SC2001: Project 3

By:
Bui Dang Nguyen
Toh Kok Soon
Ammamalai Ramesh Santhosh Kumar
Bose Samrath

Problem Definition

We're given

- n items
- Each item i has a weight w_i and a profit of p_i
- The knapsack has a maximum capacity of C
- We can take unlimited copies of the item.

Goal: We're trying to maximise the total profit without exceeding the capacity.

Identifying Possible Decisions

If Capacity $C \leq 0$, (Base Case):

- We cannot take any items:

$$P(C) = 0 \quad (\text{if } C \leq 0)$$

At any capacity C, we can make a decision:

- For each item i, that fits $(w_i \leq C)$:
    - Include at least one instance of item i.
    - If item i is included once, we gain profit p_i from it, and now we have $C - w_i$ capacity left to fill and we can fill that optimally again.

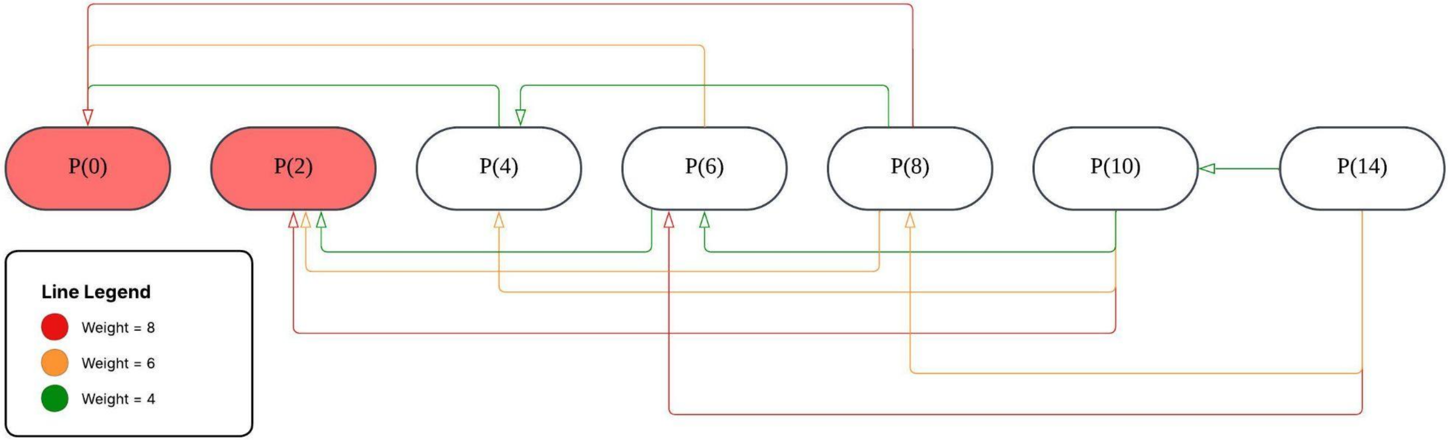
So the total profit if we pick item i is:

$$(p_i + P(C - w_i))$$

Recursive Definition

$$P(C) = \begin{cases} 0, & \text{if } C \leq 0 \\ \max\limits_{i:\, w_i \leq C} \left(p_i + P(C - w_i)\right), & \text{if } C > 0 \end{cases}$$

Subproblem (DAG)



- We stop when capacity reaches 0 or when the capacity is lower than the minimum weight:
  - P(0) and P(2)

Basic Implementation

```python
def knapsack(C, weights, profits):
    dp = [0] * (C + 1)
    for c in range(1, C + 1):
        for w, p in zip(weights, profits):
            if w <= c:
                dp[c] = max(dp[c], dp[c - w] + p)
    return dp[C]
```

Explanation

This dynamic-programming algorithm finds the maximum profit for a knapsack that can include **unlimited copies** of each item.

1. Create an array `dp` of size `C + 1`, where `dp[c]` represents the best profit achievable for capacity `c`.
2. Initialize all values to 0, since a knapsack of capacity 0 yields zero profit.
3. For each capacity `c` from 1 to `C`:
   - For every item `(w, p)`:

     If the item fits

$$(w \leq c), update:$$

$$dp[c] = \max(dp[c], dp[c - w] + p).$$

- This relation ensures smaller subproblems are reused, building up to larger capacities.

4. After filling the array, `dp[C]` holds the **maximum profit** achievable.

Complexity:

- Time = O(C × n)
- Space = O(C)

This method is efficient, concise, and ideal when we only need the maximum profit (not the exact combination of items).

Result

```
Case 1: C = 14, weights = [4, 6, 8], profits = [7, 6, 9]
Result: 21
Case 2: C = 14, weights = [5, 6, 8], profits = [7, 6, 9]
Result: 16
Case 3: C = 50, weights = [4, 5, 6], profits = [7, 6, 9]
Result: 86
```

Bonus Implementation

Intuition

- In the case where C (the maximum capacity of the knapsack) is very large, intuition tells us:
  - Stuff it with most cost-efficient (i.e. profit/weight ratio)
- But how correct is this?
  - Following slides will prove that the above method hold true until the remaining capacity is $\leq M^2$

Theorem

Assume that items are numbered from 1 to n.

Let j be the item that has the highest profit over weight ratio (that is, $\frac{p_j}{w_j}$ is the highest of all items).

Let the knapsack capacity be $C > w_j^2$, and M be the maximum weight of any item.

We prove that there exists an optimal solution for this instance of unbounded knapsack that contains less than w_j non-j items.

Proof By Contradiction

Let S be an optimal multiset of items for capacity C, chosen in a way that the number of non-j items in S is the minimum out of all optimal solutions.

Suppose the number of non-j items in S is $k \geq w_j$. We will derive a contradiction by proving that there's still a way to replace some of these items with copies of item j.

Write out the list of weights of all non-j items in S in arbitrary order as

$$a_1, a_2, \ldots, a_k,$$

where $1 \leq a_i \leq M$ for all $1 \leq i \leq k$.

Consider the values

$$\begin{aligned} r_1 &= (a_1) \bmod w_j \\ r_2 &= (a_1 + a_2) \bmod w_j \\ &\cdots \\ r_k &= (a_1 + a_2 + \cdots + a_k) \bmod w_j \end{aligned}$$

Because $k \geq w_j$, either

- some $r_i = 0$, which means $a_1, a_2, \ldots, a_i$ sum to a multiple of w_j, or
- there exists some $u < v$ such that $r_u = r_v$ (pigeonhole principle on $w_j - 1$ possible values), which means $a_{u+1}, \ldots, a_v$ sum to a multiple of w_j.

Proof By Contradiction (Continued)

Either case, there exists some nonempty subset T of S whose weights sum to $t \cdot w_j$, where t is an integer ≥ 1.

Because j has the highest profit-to-weight ratio, for items in T, we have

$$\sum_{i \in T} p_i = \sum_{i \in T} w_i \cdot \frac{p_i}{w_i} \leq \sum_{i \in T} w_i \cdot \frac{p_j}{w_j} = t \cdot p_j.$$

Thus, we can replace T with t copies of item j, maintaining the same total weight while achieving at least the same total profit:

- If the replacement strictly increases the total profit, that contradicts the optimality of S.
- If the replacement keeps the total profit the same, that contradicts the minimality of non-j items of S.

Either case, we reach a contradiction.

Therefore, our assumption is false, so there exists an optimal solution with less than w_j non-j items.

Application

From the theorem, we know that there exists a solution whose total weight of non-j items is no more than $(w_j - 1) \cdot M < M^2$.

We can apply this fact to create a $O(n \cdot M^2)$ solution as follows:

- Step 1: Find the optimal item j. Tiebreak arbitrarily.
- Step 2: Let $k = \max(0, \lceil \frac{C - w_j^2}{w_j} \rceil)$. Fill the knapsack with k copies of item j, and simultaneously decrease capacity C by $k \cdot w_j$.
- Step 3: Use the bottom-up DP we discussed earlier on the remaining capacity, which is now $\leq w_j^2$.

The complexity of this algorithm is $O(n \cdot w_j^2) = O(n \cdot M^2)$.

This is also sometimes written as $O(M^3)$ when $n = O(M)$ (there exist an item for basically every weight from 1 to M).

With little adjustments, the same idea can also be used for an $O(n \cdot V^2)$ algorithm, where V is the maximum profit of any item.

Implementation

```
def knapsack_2(C, weights, profits):
    best_p, best_w = profits[0], weights[0]
    for w, p in zip(weights, profits):
        if p * best_w > best_p * w:
            best_w, best_p = w, p
        elif p * best_w == best_p * w and w < best_w:
            best_w, best_p = w, p

    k = max(0, (C - best_w ** 2) // best_w)

    C -= k * best_w

    return knapsack(C, weights, profits) + k * best_p
```

Result

- Shows identical result

```
if __name__ == "__main__":
    cases = [
        (14, [4, 6, 8], [7, 6, 9]),
        (14, [5, 6, 8], [7, 6, 9]),
        (50, [4, 5, 6], [7, 6, 9])
    ]
    for i, (C, w, p) in enumerate(cases, 1):
        print(f"Case {i}: C = {C}, weights = {w}, profits = {p}")
        print(f"Solution 1: {knapsack(C, w, p)}")
        print(f"Solution 2: {knapsack_2(C, w, p)}")
```

```
Case 1: C = 14, weights = [4, 6, 8], profits = [7, 6, 9]
Solution 1: 21
Solution 2: 21
Case 2: C = 14, weights = [5, 6, 8], profits = [7, 6, 9]
Solution 1: 16
Solution 2: 16
Case 3: C = 50, weights = [4, 5, 6], profits = [7, 6, 9]
Solution 1: 86
Solution 2: 86
```

Thank You

Detailed Explanation and Implementation Code Here:
https://colab.research.google.com/drive/1UALUuhPZ0JJgIQcg6k83040dhfjO6Xbq?usp=sharing