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Novel design of Morlet wavelet neural network for solving second order Lane–Emden equation

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Abstract

In this study, a novel computational paradigm based on Morlet wavelet neural network (MWNN) optimized with integrated strength of genetic algorithm (GAs) and Interior-point algorithm (IPA) is presented for solving second order Lane–Emden equation (LEE). The solution of the LEE is performed by using modelling of the system with MWNNs aided with a hybrid combination of global search of GAs and an efficient local search of IPA. Three variants of the LEE have been numerically evaluated and their comparison with exact solutions demonstrates the correctness of the presented methodology. The statistical analyses are performed to establish the accuracy and convergence via the Theil's inequality coefficient, mean absolute deviation, and Nash Sutcliffe efficiency based metrics.

Keywords: Lane–Emden equation; Artificial neural networks; Singular; Genetic algorithm; Nonlinear; Interior-point algorithm

1. Introduction

Recently, the study of the second order Lane–Emden equation (LEE) attained the attention of many researchers due to its vast applications in applied sciences, technological and engineering fields. The standard LEE takes the form as [7,23,46,50]:

$$\begin{cases} \omega''(t) + \dfrac{\chi}{t}\omega'(t) + f(t,\omega) = u(t), 0 < t \le 1, \chi \ge 0, \\ \omega(0) = a_1, \ \ \omega'(0) = a_2, \end{cases} \tag{1}$$

where a_1 and a_2 are constants, the real valued continuous function is $f(t,\omega)$ and $u(t) \in C\ [0, 1]$. The LEE has a great history; the astrophysicists Lane and Emden used this equation in their pioneer work in the thermodynamics using classical-laws [29]. The LLE is applied in the modelling of several phenomena like as astrophysics, mathematical

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physics, stellar structure theory and isothermal gas spheres [15,27,44]. Solving the LLE is always very challengeable and interesting for the researchers due to the singularity at the origin. Many researchers applied different techniques to solve LEE in their research studies, e.g., the approximate solutions of the LEE. Shawagfeh [42] and Wazwaz [48] applied Adomian methods, Parand et al. [28] suggested an algorithm using the Hermite functions based collocation technique, Bender et al. [9] proposed the perturbation techniques, Nouh [26] used Padé approximation and Euler–Abel transformation of the power series convergence solution of the LLE, Ramos [37] suggested piecewise linearization method for the solution of LEE while Mandelzweig and Tabakin [17] used Bellman and Kalaba's quasilinearization technique.

It is clear that the LEE previously solved by many techniques and every method has its own strength and efficiency, as well as, limitations over one another. The present technique is based on stochastic solvers for the solutions of linear and nonlinear systems by the neural network's modelling strength and optimization ability of the networks [4,6]. Some recent applications of neural network and evolutionary computing are Pseudo-almost periodic solution [5], nonlinear singular Thomas–Fermi systems [38], nonlinear Troesch's problem [35], doubly singular nonlinear systems [32], nonlinear predator prey model [45], nonlinear model for wire coating analysis [24], heartbeat model [34], distribution of heat phenomena [36], control system [14], power [30], cell biology [40] and energy [19].

The aim of the present work is two folded as:

- Introducing the layer structure design of Morlet wavelet neural networks (MWNNs) to solve the LEE.
- Present the solution of LEE for better understanding of the system dynamics by exploitation of stochastic numerical technique.

The intention of the present work is to solve the model of LEE given in Eq. (1) with the help of intelligent computing based on the strength of MWNNs optimized with genetic algorithm (GA) supported by interior-point algorithm (IPA), i.e., MWNN-GA-IPA. Some main features of the present scheme MWNN-GA-IPA are briefly summarized as follows:

- A new layer structure design of MWNNs is presented for effective, reliable and accurate modelling of variants of LEE.
- Exploration and manipulation in MWNNs with integrated evolutionary heuristics to find the accurate, consistent and robust numerical results for the LEE.
- The consistent overlapping solutions with exact/desire results established the precision and stability of the MWNN-GA-IPA based numerical solver.
- Validation of working of MWNN-GA-IPA through the performance measures based statistics with the help of mean absolute deviation (MAD), Theil's inequality coefficient (TIC) and Nash Sutcliffe efficiency (NSE) based indices.
- The advantages of proposed methodology MWNN-GA-IPA are ease in smooth implementation and exhaustive applications by utilizing the neural network modelling with Morlet wavelet based activation function to handle effectively the complicated, nonlinear, singular and stiff Lane–Emden based systems.

Information in the rest of the paper is presented as follows. The designed scheme MWNN-GA-IPA is presented in Section 2, simulations and results are provided in Section 3 while conclusions are provided in Section 4.

2. Designed methodology

The design of MWNNs is presented here for solving second order LEE. The construction of fitness function using the MWNNs, and optimization with GA-IPA is also described. The graphical illustration of the designed methodology MWNN-GA-IPA is provided in Fig. 1.

2.1. Design Morlet wavelet neural networks

The ability of ANN based models for providing the consistent solution is arising in many fields. The mathematical model of the LEE given in the model (1) is expressed with feed-forward MWNNs in the form of the approximate

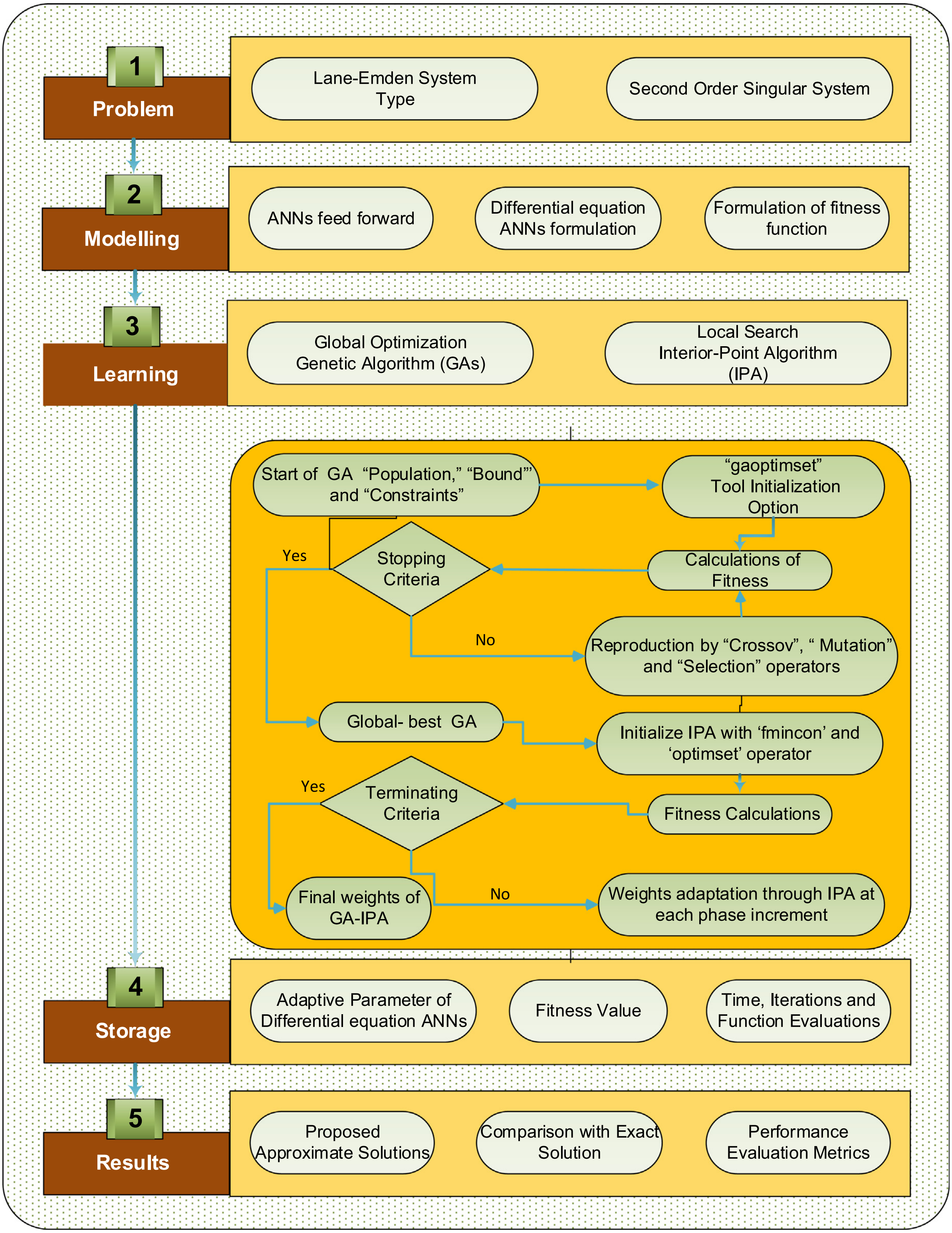


Fig. 1. Schematic of MWNN-GA-IPA for solving the LEEs.

solution and its respective nth order derivatives as:

$$\begin{aligned}\hat{\omega}(t) &= \sum_{i=1}^{m} \varphi_i\, p(w_i t + \beta_i)\\ \hat{\omega}^{(\mathrm{n})} &= \sum_{i=1}^{m} \varphi_i\, p^{(\mathrm{n})}(w_i t + \beta_i)\end{aligned} \tag{2}$$

The network with m neurons while $[\varphi, w, \beta]$ is one set of neuron represented with unknown weight vectors $\boldsymbol{W} = [\boldsymbol{\varphi}, \boldsymbol{w}, \boldsymbol{\beta}]$ as:

$$\boldsymbol{\varphi} = [\varphi_1, \varphi_2, \ldots, \varphi_m],\ \boldsymbol{w} = [w_1, w_2, \ldots, w_m] \text{ and } \boldsymbol{\beta} = [\beta_1, \beta_2, \ldots, \beta_m]$$

The Morlet wavelet neural network has not been designed before for solving the LEE. The Morlet wavelet function is actually the product of cos of $\frac{7}{4}t$ with the Gaussian function and given as:

$$p(t) = \cos\left(\frac{7}{4}t\right) e^{\left(-\frac{t^2}{2}\right)}. \tag{3}$$

Using the Morlet activation function, Eq. (3) becomes

$$\begin{aligned}\hat{\omega}(t) &= \sum_{i=1}^{m} \varphi_i \cos\left(\frac{7}{4}(w_i t + \beta_i)\right) e^{-\frac{(w_i t+\beta_i)^2}{2}},\\ \hat{\omega}'(t) &= \sum_{i=1}^{m} -\varphi_i w_i e^{-\frac{(w_i t+\beta_i)^2}{2}} \left(\sin\left\{\frac{7}{4}(w_i t + \beta_i)\right\} + (w_i t + \beta_i)\cos\left\{\frac{7}{4}(w_i t + \beta_i)\right\}\right),\\ \hat{\omega}''(t) &= \sum_{i=1}^{m} -\varphi_i w_i^2 e^{-\frac{(w_i t+\beta_i)^2}{2}} \begin{pmatrix} -3.0625\cos\left\{\frac{7}{4}(w_i t + \beta_i)\right\} + \frac{7}{2}(w_i t + \beta_i)\sin\left\{\frac{7}{4}(w_i t + \beta_i)\right\} \\ +\left\{-1 + (w_i t + \beta_i)^2\right\}\cos\left\{\frac{7}{4}(w_i t + \beta_i)\right\} \end{pmatrix}.\end{aligned} \tag{4}$$

Using the networks (4), one can formulate the architecture for LEE, while objective function E in mean squared error sense is defined as a merit function as:

$$E = E_1 + E_2, \tag{5}$$

where E_1 is used as an unsupervised error while E_2 is used for initial conditions. The E_1 is expressed as follows:

$$E_1 = \frac{1}{N}\sum_{k=1}^{N}\left(\hat{\omega}_k'' + \frac{\chi}{t_k}\hat{\omega}_k' + f(t_k, \hat{\omega}_k) - u_k\right)^2, \tag{6}$$

with $N = \frac{1}{h},\ \ \hat{\omega}_k = \omega(t_k),\ \ u_k = u(t_k),\ \ t_k = kh$

Accordingly, E_2 is given as:

$$E_2 = \frac{1}{2}\left((\hat{y}_0)^2 + (\hat{y}_N)^2\right) \tag{7}$$

The solution of the LEE can be found with the weights, such that error function $E \to 0$, then, the estimated results match with the exact solutions of the Lane–Emden model, *i.e.*, $\hat{\omega}(t) \to \omega(t)$.

2.2. Optimization of MWNNs weights

The optimization of an MWNNs based models of differential equation for singular LEE is achieved with the help of hybrid computing of GA-IPA.

Genetic Algorithms (GAs) are global search algorithm modelled on the evolutionary concepts of natural genetic process in human beings. It is an intelligent random search method used to solve optimization problems. For the near to the best solution of the model, the GAs work with its reproduction tools via crossover, selection and mutation operations. Recently GAs are used for monitoring of virtual machines in cloud environment [8], dynamic product routing in agile manufacturing environment [16], simulation for variable transmission [11], route optimization of

Table 1
Pseudo code of MWNN-GA-IPA for solving LEEs.

```
Genetic Algorithm: start
      Inputs: The individual signified with equal number of entries as
      decision variables:
            W = [φ, w, β]
      Population: Created with set of W as follows
            P = [W_1, W_2, ..., W_n]^t
            φ = [φ_1, φ_2, ..., φ_m], w = [w_1, w_2, ..., w_m] and β = [β_1, β_2, ..., β_m]
      Output: The Global Best weights of genetic algorithm W_GA-Best
      Initialize: Formulated W with real numbers and the initial P.
      Fitness Evaluation: Find the fitness E of each W in P using equations
      (5-7)
      Terminate: Stop GA optimization for any of the following
         • "Fitness" E leas than equal to 10^-14,
         • "TolFun" less than equal to 10^-12,
         • "TolCon" less than equal to 10^-12,
         • "StallGenLimit" equal to 90
         • "Generations" equal to 130,
         • Other default settings
      Go to the storage step else continues.
      Rank Scaling: Scale the rank of each W in P according to E value
      Reproduction: Create P for each generation by Selection: "@uniform",
      Crossover: "@heuristic", Mutations: "@aptfeasible" and Elitism
      Storage: Store W_GA-Best, E, time, generations and function counts.
End of Genetic algorithm
IPA based local search: Start
      Inputs: W_GA-Best as a initial bias weights
      Output: Weight vector of GA-IPA, i.e., W_GA-IPA
      Initialization: Start-point as W_GA-Best, Assignments for bounds,
      iterations and other settings
      Termination: Adaptation process ends for any of the following
      conditions:
      'Fitness' E less than equal to 10^-15,
      'total Iterations' equal to 600,
      'TolFun' less than equal to 10^-20,
      'TolX' less than equal to 10^-22,
      'TolCon' less than equal to 10^-22, and
      'MaxFunEvals' less than equal to 220000
      While (Terminate required fulfilled)
            Fitness evaluation: To evaluate E of W using equations (5-7)
            Fine tunning: Use "fmincon" routine algorithm interior-point for
            IPA. Update parameters of W for each iteration of IPA and
            calculate E of modified W using equations (5-7)
      Store
            Save weight vector W_GA-IPA, E, time, iterations and function counts.
End IPA procedure
```

municipal solid waste collection [2], dynamic stability margin evaluation of multi-machine power systems [25], diagonalization of symmetric and Hermitian matrices [47], for detecting communities in social networks [39] and for the two-stage capacitated facility location problems [10].

Interior-point algorithm is a local search algorithm used for optimization of convex constrained/unconstrained tasks. In recent years, the said local search based optimization methods are applied in linear programming [49], for non-smooth contact dynamics [18], for atomic norm soft thresholding [12], for symmetric cone optimization [41] and for nonlinear optimization with a quasi-tangential sub-problem [43]. Integrated strength of hybrid computing GA-IPA is utilized to get the decision variables of the MWNNs to solve the variants of LEE (see Table 1).

3. Performance indices

The performance operators with their global updated form, i.e., TIC, MAD and ENCE, as well as, GTIC, GMAD and GENSE, are implemented to access the working of the presented MWNN-GA-IPA. The formulation of these indices is given as follows:

$$\mathrm{MAD} = \frac{1}{n}\sum_{m=1}^{n}\left|\omega_m - \hat{\omega}_m\right|, \tag{8}$$

$$\mathrm{TIC} = \frac{\sqrt{\frac{1}{n}\sum_{m=1}^{n}\left(\omega_m - \hat{\omega}_m\right)^2}}{\left(\sqrt{\frac{1}{n}\sum_{m=1}^{n}\omega_m^2} + \sqrt{\frac{1}{n}\sum_{m=1}^{n}\hat{\omega}_m^2}\right)} \tag{9}$$

$$\mathrm{NSE} = \left\{1 - \frac{\sum_{m=1}^{n}\left(\omega_m - \hat{\omega}_m\right)^2}{\sum_{m=1}^{n}\left(\omega_m - \overline{\omega}_m\right)^2}, \qquad \overline{\omega}_m = \frac{1}{n}\sum_{m=1}^{n}\omega_m \right. \tag{10}$$

$$\mathrm{ENSE} = 1 - \mathrm{NSE} \tag{11}$$

where n denotes the grid points.

4. Simulations and results

In this section, numerical implementations of MWNN-GA-IPA have been presented for solving second order singular LEEs.

Problem 1. Consider the non-homogeneous LEE is:

$$\begin{cases} \omega''(t) + \dfrac{2}{t}\omega'(t) + \omega(t) = 6 + 12t + 2t^2 + t^3, \\ \omega(0) = 0, \omega'(0) = 0. \end{cases} \tag{12}$$

The exact solution is $t^2 + t^3$ while the respective objective function is given as:

$$E = \frac{1}{N}\sum_{m=0}^{N}\left(\left(t_m\widehat{\omega}''_m + 2\hat{\omega}'_m + t_m\hat{\omega}_m - 6t_m - 12t_m^2 - 2t_m^3 - t_m^4\right)^2\right) + \frac{1}{2}\left(\left(\hat{\omega}_0\right)^2 + \left(\hat{\omega}'_0\right)^2\right) \tag{13}$$

Problem 2. Consider the second singular system as:

$$\begin{cases} \omega''(t) + \dfrac{2}{t}\omega'(t) = 1, \\ \omega(0) = 1, \omega'(0) = 0. \end{cases} \tag{14}$$

The exact solution is $\frac{1}{6}t^2$ while the respective objective function of Eq. (14) is given as:

$$E = \frac{1}{N}\sum_{m=0}^{N}\left(\left(t_m\widehat{\omega}''_m + 2\hat{\omega}'_m - 1\right)^2\right) + \frac{1}{2}\left(\left(\hat{\omega}_0\right)^2 + \left(\hat{\omega}'_0\right)^2\right) \tag{15}$$

Problem 3. Consider the inhomogeneous doubly singular system as:

$$\begin{cases} \omega''(t) + \dfrac{1}{t}\omega'(t) - 9\omega(t) = -9 - 9t^3 + \dfrac{1}{t}, \\ \omega(0) = 1, \omega(1) = 3. \end{cases} \tag{16}$$

The exact solution is $1 + t + t^3$ and the objective function of Eq. (16) is written as:

$$E = \frac{1}{N}\sum_{m=0}^{N}\left(\left(t_m\widehat{\omega}''_m + \hat{\omega}'_m - 9t_m\hat{\omega}_m + 9t_m + 9t_m^4 - 1\right)^2\right) + \frac{1}{2}\left(\left(\hat{\omega}_0 - 1\right)^2 + \left(\hat{\omega}_N - 3\right)^2\right) \tag{17}$$

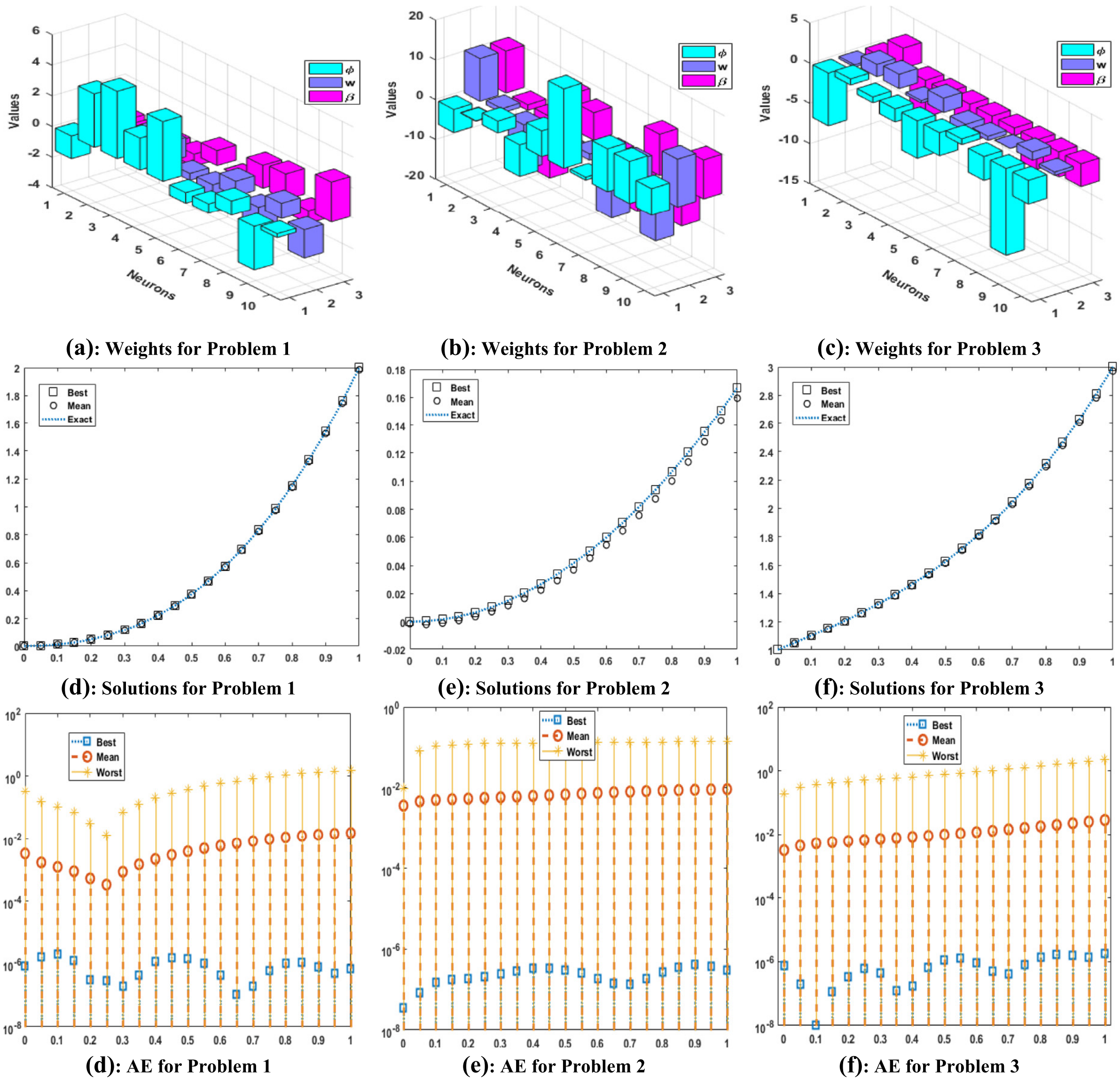


Fig. 2. Weights on MWNNs, approximate solution and AE for each LEE presented in Problems 1–3.

Optimization procedure MWNN-GA-IPA based solution of the singular second-order LEE is determined for 100 distinct runs with the help of adaptable network parameters. These sets of weights as presented in Fig. 2 are used to find the approximate solution of the LEEs. The mathematical form for three respective problems is derived as follows:

$$\begin{aligned}\hat{\omega}_{p-1} = & -1.4826\cos\left[1.75(0.6042t-0.3991)\right]e^{-\frac{(0.6042t-0.3991)^2}{2}} \\ & +3.4890\cos\left[1.75(0.2107t-0.4454)\right]e^{-\frac{(0.2107t-0.4454)^2}{2}} \\ & +\cdots+0.2564\cos\left[1.75(-1.8543t+2.6213)\right]e^{-\frac{(-1.8543t+2.6213)^2}{2}}.\end{aligned} \tag{18}$$

$$\begin{aligned}\hat{\omega}_{p-2} = & -5.8170\cos\left[1.75(10.7387t+10.5499)\right]e^{-\frac{(10.7387t+10.5499)^2}{2}} \\ & +0.1166\cos\left[1.75(1.0941t-1.4023)\right]e^{-\frac{(1.0941t-1.4023)^2}{2}} \\ & +\cdots+6.7393\cos\left[1.75(12.1091t+9.8588)\right]e^{-\frac{(12.1091t+9.8588)^2}{2}}.\end{aligned} \tag{19}$$

Table 2
Results of statistical indicators for nonlinear singular LEEs.

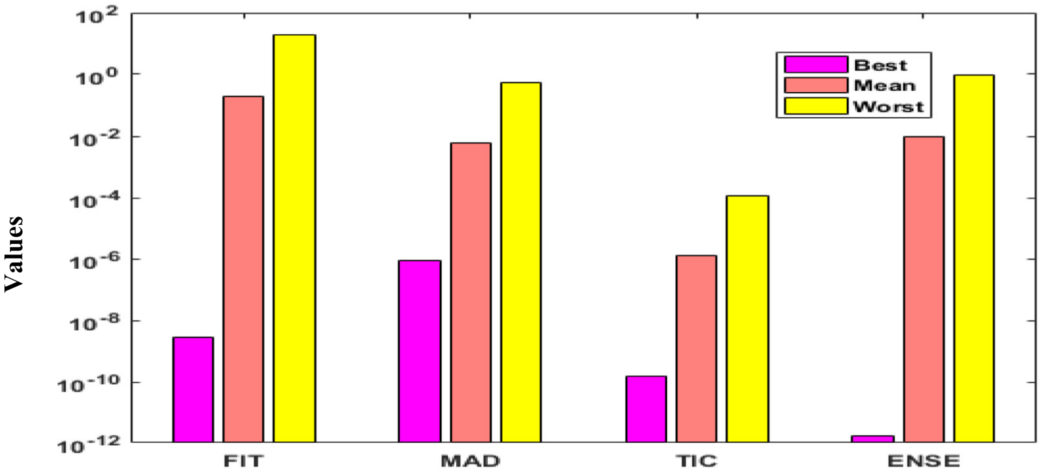
t	Problem-1			Problem-2			Problem-3		
	Min	Mean	SD	Min	Mean	SD	Min	Mean	SD
0	1.8E−07	3.3E−03	3.2E−02	4.5E−10	3.6E−03	7.4E−03	2.0E−07	3.1E−03	2.0E−02
0.05	1.7E−08	1.7E−03	1.6E−02	8.4E−08	4.6E−03	1.1E−02	2.0E−07	4.4E−03	3.1E−02
0.10	1.1E−06	1.2E−03	1.0E−02	1.5E−07	5.0E−03	1.4E−02	1.0E−08	5.2E−03	3.8E−02
0.15	7.8E−07	8.9E−04	6.8E−03	7.1E−09	5.2E−03	1.4E−02	7.5E−08	5.7E−03	4.3E−02
0.20	3.2E−07	5.3E−04	3.2E−03	5.9E−08	5.4E−03	1.5E−02	4.9E−08	6.1E−03	4.7E−02
0.25	2.9E−07	3.4E−04	1.4E−03	2.1E−07	5.6E−03	1.5E−02	5.0E−07	6.5E−03	5.1E−02
0.30	1.9E−07	8.6E−04	6.7E−03	2.2E−07	5.8E−03	1.5E−02	2.6E−07	7.1E−03	5.5E−02
0.35	4.4E−07	1.5E−03	1.3E−02	8.6E−08	6.1E−03	1.6E−02	5.5E−08	7.6E−03	6.0E−02
0.40	3.3E−07	2.2E−03	2.0E−02	3.9E−08	6.3E−03	1.6E−02	5.5E−08	8.3E−03	6.5E−02
0.45	6.6E−07	3.0E−03	2.8E−02	5.0E−08	6.6E−03	1.6E−02	1.3E−08	9.0E−03	7.1E−02
0.50	5.9E−07	3.9E−03	3.7E−02	2.1E−07	6.9E−03	1.7E−02	8.4E−08	9.8E−03	7.8E−02
0.55	3.8E−07	4.9E−03	4.7E−02	2.5E−07	7.2E−03	1.7E−02	3.9E−08	1.1E−02	8.5E−02
0.60	4.4E−07	5.9E−03	5.7E−02	1.8E−07	7.5E−03	1.8E−02	3.2E−08	1.2E−02	9.4E−02
0.65	9.8E−08	7.1E−03	6.9E−02	1.4E−07	7.8E−03	1.8E−02	2.6E−08	1.3E−02	1.0E−01
0.70	1.9E−07	8.3E−03	8.1E−02	8.2E−08	8.0E−03	1.9E−02	3.7E−09	1.4E−02	1.1E−01
0.75	4.5E−07	9.6E−03	9.4E−02	1.9E−07	8.3E−03	2.0E−02	5.2E−07	1.6E−02	1.3E−01
0.80	9.1E−08	1.1E−02	1.1E−01	2.7E−07	8.5E−03	2.1E−02	7.0E−08	1.8E−02	1.4E−01
0.90	1.8E−07	1.2E−02	1.2E−01	3.6E−07	8.8E−03	2.1E−02	9.2E−08	2.0E−02	1.6E−01
1	2.7E−07	1.3E−02	1.3E−01	4.0E−07	8.9E−03	2.2E−02	3.5E−08	2.3E−02	1.8E−01

$$
\begin{aligned}
\hat{\omega}_{p-3} = & -6.5563\cos\left[1.75(-0.2803t-0.9916)\right]e^{-\frac{(-0.2803t-0.9916)^2}{2}} \\
& +0.7546\cos\left[1.75(1.5309t+2.5106)\right]e^{-\frac{(1.5309t+2.5106)^2}{2}} \\
& +\cdots-2.9534\cos\left[1.75(-0.3994t-2.8392)\right]e^{-\frac{(-0.3994t-2.8392)^2}{2}}.
\end{aligned} \tag{20}
$$

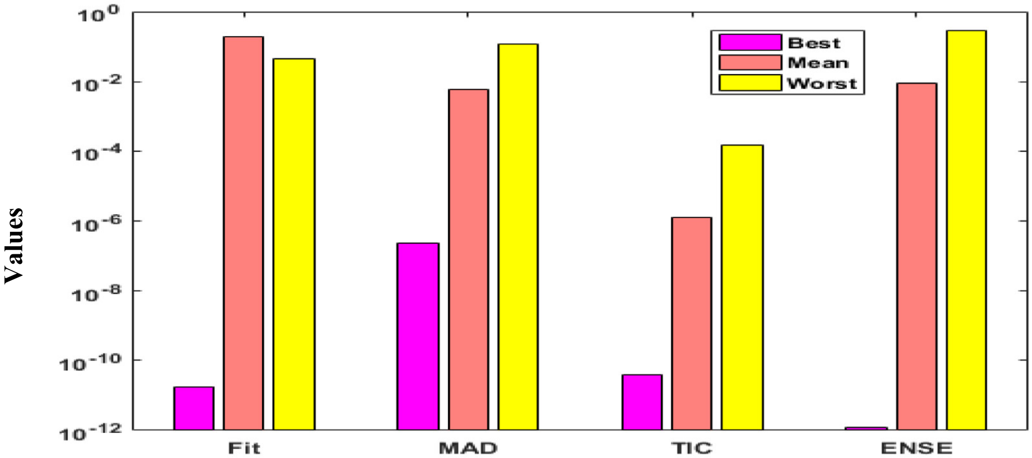
The comparison of results along absolute error plots for Problems 1–3 is also illustrated in Fig. 2. The overlapping of the present solutions with the average and desire solutions for Problems 1–3 show the correctness of the designed scheme, as well as, indicating the worth of the designed neural networks. The third part of Fig. 2 depends upon absolute error (AE) used to access the similarities of the results for three Problems 1–3. The best values and mean values for Problems 1–3 lie in the range of 10^{-06} to 10^{-08}, 10^{-02} to 10^{-04}, while the worst values for all the problems are close to 10^{-02}. Comparison of the result based on best, mean and exact solutions is also plotted in Fig. 3. for the MAD, fitness, ENSE and TIC based performance measures. The best level of the accuracy lies in the ranges of 10^{-08} to 10^{-12}. The calculated results substantiated the value and efficiency of MWNN-GA-IPA scheme. In Fig. 4, we present the fitness values as well as MAD parameter along with the histogram plots for LEEs given in Problems 1–3. It is evident that around 75% of trials attained high level of the accuracy measure on fitness and MAD indices for Problems 1–3. In Fig. 5, results for TIC and ENSE indices are plotted. The performance of MWNN-GA-IPA scheme is found reasonable accurate for these statistical metrics.

The further analysis of the precision of the present scheme MWNN-GA-IPA on statistical observations is tabulated in terms of minimum (Min), mean (Mean) and standard deviation (SD) based indices. These performance operators based results for LEEs are presented in Table 2 for $\hat{\omega}(t)$. The Min index lies between 10^{-06} to 10^{-08} for LEE in Problem 1, 10^{-07} to 10^{-10} for LEE in Problem 2 and 10^{-07} to 10^{-08} for LEE in Problem 3, and accordingly the Mean index lies between 10^{-02} to 10^{-4} for each LEE presented in Problems 1–3. While, the SD values also lie in good ranges, i.e., very small values. In Table 3, results of global indices based on GMAD, GFIT, GENSE and GTIC for 100 runs are tabulated and their values are found to be near to perfect modelling results for all these measures. To find the percentage based convergence performance, the analyses are made of different indicators and results are presented in Table 4. The large number of independent runs attain reasonable accurate level of accuracy, while for comparatively harder/stiff standards, the number of runs decreased considerably, but a proficient number of runs is still there to achieve that level of accuracy.

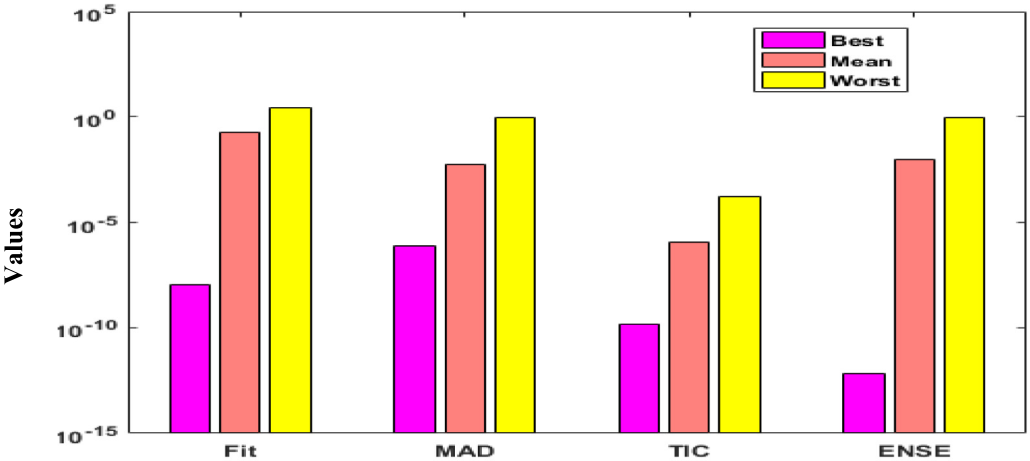
For the proposed algorithm MWNN-GA-IPA, the computational complexity parameters by means of the implementation/execution time, number of cycles/iterations and counts of fitness function evaluations in order



(a) Result of performance indication for LEE in Problem 1



(b) Result of performance indication for LEE in Problem 2



(c) Result of performance indication for LEE in Problem 3

Fig. 3. Performance measures for LEEs presented in Problems 1–3.

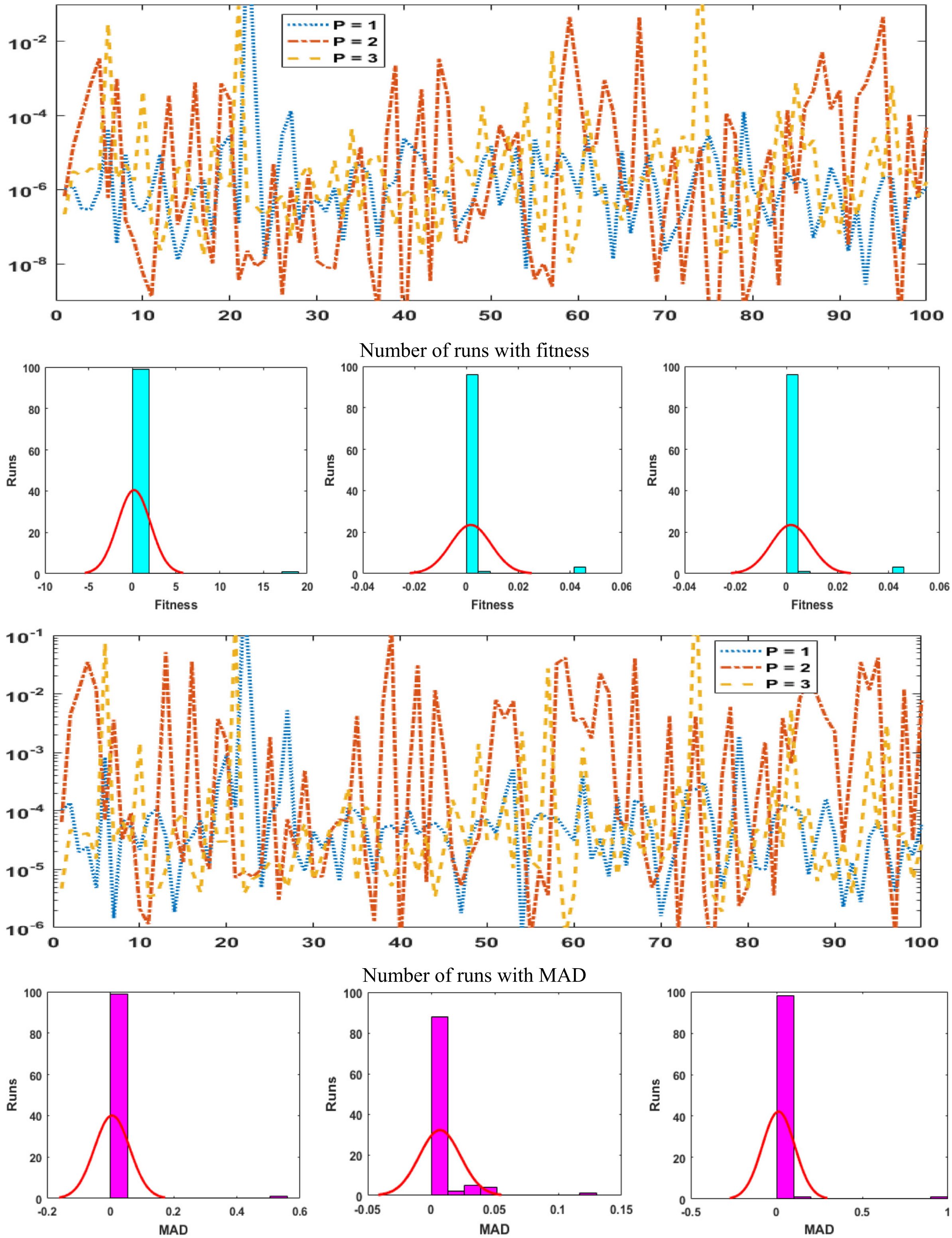


Fig. 4. Statistical observations on fitness and MAD along with the histogram studies for MWNN-GA-IPA for LEEs given in Problems 1–3.

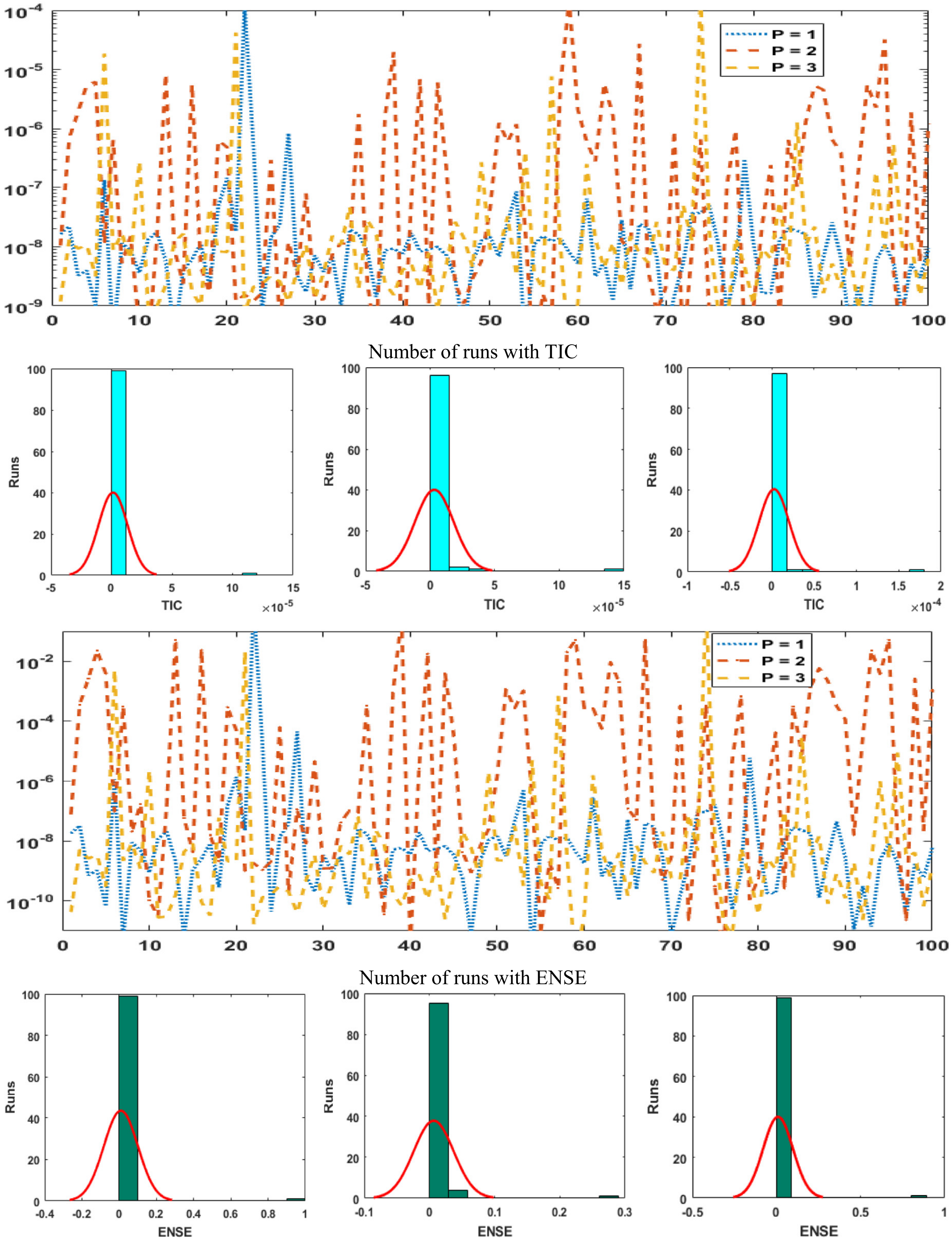


Fig. 5. Statistical observations on TIC and ENSE along with the histogram studies for MWNN-GA-IPA for LEEs given in Problems 1–3.

Table 3
Results of global indices for each LEE.

Index	Problem	GFIT		GMAD		GTIC		GENSE	
		Values	SD	Values	SD	Values	SD	Values	SD
$\hat{\omega}(t)$	1	1.9E−01	1.9E+00	5.8E−03	5.6E−02	7.6E−03	7.4E−02	9.2E−03	9.2E−02
	2	1.7E−03	7.8E−03	6.9E−03	1.6E−02	7.4E−03	1.7E−02	6.5E−03	3.1E−02
	3	2.7E−02	2.6E−01	1.2E−02	9.5E−02	1.4E−02	1.1E−01	9.2E−03	8.9E−02

Table 4
Convergence analysis for each LEE.

Index	Problem	FIT $\leq$			MAD $\leq$			TIC $\leq$			ENSE $\leq$		
		10^{-03}	10^{-04}	10^{-05}	10^{-03}	10^{-04}	10^{-05}	10^{-03}	10^{-04}	10^{-05}	10^{-06}	10^{-07}	10^{-08}
$\hat{\omega}(t)$	1	99	96	76	93	52	10	49	8	0	94	87	53
	2	83	69	56	55	39	14	39	13	2	52	39	29
	3	94	89	74	89	66	13	49	3	0	89	85	70

Table 5
Complexity analysis for each LEE.

Problem	Time of execution		Generation/Iteration		Function counts	
	Mean	SD	Mean	SD	Mean	SD
1	16.5469037	0.75671649	669.87	1.300	47 501.06	147.864654
2	15.5009769	2.81516155	621.64	124.472586	44 521.12	7637.41535
3	16.1100534	0.91439076	662.36	36.6442135	47 061.91	2270.80138

to obtained the decision variables are listed in Table 5 for all three nonlinear singular LEEs. It is seen that MWNN-GA-IPA takes around 16 s of time, 650 cycles and 44 500 function evaluations for LEEs.

5. Conclusions

The solutions of singular nonlinear differential equations are always a difficult task for any traditional, as well as, modern numerical techniques. The presented study in this work exploits the evolutionary computing paradigm based an alternate, accurate and reliable numerical solver MWNN-GA-IPA to solve such a challenging model using the Morlet wavelet neural network. Now it can be settled through the analysis on single and multiple runs of MWNN-GA-IPA that the singular second order LEE can be readily and efficiently solved by using the proposed stochastic computing methodology.

In future, the proposed stochastic computing solver MWNN-GA-IPA looks promising methodology to be exploited as alternate, accurate, reliable and robust computing framework for solving variety of the problems arising in astrophysics, atomic physics, plasma physics, nonlinear optic, electric machines, nanotechnology, fuel ignition model, fluid dynamics, bioinformatics and financial mathematics, see [1,3,13,20–22,31,33] and reference mentioned in them.

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