

ARBITRARY RECURSION

Alessandro Coglio

Kestrel Institute

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One Recursive Call

arbitrary, possibly non-terminating, recursive "definition" (e.g. a program-mode function):

$$f(\bar{x}) \stackrel{?}{=} \underline{\text{if}} \ a(\bar{x}) \ \underline{\text{then}} \ b(\bar{x}) \ \underline{\text{else}} \ c(\bar{x}, f(\bar{d}(\bar{x}))) \quad \bar{x} = (x_1, \dots, x_n) \quad \bar{d}(\bar{x}) = (d_1(\bar{x}), \dots, d_n(\bar{x})) \quad n > 0$$

$$a \in \mathcal{U}^n \quad b: \mathcal{U}^n \rightarrow \mathcal{U}^m \quad d_i: \mathcal{U}^n \rightarrow \mathcal{U} \quad c: \mathcal{U}^n \times \mathcal{U}^m \rightarrow \mathcal{U}^m \quad m > 0 \quad (\text{number of results})$$

$$t(\bar{x}) \triangleq [\exists k \in \mathbb{N}. a(\bar{d}^k(\bar{x}))] \quad - \text{f terminates on } \bar{x}$$

always-terminating "version" of f (a logic-mode function, if a, b, c, d are logic-mode):

$$\hat{f}(\bar{x}) \triangleq \underline{\text{if}} \ t(\bar{x}) \ \underline{\text{then}} \ [\underline{\text{if}} \ a(\bar{x}) \ \underline{\text{then}} \ b(\bar{x}) \ \underline{\text{else}} \ c(\bar{x}, \hat{f}(\bar{d}(\bar{x})))] \ \underline{\text{else}} \ \dots \text{any value (irrelevant)}$$

$$\mu_{\hat{f}}(\bar{x}) \triangleq \begin{cases} \text{if } t(\bar{x}) & \underline{\text{then}} \min \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \\ \underline{\text{else}} & \dots \end{cases} \quad \text{any natural number (irrelevant)} \quad \mu_{\hat{f}}: \mathcal{U}^n \rightarrow \mathbb{N}$$

$$\angle_f \triangleq \angle \subseteq \mathbb{N} \times \mathbb{N}$$

$$\vdash \boxed{\mu\text{-end}} \quad t(\bar{x}) \Rightarrow a(\bar{d}^{M_{\hat{f}}(\bar{x})}(\bar{x}))$$

$\mu\text{-end} \quad t(\bar{x}) \Rightarrow a(\bar{d}^{\mu_f^*(\bar{x})}(\bar{x}))$
 $t(\bar{x}) \xrightarrow{\delta_{\mu_f}} \mu_f(\bar{x}) = \min \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow \mu_f(x) \in \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow a(\bar{d}^{\mu_f^*(\bar{x})}(\bar{x}))$
 QED

$$\vdash \boxed{\mu\text{-min}} \ a(\bar{d}^l(\bar{x})) \Rightarrow l \geq \mu_f^*(\bar{x})$$

$$\begin{array}{l} \vdash \boxed{\mu\text{-min}} \quad a(\bar{d}^l(\bar{x})) \Rightarrow l \geq \hat{\mu}_f(\bar{x}) \\ a(\bar{d}^l(\bar{x})) \xrightarrow{\delta_t} t(\bar{x}) \xrightarrow{\delta_{\hat{\mu}_f}} \hat{\mu}_f(\bar{x}) = \min \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \\ \quad \searrow \quad \quad \quad \hookrightarrow l \in \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow l \geq \hat{\mu}_f(\bar{x}) \\ \text{QED} \end{array}$$

$$\vdash_{\mathcal{L}} [\tau_{\hat{f}}] \quad t(\bar{x}) \wedge \neg a(\bar{x}) \Rightarrow \mu_{\hat{f}}(d(\bar{x})) <_{\hat{f}} \mu_{\hat{f}}(\bar{x})$$

$$\begin{array}{l} \vdash [\tau_{\hat{f}}] \quad t(\bar{x}) \wedge \neg a(\bar{x}) \Rightarrow \mu_{\hat{f}}(\bar{d}(\bar{x})) <_{\hat{f}} \mu_{\hat{f}}(\bar{x}) \\ \quad t(\bar{x}) \xrightarrow{\mu\text{-end}} a(\bar{d}^{\mu_{\hat{f}}(\bar{x})}(\bar{x})) \longrightarrow a(\bar{d}^{\mu_{\hat{f}}(\bar{x})-1}(\bar{d}(\bar{x}))) \xrightarrow[\mu\text{-min}]{L := \mu_{\hat{f}}(\bar{x})-1, \bar{x} := \bar{d}(\bar{x})} \mu_{\hat{f}}(\bar{x})-1 \geq \mu_{\hat{f}}(\bar{d}(\bar{x})) \longrightarrow \mu_{\hat{f}}(\bar{x}) \geq \mu_{\hat{f}}(\bar{d}(\bar{x}))+1 > \mu_{\hat{f}}(\bar{d}(\bar{x})) \xrightarrow{\delta <_{\hat{f}}} \mu_{\hat{f}}(\bar{d}(\bar{x})) <_{\hat{f}} \mu_{\hat{f}}(\bar{x}) \\ \quad \neg a(\bar{x}) \longrightarrow \neg a(\bar{d}^0(\bar{x})) \longrightarrow \mu_{\hat{f}}(\bar{x}) \neq 0 \end{array}$$

alternative recursive definition of the measure function:

$\varphi(u) \triangleq$ if $u \in \mathbb{N}$ then u else 0 — fixing function for \mathbb{N}

$\varepsilon_t(\bar{x}) \triangleq \varepsilon k. a(\bar{d}^{\varphi(k)}(\bar{x}))$ — witness of t , for slightly modified definition $t(\bar{x}) \triangleq [\exists k. a(\bar{d}^{\varphi(k)}(\bar{x}))]$

$v(\bar{x}, k) \triangleq$ let $\tilde{k} = \varphi(k)$ in if $a(\bar{d}^{\tilde{k}}(\bar{x})) \vee \tilde{k} \geq \varphi(\varepsilon_t(\bar{x}))$ then \tilde{k} else $v(\bar{x}, \tilde{k}+1)$ } recursively find min k such that $a(\bar{d}^k(\bar{x}))$, if $t(\bar{x})$;
 $\mu_v(\bar{x}, k) \triangleq \varphi(\varepsilon_t(\bar{x}) - \varphi(k))$ } stop at $\varphi(\varepsilon_t(\bar{x}))$ anyhow; min k is always found
 $\prec_v \triangleq < \subseteq \mathbb{N} \times \mathbb{N}$ } if $t(\bar{x})$, because $\min k \leq \varphi(\varepsilon_t(\bar{x}))$

$\vdash \boxed{\tau_v} \neg a(\bar{d}^{\varphi(k)}(\bar{x})) \wedge \varphi(k) < \varphi(\varepsilon_t(\bar{x})) \Rightarrow \mu_v(\bar{x}, \varphi(k)+1) \prec_v \mu_v(\bar{x}, k)$

$\mu_v(\bar{x}, \varphi(k)+1) \stackrel{\delta_{\mu_v}}{=} \varphi(\varepsilon_t(\bar{x}) - \varphi(\varphi(k)+1)) \stackrel{\delta_{\varphi}}{=} \varphi(\varepsilon_t(\bar{x}) - \varphi(k) - 1) \stackrel{\delta_{\varphi}}{=} \varepsilon_t(\bar{x}) - \varphi(k) - 1$
 $\varphi(k) < \varphi(\varepsilon_t(\bar{x})) \Rightarrow \varphi(\varepsilon_t(\bar{x})) \neq 0 \xrightarrow{\delta_{\varphi}} \varepsilon_t(\bar{x}) \in \mathbb{N} \xrightarrow{\delta_{\varphi}} \varphi(k) < \varepsilon_t(\bar{x}) \rightarrow \varepsilon_t(\bar{x}) - \varphi(k) > 0$
 $\mu_v(\bar{x}, k) \stackrel{\delta_{\mu_v}}{=} \varphi(\varepsilon_t(\bar{x}) - \varphi(k)) \stackrel{\delta_{\varphi}}{=} \varepsilon_t(\bar{x}) - \varphi(k) \longrightarrow \mu_v(\bar{x}, \varphi(k)+1) \prec_v \mu_v(\bar{x}, k) \xleftarrow{\delta_{\prec_v}}$

QED

$\vdash \boxed{vN} v(\bar{x}, k) \in \mathbb{N}$

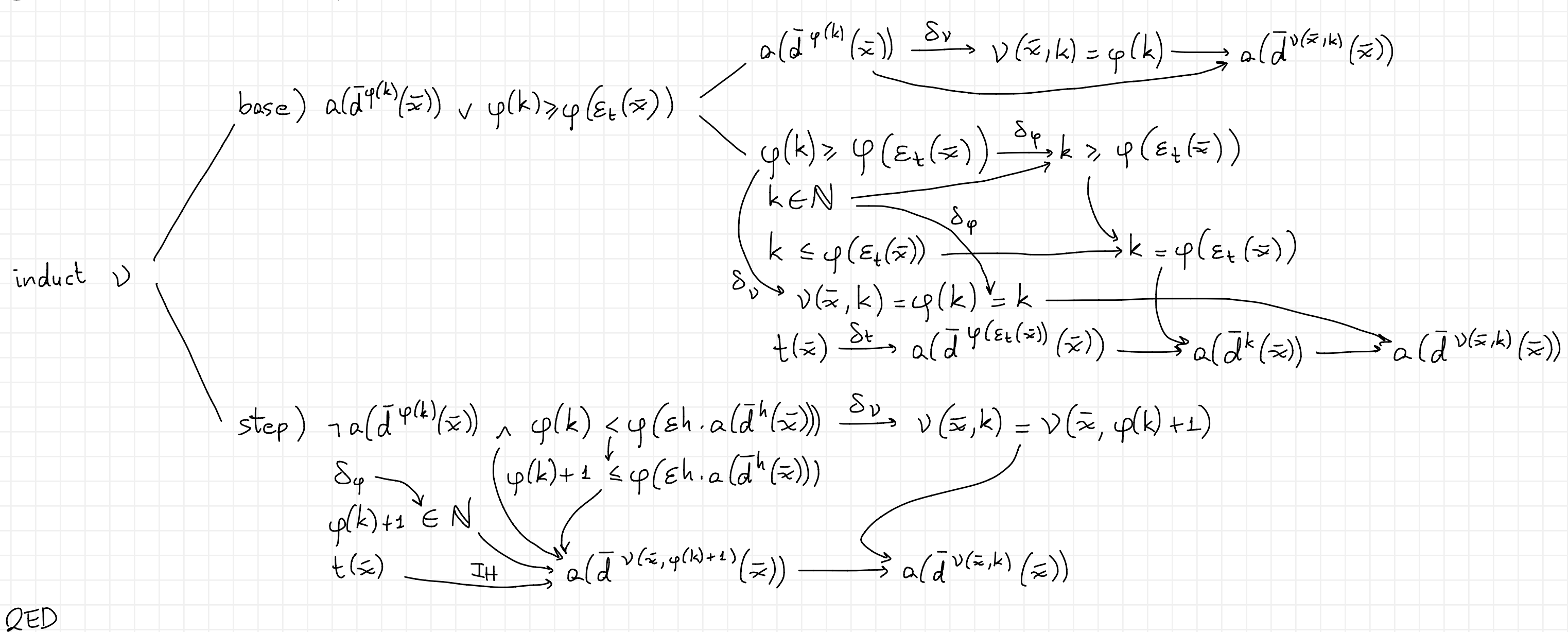
induct v
 base) $\delta_v \rightarrow v(\bar{x}, k) = \varphi(k) \rightarrow v(\bar{x}, k) \in \mathbb{N}$
 $\delta_{\varphi} \rightarrow \varphi(k) \in \mathbb{N} \rightarrow v(\bar{x}, k) \in \mathbb{N}$
 step) $\delta_v \rightarrow v(\bar{x}, k) = v(\bar{x}, \varphi(k)+1) \rightarrow v(\bar{x}, k) \in \mathbb{N}$
 $IH \rightarrow v(\bar{x}, \varphi(k)+1) \in \mathbb{N} \rightarrow v(\bar{x}, k) \in \mathbb{N}$

QED

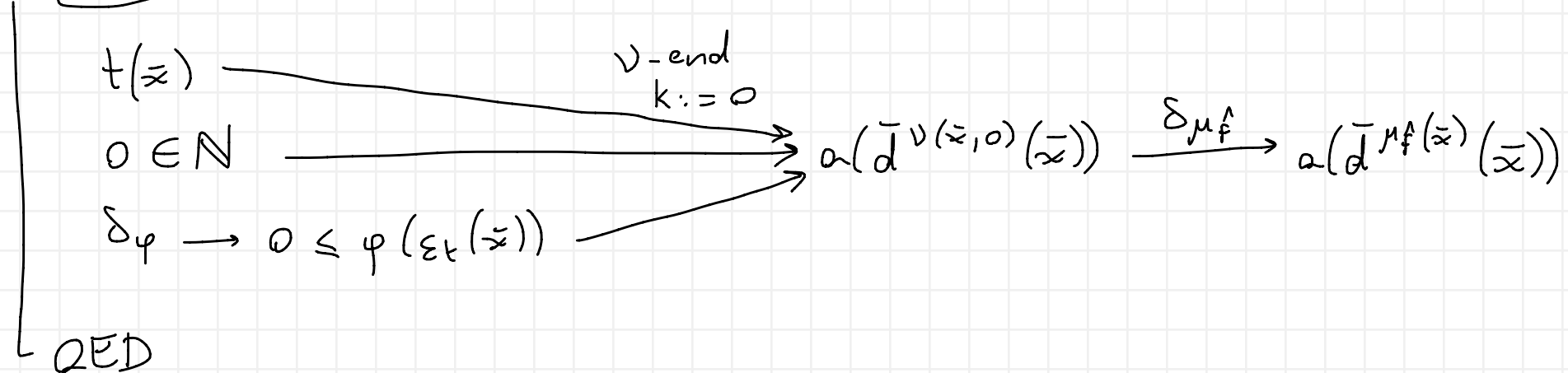
$\mu_{\hat{f}}(\bar{x}) \triangleq v(\bar{x}, 0)$ — alternative definition of $\mu_{\hat{f}}$

$\mu_{\hat{f}} : \mathcal{U}^n \rightarrow \mathbb{N} \quad (\leq vN)$

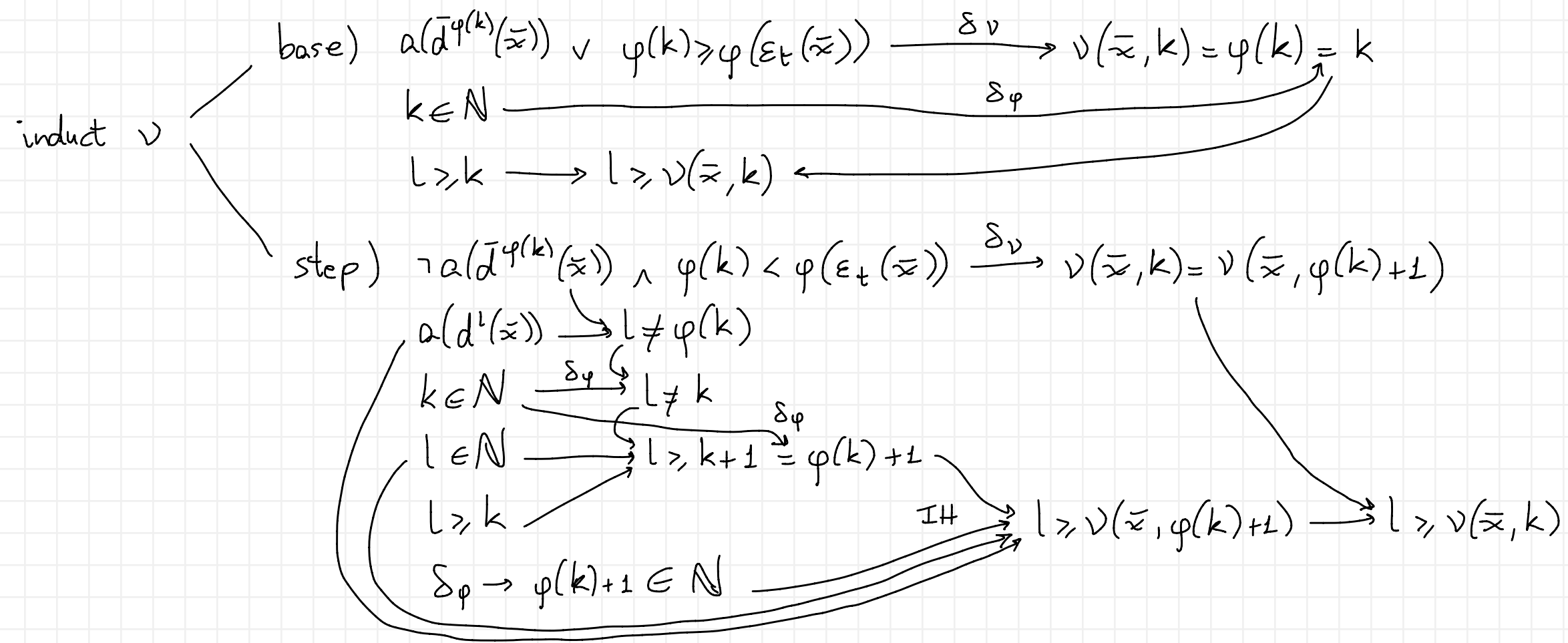
$\vdash \boxed{v\text{-end}} \quad t(\bar{x}) \wedge k \in \mathbb{N} \wedge k \leq \varphi(\varepsilon_t(\bar{x})) \Rightarrow a(\bar{d}^{v(\bar{x},k)}(\bar{x}))$



$\vdash \boxed{\mu\text{-end}} \quad t(\bar{x}) \Rightarrow a(\bar{d}^{\mu_f(\bar{x})}(\bar{x}))$ — alternative proof for the alternative definition of μ_f

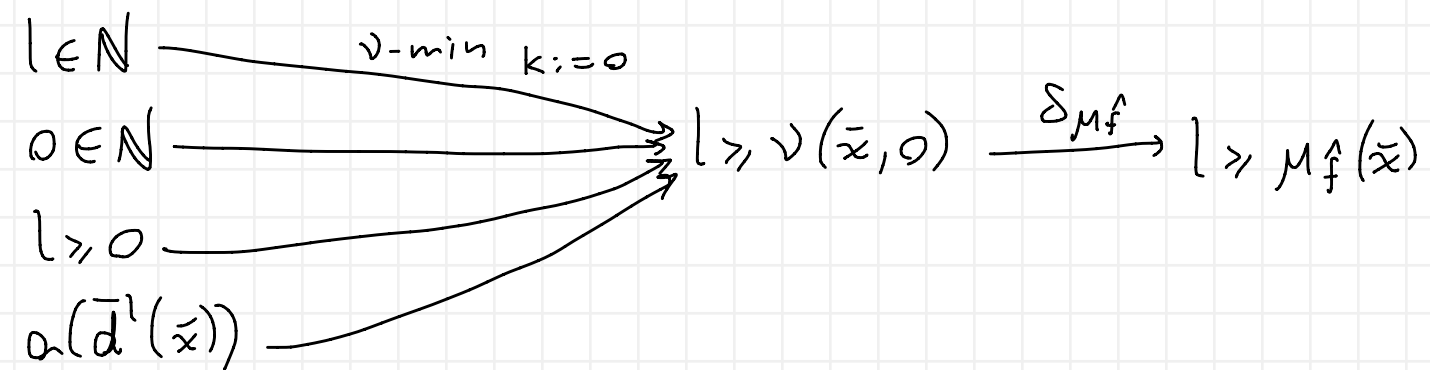


⊢ v-min $l \in \mathbb{N}, k \in \mathbb{N}, l \geq k, a(d^l(\bar{x})) \Rightarrow l \geq v(\bar{x}, k)$



QED

⊢ mu-min $a(d^l(\bar{x})) \Rightarrow l \geq \mu_f(\bar{x})$ — alternative proof for the alternative definition of μ_f



QED

τ_f proved as before using μ -end and μ -min