Introduction to Quantitative Methods in Biology

Midterm (March 8, 2021)

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Helpful information

Geometric Distribution

$\Pr(x) = \theta(1-\theta)^x$ (the probability of x failures before a success on the $x+1$ try, when a success has a probability of θ)

Exponential Distribution

PDF: $f(x) = \lambda e^{-\lambda x}$, CDF: $F(x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}$

Binomial Distribution

$$\Pr(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$E(X) = \sum x p(x)$

$Var(X) = E(X^2) - E(x)^2$

$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Question 1. The country of Lilliput has a lottery that works as follows. During the week, participants can pay a guilder to buy a lottery ticket. The participant then selects five numbers from a total of 40 numbers. Every Sunday, the lottery is held and five numbers are selected at random from an urn in which all 40 numbers are written on very small (in fact, Lilliputian-sized) sheets of paper. To win, a Lilliputian must choose all five of the numbers correctly. (That is to say that the five numbers the Lilliputian chose on his lottery ticket match the five numbers chosen from the urn.) What is the probability that a Lilliputian with a single ticket wins the jackpot?

are the 40 numbers unique?
replacement?

order doesn't matter

$$P(X) = \boxed{\frac{1}{\binom{40}{5}}} = \frac{1}{{}_{40}C_5}$$

assuming unique numbers,
and no replacement

Question 2. Jane rolls a fair six-sided die repeatedly until a six shows. What is the probability that the first six appears on the first roll? What is the probability that she has to roll the die three times to obtain the first six? (That is, the first six appears on the third roll.)

6_1

$$\boxed{P(6_1) = \frac{1}{6}}$$

$$\boxed{P(6_3) = \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)^1}$$

Question 3. A box contains 10,000 coins. One of the coins has heads on both sides but all the other coins are fair coins. You choose at random one of the coins. You toss this coin 15 times, with the result that all 15 tosses result in heads! Use Bayes' rule to find the probability that you have chosen the two-headed coin. (Assume that you are not allowed to peak at the coin to see if it actually has two heads!)

$$P(\text{unfair}) = \frac{1}{10{,}000}$$

$$P(\text{fair}) = \frac{9{,}999}{10{,}000}$$

$$P(\text{unfair} \mid 15H) = \frac{P(15H \mid \text{unfair})\,P(\text{unfair})}{P(15H)}$$

$$P(15H \mid \text{unfair}) = 1$$

$$P(15H) = P(\text{unfair} \cap 15H) \cup P(\text{fair} \cap 15H) =$$

$$= P(15H \mid \text{unfair})P(\text{unfair}) + P(15H \mid \text{fair})P(\text{fair}) =$$

$$= \frac{1}{10{,}000} + \left(\frac{1}{2}\right)^{15}\left(\frac{9999}{10{,}000}\right) =$$

$$= 0.0001305$$

$$P(\text{unfair} \mid 15H) = \frac{\frac{1}{10{,}000}}{0.0001305} =$$

$$= \boxed{0.766}$$

Question 4. The number of miles that a particular car can run before its battery wears out is exponentially distributed with rate $\lambda = 1/10000$ (failures per mile). The owner of the car needs to take a 5000-mile trip. What is the probability that he will be able to complete the trip without having to replace the car battery?

$$\exp(1/10000)$$

$$P(5000) = \lambda e^{-\lambda x} = \frac{1}{10000} e^{-\frac{1}{10\cancel{000}}(5\cancel{000})} =$$

$$= \boxed{\frac{1}{10000} e^{-\frac{1}{10}}}$$

Question 5. In D&D, one sometimes uses a four-sided die (d4; for example, when rolling damage from a light weapon). Such a die has the numbers 1, 2, 3, and 4 etched on the facets. Assuming that such a die is fair, what is the expected value of a roll? What is the variance?

$$E(X) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} =$$

$$= \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} = \frac{10}{4} = \frac{5}{2} = \boxed{2.5}$$

$$\sigma^2 = \frac{1}{4}\left[(1-2.5)^2 + (2-2.5)^2 + (3-2.5)^2 + (4-2.5)^2\right]$$

whatever that is

Question 6. The starship Enterprise is planning a surprise attack against the Klingons at 0100 hours. The possibility of encountering a radiation storm, though, is causing Captain Kirk and Mr. Spock to reassess their strategy. According to Spock's calculations, the probability of the Enterprise encountering a severe radiation storm at 0100 hours is 0.6, a moderate storm is 0.3, and no storm at all is 0.1. Captain Kirk feels the probability of the attack being a success is 0.8 if there is a severe storm, 0.7 if there is a moderate storm, and 0.2 if there is no storm. Spock claims that the attack would be a tactical misadventure if its probability of success were not at least 0.7306.

Should they attack? (Hint: Let the events A_1, A_2, and A_3 be the events "severe radiation storm", "moderate radiation storm", and "no radiation storm", respectively, and let B be the event "attack is a success.") Show your work.

$P(A_1) = 0.6 \quad P(A_2) = 0.3 \quad P(A_3) = 0.1$

$P(B|A_1) = 0.8 \quad P(B|A_2) = 0.7 \quad P(B|A_3) = 0.2$

don't attack if $P(B) < 0.7306$

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) =$$

$$= 0.8(0.6) + 0.7(0.3) + 0.2(0.1) =$$

$$= 0.71$$

No, don't attack

Question 7. It is estimated that 50% of emails are spam emails. Some software has been applied to filter these spam emails before they reach your inbox. A certain brand of software claims that it can detect 99% of spam emails, and the probability for a false positive (a non-spam email detected as spam) is 5%.

If an email is detected as spam, what is the probability that it is in fact a non-spam email?

[Hint: I would denote the event that the e-mail is flagged as spam as F and the events that the e-mail is spam as S and not spam as S^C. The question asks for $P(S^C \mid F)$.]

$P(\text{detect}+ \mid \text{spam}+) = 0.99$ $\quad p(\text{spam}+) = 0.5$

$P(\text{detect}+ \mid \text{spam}-) = 0.05$ $\quad p(\text{spam}-) = 0.5$

$$P(\text{spam}- \mid \text{detect}+) = ?$$

$$= \frac{P(\text{detect}+ \mid \text{spam}-)\,P(\text{spam}-)}{p(\text{detect}+)}$$

$$p(\text{detect}+) = p(\text{detect}+ \mid \text{spam}+)\,p(\text{spam}+) + p(\text{detect}+ \mid \text{spam}-)\,p(\text{spam}-) =$$

$$= 0.99(0.5) + 0.05(0.5) = 0.52$$

$$P(\text{spam}- \mid \text{detect}+) = \frac{0.05(0.5)}{0.52} = \boxed{0.04807}$$