LA.

Hoffman & Kunze

Syllabus.

* System of linear equations
* vector spaces
* Linear transformations.
* polynomial theory.

* System of linear equations

$$\left.\begin{array}{l} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = y_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = y_m \end{array}\right\} m \text{ eq}^n\text{s.}$$

Goal is to find $(x_1 \ldots x_n)$ s.t they satisfy all above eq^n.

soln of linear eq^n.

A_{ij} $\forall$ $1 \le i \le m$, $1 \le j \le n$, $A_{ij} \in F$ $\in$ field

$y_j \in F$

Field :- let us take a set F and consider two binary operations '+', '·'

(addn) (multiplication)

we call F as a field if following conditions are satisfied.

(A1) $(F, +)$ is an abelian group.

(i) closure

(ii) Associative if $x, y, z \in F$

$(x+y)+z = x+(y+z)$

(iii) for each $x \in F$, $\exists$ a unique element $0 \in F$ s.t. $x+0 = x = 0+x$

additive identity.

(iv) for each $x \in F$ $\exists$ unique element $(-x) \in F$ s.t $x + (-x) = 0$

additive inverse.

(V) Commutativity

for each $x, y \in F$, $x + y = y + x$.

(A2) $(F, *)$ is an abelian group for all non zero $x \in F$

(i) $x(y.z) = (x.y)(z)$

(ii) for each $x \in F$, $\exists$ unique element $1 \in F$ s.t. $x.1 = 1.x = x$

$\uparrow$ multiplicative identity.

inverse for 0 are not exists (since we are checking for only nonzero)

(iii) for each $x \in F$ $\exists$ unique element $x^{-1} \in F$ s.t. $x.x^{-1} = 1 = x^{-1}x$.

(iv) $x.y = y.x$.

(A3) $x.(y+z) = x.y + x.z$ for all $x, y, z \in F$

examples :- (i) $\mathbb{R}$ set of all real numbers

(ii) Q set of all rational numbers (p/q)

(iii) C set of complex numbers

{ Subfield $\rightarrow$ set of complex no $\supset$ set of real no
$\therefore$ subfield. }

e.g of which are Not field

$Z_+ = \{1, 2, \dots\}$

$N = \{0, 1, 2 \dots\}$

$F = \{0, 1\}$ Binary field.

'+' $\rightarrow$ binary addⁿ

+	0	1
0	0	1
1	1	0

'.' $\rightarrow$ binary multiplicⁿ

.	0	1
0	0	0
1	0	1

$1+1=0$ field of characteristic 2.

$\underbrace{1+1+\dots+1}_{n \text{ times}}=0$ field of characteristic n.

{this won't happen in real or complex no}
because there will always be some no after

* characteristic of real number
if there exists no 'n' s.t.

$1+\dots+1=0$

such a field has characteristic 0.

in this course we will only deal with fields with characteristics 0.

$$\left.\begin{array}{l} A_{11}x_1+\dots+A_{1n}x_n=y_1 \\ \vdots \\ A_{m1}x_1+\dots+A_{mn}x_n=y_m \end{array}\right\} \text{①} \quad \begin{array}{l} \times c_1 \\ \\ \times c_n \end{array}$$

system of m linear eq^ns with n unknown
if $y_i=0 \ \forall i$, then the system is called homogeneous.

$$2x_1-x_2+x_3=0 \quad (r_1)$$
$$x_1+3x_2+4x_3=0 \quad (r_2)$$

elimination :- (a) $r_1+(-2)r_2$

$$-7x_2-7x_3=0 \Rightarrow x_2=-x_3$$

(b) : $3(r_1)+(r_2)$

$$7x_1+7x_3=0 \Rightarrow x_1=-x_3$$

$(x_1, x_2, x_3)=(c, c, -c)$ for all $c \in F$

$AX = Y$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ \vdots \\ \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} y_1 \\ \vdots \\ \\ y_m \end{bmatrix}_{m\times 1}$$

$(c_1A_{11} + c_2A_{21} + \cdots + c_mA_{m1})x_1 + \cdots$ Eq

$(c_1A_{12} + c_2A_{22} + \cdots + c_mA_{m2})x_2 + \cdots$

$\cdots$

$(c_1A_{1n} + c_2A_{2n} + \cdots + c_mA_{mn})x_n = c_1y_1 + c_2y_2 + \cdots$
c_my_n

<u>lemma 1</u> :- A set of linear eqn obtained by linear combinations of set (1) of linear eqn will have exactly the same solutions.

* Suppose we have two set of equations

$AX = 0 \longrightarrow 1(a)$ > equivalent system of eqn

$BX = 0 \longrightarrow 1(b)$

$A, B \in M_{m\times n}$ → all matrices of the $m\times n$ with elements from f.

then both eqn have exactly the same soln if A is linear combn of B & B is linear combn of A

* <u>elementary row operations</u>

(i) multiplying any row by a nonzero scalar $c \in f$

(ii) multiplying s^{th} row by c and adding to row r. where $r \neq s$. $(r \to r + c(s))$

(iii) interchange of rows.

$e : (i, j) \to F$

(i) $e(A)_{ij} = A_{ij}$ if $i \neq r$ (बाकी रो बदलेगा नहीं)

& $e(A)_{rj} = c A_{rj}$

(ii) $e(A)_{ij} = A_{ij} \quad \forall\, i \neq r$

$e(A)_{rj} = A_{rj} + c A_{sj}$, where $r \neq s$.

(iii) $e(A)_{ij} = A_{ij}$ if $i \neq r$ & $i \neq s$.

$e(A)_{rj} = A_{sj} \quad e(A)_{sj} = A_{rj}$

operations are defined for a fixed no. of row
{5×5 matrix, you can't change 5th row with 6th}

<u>Th^m</u> :- to each elementary op^n e, $\exists$ another elementary row op^n $\tilde{e}$ of the same type, s.t. $e(\tilde{e}(A)) = A = \tilde{e}(e(A))$

<u>proof</u> :- ① suppose $e(A)_{ij} = A_{ij} \quad i \neq r$

$e(A)_{rj} = c A_{rj}$

$\tilde{e}(A)_{ij} = A_{ij} \quad i \neq r$

$\tilde{e}(A)_{rj} = c^{-1} A_{rj}$

$\tilde{e}(e(A)) = A$

② $e(A)_{ij} = A_{ij} \quad i \neq r$ | $\tilde{e}(A)_{ij} = A_{ij}$ if $i \neq r$

$e(A)_{rj} = A_{rj} + c A_{sj}$ | $\tilde{e}(A)_{rj} = A_{rj} - c A_{sj}$

<u>$e = \tilde{e}$.</u>

$A = A_0 \xrightarrow{e_{11}} A_1 \xrightarrow{e_{12}} A_2 \cdots \xrightarrow{e_{1n}} A_n = B$.

$A = A_0 \xleftarrow[\tilde{e}_{11} \quad \tilde{e}_{12}]{} A_{n-1} \xleftarrow{\tilde{e}_{1n}}$

B has rows which are linear combinations of rows of A.

Rows of A are linear comb^n of rows of B.

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defin^n : Row equivalence

Let $A, B \in M_{m \times n}(F)$ then B is said to be row equivalent to A if it can be obtained via finite number of elementary row op^n on A

lemma - Row equivalence is an equivalence relation R.

(i) $A R A$ Reflexive

(ii) $A R B$, $B R A$ symmetric

(iii) $A R B$, $B R C \Rightarrow A R C$ transitivity.

$\downarrow$

Row equivalence relation

* Row reduced form (defin^n)

Suppose you are given a system of m linear eq^n in n unknown i.e. $Ax = 0$

then A will be called to be in row reduced form if

(i) each non-zero entry in each nonzero row of A is 1.

(ii) each column containing the leading nonzero element of some row contains all other terms zero.

$Ax = 0$, defined over field of rational no's.

E.g. $$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}_{3\times 4} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}$$

$(r_2 - r_1, \ r_3 - 2r_1)$

$-3/2 + \frac{2/9}{3}$ $\quad -1/3$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 9/2 & -3/2 & -2 \\ 0 & 7 & -4 & 3 \end{bmatrix} \xrightarrow{2/9\, r_2} \begin{bmatrix} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1/3 & -4/9 \\ 0 & 7 & -4 & 3 \end{bmatrix}$$

$r_1 + 1/2\, r_2$.
$r_3 - 7 r_2$

$$\begin{bmatrix} 1 & 0 & 4/3 & 7/9 \\ 0 & 1 & -1/3 & -4/9 \\ 0 & 0 & -5/3 & 55/9 \end{bmatrix} \xrightarrow[r_2 + 1/3 r_3]{r_1 - 4/3 r_3} \begin{bmatrix} 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \\ 0 & 0 & 1 & -4/3 \end{bmatrix}$$

$\rightarrow A'$

$A'X = 0 \longleftrightarrow AX = 0$

$\cancel{+ 17/3\, x_4 = 0.}$
$\cancel{- 5/3\, x_4 = 0}$
$\cancel{- 11/3\, x_4 = 0.}$

$x_1 = -17/3\, x_4$
$x_2 = 5/3\, x_4$
$x_3 = 11/3\, x_4$
several Nontrivial soln
$(-17/3\, c,\ 5/3\, c,\ 11/3\, c)$.

* Therom every $m \times n$ matrix over field F is row equivalent to a row reduced matrix

* Row reduced echelon matrices
A matrix R is said to be row reduced echelon matrix if
(i) R is row reduced matrix.
(ii) every row of R that has all zero entries appears below rows which has non zero entries.

(iii) if rows $1, \ldots r$ have nonzero entries occur in columns $k_1, \ldots k_r$ resp then $k_1 < k_2 \ldots < k_r$

Suppose $Ax = 0 \leftrightarrow Rx = 0$

Row reduced echelon form

Suppose R has only $r \leq m$ non zero rows.

$Ax = 0$ has only r non trivial eq^{n}s.

Suppose that non-zero entry occur at columns $k_1 \ldots k_r$ ($k_1 < k_2 \ldots$)

$x_{k_1} + \sum$

Therom: if A is $n \times n$ square matrix, then A is row equivalent to $n \times n$ identity matrix iff $AX = 0$ has trivial sol^n.

$\Rightarrow$ suppose $A \longrightarrow R = I = \begin{bmatrix} 1 & 0 & \cdots & \\ 0 & 1 & \cdots & \vdots \\ 0 & \cdot & 1 & \cdot \end{bmatrix}$ (identity)

Suppose $AX = 0$ has $X = 0$ as sol^n.

$\Downarrow$ Row reduced form

$RX = 0$. $\longrightarrow$ R has r nonzero rows

if $r < n \Rightarrow$ A has nontrivial solutions

$\Downarrow$

$r \geq n$ but by defi^n $r < n$

$\Rightarrow \boxed{r = n}$.

* suppose $AX = Y$ this system might not have a sol^n.
{not even trivial {0, 0...}}

Augmented matrix

$$A' = \left[\begin{array}{c|c} & y_1 \\ A & \vdots \\ & y_m \end{array}\right]_{m \times (n+1)}$$

if I perform row reduction of^n A' we get R'

$$R' = \left[\begin{array}{cccc|c} 1 & 0 & \cdot\,0 & 0 & z_1 \\ 0 & 1 & \cdots & 0 & z_2 \\ 0 & 0 & 0 & 1 & \vdots \\ 0. & 0 & \cdot\cdot\,0 & 0 & z_n \end{array}\right]$$

$$z_i = \sum_{i=1}^{m} c_{ij} y_j$$

$AX = Y \Rightarrow RX = Z.$

How to find solⁿ from here

Suppose R has r non-zero rows containing nonzero entries from k_1 to k_r column resp.

$$x_{k_1} + \sum_{j=1}^{n-r} c_{1j} u_j = z_1$$

$$\vdots$$

$$x_{k_r} + \sum_{j=1}^{n-r} c_{rj} u_j = z_r$$

Conditions for $AX = Y \Rightarrow RX = Z$ to have solⁿ $z_i = 0 \ \forall \ i > r$.

$AX = Y_i$

$$A = \left[\begin{array}{ccc|c} 1 & -2 & 1 & y_1 \\ 2 & 1 & 1 & y_2 \\ 0 & 5 & -1 & y_3 \end{array}\right]$$

$\downarrow$ row reduction

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/5 & (y_1 + 2y_2)/5 \\ 0 & 1 & -1/5 & (y_2 - 2y_1)/5 \\ 0 & 0 & 0 & y_3 - y_2 + 2y_1 \end{array}\right]$$

$$x_1 + \tfrac{3}{5} x_3 = \frac{y_1 + 2y_2}{5}$$

$$x_2 - \tfrac{1}{5} x_3 = \frac{y_2 - 2y_1}{5}$$

$$e_3 \leftarrow \boxed{y_3 - y_2 + 2y_1 = 0} \quad \text{—— (1)}$$

if ① is satisfied then

$$x_1 = -3/_5 c + \frac{y_1 + 2y_2}{5}$$

$$x_2 = 1/_5 c + \frac{y_2 - 2y_1}{5}$$

$c \in F.$

Goal :- how do we write row reductions as matrix multiplications

suppose B is an $n \times p$ matrix over field F with rows $B_1 \dots B_n$

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \qquad \begin{aligned} B_1 &= (B_{11}, B_{12}, \dots B_{1p}) \\ B_n &= (B_{n1} \; B_{n2} \dots B_{np}) \end{aligned}$$

Now from B you construct a matrix C with rows $\alpha_1 \dots \alpha_m$ by.

$$\alpha_i = A_{i1} B_1 + A_{i2} B_2 + \dots + A_{in} B_n$$

A_{ij} are some coefficient that are known $(i = 1 \dots m)$

$A_{i1} B_{11} + A_{i2} B.$

$$\alpha_i = \left(\sum_{r=1}^{n} A_{ir} B_{r1} \dots \sum_{r=1}^{n} A_{ir} B_{r2} \right)$$

$$(C_{i1} \; C_{i2} \; C_{ip}) = \left(\sum_{r=1}^{r} A_{1r} B \dots \sum_{r=1}^{n} A_{ir} B_{rj} \right.$$

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$

$A \to m \times n$

$B \to n \times p \Rightarrow C$ is $m \times p$

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj} \qquad C = AB$$

$B = n \times p$ matrix

$A = m \times n$ matrix

$$(C_{i1}, \ldots C_{ip}) = \left(\sum_r A_{ir} B_{r1}, \ldots \sum_r A_{ir} B_{rp} \right)$$

$$r_i = A_{i1} B_1 + \ldots + A_{in} B_n$$

B_i's are rows of B.

$$\boxed{C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj} \iff C = \underset{\uparrow}{A \cdot B}}$$

matrix product

this is defined only when
columns of A = # rows of B.

$$A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix}_{2\times 3}$$

first row of A.B

$$r_1 = [-1(5) + 0(15) \quad 1(-1) + 0(4) \quad 1(2) + 0(8)]$$

$$AB \neq BA$$

Therem :- let A, B, C are three matrices over field F such that B·C & A·(B·C) are defined then so are the product A·B & (A·B)·C are defined further A·(B·C) = (A·B)·C.

⟹ you assume without loss of generality that B is $n \times p$ matrix

① if B.C is defined then C will have p rows

② if A·(B·C) is also defined A will have n columns since (B·C) had n rows.

The premise of thm is true if

A = $m \times n$ matrix
B = $n \times p$ matrix
C = $p \times r$ matrix

$$\left((A \cdot B)_{m \times p} \cdot C_{p \times r}\right)$$

$\underline{m \times r}$.

$\forall i, j$

$$[A \cdot (B \cdot C)]_{ij} = \sum_{k=1}^{n} A_{ik} (B \cdot C)_{kj}$$

$$= \sum_{k=1}^{n} A_{ik} \sum_{l=1}^{p} B_{kl} C_{lj}$$

$$= \sum_{l=1}^{p} \left(\sum_{k=1}^{n} A_{ik} B_{kl} \right) C_{lj}$$

$$= \sum_{l=1}^{p} (A \cdot B)_{il} C_{lj} = [(A \cdot B) \cdot C]_{ij}$$

A·(BC) = (AB)·C.

total matrix is some linear combination defined by matrix (A B) of matrix C.

A·(B·C) defined by B.

linear combn of C

linear combn of linear combn of C.

eg Suppose $AX=0$ when A is an $m\times n$ matrix over field F.
Let S be the set of all solutions of $AX=0$

Claim is S is a vector space

$$AX=0 \xrightarrow[\text{transformation}]{\text{Row}} RX=0$$

row reduced echelon form matrix

Assume R has r nonzero leading entries & they appear at positions $k_1 \dots k_r$.

$$x_{k_1} + \sum_{j\in J} c_{ij} x_j = 0 \quad \dots \quad x_{k_r} + \sum_{j\in J} c_{rj} x_j \qquad J=\{1,2 \dots m\}-\{k_{r_1}\dots k_{r_m}\}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$$

$k_1 = 1$, $k_2 = 1$

R has only two nonzero rows

$J = \{1,2,3,4\} - \{1,2\} \Rightarrow \{3,4\}$

$$x_1 + \sum_{j\in(3,4)} c_{1j} x_j = 0 \Rightarrow x_1 = 0$$

$$x_2 + \sum_{j\in 3,4)} c_{2j} x_j = 0 = 0 \Rightarrow x_2 + (x_3 + x_4) = 0$$

$$x_{k_1} + \sum_{j\in J} c_{ij} x_j = 0.$$

$$\vdots \qquad J = (1 \dots m) - \{k \dots k_r\}$$

$$x_{k_r} + \sum_{j\in J} c_{rj} x_j = 0$$

all sol^n of $Rx=0 \leftrightarrow$ arbitrary choice of $x_j \ \forall \ j\in J$

each choice $\{x_j\}$ is one sol^n $\begin{pmatrix} x_1 \\ \vdots \end{pmatrix}$

let $V = M^n$ or S^n or H^n

field : F.

vector addition $(A+B)_{ij} = A_{ij} + B_{ij}$

Scalar multiplication

$$(cA)_{ij} = cA_{ij}$$

<u>Claim</u> :- M^n, S^n, H^n are vector spaces over F.

* <u>Subspaces</u>: let V be a vector space over field F. Then A subset W of V is called subspace if W itself is a vector space with same rule of vector addition & scalar multiplicaⁿ as in V.

M^n, S^n, H^n

$S^n \subset M^n$

$H^n \subset M^n$

<u>Thᵐ</u> :- A nonempty subset W of a vector space V is a subspace of V iff for each pair of vectors $x, y \in W$ & each scalar $c \in F$, $cx + y \in W$

<u>proof</u> :- if $\forall x, y \in W$ & $c \in F$, $cx + y \in W$

Trivial <u>direction</u> $\rightarrow$ if W is a subspace then for each $c \in F$ & $x \in W$ $cx \in W$.

$cx \in W$, $y \in W$, $cx + y \in W$

<u>Nontrivial</u> :- Assume $\forall x, y \in W$ & $c \in F$

$cx + y \in W$

since W is non-empty $\exists x$ s.t.

$(-1)x + x \in W$ (from assumption)

$0 \in W$

similarly

$$cx = cx + 0 \text{ because } x \,\&\, 0 \in W$$

$$\Downarrow$$

$$cx \in W$$

Defi^n :- (elementary matrix)

An $m \times m$ matrix is said to be an elementary matrix if it can be obtained from $m \times m$ identity matrix by means of a single elementary row op^n.

$c \neq 0$

e.g. :- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{type 1}} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$

type 2: $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$

type 3: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Therom let e be an elementary row reduction op^n let E be the elementary matrix of size $m \times m$ i.e. $E = e(I_m)$ then

$$e(A) = EA.$$

→ proof :- R.H.S

$$(EA)_{ij} = \sum_{r=1}^{n} E_{ir} A_{rj}$$

type I

$$E_{ir} = (e(I))_{ir}.$$

let e be of type 2.

$$\begin{cases} \delta_{ir} & \text{if } i \neq s \\ \delta_{sr} + c\,\delta_{tr} & \text{if } i = s; \end{cases}$$

$$(EA)_{ij} = \sum_r E_{ir} A_{rj}$$

$$\begin{cases} \sum_r \delta_{ir} A_{rj} & \text{if } i \neq s \\ \sum_r (\delta_{sr} + c\,\delta_{tr}) A_{rj} & \text{if } i = s. \end{cases}$$

each choice $\{x_j\}$ is one sol^n

In particular

$$E_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}_{|J| \times 1} \leftarrow j^{th} \text{ row.}$$

$\{E_j\}$ = then set of vectors is also a sol^n

$$\begin{pmatrix} x_1 \\ \vdots \\ x_J \end{pmatrix}_{|J| \times 1} = x_1 E_1 + \ldots + x_J E_J$$

$\Rightarrow$ $\{E_j\}$ is a basis for all sol^n.

e.g :- consider a field F. then consider the set $V = F^n = \underbrace{F \times F \cdots \times F}_{n \text{ times}} = \{(x_1 \cdots x_n) : x_i \in F \ \forall i\}$

vector addition :-

$x + y = (x_1 + y_1, \ldots x_n + y_n)$; where

n tuple

$x = (x_1, \ldots x_n) \in F^n$

$y = (y_1, \ldots y_n) \in F^n$

Scalar multiplication :- $\forall c \in F, x \in F^n$.

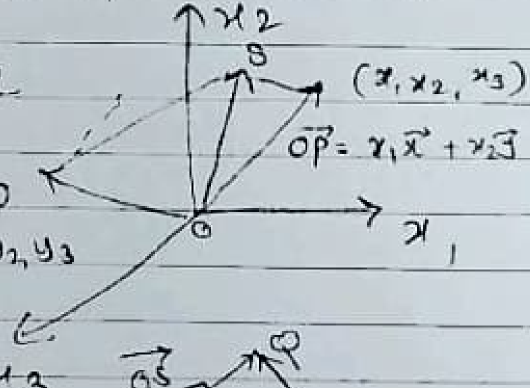
$cx = (cx_1 \ldots cx_n)$

claim :- F^n is a vector space over field F.

for n = 3

$F^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$

vector has a length & direction



$\vec{OS} = \vec{OP} + \vec{OQ}$

$= (x_1 + y_1)\vec{x_1} + (x_2 + y_2)\vec{x_2} + (x_3 + y_3)\vec{x_3}$

e.g. Consider a set of all $n \times n$ matrices over the field F $\rightarrow M^n$

* Symmetric matrices :- An $n \times n$ matrices over F is said to be

(a) Symmetric if $A_{ij} = A_{ji}$ $\quad S^n \quad (A^T = A)$

(b) Hermitian if $A_{ij}^* = A_{ji}$ $\quad H^n \quad (A^{-1} = A)$

last row of R cannot be 0.

$\rightarrow$ $\underset{m n}{R} = \underset{m n}{P}^{\text{invertible}} A_{n \times n}$

$Rx = E \longleftrightarrow PAx = E$.

$\longleftrightarrow Ax = P^{-1}(E)$.

$\Longrightarrow$ since $Ax = (P^{-1}E)$ has a sol^n.

therefore $Rx = E$ will also have a sol^n

R cannot have last row zero. $\Rightarrow R = I$.

Chapter 2 :-

* Vector spaces :- a vector space consist of following
  - (i) A set ~~of~~ V of vectors
  - (ii) A field f
  - (iii) An operation called vector addition, which associates each vector $x, y \in V$ and another vector $x + y \in V$, such that
    - (a) $x + y = y + x$
    - (b) $(x+y) + z = x(y+z)$
    - (c) $\exists$ a unique 'zero vector 0 in ~~vectors~~ V such that $x + 0 = 0$.
    - (d) $\exists$ a unique vector $-x \in V$ $\forall x \in V$ such that $x + (-x) = 0$.
  - (iv) An operation called scalar multiplication, which associates with each scalar $c \in F$ & each vector $x \in V$ another vector $cx \in V$ s.t.
    - (a) $1 . x = x$
    - (b) $(c_1 c_2) x = c_1 (c_2 x)$
    - (c) $c(x+y) = cx + cy$
    - (d) $(c_1 + c_2) x = c_1 x + c_2 x$

then V is a vector space over field f.

$r_1 = r_1 - \frac{1}{2} r_2$

$r_3 = r_3 - \frac{1}{2} r_2$

$$\begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1 \\ 0 & 0 & 1/180 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 0 \\ -6 & 12 & 0 \\ 30 & -180 & 180 \end{bmatrix}$$

$r_1 = r_1 + \frac{1}{6} r_3$ $\qquad -6 - \frac{\cancel{180}^{30}}{6}$ $\qquad -6$ $\qquad 12 + 180 = 192$

$r_2 = r_2 - r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

(A^{-1})

Thm:- let A be $n \times n$ matrix. then following are equivalent (i) A is invertible

(ii) $Ax = 0$ has only trivial sol^n

(iii) $Ax = Y$ has a sol^n X for each $n \times 1$ matrix Y. $\rightarrow$ (Not identically 0)

proof:- (i) $\rightarrow$ (ii) trivial $\quad X = A^{-1}(0) = 0$

(ii) $\rightarrow$ (i) $\quad Ax = 0$ has trivial sol^n $\Rightarrow$ A can be reduced to $R = I$ $\Rightarrow$ A is invertible.

(i) $\rightarrow$ (iii) $\quad X = A^{-1} Y$

(iii) $\rightarrow$ (i) :- suppose $Ax = Y$ has a sol^n for all Y

claim if we can prove that A is row reduced to I then we are done

let us assume that A is row reduced to some R (row reduced echelon form)

$\Rightarrow Rx = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$ will also have a sol^n. $\Longrightarrow$

Thm.- if A is an $n \times n$ matrix then following statements are equivalent

1) A is invertible
2) A is row equivalent to $n \times n$ identity matrix
3) A is product of elementary matrices

$\Rightarrow$

proof:- let R be a row reduced echelon form matrix which is equivalent to A, i.e.

$R = E_K E_{K-1} \cdots E_1 A$

elementary matrix

(sequence of elementary matrices)

all elementary matrices are invertible.

$$E_K^{-1} R = E_{K-1} \cdots E_1 A.$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_K^{-1}}_{\text{this is invertible.}} R.$$

A is invertible iff. R is invertible

$\Rightarrow$ R cannot have any 0 rows.

$\Downarrow$

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$$

$\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_K^{-1}$

inverse of elementary matrix is also elementary.

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$c=1$, $x+y \in W$

Linear Combination :- A vector $y \in V$ is said to be linear combination of vector $x_1, \ldots x_n \in V$ provided $\exists\ c_1 \ldots c_n \in F$ s.t. $y = c_1x_1 + \ldots\ldots + c_nx_n$

$$= \sum_{i=1}^{n} c_i x_i$$

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* **Th^m** :- Let a V be a vector space over field F. then intersection of finite number of subspaces of V is also a subspace.

proof :- Let $\{W_\alpha\}$ be a collection of subspaces of V

$W = \bigcap_\alpha W_\alpha$ (intersection) (only common elements bet^n each α will belong to W)

(i) since W_α are subspaces of V then $0 \in W_\alpha$, for all $\alpha \Rightarrow 0 \in W$ Hence W is non- ~~for all~~ empty & contains 0.

$cx+y$ where $c \in F$ $x, y \in W$

assuming $x, y \in W$ & $c \in F$

then by defi^n $x, y \in W_\alpha, \forall \alpha$

since W_α is a subspace $\Rightarrow cx+y \in W_\alpha\ \forall \alpha$

$$\Downarrow$$

$$cx+y \in W$$

which together implies $\longrightarrow$ W is a subspace of V.

Q. What is the smallest subspace containing a set S?

$\rightarrow$ {Notion of span of some vector}

Let S be a set of vectors in a vector space V over field F.

the subspace spanned by 'S' is defined as the intersection of all subspaces of V

* dimension of a vector space

defⁿ (linear dependence)
Let V be a vector space over F. Consider a subset $S = \{x_1 \dots x_n\}$ of V, then S is said to be linearly dependent if $\exists\, c_1 \dots c_n$ (not all zero) s.t. $\sum_{i=1}^{n} c_i x_i = 0$

otherwise S is called linearly independent.

$\Longrightarrow$

$V = \mathbb{R}^2$

$S = \{(0,1)(1,2)\}$

(ordered set)

$x_1 + x_2 + 0x_3 = 0$

$c_1 = 1, c_2 = 1$ ⇓ $x_1 = -x_2$

if possible we can derive one vector from other then both are lin. combⁿ of each other

$(0,0) = 0 = c_1(0,1) + c_2(1,2)$
$= (c_2, c_1 + 2c_2)$ ⇒ S is a linearly independent set.
$c_2 = 0,\ c_1 + 2c_2 = 0 \Rightarrow c_1 = 0$

$S = \{(0,1)(1,2)(1,0)\}$ ← Now this becomes lin. dep because S is defined in 2D plane, more than 2 vectors will always be dependent

* Basis of a vector space.

Let V be a vector space over F. A basis for V is a linearly independent set of vectors in V which span V. The space is finite dimensional if it has finite basis.

{ basis of vector space is not unique. there can be multiple basis of a v.s. containing 2 non parallel vector in basis.

$e_1 = (1, 0)$ [illegible]

$$= \sum_{i=1}^{n} c_i x_i \qquad c_i = f_i(c_i', d_i)$$

$\Rightarrow cx+y \in L \Rightarrow L$ is a subspace of V.

$S \subseteq L$ also $W \subseteq L$

the subspace W spanned by S is intersection of all subspaces containing S.

$\Rightarrow$ elements of W are such that for given $x, y \in W$ & $c \in F$, $cx+y \in W$

$\Downarrow$

W contains all linear combinations of atleast S.

$L \subseteq W$

Ex: $V \equiv R^2 = \{(x, y) ;\ x \in R, y \in R\}$

(i) $S = \{(x, y) ;\ x^2+y^2 = 1,\ x \in R,\ y \in R\}$

(ii) $S = \{(x, y) ;\ x+y = 0\ \ x, y \in Z_n\}$

$x, y = \pm 1 \ldots \pm n$

What is span of S

defn: Let $S_1, S_2 \ldots S_k$ are subsets of a vector space V over field F

then set of all sums $x_1 + \ldots + x_k$ of vectors $x_i \in S_i\ \forall i = 1 \ldots k$ is called "sum of subsets" & we denote it by $\tilde{S} = \sum_{i=1}^{k} S_i$

$$\tilde{S} = \{x_1 + \ldots + x_k : x_i \in S_i\}$$

Theorem: If $W_1 \ldots W_k$ are subspaces of V $\forall i$ then $\tilde{W} = W_1 + \ldots + W_k$ is a subspace containing each W_i, & W is spanned by union of $\{W_i\}$

{ proof same as one of the previous proof }
{ exercise }.

7/10/23.

lec - 4.

Theorm :- let A be an $n \times n$ matrix, then following are equivalent
(i) A is invertible
(ii) A can be row reduced to identity
(iii) A can be written as product of elementary matrices.

from (ii) $R = I = \underbrace{E_m \ldots E_1}A$.
applying elementary trans^n A to I converted

$A^{-1} = E_m \ldots E_1 (I)$

Corollary : if A is an invertible (square) matrix and if a sequence $E_1 \ldots E_m$ of elementary transformations reduces A to I i.e. $I = E_m \ldots E_1 A$. then same sequence of elementary transformations maps I to A^{-1} i.e.
$A^{-1} = E_m \ldots E_1 (I)$.

e.g. $A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow r_2 = r_2 - \frac{1}{2} r_1$
$r_3 = r_3 - \frac{1}{3} r_1$

$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 1/12 & 4/45 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$

$12 \times r_2$

$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 1/12 & 4/45 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 12 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$

containing set S. when S is a finite set i.e. $S = \{x_1, \ldots, x_n\}$ we say subspace is spanned by vectors $x_1 \ldots x_n$.

Thm:- the subspace spanned by a finite set S is equal to the set of all linear combination of S
$S = \{x_1, \ldots, x_n\}$
Any linear combn in S will be of this form

$$\sum_{i=1}^{n} c_i x_i \quad \text{when } c_i \in F$$

$$L = \left\{ \sum_{i=1}^{n} c_i x_i : c_i \in F \ \forall i \right\}$$

proof:- Note that $S = \{x_1 \ldots x_n\}$, $L = \left\{ \sum_{i=1}^{n} c_i x_i : c_i \in F \ \forall i \right\}$

if L pick $c_1 = 1$, $c_i = 0 \ \forall i \neq 1$ $\sum_i c_i x_i = x_1$

means if $c_j = 1$, $c_i = 0 \ \forall i \neq j$, $\sum_i c_i x_i = x_j$

$S \subseteq L$, L is nonempty

Now $x, y \in L$ & $c \in F$ then

$$x = \sum_{i=1}^{m_1} c_i x_i, \quad c_i \in F$$

$$y = \sum_{i=1}^{m_2} d_i x_i, \quad d_i \in F$$

$$cx + y = c \sum_{i=1}^{m_1} c_i x_i + \sum_{i=1}^{m_2} d_i x_i$$

$$= \sum_{i=1}^{m_1} \underbrace{(c c_i)}_{c_i'} x_i + \sum_{i=1}^{m_2} d_i x_i, \quad \text{when } c_i' = c c_i$$

Corollary :- let A, B be $m \times n$ matrices over field F then B is row equivalent to A iff $B = PA$ where P is a product of $m \times m$ elementary matrices.

$B = E_m \dots E_1 A$
$= (E_m \dots E_1) A$.

Suppose $B = PA$ product of elementary matrices
$\Rightarrow$ B is row equivalent to A
A is row equivalent to B or $\exists Q$ (product of elementary matrices)
such that $A = QB$
if $A = I \Rightarrow B = P$ $QP = I \Rightarrow Q = P^{-1}$ or $P = Q^{-1}$

defn :- Inverse of a matrix let A be $n \times n$ matrix over F, then, a) An $n \times n$ matrix B s.t $BA = I_n$ is called left inverse of A.
(b) if $AB = I_n$ .. right inverse of A.
(c) .. $AB = BA = I_n$ two sided inverse of A.

Thm :- (1) if A has left inverse B & right inverse then $B = C$.
(2) if A is invertible then A^{-1} is also invertible in fact $(A^{-1})^{-1} = A$
(3) if A, B are invertible then AB is also invertible $(AB)^{-1} = B^{-1}A^{-1}$
(4) every elementary matrix is invertible