Nonlinear Dyn (2009) 58: 295–306
DOI 10.1007/s11071-009-9481-4

ORIGINAL ARTICLE

Nonlinear dynamic analysis of Timoshenko beams by BEM. Part I: Theory and numerical implementation

E.J. Sapountzakis · J.A. Dourakopoulos

Received: 2 December 2008 / Accepted: 2 February 2009 / Published online: 21 February 2009

Abstract In this two-part contribution, a boundary element method is developed for the nonlinear dynamic analysis of beams of arbitrary doubly symmetric simply or multiply connected constant cross section, undergoing moderate large displacements and small deformations under general boundary conditions, taking into account the effects of shear deformation and rotary inertia. Part I is devoted to the theoretical developments and their numerical implementation and Part II discusses analytical and numerical results obtained from both analytical or numerical research efforts from the literature and the proposed method. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to two stress functions and solved using the Analog Equation Method, a BEM based method. Application of the boundary element technique yields a nonlinear coupled system of equations of motion. The solution of this system is accomplished iteratively by employing the average acceleration method in combination with the modified Newton–Raphson method. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The proposed model takes into account the coupling effects of bending and shear deformations along the member, as well as the shear forces along the span induced by the applied axial loading.

Keywords Nonlinear dynamic analysis · Moderate large displacements · Timoshenko beam · Shear center · Shear deformation coefficients · Boundary element method

E.J. Sapountzakis (✉) · J.A. Dourakopoulos
Institute of Structural Analysis and Aseismic Research, School of Civil Engineering, National Technical University of Athens, Zografou Campus, 157 80 Athens, Greece
e-mail: cvsapoun@central.ntua.gr

J.A. Dourakopoulos
e-mail: jdour@mail.ntua.gr

1 Introduction

The study of nonlinear effects on the dynamic analysis of structural elements is essential in aerospace, civil and mechanical engineering applications, wherein weight saving is of paramount importance. This nonlinearity results from retaining the square of the slope in the strain–displacement relations (intermediate nonlinear theory), avoiding in this way the inaccuracies arising from a linearized second-order analysis. Thus, the aforementioned study takes into account the influence of the action of axial, lateral forces and end-moments on the deformed shape of the structural element. Moreover, due to the intensive use of materials having relatively low transverse shear stiffness (i.e., high $EI/\kappa AG$, such as those corresponding to composite materials of members with open webs) and short span members (like seismic isolators made of elastomeric materials), as well as the need for beam members with high natural frequencies (including those of the higher-modes of vibration), the error incurred from the ignorance of the effect of shear deformation may be substantial, particularly in the case of heavy lateral loading. The Timoshenko beam theory, which includes shear deformation and rotary inertia effects, has an extended range of application, as it allows treatment of deep beams, short- and thin-webbed beams, and beams where higher modes are excited. However, it introduces some complications not found in the elementary Bernoulli–Euler formulation.

When the deflections of the structure are small, a wide range of linear analysis tools, such as modal analysis, can be used, and some analytical results are possible. As the deflections become larger, the induced geometric nonlinearities result in effects that are not observed in linear systems. In such situations the possibility of an analytical solution method is significantly reduced and is restricted to special cases of beam boundary conditions or loading. During the past few years, the nonlinear dynamic analysis of the classical Bernoulli–Euler beam undergoing large deflections has received a good amount of attention in the literature employing semi-analytical solutions [1, 2], an exact successive integration and iteration technique [3, 4], a modeshape technique [5], the Hamilton principle and a perturbation procedure [6], the Finite Element Method [7–13] and the Boundary Element Method [14, 15].

On the other hand, the effects of shear deformation and rotary inertia have been taken into account in linear dynamic analysis of beams [16, 17], in linearized dynamic or stability analysis [18–22] ignoring the squares of the derivatives of the deflections in the normal strain component and the axial differential equation of equilibrium and in free vibrations of beams with special boundary conditions performing a nonlinear dynamic analysis. More specifically for this latter case, Rao et al. [23], employing the finite element method and polynomial expressions for the displacement components, presented the large amplitude free vibrations of slender beams. Foda [24], utilizing the method of multiple scales, considered the large amplitude free vibration of a simply supported Timoshenko beam. For the same beam, Zhong and Guo [25] employed the Differential Quadrature Method, while Guo et al. [26] studied the effects of three cases of boundary conditions. Zhong and Liao [27], using a spline-based differential quadrature method, presented the higher-order nonlinear free vibrations of Timoshenko beams with immovable ends, while Liao and Zhong [28], employing the Differential Quadrature Method, studied the nonlinear flexural free vibrations of tapered Timoshenko beams with two simply supported, or clamped, ends. Moreover, the nonlinear dynamic analysis of a flexible Timoshenko beam, with geometrical nonlinearities subjected to initial conditions employing the Galerkin's method [29], the finite element method [30] and the Lagrange's equation based on the expression of the kinetic and potential energies in terms of generalized coordinates [31, 32], is also presented. Finally, in [33] a more general approach than the Timoshenko beam theory, allowing for more general modes of cross-section deformation has been presented, employing the finite element method based on the nonlinear absolute nodal coordinate formulation. The boundary element method has not yet been used for the nonlinear dynamic analysis of Timoshenko beams.

In this paper, a boundary element method is developed for the nonlinear dynamic analysis of beams of arbitrary doubly symmetric simply or multiply connected constant cross section, undergoing moderate large displacements and small deformations under general boundary conditions, taking into account the effects of shear deformation and rotary inertia. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions, as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to two stress functions, and solved using the Analog Equation Method [34], a BEM based method. Application of the boundary element technique yields a nonlinear coupled system of equations of motion. The solution of this system is accomplished iteratively by employing the average acceleration method in combination with the modified Newton–Raphson method [35, 36]. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The essential features and novel aspects of the present formulation compared with the previous ones are summarized as follows.

(i) The beam is subjected to an arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading.
(ii) The beam is supported by the most general nonlinear boundary conditions including elastic support or restraint, while its cross section is an arbitrary doubly symmetric one.
(iii) Shear deformation effect and rotary inertia are taken into account on the nonlinear dynamic analysis of beams subjected to arbitrary loading and boundary conditions.
(iv) The proposed model takes into account the coupling effects of bending and shear deformations along the member, as well as shear forces along the span induced by the applied axial loading. The employed method for these coupling effects is based on the Timoshenko approach and is described in [37] as compared with the Engesser or Haringx methods [19, 37]. It is worth here noting that the Haringx method, described in detail by Aristizabal-Ochoa in [16, 18, 20], retains the advantage of analyzing the phenomena of instability under tensile axial forces in members with low shear stiffness [38], which cannot be captured by the Timoshenko approach followed in this investigation.
(v) The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko's [39] and Cowper's [40] definitions, for which several authors [41, 42] have pointed out that one obtains unsatisfactory results or definitions given by other researchers [43, 44], for which these factors take negative values.
(vi) The effect of the material's Poisson ratio ν is taken into account in the evaluation of shear deformation coefficients.
(vii) The proposed method employs a BEM approach (requiring boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

2 Statement of the problem

Let us consider a prismatic beam of length l (Fig. 1), of constant arbitrary doubly symmetric cross-section of area A. The homogeneous isotropic and linearly elastic material of the beam cross section, with modulus of elasticity E, shear modulus G and Poisson ratio υ, occupies the two-dimensional multiply connected region Ω of the y,z plane and is bounded by the Γ_j ($j = 1, 2, \ldots, K$) boundary curves, which are piecewise smooth, i.e., they may have a finite number of corners. In Fig. 1(b) Cyz is the principal bending coordinate system through the cross section's centroid. The beam is subjected to the combined action of the arbitrarily distributed or concentrated time-dependent axial loading $p_x = p_x(x,t)$, transverse loading $p_y = p_y(x,t)$, $p_z = p_z(x,t)$ acting in the y and z directions, respectively, and bending moments $m_y = m_y(x,t)$, $m_z = m_z(x,t)$ along y and z axes, respectively (Fig. 1(a)).

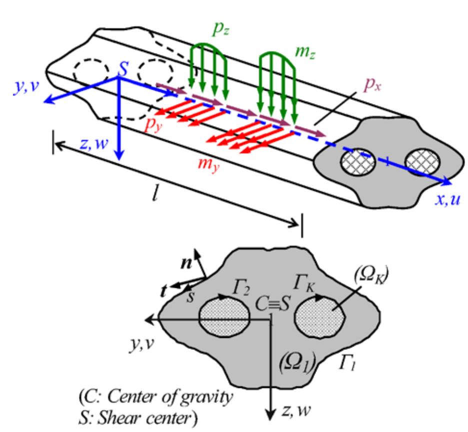
Under the action of the aforementioned loading, the displacement field of the beam, taking into account shear deformation effect, is given as

$$\bar{u}(x,y,z,t) = u(x,t) - y\theta_z(x,t) + z\theta_y(x,t), \tag{1a}$$

$$\bar{v}(x,t) = v(x,t), \tag{1b}$$

$$\bar{w}(x,t) = w(x,t), \tag{1c}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u(x,t)$, $v(x,t)$, $w(x,t)$ are the corresponding components of the centroid C and $\theta_y(x,t)$, $\theta_z(x,t)$ are



(a)
(b)

Fig. 1 Prismatic beam in axial-flexural loading (**a**) with an arbitrary doubly symmetric cross section occupying the two-dimensional region Ω (**b**) the angles of rotation due to bending of the cross section with respect to its centroid.

Employing the strain–displacement relations of the three-dimensional elasticity for moderate large displacements and small deformations [45, 46], the following strain components can be easily obtained:

$$\varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \bar{v}}{\partial x}\right)^2 + \left(\frac{\partial \bar{w}}{\partial x}\right)^2\right], \tag{2a}$$

$$\gamma_{xz} = \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} + \left(\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\frac{\partial \bar{w}}{\partial z}\right), \tag{2b}$$

$$\gamma_{xy} = \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} + \left(\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial x}\frac{\partial \bar{w}}{\partial y}\right), \tag{2c}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0, \tag{2d}$$

where it has been assumed that for moderate displacements, $(\partial\bar{u}/\partial x)^2 \ll \partial\bar{u}/\partial x$, $(\partial\bar{u}/\partial x)(\partial\bar{u}/\partial z) \ll (\partial\bar{u}/\partial x) + (\partial\bar{u}/\partial z)$, $(\partial\bar{u}/\partial x)(\partial\bar{u}/\partial y) \ll (\partial\bar{u}/\partial x) + (\partial\bar{u}/\partial y)$. Substituting the displacement components (1) to the strain–displacement relations (2), the strain components can be written as

$$\varepsilon_{xx} = u' + z\theta_y' - y\theta_z' + \frac{1}{2}\left(v'^2 + w'^2\right), \tag{3a}$$

$$\gamma_{xy} = v' - \theta_z, \tag{3b}$$

$$\gamma_{xz} = w' + \theta_y, \tag{3c}$$

where γ_{xy}, γ_{xz} are the additional angles of rotation of the cross section due to shear deformation.

Considering strains to be small, employing the second Piola–Kirchhoff stress tensor and assuming an isotropic and homogeneous material, the stress components are defined in terms of the strain ones as

$$\begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

or, employing (3), as

$$S_{xx} = E\left[u' + z\theta_y' - y\theta_z' + \frac{1}{2}\left(v'^2 + w'^2\right)\right], \tag{5a}$$

$$S_{xy} = G \cdot (v' - \theta_z), \tag{5b}$$

$$S_{xz} = G \cdot (w' + \theta_y). \tag{5c}$$

On the basis of the Hamilton principle, the variations of the Lagrangian equation defined as

$$\delta \int_{t_1}^{t_2} (U - K - W_{\text{ext}})\, dt = 0 \tag{6}$$

and expressed as a function of the stress resultants acting on the cross section of the beam in the deformed state provide the governing equations and the boundary conditions of the beam subjected to nonlinear vibrations. In (6), $\delta(\cdot)$ denotes variation of quantities, while U, K, and W_{ext} are the strain energy, the kinetic energy, and the external load work, respectively, given by

$$\delta U = \int_V (S_{xx}\delta\varepsilon_{xx} + S_{xy}\delta\gamma_{xy} + S_{xz}\delta\gamma_{xz})\, dV, \tag{7a}$$

$$\delta K = \int_V \rho\left(\dot{\bar{u}}\delta\dot{\bar{u}} + \dot{\bar{v}}\delta\dot{\bar{v}} + \dot{\bar{w}}\delta\dot{\bar{w}}\right) dV,$$

$$\delta W_{\text{ext}} = \int_l (p_x\delta u + p_y\delta v + p_z\delta w + m_y\delta\theta_y + m_z\delta\theta_z)\, dx. \tag{7b}$$

Moreover, the stress resultants of the beam are given by

$$N = \int_\Omega S_{xx}\, d\Omega, \tag{8a}$$

$$M_y = \int_\Omega S_{xx} z\, d\Omega, \tag{8b}$$

$$M_z = -\int_\Omega S_{xx} y\, d\Omega, \tag{8c}$$

$$Q_y = \int_{A_y} S_{xy}\, d\Omega, \tag{8d}$$

$$Q_z = \int_{A_z} S_{xz}\, d\Omega. \tag{8e}$$

Substituting the expressions of the stress components (5) into (8), the stress resultants are obtained as

$$N = EA\left[u' + \frac{1}{2}\left(v'^2 + w'^2\right)\right], \tag{9a}$$

$$M_y = EI_y\theta_y', \tag{9b}$$

$$M_z = EI_z\theta_z', \tag{9c}$$

$$Q_y = GA_y\gamma_{xy}, \tag{9d}$$

$$Q_z = GA_z\gamma_{xz}, \tag{9e}$$

where A is the cross section area, I_y, I_z the moments of inertia with respect to the principal bending axes, given as

$$A=\int_{\Omega} d\Omega, \tag{10}$$

$$I_y=\int_{\Omega} z^2\,d\Omega, \tag{11a}$$

$$I_z=\int_{\Omega} y^2\,d\Omega, \tag{11b}$$

and GA_y, GA_z are its shear rigidities by the Timoshenko beam theory, where

$$A_z=\kappa_z A=\frac{1}{a_z}A, \tag{12a}$$

$$A_y=\kappa_y A=\frac{1}{a_y}A \tag{12b}$$

are the shear areas with respect to y, z axes, respectively, with κ_y, κ_z the shear correction factors and a_y, a_z the shear deformation coefficients. Substituting the stress components given in (5) and the strain resultants given in (3) to the strain energy variation δE_{int} (7a) and employing (6), the equilibrium equations of the beam are derived as

$$-EA(u''+w'w''+v'v'')+\rho A\ddot{u}=p_x, \tag{13a}$$

$$-(Nv')'+\rho A\ddot{v}-GA_y(v''-\theta_z')=p_y, \tag{13b}$$

$$-EI_z\theta_z''+\rho I_z\ddot{\theta}_z-GA_y(v'-\theta_z)=m_z, \tag{13c}$$

$$-(Nw')'+\rho A\ddot{w}-GA_z(w''+\theta_y')=p_z, \tag{13d}$$

$$-EI_y\theta_y''+\rho I_y\ddot{\theta}_y+GA_z(w'+\theta_y)=m_y. \tag{13e}$$

Combining (13b), (13c) and (13d), (13e), the following differential equations with respect to u, v, w are derived:

$$-EA(u''+w'w''+v'v'')+\rho A\ddot{u}=p_x, \tag{14a}$$

$$\begin{aligned}&EI_z v''''-\rho I_z\left(\frac{Ea_y}{G}+1\right)\frac{\partial^2\ddot{v}}{\partial x^2}\\&\quad+\rho A\ddot{v}+\frac{EI_z}{GA_y}(Nv')'''-(Nv')'\\&\quad-\frac{\rho I_z}{GA_y}\left(\frac{\partial^2(Nv')'}{\partial t^2}-\rho A\ddddot{v}\right)\\&=p_y-\frac{EI_z}{GA_y}p_y''+\frac{\rho I_z}{GA_y}\ddot{p}_y-m_z',\end{aligned} \tag{14b}$$

$$\begin{aligned}&EI_y w''''-\rho I_y\left(\frac{Ea_z}{G}+1\right)\frac{\partial^2\ddot{w}}{\partial x^2}\\&\quad+\rho A\ddot{w}+\frac{EI_y}{GA_z}(Nw')'''-(Nw')'\\&\quad-\frac{\rho I_y}{GA_z}\left(\frac{\partial^2(Nw')'}{\partial t^2}-\rho A\ddddot{w}\right)\\&=p_z-\frac{EI_y}{GA_z}p_z''+\frac{\rho I_y}{GA_z}\ddot{p}_z+m_y'.\end{aligned} \tag{14c}$$

Equations (14) constitute the governing differential equations of a Timoshenko beam subjected to nonlinear vibrations due to the combined action of time-dependent axial, transverse loading and bending moments. These equations are also subjected to the pertinent boundary conditions of the problem, which are given as

$$a_1u(x,t)+\alpha_2N(x,t)=\alpha_3, \tag{15}$$

$$\beta_1v(x,t)+\beta_2V_y(x,t)=\beta_3, \tag{16a}$$

$$\bar{\beta}_1\theta_z(x,t)+\bar{\beta}_2M_z(x,t)=\bar{\beta}_3, \tag{16b}$$

$$\gamma_1w(x,t)+\gamma_2V_z(x,t)=\gamma_3, \tag{17a}$$

$$\bar{\gamma}_1\theta_y(x,t)+\bar{\gamma}_2M_y(x,t)=\bar{\gamma}_3, \tag{17b}$$

at the beam ends $x=0,l$, together with the initial conditions

$$u(x,0)=\bar{u}_0(x), \tag{18a}$$

$$\dot{u}(x,0)=\dot{\bar{u}}_0(x), \tag{18b}$$

$$v(x,0)=\bar{v}_0(x), \tag{19a}$$

$$\dot{v}(x,0)=\dot{\bar{v}}_0(x), \tag{19b}$$

$$w(x,0)=\bar{w}_0(x), \tag{20a}$$

$$\dot{w}(x,0)=\dot{\bar{w}}_0(x), \tag{20b}$$

where $\bar{u}_0(x)$, $\bar{v}_0(x)$, $\bar{w}_0(x)$, $\dot{\bar{u}}_0(x)$, $\dot{\bar{v}}_0(x)$ and $\dot{\bar{w}}_0(x)$ are prescribed functions. In (16), (17) V_y, V_z and M_y, M_z are the reactions and bending moments at the boundary with respect to y, z, respectively, which together with the angles of rotation due to bending, θ_y, θ_z, are given by the following relations:

$$\begin{aligned}V_y&=Nv'-EI_zv'''\\&\quad-\frac{EI_z}{GA_y}\left[Nv'''-\rho A\frac{\partial\ddot{v}}{\partial x}\right]+\rho I_z\ddot{\theta}_z,\end{aligned} \tag{21a}$$

$$V_z=Nw'-EI_yw'''$$

$$-\frac{EI_y}{GA_z}\left[Nw''' - \rho A\frac{\partial \ddot{w}}{\partial x}\right] - \rho I_y\ddot{\theta}_y, \tag{21b}$$

$$M_z = EI_z v'' + \frac{EI_z}{GA_y}[Nv'' - \rho A\ddot{v}], \tag{21c}$$

$$M_y = -EI_y w'' - \frac{EI_y}{GA_z}[Nw'' - \rho A\ddot{w}], \tag{21d}$$

$$\theta_y = \frac{EI_y}{G^2A_z^2}\left(\rho A\frac{\partial \ddot{w}}{\partial x} - Nw'''\right) - \frac{1}{GA_z}(EI_y w''' + \rho I_y\ddot{\theta}_y + GA_z w'), \tag{21e}$$

$$\theta_z = \frac{EI_z}{G^2A_y^2}\left(Nv''' - \rho A\frac{\partial \ddot{v}}{\partial x}\right) + \frac{1}{GA_y}(EI_z v''' - \rho I_z\ddot{\theta}_z + GA_y v'). \tag{21f}$$

Finally, α_k, $\bar{\alpha}_k$, β_k, $\bar{\beta}_k$, γ_k, $\bar{\gamma}_k$ $(k = 1, 2, 3)$ are functions specified at the beam ends $x = 0, l$. Equations (15)–(17) describe the most general nonlinear boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g., for a clamped edge it is $\alpha_1 = \beta_1 = \gamma_1 = 1$, $\overline{\alpha}_1 = \overline{\beta}_1 = \overline{\gamma}_1 = 1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \overline{\alpha}_2 = \overline{\alpha}_3 = \overline{\beta}_2 = \overline{\beta}_3 = \overline{\gamma}_2 = \overline{\gamma}_3 = 0$).

The solution of the initial boundary value problem given from (14), subjected to the boundary conditions (15)–(17) and the initial conditions (18)–(20) which represent the nonlinear flexural dynamic analysis of Timoshenko beams, presumes the evaluation of the shear deformation coefficients a_y, a_z, corresponding to the principal coordinate system Cyz. These coefficients are established equating the approximate formula of the shear strain energy per unit length [42]:

$$U_{\text{appr.}} = \frac{a_y Q_y^2}{2AG} + \frac{a_z Q_z^2}{2AG} \tag{22}$$

with the exact one given from

$$U_{\text{exact}} = \int_{\Omega} \frac{(\tau_{xz})^2 + (\tau_{xy})^2}{2G} d\Omega, \tag{23}$$

and are obtained as [47]

$$a_y = \frac{1}{\kappa_y} = \frac{A}{\Delta^2}\int_{\Omega}\big[(\nabla\Theta) - \mathbf{e}\big]\cdot\big[(\nabla\Theta) - \mathbf{e}\big]\, d\Omega, \tag{24a}$$

$$a_z = \frac{1}{\kappa_z} = \frac{A}{\Delta^2}\int_{\Omega}\big[(\nabla\Phi) - \mathbf{d}\big]\cdot\big[(\nabla\Phi) - \mathbf{d}\big]\, d\Omega, \tag{24b}$$

where $(\tau_{xz})_j$, $(\tau_{xy})_j$ are the transverse (direct) shear stress components, $(\nabla) \equiv \mathbf{i}_y(\partial/\partial y) + \mathbf{i}_z(\partial/\partial z)$ is a symbolic vector with $\mathbf{i}_y$, $\mathbf{i}_z$ the unit vectors along y and z axes, respectively, and Δ is given from

$$\Delta = 2(1+\nu)\mathrm{I}_y\mathrm{I}_z, \tag{25}$$

ν is the Poisson ratio of the cross-sectional material, $\mathbf{e}$ and $\mathbf{d}$ are vectors defined as

$$\mathbf{e} = \left(\nu I_y \frac{y^2 - z^2}{2}\right)\mathbf{i}_y + \nu I_y yz\mathbf{i}_z, \tag{26a}$$

$$\mathbf{d} = \nu I_z yz\mathbf{i}_y - \left(\nu I_z \frac{y^2 - z^2}{2}\right)\mathbf{i}_z, \tag{26b}$$

and $\Theta(y,z)$, $\Phi(y,z)$ are stress functions, which are evaluated from the solution of the following Neumann type boundary value problems [47]:

$$\nabla^2\Theta = -2I_y y \quad \text{in } \Omega, \tag{27a}$$

$$\frac{\partial\Theta}{\partial n} = \mathbf{n}\cdot\mathbf{e} \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j, \tag{27b}$$

$$\nabla^2\Phi = -2I_z z \quad \text{in } \Omega, \tag{28a}$$

$$\frac{\partial\Phi}{\partial n} = \mathbf{n}\cdot\mathbf{d} \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j, \tag{28b}$$

where $\mathbf{n}$ is the outward normal vector to the boundary Γ. In the case of negligible shear deformations, $a_z = a_y = 0$. It is also worth here noting that the boundary conditions (27b), (28b) have been derived from the physical consideration that the traction vector in the direction of the normal vector $\mathbf{n}$ vanishes on the free surface of the beam.

3 Integral representations—numerical solution

According to the precedent analysis, the nonlinear flexural dynamic analysis of Timoshenko beams, undergoing moderate large displacements and small deformations, reduces in establishing the displacement components $u(x,t)$ and $v(x,t)$, $w(x,t)$ having continuous derivatives up to the second order and up to the fourth order with respect to x, respectively, and also having derivatives up to the second order with respect to t (ignoring the inertia terms of the fourth order [48]). Moreover, these displacement components must satisfy the coupled governing differential equations (14) inside the beam, the boundary conditions (15)–(17) at the beam ends $x=0,l$ and the initial conditions (18)–(20). Equations (14) are solved using the Analog Equation Method [34] as it is developed for hyperbolic differential equations [49].

3.1 For the transverse displacements v, w

Let $v(x,t)$, $w(x,t)$ be the sought solution of the aforementioned boundary value problem. Setting $u_2(x,t)=v(x,t)$, $u_3(x,t)=w(x,t)$ and differentiating these functions four times with respect to x yields

$$\frac{\partial^4 u_i}{\partial x^4}=q_i(x,t)\quad (i=2,3). \tag{29}$$

Equations (29) are quasi-static: that is, the time variable appears as a parameter. They indicate that the solution of (14b), (14c) can be established by solving (29) under the same boundary conditions (16)–(17), provided that the fictitious load distributions $q_i(x,t)$ $(i=2,3)$ are first established. These distributions can be determined using BEM as follows.

The solution of (29) is given in integral form as

$$\begin{aligned}u_i(x,t)&=\int_0^l q_i(\xi,t)u^*\,d\xi\\&\quad-\left[u^*\frac{\partial^3 u_i}{\partial x^3}-\frac{du^*}{dx}\frac{\partial^2 u_i}{\partial x^2}+\frac{d^2u^*}{dx^2}\frac{\partial u_i}{\partial x}\right.\\&\quad\left.-\frac{d^3u^*}{dx^3}u_i\right]_0^l\end{aligned} \tag{30}$$

where u^* is the fundamental solution given as

$$u^*=\frac{1}{12}l^3\left(2+\left|\frac{r}{l}\right|^3-3\left|\frac{r}{l}\right|^2\right) \tag{31}$$

with $r=x-\xi$, x, ξ points of the beam, which is a particular singular solution of the equation

$$\frac{d^4u^*}{dx^4}=\delta(x-\xi). \tag{32}$$

Employing (31), the integral representation (30) can be written as

$$\begin{aligned}u_i(x,t)&=\int_0^l q_i(\xi,t)\Lambda_4(r)\,d\xi\\&\quad-\left[\Lambda_4(r)\frac{\partial^3 u_i}{\partial x^3}+\Lambda_3(r)\frac{\partial^2 u_i}{\partial x^2}\right.\\&\quad\left.+\Lambda_2(r)\frac{\partial u_i}{\partial x}+\Lambda_1(r)u_i\right]_0^l,\end{aligned} \tag{33}$$

where the kernels $\Lambda_j(r)$ $(j=1,2,3,4)$ are given as

$$\Lambda_1(r)=-\frac{1}{2}\operatorname{sgn}\frac{r}{l}, \tag{34a}$$

$$\Lambda_2(r)=-\frac{1}{2}l\left(1-\left|\frac{r}{l}\right|\right), \tag{34b}$$

$$\Lambda_3(r)=-\frac{1}{4}l^2\left|\frac{r}{l}\right|\left(\left|\frac{r}{l}\right|-2\right)\operatorname{sgn}\frac{r}{l}, \tag{34c}$$

$$\Lambda_4(r)=\frac{1}{12}l^3\left(2+\left|\frac{r}{l}\right|^3-3\left|\frac{r}{l}\right|^2\right). \tag{34d}$$

Notice that in (33) for the line integral it is $r=x-\xi$, x, ξ points inside the beam, whereas for the rest terms it is $r=x-\zeta$, x inside the beam, ζ at the beam ends $0,l$.

Differentiation of (33) with respect to x results in the integral representations of the derivatives of u_i $(i=2,3)$ as

$$\begin{aligned}\frac{\partial u_i(x,t)}{\partial x}&=\int_0^l q_i(\xi,t)\Lambda_3(r)\,d\xi\\&\quad-\left[\Lambda_3(r)\frac{\partial^3 u_i}{\partial x^3}+\Lambda_2(r)\frac{\partial^2 u_i}{\partial x^2}\right.\\&\quad\left.+\Lambda_1(r)\frac{\partial u_i}{\partial x}\right]_0^l,\end{aligned} \tag{35a}$$

$$\begin{aligned}\frac{\partial^2 u_i(x,t)}{\partial x^2}&=\int_0^l q_i(\xi,t)\Lambda_2(r)\,d\xi\\&\quad-\left[\Lambda_2(r)\frac{\partial^3 u_i}{\partial x^3}+\Lambda_1(r)\frac{\partial^2 u_i}{\partial x^2}\right]_0^l,\end{aligned} \tag{35b}$$

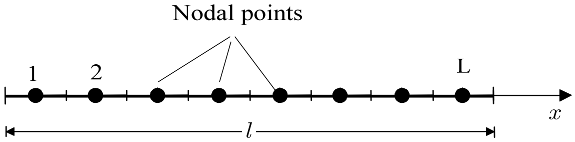


Fig. 2 Discretization of the beam interval and distribution of the nodal points

$$\frac{\partial^3 u_i(x,t)}{\partial x^3} = \int_0^l q_i(\xi,t)\Lambda_1(r)\,d\xi - \left[\Lambda_1(r)\frac{\partial^3 u_i}{\partial x^3}\right]_0^l, \tag{35c}$$

$$\frac{\partial^4 u_i(x,t)}{\partial x^4} = q_i(x,t). \tag{35d}$$

The integral representations (33) and (35a), when applied to the beam ends $(0,l)$, together with the boundary conditions (16)–(17) are employed to express the unknown boundary quantities $u_i(\zeta,t)$, $u_{i,x}(\zeta,t)$, $u_{i,xx}(\zeta,t)$ and $u_{i,xxx}(\zeta,t)$ $(\zeta = 0,l)$ in terms of q_i. This is accomplished numerically as follows.

The interval $(0,l)$ is divided into L equal elements (Fig. 2), on which $q_i(x,t)$ is assumed to vary according to a certain law (constant, linear, parabolic, etc.). The constant element assumption is employed here as the numerical implementation becomes very simple and the obtained results are very good. Employing the aforementioned procedure for the coupled boundary conditions (16)–(17), the following set of nonlinear equations is obtained:

$$\begin{bmatrix} \mathbf{D}_{11} & 0 & \mathbf{D}_{13} & \mathbf{D}_{14} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23} & 0 \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{E}_{41} & \mathbf{E}_{42} & \mathbf{E}_{43} & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_{2,xxx} \\ \hat{\mathbf{u}}_{2,xx} \\ \hat{\mathbf{u}}_{2,x} \\ \hat{\mathbf{u}}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\beta}_3 \\ \bar{\boldsymbol{\beta}}_3 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{Bmatrix} \mathbf{q}_2, \tag{36a}$$

$$\begin{bmatrix} \mathbf{G}_{11} & 0 & \mathbf{G}_{13} & \mathbf{G}_{14} \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} & 0 \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{E}_{41} & \mathbf{E}_{42} & \mathbf{E}_{43} & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_{3,xxx} \\ \hat{\mathbf{u}}_{3,xx} \\ \hat{\mathbf{u}}_{3,x} \\ \hat{\mathbf{u}}_3 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\gamma}_3 \\ \bar{\boldsymbol{\gamma}}_3 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{Bmatrix} \mathbf{q}_3, \tag{36b}$$

where $\mathbf{D}_{11}$, $\mathbf{D}_{13}$, $\mathbf{D}_{14}$, $\mathbf{D}_{21}$, $\mathbf{D}_{22}$, $\mathbf{D}_{23}$, $\mathbf{G}_{11}$, $\mathbf{G}_{13}$, $\mathbf{G}_{14}$, $\mathbf{G}_{21}$, $\mathbf{G}_{22}$, $\mathbf{G}_{23}$ are 2×2 known square matrices including the values of the functions $\beta_j, \bar{\beta}_j, \gamma_j, \bar{\gamma}_j$ $(j = 1,2)$ of (16)–(17); $\boldsymbol{\beta}_3$, $\bar{\boldsymbol{\beta}}_3$, $\boldsymbol{\gamma}_3$, $\bar{\boldsymbol{\gamma}}_3$ are 2×1 known column matrices including the boundary values of the functions β_3, $\bar{\beta}_3$, γ_3, $\bar{\gamma}_3$ of (16)–(17); $\mathbf{E}_{jk}$ $(j = 3,4,$ $k = 1,2,3,4)$ are square 2×2 known coefficient matrices and $\mathbf{F}_j$ $(j = 3,4)$ are $2\times L$ rectangular known matrices originating from the integration of kernels on the axis of the beam. Moreover,

$$\hat{\mathbf{u}}_i = \{u_i(0,t)u_i(l,t)\}^T, \tag{37a}$$

$$\hat{\mathbf{u}}_{i,x} = \left\{\frac{\partial u_i(0,t)}{\partial x}\frac{\partial u_i(l,t)}{\partial x}\right\}^T, \tag{37b}$$

$$\hat{\mathbf{u}}_{i,xx} = \left\{\frac{\partial^2 u_i(0,t)}{\partial x^2}\frac{\partial^2 u_i(l,t)}{\partial x^2}\right\}^T, \tag{37c}$$

$$\hat{\mathbf{u}}_{i,xxx} = \left\{\frac{\partial^3 u_i(0,t)}{\partial x^3}\frac{\partial^3 u_i(l,t)}{\partial x^3}\right\}^T \tag{37d}$$

are the vectors including the two unknown boundary values of the respective boundary quantities and $\mathbf{q}_i = \{q_1^i q_2^i \cdots q_L^i\}^T$ $(i = 2,3)$ is the vector including the L unknown nodal values of the fictitious load.

Discretization of the integral representations of the displacement components u_i $(i = 2,3)$ (33) and their derivatives with respect to x (35), after elimination of the boundary quantities employing (36), gives

$$\mathbf{u}_i = \mathbf{T}_i\mathbf{q}_i + \mathbf{t}_i, \quad i = 2,3, \tag{38a}$$

$$\mathbf{u}_{i,x} = \mathbf{T}_{ix}\mathbf{q}_i + \mathbf{t}_{ix}, \quad i = 2,3, \tag{38b}$$

$$\mathbf{u}_{i,xx} = \mathbf{T}_{ixx}\mathbf{q}_i + \mathbf{t}_{ixx}, \quad i = 2,3, \tag{38c}$$

$$\mathbf{u}_{i,xxx} = \mathbf{T}_{ixxx}\mathbf{q}_i + \mathbf{t}_{ixxx}, \quad i = 2,3, \tag{38d}$$

$$\mathbf{u}_{i,xxxx} = \mathbf{q}_i, \quad i = 2,3, \tag{38e}$$

where $\mathbf{u}_i, \mathbf{u}_{i,x}, \mathbf{u}_{i,xx}, \mathbf{u}_{i,xxx}, \mathbf{u}_{i,xxxx}$ are vectors including the values of $u_i(x,t)$ and their derivatives at the L nodal points, $\mathbf{T}_i$, $\mathbf{T}_{ix}$, $\mathbf{T}_{ixx}$, $\mathbf{T}_{ixxx}$ are known $L\times L$ matrices and $\mathbf{t}_i, \mathbf{t}_{ix}, \mathbf{t}_{ixx}, \mathbf{t}_{ixxx}$ are known $L\times 1$ matrices.

In the conventional BEM, the load vectors $\mathbf{q}_i$ are known and (38) are used to evaluate $u_i(x,t)$ and their derivatives at the L nodal points. This, however, cannot be done here since $\mathbf{q}_i$ are unknown. For this purpose, $2L$ additional equations are derived, which permit the establishment of $\mathbf{q}_i$. These equations result by applying (14b), (14c) to the L collocation points, which after ignoring the inertia terms of the fourth order arising from coupling of shear deformations and rotary inertia [48], lead to the formulation of the following set of $2L$ simultaneous equations:

$$\mathbf{M}_2\ddot{\mathbf{q}}_2 + \mathbf{S}_2\dot{\mathbf{q}}_2 + \mathbf{K}_2\mathbf{q}_2 = \mathbf{f}_2, \tag{39a}$$

$$\mathbf{M}_3\ddot{\mathbf{q}}_3 + \mathbf{S}_3\dot{\mathbf{q}}_3 + \mathbf{K}_3\mathbf{q}_3 = \mathbf{f}_3, \tag{39b}$$

where the $\mathbf{M}_2, \mathbf{M}_3, \mathbf{S}_2, \mathbf{S}_3, \mathbf{K}_2, \mathbf{K}_3$ $L \times L$ matrices and the $\mathbf{f}_2, \mathbf{f}_3$ $L \times 1$ vectors are given as

$$\mathbf{M}_2 = \rho A \mathbf{T}_2 - \rho I_z \left(\frac{Ea_y}{G} + 1\right)\mathbf{T}_{2xx} - \frac{\rho I_z}{GA_y}(\mathbf{N}_x\mathbf{T}_{2x} + \mathbf{N}\mathbf{T}_{2xx}), \tag{40a}$$

$$\mathbf{S}_2 = -\frac{2\rho I_z}{GA_y}(\mathbf{N}_{xt}\mathbf{T}_{2x} + \mathbf{N}_t\mathbf{T}_{2xx}), \tag{40b}$$

$$\begin{aligned}\mathbf{K}_2 = \mathbf{EI}_z - \mathbf{N}_x\mathbf{T}_{2x} - \mathbf{N}\mathbf{T}_{2xx} &+ \frac{EI_z}{GA_y}(\mathbf{N}_{xxx}\mathbf{T}_{2x} + 3\mathbf{N}_{xx}\mathbf{T}_{2xx} \\ &+ 3\mathbf{N}_x\mathbf{T}_{2xxx} + \mathbf{N}) \\ &- \frac{\rho I_z}{GA_y}(\mathbf{N}_{xtt}\mathbf{T}_{2x} + \mathbf{N}_{tt}\mathbf{T}_{2xx}),\end{aligned} \tag{40c}$$

$$\begin{aligned}\mathbf{f}_2 = \mathbf{p}_y - \frac{EI_z}{GA_y}\mathbf{p}_{y,xx} + \frac{\rho I_z}{GA_y}\mathbf{p}_{y,tt} - \mathbf{m}_{z,x} &+ \mathbf{N}_x\mathbf{t}_{2x} + \mathbf{N}\mathbf{t}_{2xx} \\ &- \frac{EI_z}{GA_y}(\mathbf{N}_{xxx}\mathbf{t}_{2x} + 3\mathbf{N}_{xx}\mathbf{t}_{2xx} + 3\mathbf{N}_x\mathbf{t}_{2xxx}) \\ &+ \frac{\rho I_z}{GA_y}(\mathbf{N}_{xtt}\mathbf{t}_{2x} + \mathbf{N}_{tt}\mathbf{t}_{2xx}),\end{aligned} \tag{40d}$$

$$\mathbf{M}_3 = \rho A \mathbf{T}_3 - \rho I_y \left(\frac{Ea_z}{G} + 1\right)\mathbf{T}_{3xx} - \frac{\rho I_y}{GA_z}(\mathbf{N}_x\mathbf{T}_{3x} + \mathbf{N}\mathbf{T}_{3xx}), \tag{40e}$$

$$\mathbf{S}_3 = -\frac{2\rho I_y}{GA_z}(\mathbf{N}_{xt}\mathbf{T}_{3x} + \mathbf{N}_t\mathbf{T}_{3xx}), \tag{40f}$$

$$\begin{aligned}\mathbf{K}_3 = \mathbf{EI}_y - \mathbf{N}_x\mathbf{T}_{3x} - \mathbf{N}\mathbf{T}_{3xx} &+ \frac{EI_y}{GA_z}(\mathbf{N}_{xxx}\mathbf{T}_{3x} \\ &+ 3\mathbf{N}_{xx}\mathbf{T}_{3xx} + 3\mathbf{N}_x\mathbf{T}_{3xxx} + \mathbf{N}) \\ &- \frac{\rho I_y}{GA_z}(\mathbf{N}_{xtt}\mathbf{T}_{3x} + \mathbf{N}_{tt}\mathbf{T}_{3xx}),\end{aligned} \tag{40g}$$

$$\begin{aligned}\mathbf{f}_3 = \mathbf{p}_z - \frac{EI_y}{GA_z}\mathbf{p}_{z,xx} + \frac{\rho I_y}{GA_z}\mathbf{p}_{z,tt} + \mathbf{m}_{y,x} &+ \mathbf{N}_x\mathbf{t}_{3x} + \mathbf{N}\mathbf{t}_{3xx} \\ &- \frac{EI_y}{GA_z}(\mathbf{N}_{xxx}\mathbf{t}_{3x} + 3\mathbf{N}_{xx}\mathbf{t}_{3xx} + 3\mathbf{N}_x\mathbf{t}_{3xxx}) \\ &+ \frac{\rho I_y}{GA_z}(\mathbf{N}_{xtt}\mathbf{t}_{3x} + \mathbf{N}_{tt}\mathbf{t}_{3xx})\end{aligned} \tag{40h}$$

where $\mathbf{N}$, $\mathbf{N}_{km}$ $(k, m = x, t)$ are $L \times L$ diagonal matrices containing the values of the axial force and its derivatives with respect to k and m parameters at the L nodal points, $\mathbf{EI}_y$, $\mathbf{EI}_z$ are $L \times L$ diagonal matrices including the values of the corresponding quantities, at the aforementioned points, while $\mathbf{p}_y$, $\mathbf{p}_{y,xx}$, $\mathbf{p}_{y,tt}$, $\mathbf{p}_z$, $\mathbf{p}_{z,xx}$, $\mathbf{p}_{z,tt}$, $\mathbf{m}_{y,x}$ and $\mathbf{m}_{z,x}$ are $L \times 1$ vectors containing the values of the external loading and its derivatives at these points.

3.2 For the axial displacement u

Let $u_1 = u(x,t)$ be the sought solution of the boundary value problem described by (14a) and (15). Differentiating this function twice yields

$$\frac{\partial^2 u_1}{\partial x^2} = q_x(x,t). \tag{41}$$

Equation (41) indicates that the solution of the original problem can be obtained as the axial displacement of a beam with unit axial rigidity subjected to an axial fictitious load $q_x(x,t)$ under the same boundary conditions. The fictitious load is unknown.

The solution of (41) and its derivative are given in integral form as

$$u(x,t) = \int_0^l u^* q_x(\xi,t)\,d\xi - \left[u^*\frac{\partial u}{\partial x} - \frac{du^*}{dx}u\right]_0^l, \tag{42a}$$

$$\frac{\partial u(x,t)}{\partial x} = -\int_0^l \frac{du^*}{dx} q_x(\xi,t)\,d\xi - \left[-\frac{du^*}{dx}\frac{\partial u}{\partial x}\right]_0^l, \tag{42b}$$

where u^* is the fundamental solution, which is given as

$$u^* = \frac{1}{2}|r|. \tag{43}$$

Following the same procedure as in Sect. 3.1, the discretized counterpart of (42) when applied to all nodal points in the interior of the beam yields

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{q}_1 + \mathbf{t}_1, \tag{44a}$$

$$\mathbf{u}_{1,x} = \mathbf{T}_{1x}\mathbf{q}_1 + \mathbf{t}_{1x}, \tag{44b}$$

where $\mathbf{T}_1$, $\mathbf{T}_{1x}$ are known $L \times L$ matrices, similar to those mentioned before for the displacements u_2, u_3. Application of (14a) to the L collocation points, after employing (38), (44) leads to the formulation of the following system of L equations with respect to $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ fictitious load vectors:

$$\begin{aligned}&\mathbf{EAq}_1 - \rho\mathbf{AT}_1\ddot{\mathbf{q}}_1\\ &\quad = -\mathbf{p}_x - \mathbf{EA}\big[(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx})\big]_{\mathrm{dg}}(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x})\\ &\quad - \mathbf{EA}\big[(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx})\big]_{\mathrm{dg}}(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x})\end{aligned} \tag{45}$$

where $\mathbf{EA}$, $\rho\mathbf{A}$ are $L \times L$ diagonal matrices including the values of the corresponding quantities at the L nodal points. Moreover, substituting (38), (44) in (9a) the discretized counterpart of the axial force at the neutral axis of the beam is given as

$$\begin{aligned}\mathbf{N} &= \mathbf{EA}(\mathbf{T}_{1x}\mathbf{q}_1 + \mathbf{t}_{1x})\\ &\quad + \frac{1}{2}\mathbf{EA}\big[(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x})\big]_{\mathrm{dg}}(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x})\\ &\quad + \frac{1}{2}\mathbf{EA}\big[(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x})\big]_{\mathrm{dg}}(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}).\end{aligned} \tag{46}$$

Equations (39a), (39b), (45) and (46) constitute a nonlinear coupled system of equations with respect to $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ and $\mathbf{N}$ quantities. The solution of this system is accomplished iteratively by employing the average acceleration method in combination with the modified Newton–Raphson method [35, 36].

3.3 For the stress functions $\Theta(y,z)$ and $\Phi(y,z)$

The evaluation of the stress functions $\Theta(y,z)$ and $\Phi(y,z)$ is accomplished using BEM as this is presented in Sapountzakis and Mokos [47].

Moreover, since the nonlinear flexural problem of Timoshenko beams is solved by the BEM, the domain integrals for the evaluation of the area, the bending moments of inertia (10), (11) and the shear deformation coefficients (24) have to be converted to boundary line integrals, in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the moments, the product of inertia and the cross section area can be written as

$$I_y = \int_\Gamma \left(yz^2 n_y\right) ds, \tag{47a}$$

$$I_z = \int_\Gamma \left(zy^2 n_z\right) ds, \tag{47b}$$

$$A = \frac{1}{2}\int_\Gamma (yn_y + zn_z)\, ds, \tag{47c}$$

while the shear deformation coefficients a_y and a_z are obtained from the relations

$$a_y = \frac{A}{\Delta^2}\left((4v+2)I_y I_{\Theta y} + \frac{1}{4}v^2 I_y^2 I_{ed} - I_{\Theta e}\right), \tag{48a}$$

$$a_z = \frac{A}{\Delta^2}\left((4v+2)I_z I_{\Phi z} + \frac{1}{4}v^2 I_z^2 I_{ed} - I_{\Phi d}\right), \tag{48b}$$

where

$$I_{\Theta e} = \int_\Gamma \Theta(\mathbf{n}\cdot\mathbf{e})\, ds, \tag{49a}$$

$$I_{\Phi d} = \int_\Gamma \Phi(\mathbf{n}\cdot\mathbf{d})\, ds, \tag{49b}$$

$$I_{ed} = \int_\Gamma \left(y^4 zn_z + z^4 yn_y + \frac{2}{3}y^2 z^3 n_z\right) ds, \tag{49c}$$

$$I_{\Theta y} = \frac{1}{6}\int_\Gamma \left[-2I_y y^4 zn_z + \left(3\Theta n_y - y(\mathbf{n}\cdot\mathbf{e})\right)y^2\right] ds, \tag{49d}$$

$$I_{\Phi z} = \frac{1}{6}\int_\Gamma \left[-2I_z z^4 yn_y + \left(3\Phi n_z - z(\mathbf{n}\cdot\mathbf{d})\right)z^2\right] ds. \tag{49e}$$

4 Summary

In this paper, a boundary element method is developed for the nonlinear dynamic analysis of beams of arbitrary doubly symmetric simply or multiply connected constant cross section, undergoing moderate large displacements and small deformations under general boundary conditions, taking into account the effects of shear deformation and rotary inertia. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions, as well as to axial loading. The proposed model takes into account the coupling effects of bending and shear deformations along the member, as well as the shear forces along the span induced by the applied axial loading.

Acknowledgements This work has been funded by the project PENED 2003. The project is co-financed 75% of public expenditure through EC—European Social Fund and 25% of public expenditure through Ministry of Development—General Secretariat of Research and Technology and through private sector, under measure 8.3 of OPERATIONAL PROGRAM "COMPETITIVENESS" in the 3rd Community Support Program.

References

1. Küpper, T., Molitor, A.: Nonlinear vibrations of simply supported beams. Nonlinear Anal. Theory Methods Appl. **3**(1), 45–48 (1977)
2. Liu, J.Q.: Nonlinear vibration of a beam. Nonlinear Anal. Theory Methods Appl. **13**(10), 1139–1148 (1989)
3. Prathap, G., Varadan, T.K.: The large amplitude vibration of hinged beams. Comput. Struct. **l9**, 219–222 (1978)
4. Prathap, G., Varadan, T.K.: The large amplitude vibration of tapered clamped beams. J. Sound Vib. **58**(1), 87–94 (1978)
5. Sato, K.: Nonlinear free vibration of beams with clamped ends and with one end clamped, other end simply supported. Bull. J. Soc. Mech. Eng. **11**, 1027–1036 (1968)
6. Rehfield, L.W.: Nonlinear free vibrations of elastic structures. Int. J. Solids Struct. **9**, 581–590 (1973)
7. Mei, Ch.: Finite element displacement method for large amplitude free flexural vibrations of beams and plates. Comput. Struct. **3**, 163–174 (1973)
8. Bhashyam, G.R., Prathap, G.: Galerkin finite element method for non-linear beam vibrations. J. Sound Vib. **72**(2), 191–203 (1980)
9. Yang, T.Y., Saigal, S.: A simple element for static and dynamic response of beams with material and geometric nonlinearities. Int. J. Numer. Methods Eng. **20**, 851–867 (1984)
10. Mei, Ch.: Discussion of finite element formulations of nonlinear beam vibrations. Comput. Struct. **22**(1), 83–85 (1986)
11. Leung, A.Y.T., Mao, S.G.: Symplectic integration of an accurate beam finite element in non-linear vibration. Comput. Struct. **54**, 1135–1147 (1995)
12. Chang, T.P., Liu, Y.-N.: Dynamic finite element analysis of a nonlinear beam subjected to a moving load. Int. J. Solids Struct. **33**(12), 1673–1688 (1996)
13. Fotouhi, R.: Dynamic analysis of very flexible beams. J. Sound Vib. **305**, 521–533 (2007)
14. Katsikadelis, J.T., Tsiatas, G.C.: Non-linear dynamic analysis of beams with variable stiffness. J. Sound Vib. **270**, 847–863 (2004)
15. Katsikadelis, J.T., Tsiatas, G.C.: Non-linear dynamic stability of damped Beck's column with variable cross section. Int. J. Non-Linear Mech. **42**, 164–171 (2007)
16. Aristizabal-Ochoa, J.D.: On the comparison of Timoshenko and shear models in beam dynamics. ASCE J. Eng. Mech. **134**(3), 269–270 (2008)
17. Sapountzakis, E.J., Dourakopoulos, J.A.: Shear deformation effect in flexural-torsional vibrations of beams by BEM. Acta Mech. **203**, 197–221 (2009)
18. Aristizabal-Ochoa, J.D.: Timoshenko beam-column with generalized end conditions and nonclassical modes of vibration of shear beams. ASCE J. Eng. Mech. **130**(10), 1151–1159 (2004)
19. Arboleda-Monsalve, L.G., Zapata-Medina, D.G., Aristizabal-Ochoa, J.D.: Stability and natural frequencies of a weakened Timoshenko beam-column with generalized end conditions under constant axial load. J. Sound Vib. **307**(1–2), 89–112 (2007)
20. Aristizabal-Ochoa, J.D.: Static and dynamic stability of uniform shear beam-columns under generalized boundary conditions. J. Sound Vib. **307**(1–2), 69–88 (2007)
21. Arboleda-Monsalve, L.G., Zapata-Medina, D.G., Aristizabal-Ochoa, J.D.: Timoshenko beam-column with generalized end conditions on elastic foundation: dynamic-stiffness matrix and load vector. J. Sound Vib. **310**, 1057–1079 (2008)
22. Hernández-Urrea, J.A., Aristizábal-Ochoa, J.D.: Static and dynamic stability of an elastically restrained Beck column with an attached end mass. J. Sound Vib. **312**(4–5), 789–800 (2008)
23. Rao, G.V., Raju, I.S., Kanaka Raju, K.: Nonlinear vibrations of beams considering shear deformation and rotary inertia. AIAA J. **14**(5), 685–687 (1976)
24. Foda, M.A.: Influence of shear deformation and rotary inertia on nonlinear free vibration of a beam with pinned ends. Comput. Struct. **71**, 663–670 (1999)
25. Zhong, H., Guo, Q.: Nonlinear vibration analysis of Timoshenko beams using the differential quadrature method. Nonlinear Dyn. **32**, 223–234 (2003)
26. Guo, Q., Liu, X., Zhong, H.: Nonlinear vibrations of Timoshenko beams with various boundary conditions. Tsinghua Sci. Technol. **9**(2), 125–129 (2004)
27. Zhong, H., Liao, M.: Higher-order nonlinear vibration analysis of Timoshenko beams by the spline-based differential quadrature method. Shock Vib. **14**, 407–416 (2007)
28. Liao, M., Zhong, H.: Nonlinear vibration analysis of tapered Timoshenko beams. Chaos Solitons Fractals **36**, 1267–1272 (2008)
29. Doong, J.L., Chen, C.S.: Large amplitude vibration of a beam based on a higher-order deformation theory. Appl. Acoustics **25**, 281–293 (1988)
30. Lai, S.H.Y.: Nonlinear finite element modelling of a high speed rotating Timoshenko beam structure. Int. J. Mech. Sci. **36**(9), 849–861 (1994)
31. Meek, J.L., Liu, H.: Nonlinear dynamics analysis of flexible beams under large overall motions and the flexible manipulator simulation. Comput. Struct. **56**(1), 1–14 (1995)
32. Wang, R.T., Chou, T.H.: Non-linear vibration of Timoshenko beam due to a moving force and the weight of beam. J. Sound Vib. **218**(1), 117–131 (1998)
33. Maqueda, L.G., Shabana, A.A.: Poisson modes and general nonlinear constitutive models in the large displacement analysis of beams. J. Multibody Syst. Dyn. **18**(3), 375–396 (2007)
34. Katsikadelis, J.T.: The analog equation method. A boundary-only integral equation method for nonlinear static and dynamic problems in general bodies. Theor. Appl. Mech. **27**, 13–38 (2002)

35. Chang, S.Y.: Studies of Newmark method for solving nonlinear systems: (I) basic analysis. J. Chin. Inst. Eng. **27**(5), 651–662 (2004)
36. Isaacson, E., Keller, H.B.: Analysis of Numerical Methods. Wiley, New York (1966)
37. Atanackovic, T.M., Spasic, D.T.: A model for plane elastica with simple shear deformation pattern. Acta Mech. **104**, 241–253 (1994)
38. Aristizábal-Ochoa, J.D.: Tension buckling in multilayer elastomeric bearings by James M. Kelly. ASCE J. Eng. Mech. **129**(12), 1363–1368 (2003). Discussion: **131**(1), 106–108 (2005)
39. Timoshenko, S.P., Goodier, J.N.: Theory of Elasticity, 3rd edn. McGraw–Hill, New York (1984)
40. Cowper, G.R.: The shear coefficient in Timoshenko's beam theory. ASME J. Appl. Mech. **33**(2), 335–340 (1966)
41. Schramm, U., Kitis, L., Kang, W., Pilkey, W.D.: On the shear deformation coefficient in beam theory. Finite Elem. Anal. Des. **16**, 141–162 (1994)
42. Schramm, U., Rubenchik, V., Pilkey, W.D.: Beam stiffness matrix based on the elasticity equations. Int. J. Numer. Methods Eng. **40**, 211–232 (1997)
43. Stephen, N.G.: Timoshenko's shear coefficient from a beam subjected to gravity loading. ASME J. Appl. Mech. **47**, 121–127 (1980)
44. Hutchinson, J.R.: Shear coefficients for Timoshenko beam theory. ASME J. Appl. Mech. **68**, 87–92 (2001)
45. Ramm, E., Hofmann, T.J.: Stabtragwerke, Der Ingenieurbau. In: Mehlhorn, G. (ed.) Band Baustatik/Baudynamik. Ernst & Sohn, Berlin (1995)
46. Rothert, H., Gensichen, V.: Nichtlineare Stabstatik. Springer, Berlin (1987)
47. Sapountzakis, E.J., Mokos, V.G.: A BEM solution to transverse shear loading of beams. Comput. Mech. **36**, 384–397 (2005)
48. Thomson, W.T.: Theory of Vibration with Applications. Prentice Hall, Englewood Cliffs (1981)
49. Sapountzakis, E.J., Katsikadelis, J.T.: Analysis of plates reinforced with beams. Comput. Mech. **26**, 66–74 (2000)