Problem 1

Order the following function by growth rate. $n!$, $n^2 + \sqrt{n}\ log^{10}n$, $n^{1/3}$, $log^{100}n$, n^3, 2^n, $10^{\sqrt{n}}$, 2^{logn}, 2^{2logn}, $2^{\sqrt{logn}}$, 128, $128n$. Indicate which functions at the same rate (all logarithms are base 2). For example, if you are asked to order n, $2n$, $2n^2$, then your answer should be "$n = \Theta(2n), 2n = o(2n^2)$".

1. 128
$$\lim_{n\to\infty}(\frac{128}{log^{100}n}) = 0,\ 128 = \mathcal{O}(log^{100}n)$$
2. $log^{100}n$
$$\lim_{n\to\infty}(\frac{log^{100}n}{2^{\sqrt{logn}}}) = 0,\ log^{100}n = \mathcal{O}(2^{\sqrt{logn}})$$
3. $2^{\sqrt{logn}}$
$$\lim_{n\to\infty}(\frac{2^{\sqrt{logn}}}{n^{1/3}}) = 0,\ 2^{\sqrt{logn}} = \mathcal{O}(n^{1/3})$$
4. $n^{1/3}$
$$\lim_{n\to\infty}(\frac{n^{1/3}}{2^{logn}}) = 0,\ n^{1/3} = \mathcal{O}(2^{logn})$$
5. 2^{logn}
$$\lim_{n\to\infty}(\frac{2^{logn}}{128n}) = 0,\ 2^{logn} = \mathcal{O}(128n)$$
6. $128n$
$$\lim_{n\to\infty}(\frac{128n}{2^{2logn}}) = 0,\ 128n = \mathcal{O}(2^{2logn})$$
7. 2^{2logn}
$$\lim_{n\to\infty}(\frac{2^{2logn}}{n^2 + \sqrt{n}log^{10}n}) = 0,\ 2^{2logn} = \mathcal{O}(n^2 + \sqrt{n}log^{10}n)$$
8. $n^2 + \sqrt{n}log^{10}n$
$$\lim_{n\to\infty}(\frac{n^2 + \sqrt{n}log^{10}n}{n^3}) = 0,\ n^2 + \sqrt{n}log^{10}n = \mathcal{O}(n^3))$$
9. n^3
$$\lim_{n\to\infty}(\frac{n^3}{10^{\sqrt{n}}}) = 0,\ n^3 = \mathcal{O}(10^{\sqrt{n}})$$
10. $10^{\sqrt{n}}$
$$\lim_{n\to\infty}(\frac{10^{\sqrt{n}}}{2^n}) = 0,\ = 10^{\sqrt{n}}\mathcal{O}(2^n)$$
11. 2^n
$$\lim_{n\to\infty}(\frac{2^n}{n!}) = 0,\ 2^n = \mathcal{O}(n!)$$
12. $n!$
$$\lim_{n\to\infty}(\frac{n!}{2^n}) = \infty,\ n! = o(2^n)$$

Problem 2

Solve the following recurrence equations, expressing the answer in Big-Oh notation. Assume that Tn is constant for sufficiently small n.

(a) $T(n) = T(n/2) + 100$

$$a = 1,\ b = 1,\ c = 0,\ f(n) = 100$$

$$f(n) = \Theta(n^c log^k n)\ when\ c = 0, k = 0$$

$$log_b a = log_2 1 = 0\ c = log_b a$$

This is case 2.

$$T(n) = \Theta(n^{log_b a} log^{k+1} n) = \Theta(n^0 log^1 n) = \Theta(log\ n)$$

(b) $T(n) = 8T(n/2) + n^2$

$$a = 8,\ b = 2,\ c = 2,\ f(n) = n^2$$

$$log_b a = log_2 8 = 3\ c < log_b a$$

This is case 1.

$$T(n) = \Theta(n^{log_b a}) = \Theta(n^3)$$

(c) $T(n) = 8T(n/2) + n^3$

$$a = 8,\ b = 2,\ c = 3,\ f(n) = n^2$$

$$f(n) = \Theta(n^c log^k n)\ c = 3,\ k = 0$$

$$log_b a = log_2 8 = 3\ c = log_b a$$

This is case 2.

$$T(n) = \Theta(n^{log_b a} log^{k+1} n) = \Theta(n^3 log\ n)$$

(d) $T(n) = 8T(n/2) + n^4$

$$a = 8,\ b = 2,\ c = 4,\ f(n) = n^4$$

$$f(n) = \Omega(n^c)\ when\ c = 4$$

$$log_b a = log_2 8 = 3\ c > log_b a$$

$$af(\frac{n}{b}) \leq kf(n),\ wherek < 1$$

$$8(\frac{n^4}{16}) < kn^4$$

$$\frac{n^4}{2} \leq kn^4$$

This is case 3.

$$T(n) = \Theta(f(n)) = \Theta(n^4)$$

(e) $T(n) = T(n-1) + logn$

$$T(n-1) = T(n-2) + log(n-1) + log\ n$$

$$\vdots$$

$$T(n) = T(n-k) + log\ 1 + log\ 2 + \ldots + log(n-1) + log\ n$$

$$Since\, log\ n! = \Sigma log\ n$$

$$T(n) = T(n-k) + log\ n!$$

$$= \Theta(log\ n!)$$

(f) $T(n) = T(n-3) + n$

$$T(n-1) = T(n-2) + log(n-1) + log\ n$$

$$\vdots$$

$$T(n) = T(n-k) + log\ 1 + log\ 2 + \ldots + log(n-1) + log\ n$$

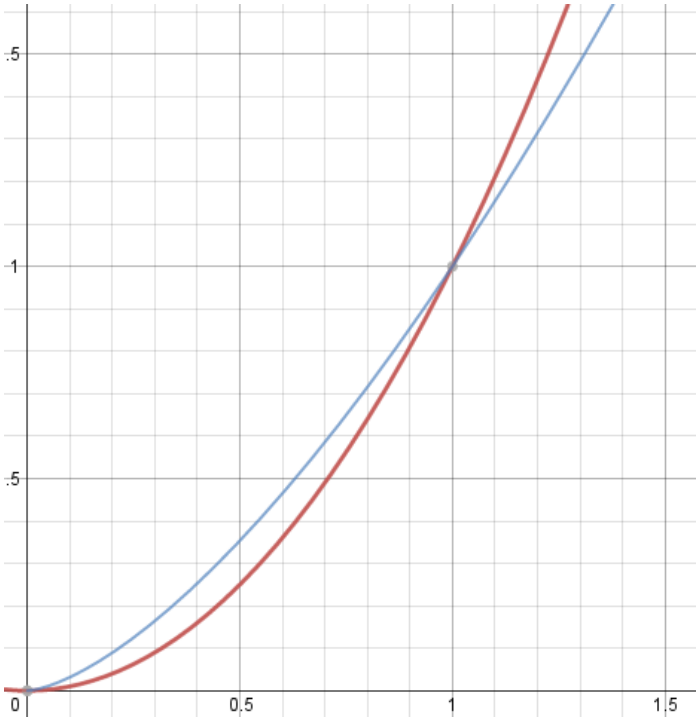
$$Since\, log\ n! = \Sigma log\ n$$

$$T(n) = T(n-k) + log\ n!$$

$$= \Theta(log\ n!)$$

Problem 3

You implemented a quadratic time algorithm for a problem P. On a test run, your algorithm takes 50 seconds for inputs of size 1000. Your classmate found a clever algorithm solving the same problem with a running time $O(n^{3/2})$. However, the faster algorithm takes 150 seconds for input of size 1000. Explain how can this happen. If you need to solve a problem of size 4000, which algorithm you should use? What about input of size 10,000? Explain your answers (assume low-order terms are negligible).



Looking at the graph the $\mathcal{O}(n^{3/2})$ function (shown in blue) initial grows at a much faster rate than the $\mathcal{O}(n^2)$ function (shown in red) hence why the "faster" algorithm takes longer for input size 1000.
For input 4,000 either algorithm could be used since input size 4,000 is still relatively small. However, for input size 10,000 the more clever algorithm might be more appropriate since according to the graph as the input size increases the growth slows in comparison to the other.

Problem 4

Recall that in the *testing safe height to drop a cellphone* problem we discussed in the class, the goal is to find out the maximum safe height to drop a cellphone without breaking it. In the class we saw that if the maximum safe height is n, then in the worst case we can perform n tests if there is only one cellphone and $2\sqrt{n}$ tests if there are two cellphones. Give an algorithm to minimize the number of tests if there are k cellphones available (assume k is constant). How many tests do you need to perform?

Starting at zero increase the drop height exponentially until a cellphone breaks.
Once a cellphone has broken perform a binary search between the last interval and the interval in which the last phone broke.

The number of tests needed for this algorithm would be $log\ n + log\ m$, where m is the size in between the last two exponential intervals that contain the maximum safe height to drop.

Problem 5

You are given a set of n numbers. Give an $O(n^2)$ algorithm to decide if there exist three numbers a, b, and c in the set such that $a + b = c$ (Hint: sort the numbers first).

```
int arr = array[n];

sort(arr);

for(int i = 0; i < n; i++) {
    int j = i;
    int k =i++;

    while(j < n && k < n) {
        if( ( (a[k] - a[j]) > a[i] {
            j++;
        }
        else if( (a[k] - a[j]) < a[i]) {
            k++;
        }
        else {
            return true;
        }
    }
}

return false;
```