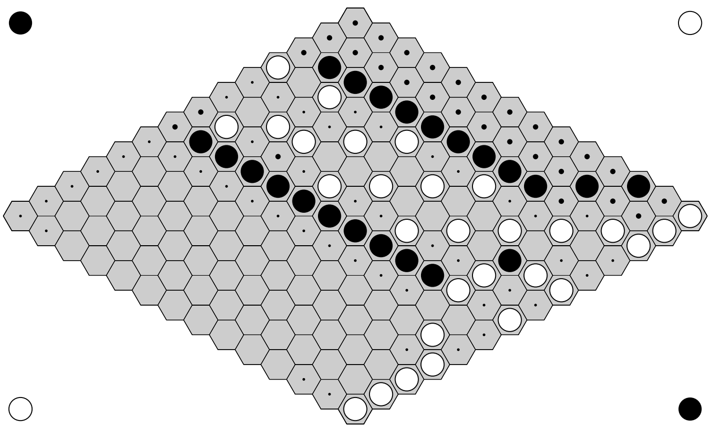

FIGURE 12. Berge's Puzzle 4: Black to play.

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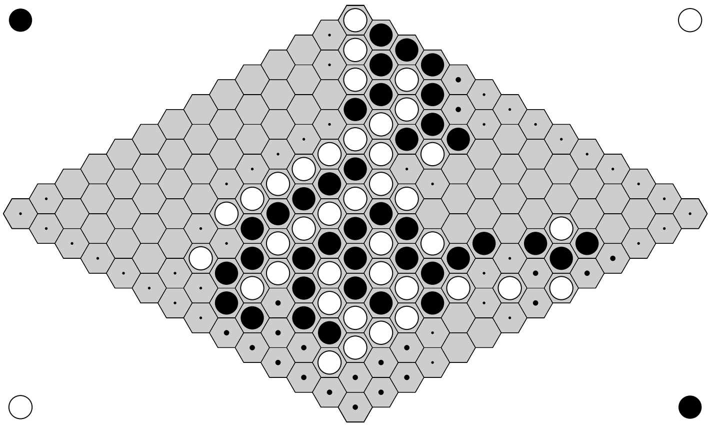

FIGURE 13. Berge's Puzzle 5: White to play.

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