# International Institute of Information Technology Hyderabad

## Modern Complexity Theory (CS1.405)

**Assignment 2**
**Deadline: October 4, 2024 (Friday), 05:00 PM**
**Venue for Submission: CSTAR, A3-110, Vindhya Block, IIIT Hyderabad**
**Total Marks: 100**

**NOTE:** It is strongly recommended that no student is allowed to copy from others.
No assignment will be taken after the deadline.
Write the following while submitting ONLY HARDCOPY:

Modern Complexity Theory (CS1.405)
Assignment 2
Name:
Roll No.:

## Questions

1. Let

   $$\text{MODEXP} = \{\langle a, b, c, p\rangle | \text{ a,b,c, and p are positive binary integers such that } a^b \equiv c \pmod{p}\}.$$

   Show that MODEXP $\in$ P.
   (Note that the most obvious algorithm doesn't run in polynomial time.
   Hint: Try it first where b is a power of 2.) **[10]**

2. Suppose a given problem X is NP-complete, which is proved through a polynomial-time reduction that maps size - n instances of SAT to size = $n^3$ instances of the problem X. Also, suppose that some genius manages to prove SAT requires $\Omega(c^n)$ time, for some constant $c > 1$. What can you conclude about the time complexity of the problem X? Justify. **[10]**

3. An oracle is a language $L \subseteq \{0,1\}^*$. An oracle Turing machine is the same as a normal Turing machine, only with the addition of a second tape, called the oracle tape. The cells on the oracle tape can contain either blanks, 0's, or 1's. Cook Reduction is a reduction computed by a deterministic polynomial time oracle Turing machine. Karp-reduction is a polynomial-time many-one reduction. Show that, if $NP \neq P$, there exists an infinite sequence of sets $\{S_1, S_2, ...\}$ in $NP \backslash P$ such that $S_{i+1}$ is Karp-reducible to $S_i$, but $S_i$ is not Cook-reducible to $S_{i+1}$.

   - Prove that a set $S$ is Karp-reducible to some set in NP if and only if $S$ is in NP.
   - If every set in NP can be Cook-reduced to some set in $NP \cap CoNP$ then $NP = CoNP$.

   **[5+5=10]**

4. Call a regular expression star-free if it does not contain any star operations. Then, let

   $$EQ_{SF-REX} = \{\langle R, S\rangle | \text{ R ans S are equivalent star-free regular expressions}\}.$$

   Show that $EQ_{SF-REX}$ is in coNP. Why does this argument fail for general regular expressions? **[10]**

5. Show that for every time constructible $t : N \to N$, if $L \in TIME(t(n))$, then there is an oblivious TM that decides $L$ in time $O(t(n)log(t(n))$. **[10]**

6. Let G represent an undirected graph. Also let

$$\text{SPATH} = \{\langle G, a, b, k\rangle | \text{ G contains a simple path of length at most k from a to b}\}$$

and

$$\text{LPATH} = \{\langle G, a, b, k\rangle | \text{ G contains a simple path of length at least k from a to b}\}$$

(a) Show that SPATH $\in$ P.

(b) Show that LPATH is NP-complete. **[(4+6)=10]**

7. Prove that if a unary language is NP-Complete then $P = NP$. **[10]**

8. In the following solitaire game, you are given an $m \times m$ board. On each of its $m^2$ positions lies either a blue stone, a red stone, or nothing at all. You play by removing stones from the board until each column contains only stones of a single color and each row contains at least one stone. You win if you achieve this objective. Winning may or may not be possible, depending upon the initial configuration. Let

$$\text{SOLITAIRE} = \{\langle G\rangle | \text{ G is a winnable game configuration}\}.$$

Prove that SOLITAIRE is NP-complete. **[10]**

9. Imagine you and your group of friends are on a covert mission. You are the brain of the entire operation. One of the areas that your team has to cross, is scattered with mines. You, being the brain of the operation, have to ensure that your team moves swiftly without setting it off and alerting the enemy (also, avoid the death of anyone in the operation). So essentially, you become the *minesweeper*, like the game. You decide to mathematically model the area and stumble across the result that $MINESWEEPER \in NP - Complete$. Your superior doubts your capabilities and decides to replace you- but you are adamant and decide to hand him the proof to prove your capabilities. Show the mathematical model and the main result of $MINESWEEPER \in NP - Complete$ you would show your superior. **[10]**

10. The last mission was easy. This time though, you have to deal with people. For some important information, you're required to meet a spy- but this spy is a bit moody. According to the spy's file, he's a big fan of the Philosopher's Football. What this means is that you could challenge the spy for a game and get the information. The problem is that you have never heard of this game! This gives room for your superior to doubt your capabilities AGAIN. But owing to your previous success, your superior gives you some time to figure it out. After studying the game, you realize that the game is a bit tricky- but you figure something out anyway. It is this- the problem of determining whether a player has a move that immediately wins the game is NP-Complete. Show this proof so that you can convince your superior and complete this mission. **[10]**

**All the best!!!**