Geometry:
2-D Transformations

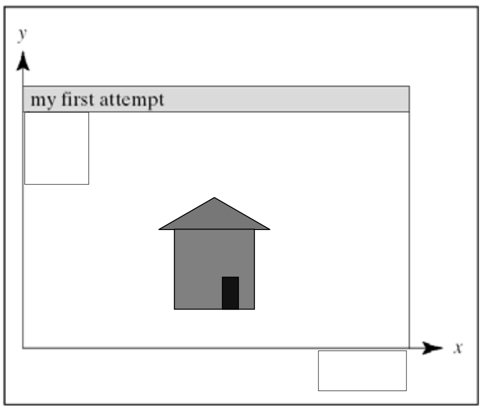
Outline

- Effects
- Mathematical representation
- OpenGL functions for applying

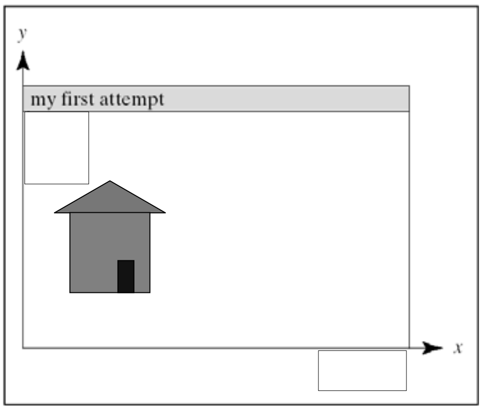
Why We Need Transformations

- Objects may have different locations, scales, and orientations
- Complex objects may be constructed by the transformation of simple objects
- Camera may have different locations and orientations

Example: Shape vs. Viewing Issues



Example: Shape vs. Viewing Issues



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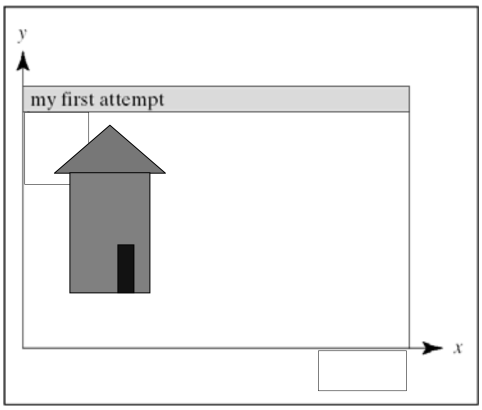
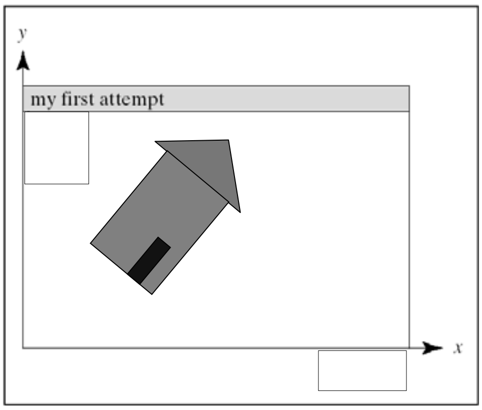
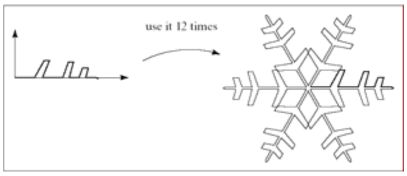


Example: Shape vs. Viewing Issues



Transformations for modeling

1. Building complex objects from simpler parts



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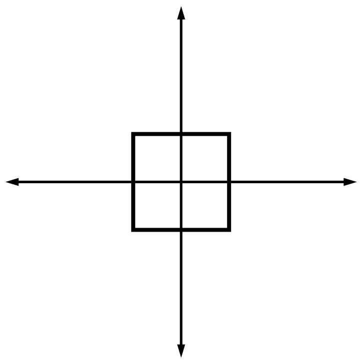
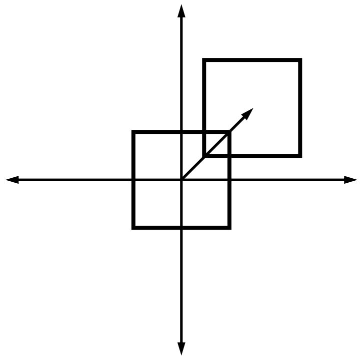
2-D Transformations

- Types
  - Translation
  - Scaling
  - Rotation
  - Shear, reflection

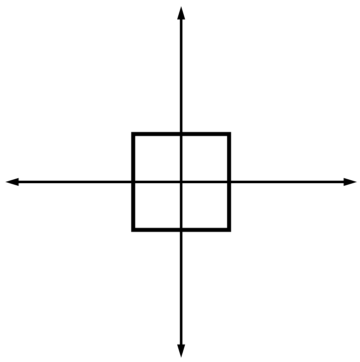
2-D Translation



2-D Translation

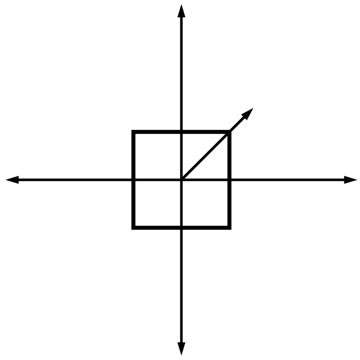


2-D Translation



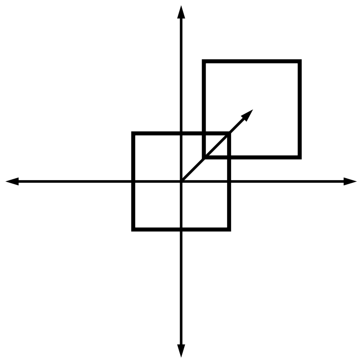
$$\begin{pmatrix} x \\ y \end{pmatrix}$$

2-D Translation



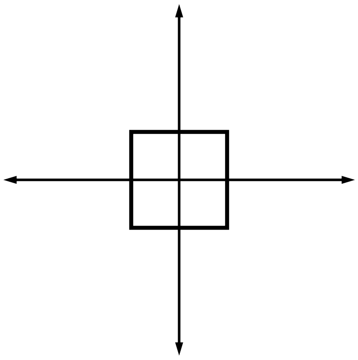
$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

2-D Translation

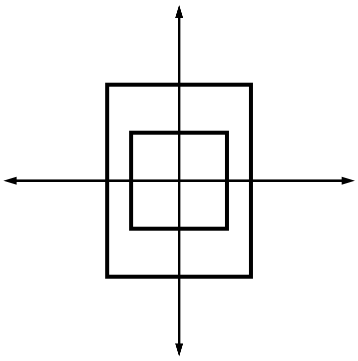


$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

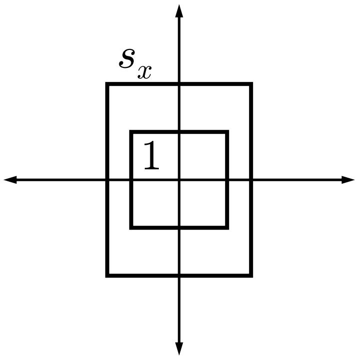
2-D Scaling



2-D Scaling

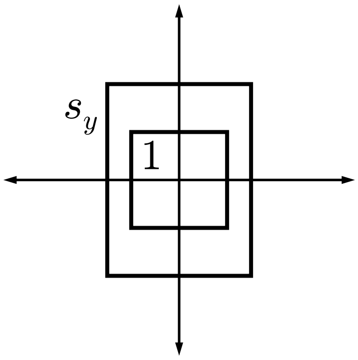


2-D Scaling



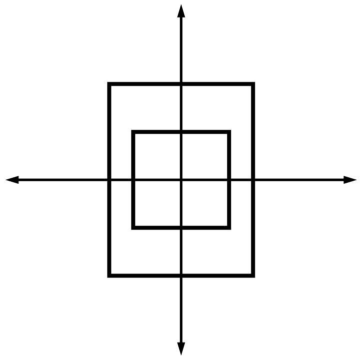
Horizontal shift proportional to horizontal position

2-D Scaling



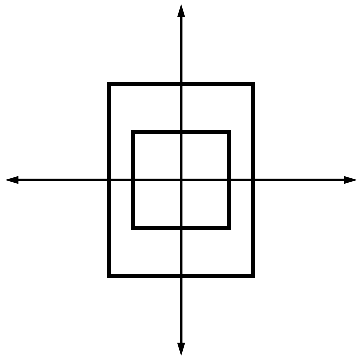
Vertical shift proportional to vertical position

2-D Scaling



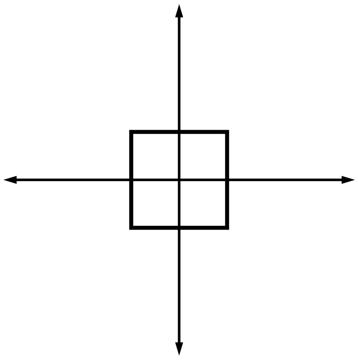
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \end{pmatrix}$$

2-D Scaling

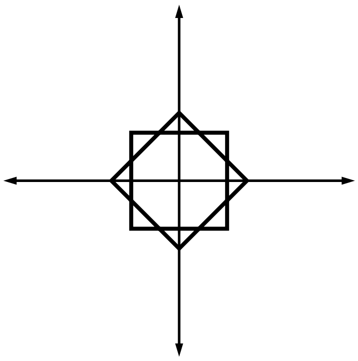


$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2-D Rotation

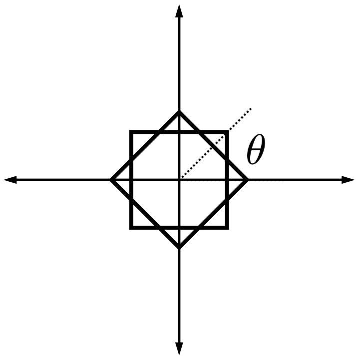


2-D Rotation



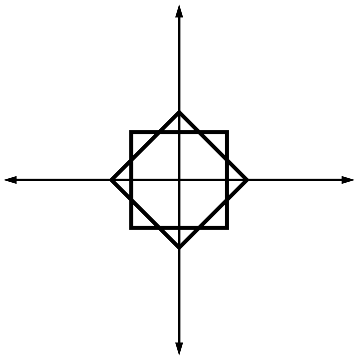
This is a counterclockwise rotation

2-D Rotation



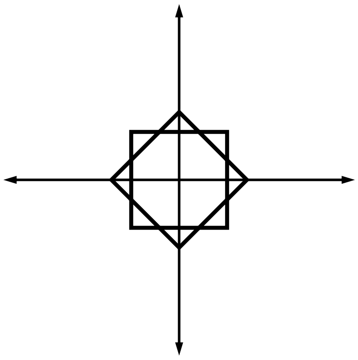
This is a counterclockwise rotation

2-D Rotation



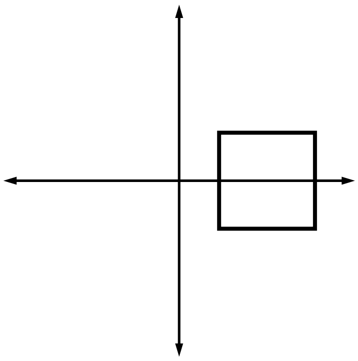
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

2-D Rotation



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2-D Rotation (uncentered)



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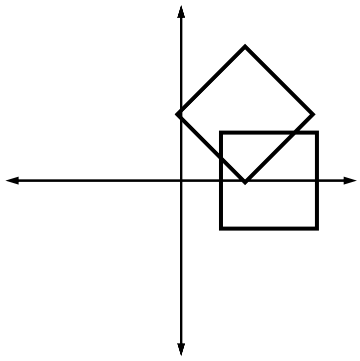
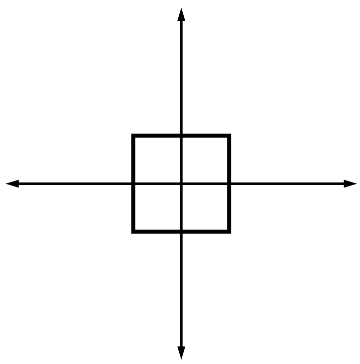
2-D Rotation (uncentered)

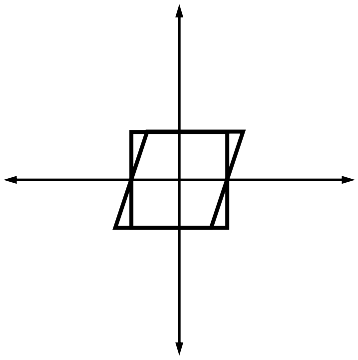


$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2-D Shear (horizontal)

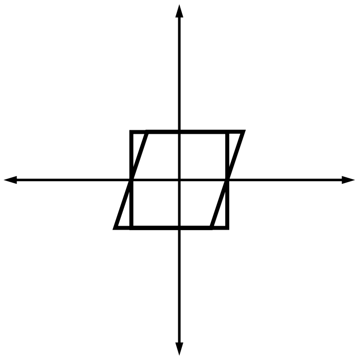


2-D Shear (horizontal)



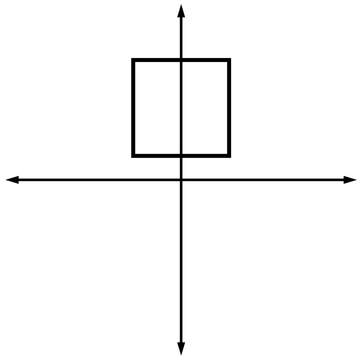
Horizontal displacement proportional to vertical position

2-D Shear (horizontal)

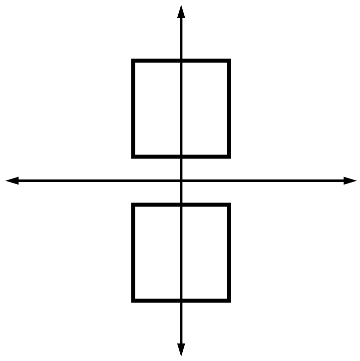


$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2-D Reflection (vertical)

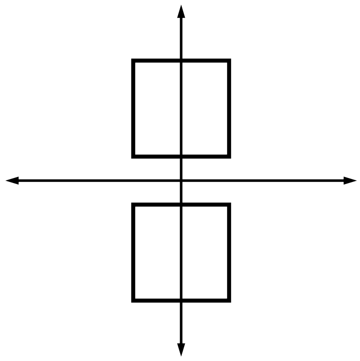


2-D Reflection (vertical)



Just a special case of scaling—"negative" scaling

2-D Reflection (vertical)



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Representing Transformations

- Note that we've defined translation as a vector addition but rotation, scaling, etc. as matrix multiplications
- It's inconvenient to have two different operations (addition and multiplication) for different forms of transformation
- It would be desirable for all transformations to be expressed in a common form

Representing Transformations

- Note that we've defined translation as a vector addition but rotation, scaling, etc. as matrix multiplications
- It's inconvenient to have two different operations (addition and multiplication) for different forms of transformation
- It would be desirable for all transformations to be expressed in a common form
  - **Solution: Homogeneous coordinates**

Homogeneous Coordinates

- Let $\mathbf{x} = (x_1, \dots, x_n)^T$ be a point in Euclidean space
- Change to *homogeneous* coordinates:

$$\mathbf{x} \Rightarrow (\mathbf{x}^T, 1)^T$$

- Defined up to scale

$$(\mathbf{x}^T, 1)^T \Rightarrow (w\mathbf{x}^T, w)^T$$

- Can go back to non-homogeneous representation as follows:

$$(\mathbf{x}^T, w)^T \Rightarrow \mathbf{x}/w$$

Homogeneous Coordinates: Translations

- 2-D translation of a point was expressed as a vector addition $\mathbf{x}' = \mathbf{x} + \mathbf{t}$
- Homogeneous coordinates allow it to be written as a multiplication by a 3 x 3 matrix:

$$\mathbf{x}' = \begin{pmatrix} \mathbf{Id} & \mathbf{t} \\ \mathbf{0}^T & \mathbf{1} \end{pmatrix} \mathbf{x}$$

Example: Translation with homogeneous coordinates

- Old way: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$
- New way:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Homogeneous Coordinates: Rotations, etc.

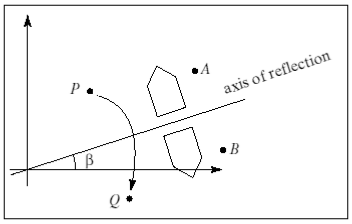
- A 2-D rotation, scaling, shear or other transformation normally expressed by a 2 x 2 matrix $\mathbf{R}$ is written in homogeneous coordinates with the following 3 x 3 matrix:

$$\mathbf{x}' = \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} \mathbf{x}$$

- The non-commutativity of matrix multiplication explains why different transformation orders give different results—i.e., $\mathbf{RT} \neq \mathbf{TR}$

2-D Transformations: Tilted Axes

- All of the scalings, reflections, etc. described so far are relative to the coordinate axes
- How can we perform a transformation relative to some **tilted** axis?
- Basic idea:
  1. Apply rotation so that tilted axis is aligned with a coordinate axis
  2. Apply desired transformation (reflection, shear, etc.) for that coordinate axis
  3. Apply inverse rotation so that tilted axis is “restored”



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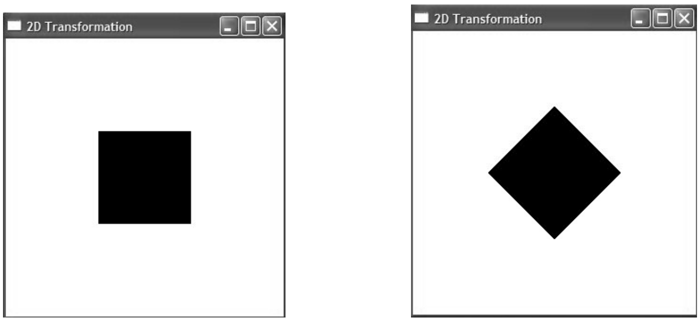
OpenGL's coordinates

- The underlying form of all points/vertices is a 4-D vector (x, y, z, w)
- If you do something in 2-D, OpenGL simply sets $z = 0$ for you
- If the scale coordinate w is not set explicitly, OpenGL sets $z = 1$ for you

2-D Transformations: OpenGL

- 2-D transformation functions
  - **`glTranslate(x, y, 0)`**
  - **`glScale(sx, sy, 0)`**
  - **`glRotate(theta, 0, 0, 1)`** (angle in degrees; direction is counterclockwise)
- Notes
  - Set **`glMatrixMode(GL_MODELVIEW)`** first
  - Transformations should be specified **before** drawing commands to be affected
  - Multiple transformations are applied in **reverse** order

Example: 2-D Translation in OpenGL



Limiting “Scope” of Transformations

- Transformations are ordinarily applied to **all** subsequent draw commands
- To limit effects, use push/pop functions:

```
glPushMatrix();
// transform
// draw affected by transform
glPopMatrix();
// draw unaffected by transform
```