GEOMETRIC 2D TRANSFORMATIONS

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Outline

- **What and Why Transformations**
- **Types of Transformations**
  1. **Translation**
  2. **Rotation**
  3. **Scaling**
- **Inverse Transformations**
- **Rotation about an Arbitrary Point**
- **Fixed Point Scaling**
- **Types of Transformations (Cont.)**
  4. **Reflection**
  5. **Shearing**

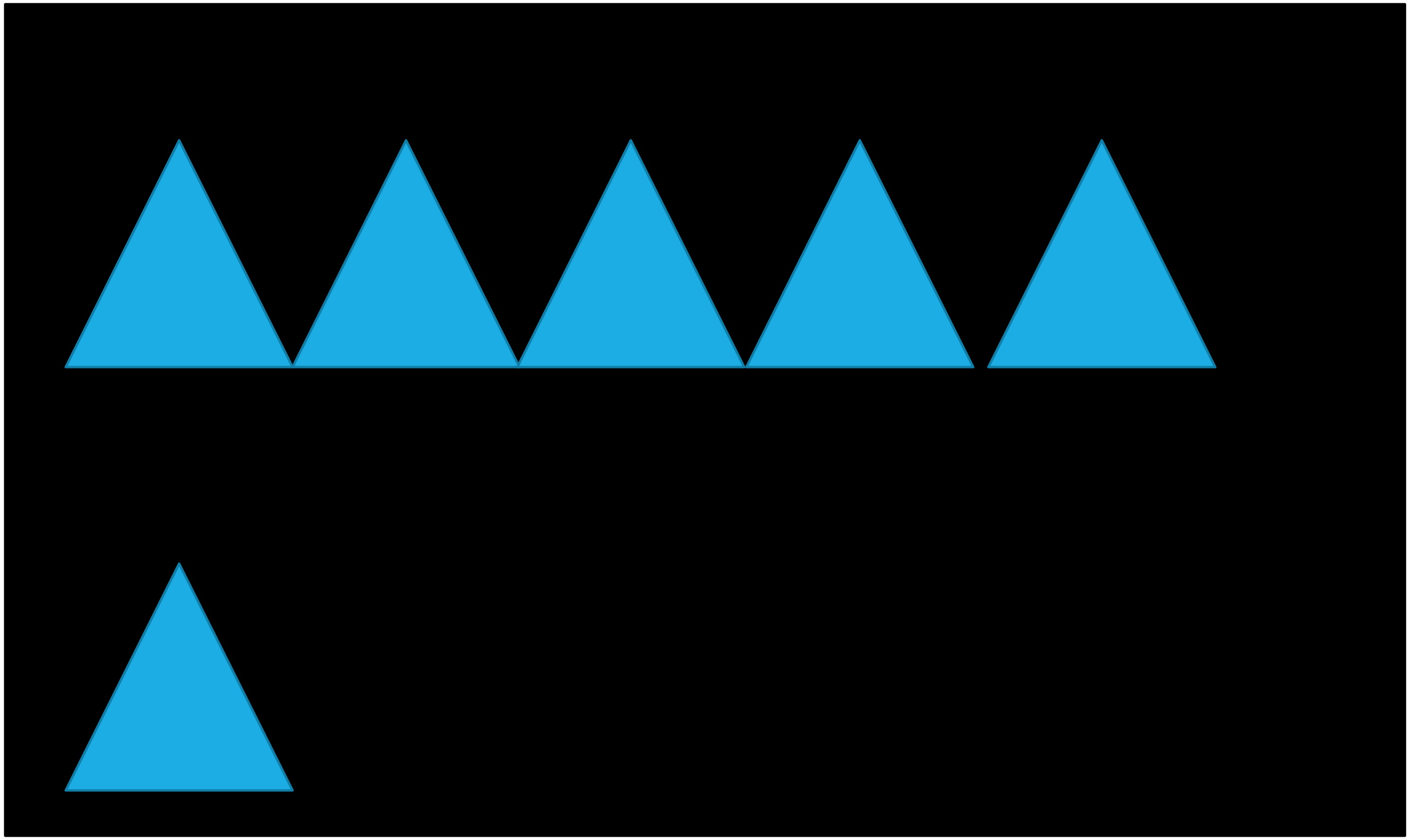
What and Why Transformations

- The operations that are applied to an object to change its **position**, **orientation**, or **size** are called Geometric Transformation
- The geometrical changes of an object from a current state to modified state

- Why the transformations is needed?

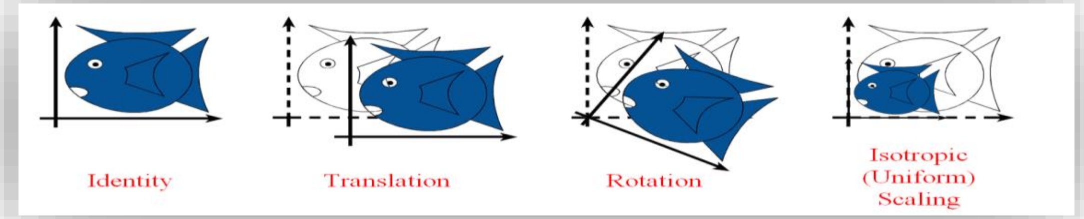
  To manipulate the initially created object and to display the modified object without having to redraw it

Different between Redrawing and Transformation

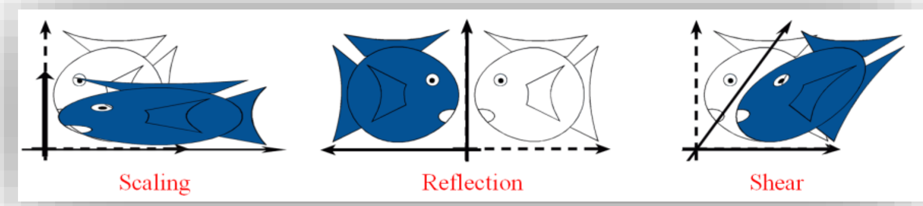


Types of Transformations

- **Geometric Transformations**
  - Translation
  - Rotation
  - Scaling

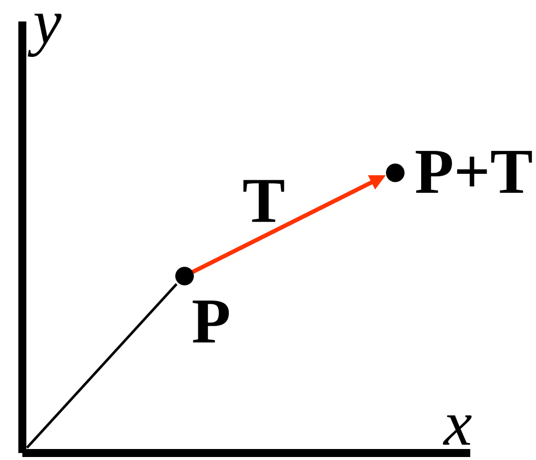


- **Linear (preserves parallel lines)**
  - Non-uniform scales, shears or skews
- **Projection (Preserves lines)**
  - Perspective projection
  - Parallel projection
- **Non-linear (Lines become curves)**
  - Twists, bends, warps, morphs



Translation

- Translation is an operation that displace points by a fixed distance in a given direction



Move point P(x, y) to P'(x', y')

$$x' = x + t_x,\ y' = y + t_y$$

OR

$$\mathbf{P}' = \mathbf{P} + \mathbf{T}, \text{ with}$$

$$\mathbf{P}' = \begin{pmatrix} x' \\ y' \end{pmatrix}, \mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathbf{T} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Translation (cont.)

With adding Homogeneous coordinates we get a matrix as following to use for translation

Which is mean we use it as the following

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

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Why Homogeneous coordinates

- **Homogenous Coordinate** system. In this system, we can represent all the transformation equations in matrix multiplication. Any Cartesian point P (X,Y) can be converted to homogenous coordinates by P' (X_h, Y_h, h).
- Unified frame work for translations, viewing, rotate ,.. etc.

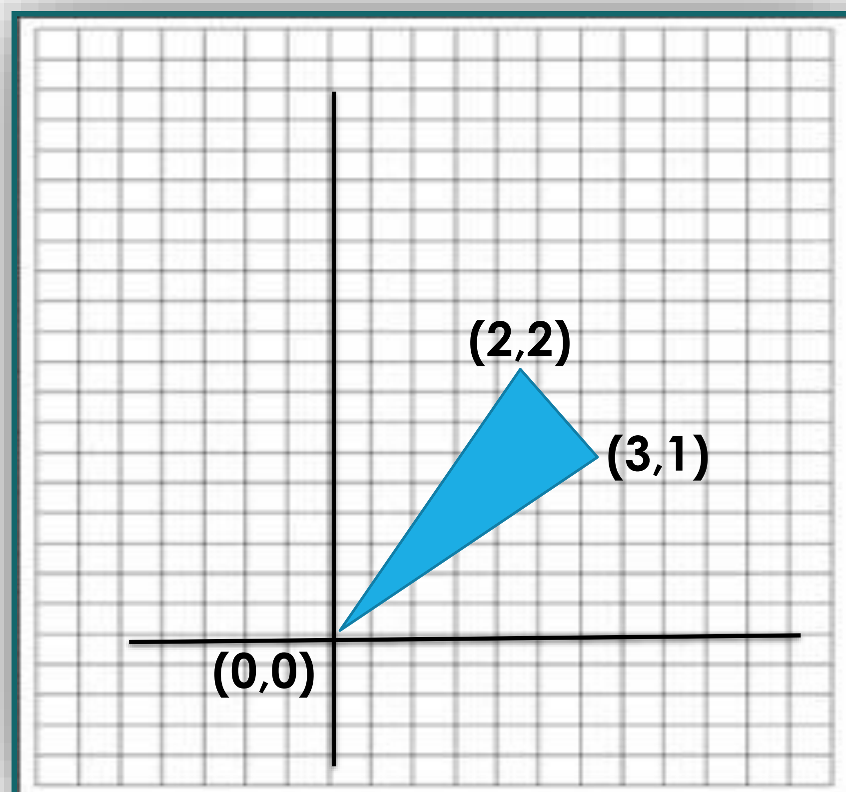
Why Homogenous Coordinate?

- Can concatenate any set of transform
- simpler formula
- Standard in graphics software and hardware

Translation: Example

- Consider a triangle with vertices A(0,0),B(2,2) and C(3,1),

translate 2 units in the **Horizontal** direction and 1 unit in the **Vertical** direction



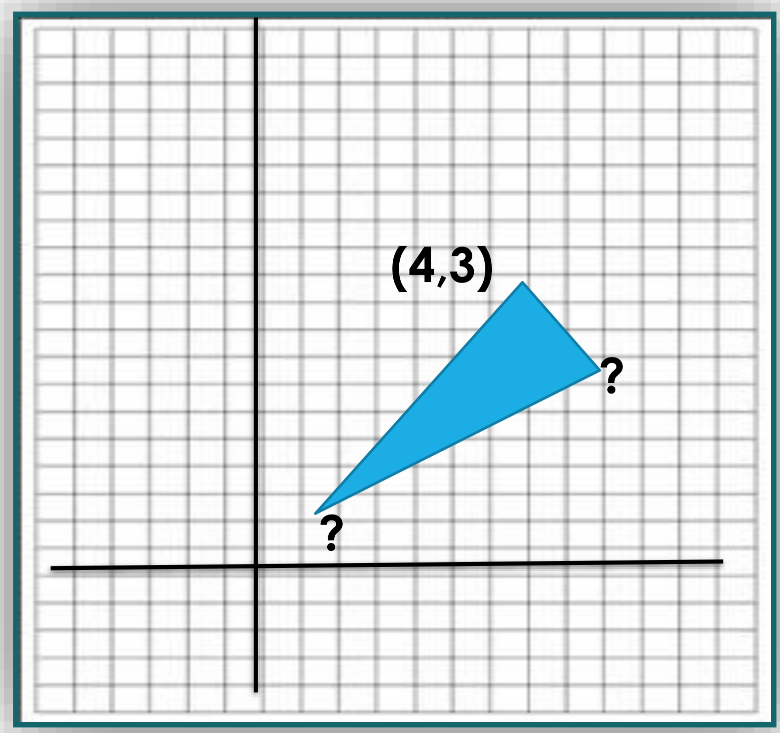
Translation: Example(cont.)

- solution

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

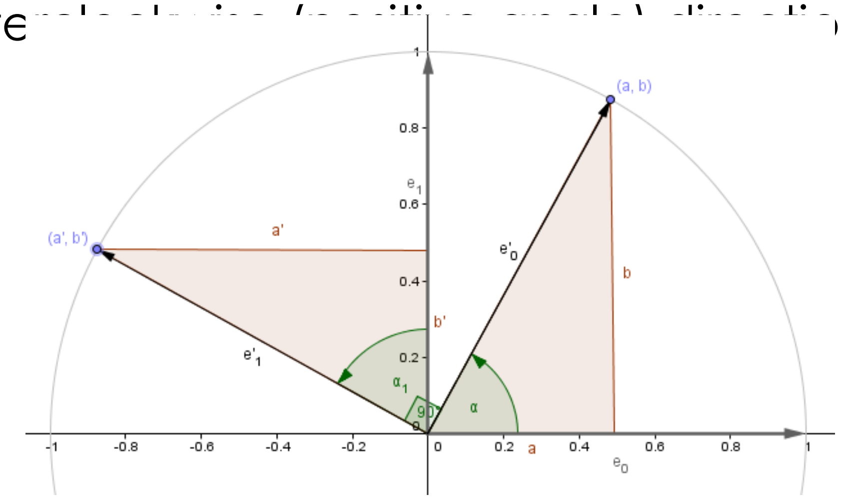
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =?$$

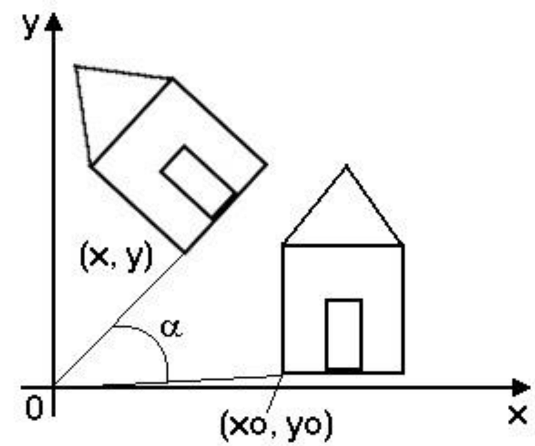
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+ & 0+ & 2 \\ 0+ & 2+ & 1 \\ 0+ & 0+ & 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$



Rotation

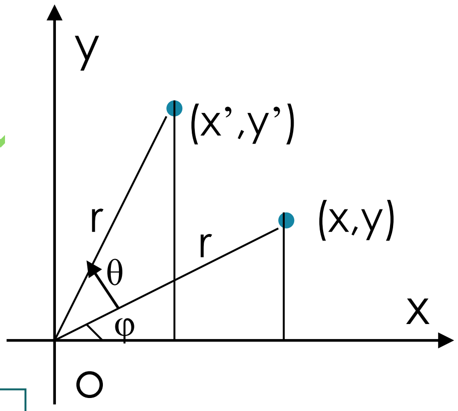
- Another useful transformation is the rotation of the object about specified pivot(axes) point
- After the object has been rotated, it still the same distance away from the pivot point
- It is possible to rotate one or more objects or the entire image
- About any point in world space in either a clockwise(negative angle) or counterclockwise (positive angle) direction





Rotation (2)

- Any point(x, y) can be represented by its radial distance (r) ,from the origin and its angle(θ)



- Rotate (x,y), angle θ, based on origin point (pivot point)

$$x' = r\cos(\theta+\phi) = \underline{r\cos\phi}\cos\theta - \underline{r\sin\phi}\sin\theta$$

$$y' = r\sin(\theta+\phi) = \underline{r\cos\phi}\sin\theta + \underline{r\sin\phi}\cos\theta$$

$$\because \quad x = r\cos\phi,\ y = r\sin\phi$$

in polar coordinates

$$\therefore \quad x' = x\cos\theta - y\sin\theta,\ y' = x\sin\theta + y\cos\theta$$

$$P' = R \bullet P$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation (cont.)

- Using Homogeneous coordinates

$$R = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} . \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Example : Rotate point (3,4) **Rotating 45° about the origin .**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} . \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

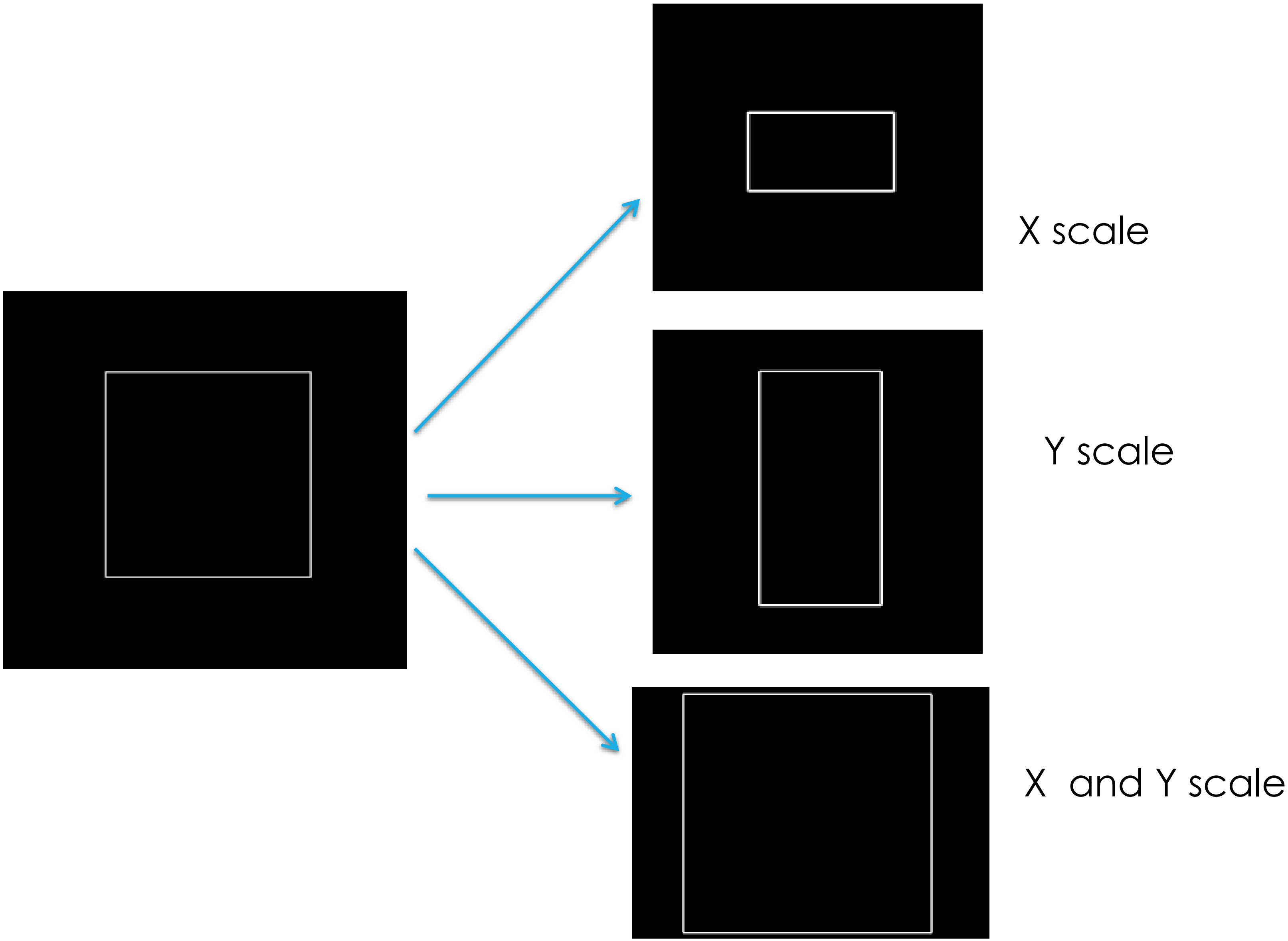
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} . \begin{bmatrix} 0.7 & -0.7 & 0 \\ 0.7 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.7 \\ 4.9 \\ 1 \end{bmatrix} .$$

Scaling

- An object can be made larger by increasing the distance between the points describing the object
- In general we can change the size of an object or entire image by multiplying the distance between points by an enlargement or reduction factor
  - this factor is called the **Scaling Factor** and the operation that change the size is called **Scaling**

$$X' = Sx \cdot X, \qquad Y' = Sy \cdot Y$$

  - if the Scaling Factor is greater than 1 is enlarge, if the factor is less than 1, the object is made smaller, a factor of 1 has no effect on the object
  - whenever scaling is performed there is one point that remain at the same location , this is called the **Fixed Point** of the scaling transformation


X scale
Y scale
X and Y scale

Scaling (cont.)

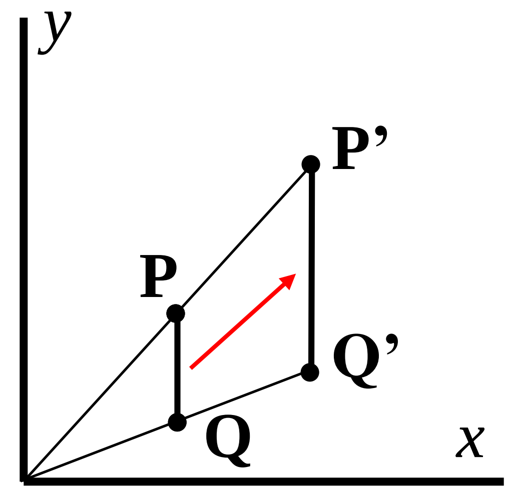
Scale with factor s_x and s_y:

$x' = s_x\, x, \quad y' = s_y\, y$

Or

$$\mathbf{P}' = \mathbf{S}\mathbf{P}, \quad \text{with}$$

$$\mathbf{P}' = \begin{pmatrix} x' \\ y' \end{pmatrix}, \mathbf{S} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \text{ and } \mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}$$

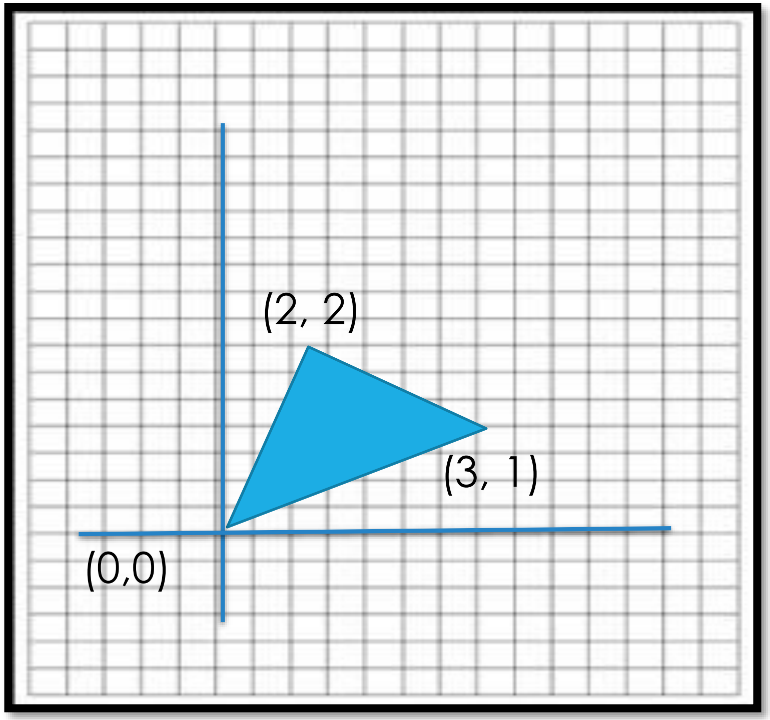


with homogeneous Coordinate

$$S = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling: Example

- Consider a triangle with vertices A(0, 0),B(2, 2) and C(3, 1),Scaling **2** units in the X axes and **2** unit in the Y axes



Solution

- $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} . \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6+ & 0+ & 0 \\ 0+ & 2+ & 0 \\ 0+ & 0+ & 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

What about A(0, 0), B(2, 2) ???

Inverse Transformations

Inverse Translation Matrix

$$\mathbf{T}^{-1}(t_x, t_y) = \mathbf{T}(-t_x, -t_y)$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Scaling Matrix

$$\mathbf{S}^{-1}(s_x, s_y) = \mathbf{S}\left(\frac{1}{s_x}, \frac{1}{s_y}\right)$$

$$S^{-1} = \begin{bmatrix} 1/S_x & 0 & 0 \\ 0 & 1/S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Rotation Matrix

$$\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta)$$

$$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

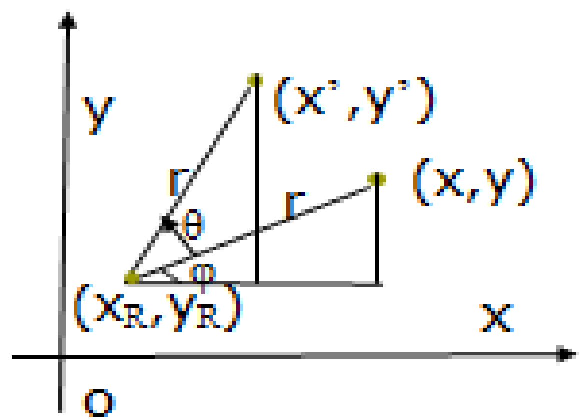
Rotation about an Arbitrary Point

- To Rotate about an arbitrary point (xR, yR) through the angle θ:

1. Translate the object to the origin by (-xR, -yR)
2. Rotate about the origin through the angle θ
3. Inverse the Translation by (xR, yR)

1 2 3

$$T \bullet R \bullet T^{-1}$$



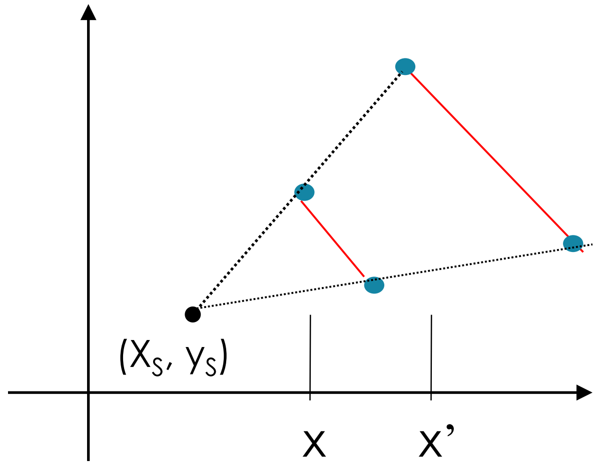
Fixed Point Scaling

- Scaling when the fixed point is not the origin (xS, yS):

1. Translate to the origin by (-xS, -yS)
2. Scaling using (Sx, Sy)
3. Inverse the Translation by (xS, yS)

1 2 3

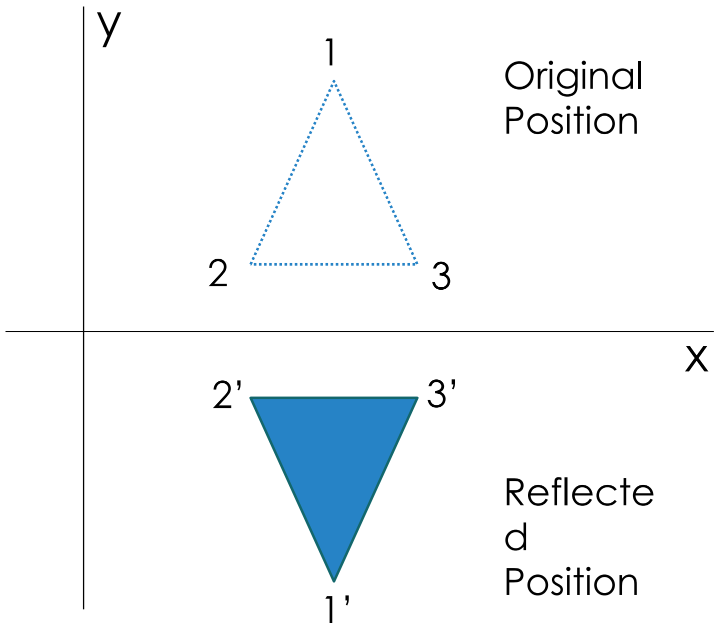
$T \bullet S \bullet T^{-1}$



Reflection

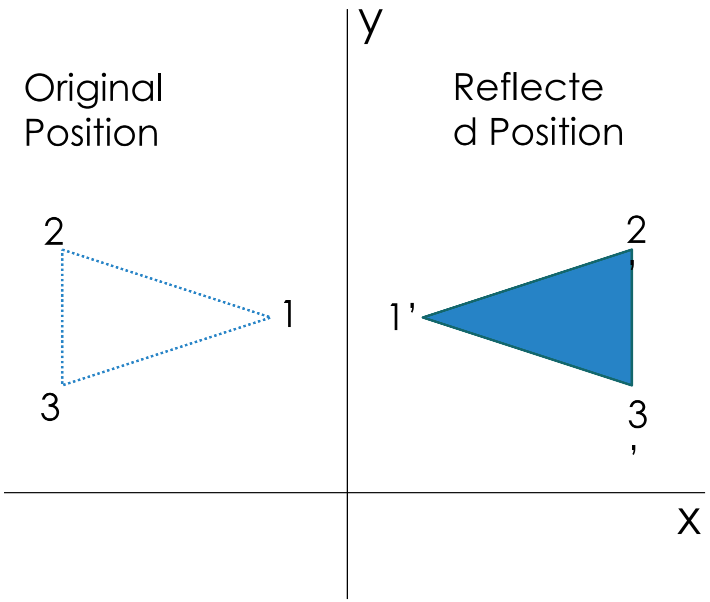
- Reflection about X-axis

$$\mathbf{Re}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Reflection about Y-axis

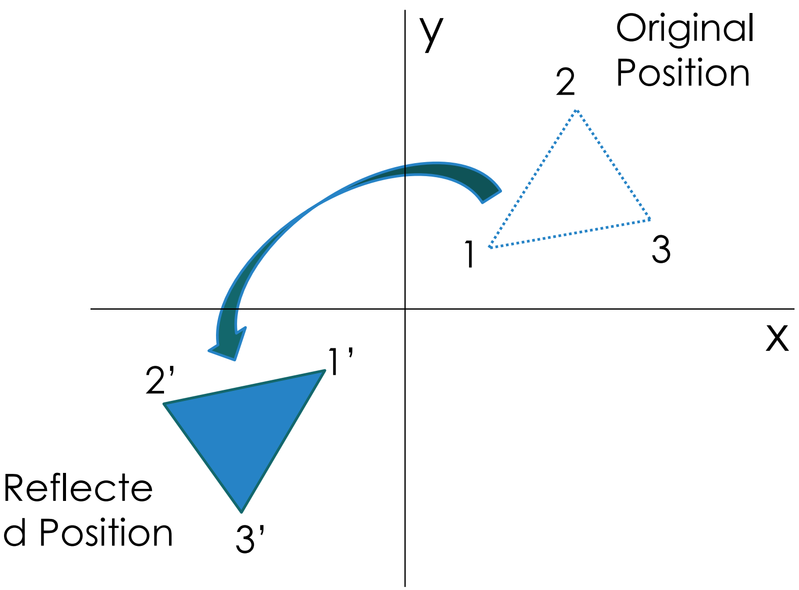
$$\mathbf{Re}_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Reflection (cont.)

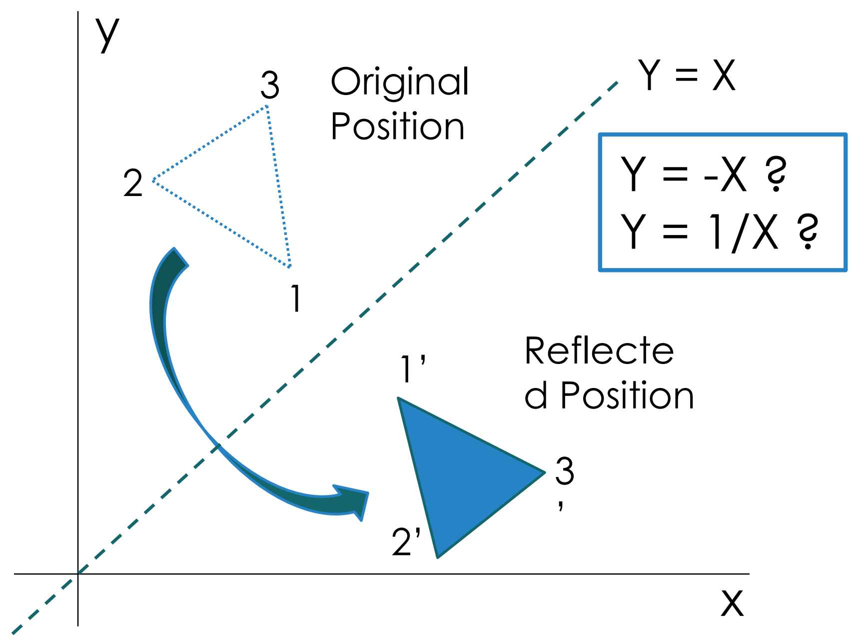
- Reflection about the Origin

$$Re = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Reflection about the line Y = X

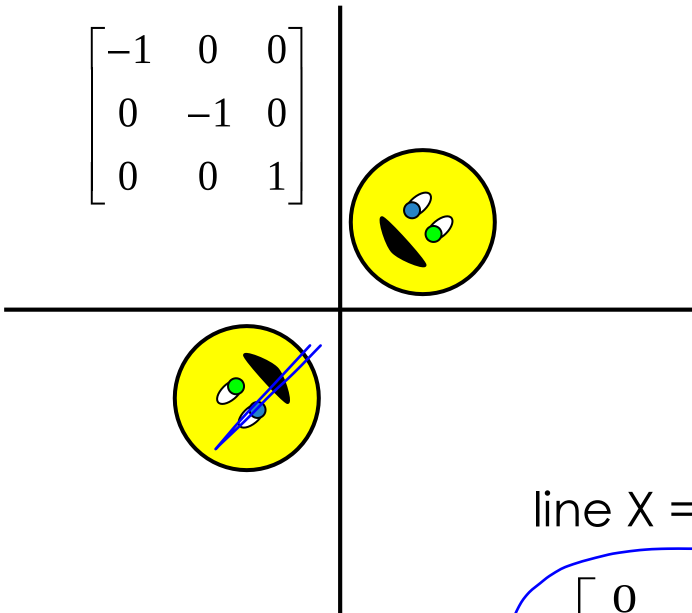
$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Reflection: Example

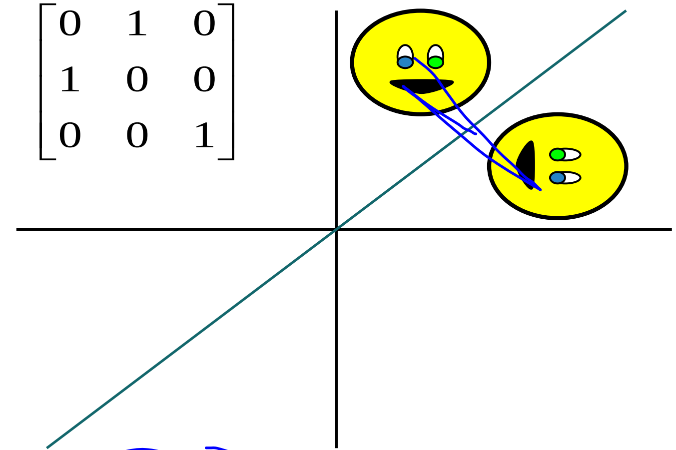
Origin

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



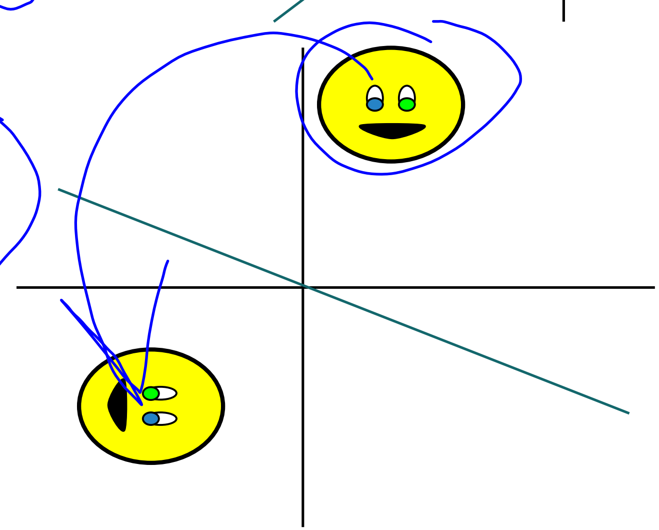
line X = Y

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



line X = -Y

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Shearing

- Distorts the shape of an object such that the transformed shape appears as if the object were composed of internal layers that had been caused to slide over each other

X-Shear

$X' = X + sh_x \cdot Y$

$Y' = Y$

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Y-Shear

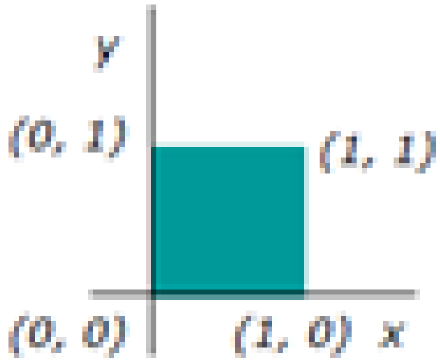
$X' = X$

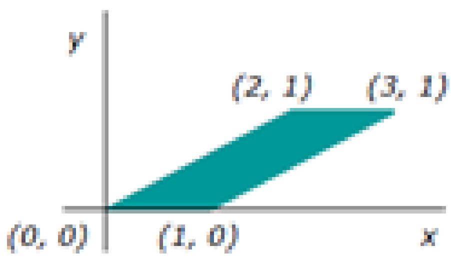
$Y' = Y + sh_y \cdot X$

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

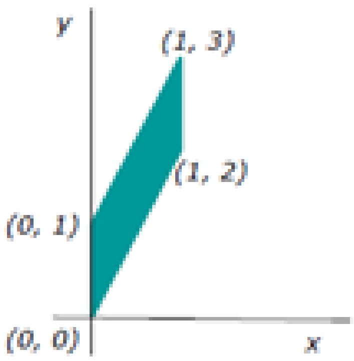
Shearing (cont.)

Without Shearing





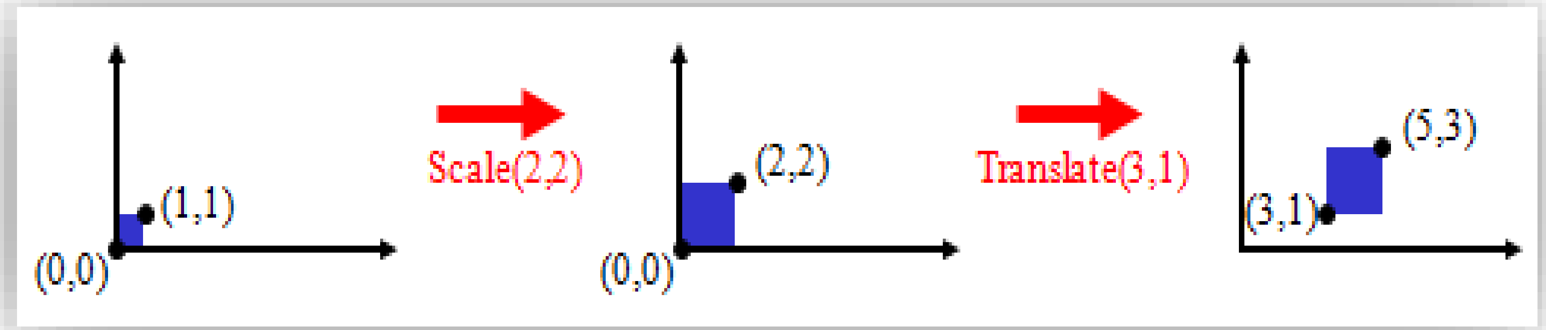
X-Shear



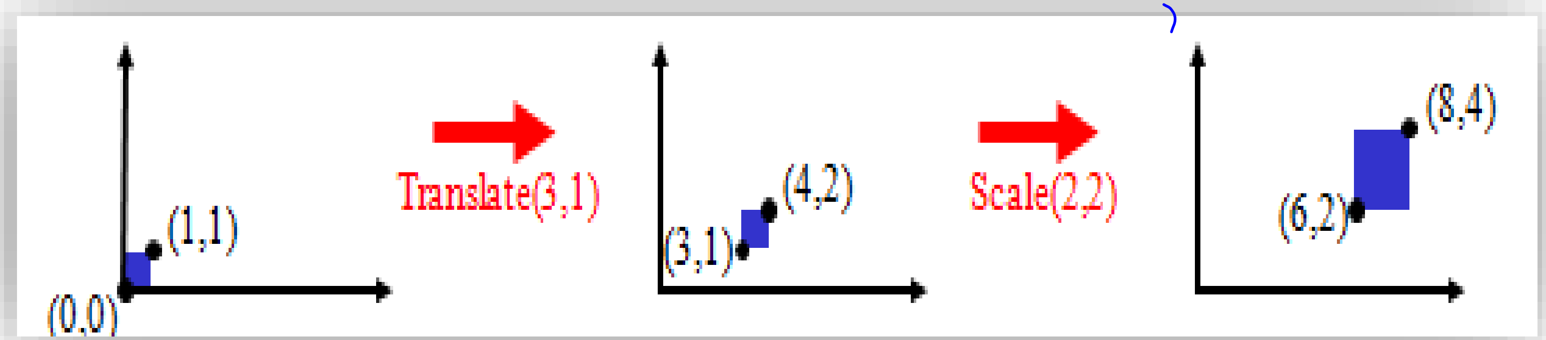
Y-Shear

Transformation: Example

Scale then **Translate**: p' = T (Sp) = T Sp



Translate then **Scale**: p' = S(Tp) = S Tp



Transformation: Example (cont.)

Scale then **Translate**: p' = T (Sp) = T Sp

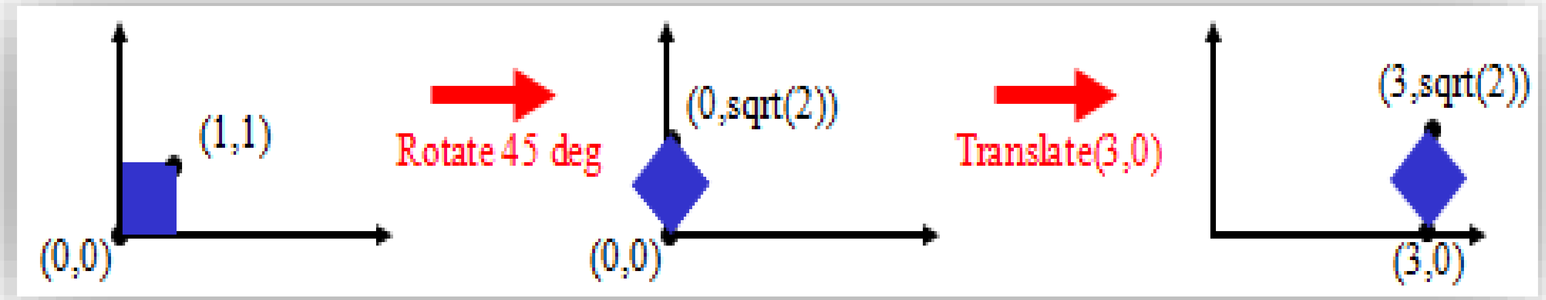
$$TS = \left[\boxed{\begin{matrix}1 & 0 & 3\end{matrix}} \atop \begin{matrix}0 & 1 & 1\\ 0 & 0 & 1\end{matrix}\right] \left[\begin{matrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{matrix}\right] \begin{bmatrix}X\\ Y\\ 1\end{bmatrix} = \begin{bmatrix}2 & 0 & 3\\ 0 & 2 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}X\\ Y\\ 1\end{bmatrix}$$

Translate then **Scale**: p' = S(Tp) = S Tp

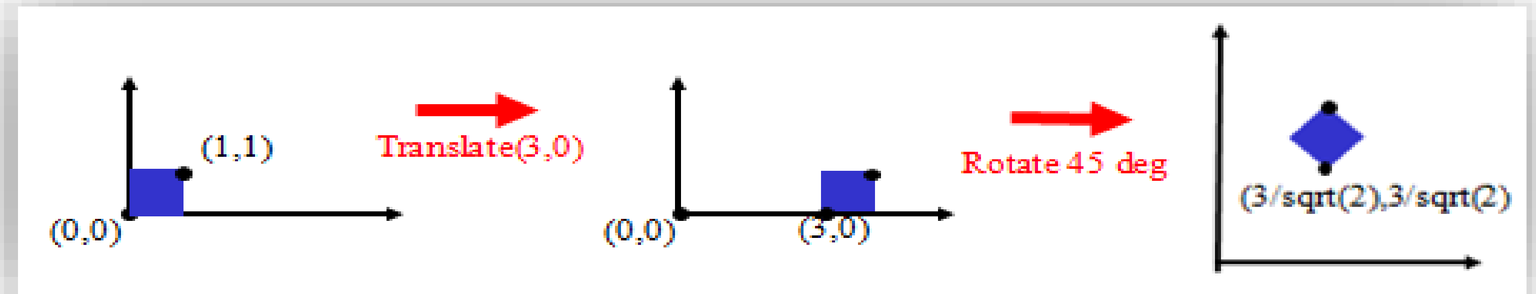
$$ST = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}X\\ Y\\ 1\end{bmatrix} = \begin{bmatrix}2 & 0 & 6\\ 0 & 2 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}X\\ Y\\ 1\end{bmatrix}$$

Transformation: Example (cont.)

Rotate then **Translate**



Translate then **Rotate**



Caution: matrix multiplication is NOT commutative!

Transformation: Example (cont.)

Rotate then **Translate**: p' = T(Rp) = T Rp

$$TR = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Translate then **Rotate**: p' = R(Tp) = R Tp

$$RT = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$