13 Limits and the Foundations of Calculus

We have developed some of the basic theorems in calculus without reference to limits. However limits are very important in mathematics and cannot be ignored. They are crucial for topics such as infinite series, improper integrals, and multivariable calculus. In this last section we shall prove that our approach to calculus is equivalent to the usual approach via limits. (The going will be easier if you review the basic properties of limits from your standard calculus text, but we shall neither prove nor use the limit theorems.)

Limits and Continuity

Let f be a function defined on some open interval containing x_0, except possibly at x_0 itself, and let l be a real number. There are two definitions of the statement

$$\lim_{x \to x_0} f(x) = l$$

Condition 1

1. Given any number $c_1 < l$, there is an interval (a_1, b_1) containing x_0 such that $c_1 < f(x)$ if $a_1 < x < b_1$ and $x \neq x_0$.
2. Given any number $c_2 > l$, there is an interval (a_2, b_2) containing x_0 such that $c_2 > f(x)$ if $a_2 < x < b_2$ and $x \neq x_0$.

Condition 2 Given any positive number ϵ, there is a positive number δ such that $|f(x) - l| < \epsilon$ whenever $|x - x_0| < \delta$ and $x \neq x_0$.

Depending upon circumstances, one or the other of these conditions may be easier to use. The following theorem shows that they are interchangeable, so either one can be used as the definition of $\lim_{x \to x_0} f(x) = l$.

Theorem 1 *For any given f, x_0, and l, condition 1 holds if and only if condition 2 does.*

Proof (a) *Condition 1 implies condition 2.* Suppose that condition 1 holds, and let $\epsilon > 0$ be given. To find an appropriate δ, we apply condition 1, with $c_1 = l - \epsilon$ and $c_2 = l + \epsilon$. By condition 1, there are intervals (a_1, b_1) and (a_2, b_2) containing x_0 such that $l - \epsilon < f(x)$ whenever $a_1 < x < b_1$ and $x \neq x_0$, and $l + \epsilon > f(x)$ whenever $a_2 < x < b_2$ and $x \neq x_0$. Now let δ be the smallest of the positive numbers $b_1 - x_0, x_0 - a_1, b_2 - x_0$, and $x_0 - a_2$. (See Fig. 13-1.) Whenever $|x - x_0| < \delta$ and $x \neq x_0$, we have

$$a_1 < x < b_1 \quad \text{and} \quad x \neq x_0 \quad \text{so} \quad l - \epsilon < f(x) \tag{1}$$

and

$$a_2 < x < b_2 \quad \text{and} \quad x \neq x_0 \quad \text{so} \quad l + \epsilon > f(x) \tag{2}$$

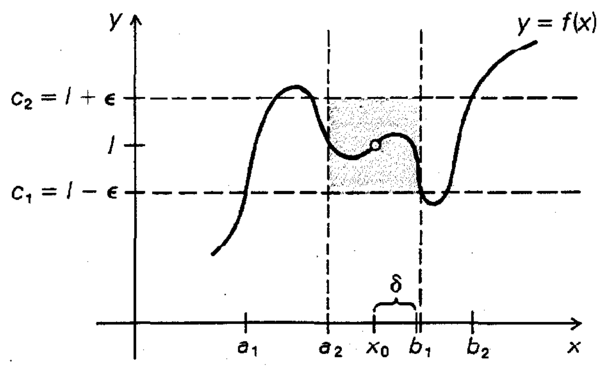

Fig. 13-1 When $|x - x_0| < \delta$ and $x \neq x_0$, $|f(x) - l| - \epsilon$.

Statements **(1)** and **(2)** together say that $l - \epsilon < f(x) < l + \epsilon$, or $|f(x) - l| < \epsilon$, which is what was required.

(b) *Condition 2 implies condition 1.* Suppose that condition 2 holds, and let $c_1 < l$ and $c_2 > l$ be given. Let ϵ be the smaller of the two positive numbers $l - c_1$ and $c_2 - l$. By condition 2, there is a positive number δ such that $|f(x) - l| < \epsilon$ whenever $|x - x_0| < \delta$ and $x \neq x_0$. Now we can verify parts 1 and 2 of condition 1, with $a_1 = b_1 = x_0 - \delta$ and $a_2 = b_2 = x_0 + \delta$. If $x_0 - \delta < x < x_0 + \delta$ and $x \neq x_0$, then $|x - x_0| < \delta$ and $x \neq x_0$, so we have $|f(x) - l| < \epsilon$; that is, $l - \epsilon < f(x) < l + \epsilon$. But this implies that $c_1 < f(x)$ and $f(x) < c_2$ (see Fig. 13-2).

The following theorem shows that our definition of continuity can be phrased in terms of limits.

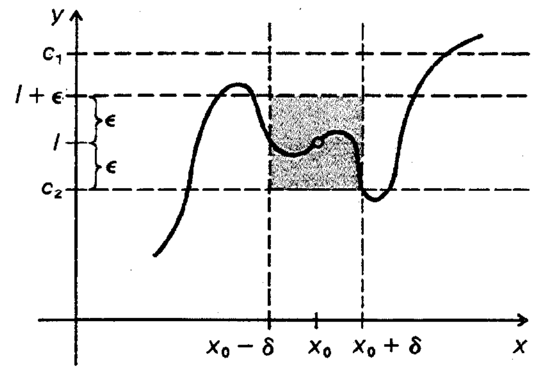

Fig. 13-2 When $x \in (x_0 - \delta, x_0 + \delta)$ and $x \neq x_0, c_1 < f(x) < c_2$.

Theorem 2 *Let f be defined on an open interval containing x_0. Then f is continuous at x_0 if and only if*

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Proof The definition of continuity given in Chapter 3 is exactly condition 1 for the statement $\lim_{x\to x_0} f(x) = f(x_0)$

Corollary *The function f is continuous at x_0 if and only if, for every positive number ϵ, there is a positive number δ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.*

Proof We have simply replaced the statement $\lim_{x\to x_0} f(x) = f(x_0)$ by its condition 2 definition. (We do not need to require that $x \neq x_0$; if $x = x_0$, $|f(x) - f(x_0)| = 0$, which is certainly less than ϵ.)

Solved Exercises

1. Using the fact that the function $f(x) = \tan x$ is continuous, show that there is a number $\delta > 0$ such that $|\tan x - 1| < 0.001$ whenever $|x - \pi/4| < \delta$.
2. Let f be defined on an open interval containing x_0, except perhaps at x_0 itself. Let

$A = \{c \mid$ there is an open interval I about x_0 such that $x \in I, x \neq x_0$ implies $f(x) > c\}$

$B = \{d \mid$ there is an open interval J about x_0 such that $x \in J, x \neq x_0$ implies $f(x) < d\}$.

Prove that $\lim_{x \to x_0} f(x) = l$ if and only if l is a transition point from A to B.

Exercises

1. Show that there is a positive number δ such that

$$\left| \frac{x-4}{x+4} - \frac{1}{3} \right| < 10^{-6}$$

whenever $|x - 8| < \delta$.

2. Is Theorem 2 valid for f with a domain which does not contain an interval about x_0? What is the definition of limit in this case?

3. Prove that limits are unique by using the definition, Solved Exercise 2, and a theorem about transitions.

4. Which of the following functions are continuous at 0?

(a) $f(x) = x \sin \frac{1}{x}, \quad x \neq 0, \quad f(0) = 0$

(b) $f(x) = \frac{1}{x} \sin x, \quad x \neq 0, \quad f(0) = 0$

(c) $f(x) = \frac{x^2}{\sin x}, \quad x \neq 0, \quad f(0) = 0$

The Derivative as a Limit of Difference Quotients

We recall the definition of the derivative given in Chapter 1.

Definition Let f be a function whose domain contains an open interval about x_0. We say that the number m_0 is the *derivative of f at x_0* provided that:

1. For every $m < m_0$, the function

$$f(x) - [f(x_0) + m(x - x_0)]$$

changes sign from negative to positive at x_0.

2. For every $m > m_0$, the function

$$f(x) - [f(x_0) + m(x - x_0)]$$

changes sign from positive to negative at x_0.

If such a number m_0 exists, we say that f is *differentiable* at x_0 and we write $m_0 = f'(x_0)$.

We will now prove that our definition of the derivative coincides with the definition found in most calculus books.

Theorem 3 Let f be a function whose domain contains an open interval about x_0. Then f is differentiable at x_0 with derivative m_0 if and only if

$$\lim_{x \to x_0} [f(x_0 + \Delta x) - f(x_0)] / \Delta x$$

exists and equals m_0.

Proof We will use the condition 1 form of the definition of limit. Suppose that $\lim_{x \to x_0} [f(x_0 + \Delta x) - f(x_0)] / \Delta x = m_0$. To verify that $f'(x_0) = m_0$, we must study the sign change at x_0 of $r(x) = f(x) - [f(x_0) + m(x - x_0)]$ and see how it depends on m.

First assume that $m < m_0$. Since the limit of difference quotients is m_0, there is an interval (a, b) containing zero such that

$$m < \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

whenever $a < \Delta x < b$, $\Delta x \neq 0$. Writing x for $x_0 + \Delta x$, we have

$$m < \frac{f(x) - f(x_0)}{x - x_0} \tag{3}$$

whenever $x_0 + a < x < x_0 + b, x \neq x_0$—that is, whenever $x_0 + a < x < x_0$ or $x_0 < x < x_0 + b$.

In case $x_0 + a < x < x_0$, we have $x - x_0 < 0$, and so equation (3) can be transformed to

$$\begin{aligned} m(x - x_0) &> f(x) - f(x_0) \\ 0 &> f(x) - [f(x_0) + m(x - x_0)] \end{aligned}$$

When $x_0 < x < x + b$, we have $x - x_0 > 0$, so equation (3) becomes

$$\begin{aligned} m(x - x_0) &< f(x) - f(x_0) \\ 0 &< f(x) - [f(x_0) + m(x - x_0)] \end{aligned}$$

In other words, $f(x) - [f(x_0) + m(x - x_0)]$ changes sign from negative to positive at x_0.

Similarly, if $m > m_0$, we can use part 2 of the condition 1 definition of limit to show that $f(x) - [f(x_0) + m(x - x_0)]$ changes sign from positive to negative at x_0. This completes the proof that $f'(x_0) = m_0$.

Next we show that if $f'(x_0) = m_0$, then $\lim_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)]/\Delta x = m_0$. This is mostly a matter of reversing the steps in the first half of the proof, with slightly different notation. Let $c_1 < m_0$. To find an interval (a, b) containing zero such that

$$c_1 < \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{4}$$

whenever $a < \Delta x < b$, $\Delta x \neq 0$, we use the fact that $f(x) - [f(x_0) + c_1(x - x_0)]$ changes sign from negative to positive at x_0. There is an interval (a_1, b_1) containing x_0 such that $f(x) - [f(x_0) + c_1(x - x_0)]$ is negative when $a_1 < x < x_0$ and positive when $x_0 < x < b_1$. Let $a = a_1 - x_0 < 0$ and $b = b_1 - x_0 > 0$. If $a < \Delta x < 0$, we have $a_1 < x_0 + \Delta x < x_0$, and so

$$\begin{aligned} 0 &> f(x_0 + \Delta x) - [f(x_0) + c_1 \Delta x] \\ c_1 \Delta x &> f(x_0 + \Delta x) - f(x_0) \\ c_1 &< \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad (\text{since } \Delta x < 0) \end{aligned}$$

which is just equation (4). If $0 < \Delta x < b$, we have $x_0 < x_0 + \Delta x < b_1$, and so

$$\begin{aligned} 0 &< f(x_0 + \Delta x) - [f(x_0) + c_1 \Delta x] \\ c_1 \Delta x &< f(x_0 + \Delta x) - f(x_0) \\ c_1 &< \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \end{aligned}$$

which is equation (4) again.

Similarly, if $c_2 > m_0$, there is an interval (a, b) containing zero such that $c_2 > [f(x_0 + \Delta x) - f(x_0)]/\Delta x$ whenever $a < \Delta x < b$, $\Delta x \neq 0$. Thus we have shown that $\lim_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)]/\Delta x = m_0$.

Combining Theorems 1 and 3, we can now give an ϵ-δ characterization of the derivative.

Corollary *Let f be defined on an open interval containing x_0. Then f is differentiable at x_0 with derivative $f'(x_0)$ if and only if, for every positive number ϵ, there is a positive number δ such that*

$$\left|\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}-f'(x_0)\right|<\epsilon$$

whenever $|\Delta x|<\delta$ and $\Delta x \neq 0$.

Proof We have just rephrased the statement

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0)$$

using the ϵ-δ definition of limit.

Solved Exercises

3. If f is differentiable at x_0, what is $\lim_{x\to x_0}[f(x)-f(x_0)]/(x-x_0)$?

4. Let f be defined near x_0, and define the function $g(\Delta x)$ by

$$g(\Delta x)=\begin{cases}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x} & \Delta x\neq 0\\ m_0 & \Delta x=0\end{cases}$$

where m_0 is some number.
Show that $f'(x_0)=m_0$ if and only if g is continuous at 0.

Exercises

5. Find $\lim_{x\to 2}(x^2+4x+3-15)/(x-2)$.

6. Prove Theorem 3 using the ϵ-δ definition of the derivative, and draw pictures to illustrate your constructions.

7. (a) Suppose that $f'(x_0) = g'(x_0) \neq 0$. Find $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{g(x) - g(x_0)}$.

   (b) Find $\lim\limits_{x \to 1} \dfrac{2x^3 - 2}{3x^2 - 3}$. (c) Find $\lim\limits_{x \to 1} \dfrac{x^n - 1}{x^m - 1}$.

8. Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{1 - x^2} - 1}{x}$:

   (a) By recognizing the limit to be a derivative.

   (b) By rationalizing.

9. Evaluate the following limit by recognizing the limit to be a derivative:

$$\lim_{x \to \pi/4} \frac{\sin x - (\sqrt{2}/2)}{x - (\pi/4)}$$

The Integral as a Limit of Sums

In this section, we shall need the notion of a limit of a sequence. (See your calculus text for examples and discussion.)

Definition Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers and let l be a real number. We say that l is the *limit* of the sequence and we write $\lim\limits_{n \to \infty} a_n = l$ if, for every $\epsilon > 0$, there is a number N such that $|a_n - l| < \epsilon$ for all $n \geqslant N$.

Now let f be defined on an interval $[a, b]$. In our definition of the integral $\int_a^b f(t)\,dt$ in Chapter 12, we considered partitions $(t_0, t_1, \ldots, t_n)$ of $[a, b]$ and lower sums:

$$\sum_{i=1}^{n} k_i(t_i - t_{i-1}) \quad \text{where } k_i < f(t) \text{ for all } t \text{ in } (t_{i-1}, t_i)$$

and upper sums:

$$\sum_{i=1}^{n} l_i(t_i - t_{i-1}) \quad \text{where } f(t) < l_i \text{ for all } t \text{ in } (t_{i-1}, t_i)$$

The integral was then defined to be the transition point between upper and lower sums, i.e., that number S, if it exists, for which every $s < S$ is a lower sum and every $s > S$ is an upper sum.

We also may consider sums of the form

$$S_n = \sum_{i=1}^{n} f(c_i)(t_i - t_{i-1}) \quad \text{where } c_i \text{ in } [t_{i-1}, t_i]$$

called *Riemann sums*. The integral can be defined as a limit of Riemann sums. This is reasonable since any Riemann sum associated to a given partition lies between any upper and lower sums for that partition. (See Fig. 13-3.) The following is a precise statement, showing that the limit approach coincides with the method of exhaustion.

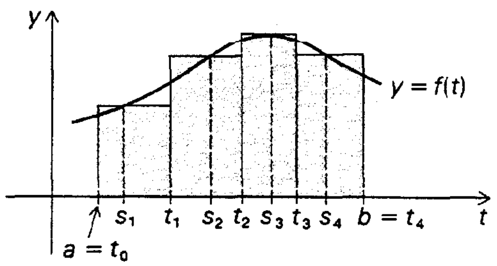

Fig. 13-3 Illustrating a Riemann sum.

Theorem 4 Let f be a bounded function on $[a, b]$.

1. *Assume that f is integrable and that the maximum of the numbers $\Delta t_i = t_i - t_{i-1}$ goes to zero as $n \to \infty$. Then for any choice of c_i,*

$$\lim_{n\to\infty} S_n = \int_a^b f(t)\,dt$$

2. *Suppose that for every choice of c_i and t_i with the maximum of Δt_i tending to zero as $n \to \infty$, the limit $\lim_{n\to\infty} S_n = S$. Then f is integrable with integral S.*

Let U and L denote the set of upper and lower sums for f, respectively. We have shown in Chapter 12 that if l is in L and u is in U, then $l \leqslant u$ and that $L = (-\infty, S_1)$ or $(-\infty, S_1]$ and $U = (S_2, \infty)$ or $[S_2, \infty)$. Integrability amounts to the requirement that $S_1 = S_2$. We shall need the following lemma before we proceed to the proof of Theorem 4.

Lemma *Let $a = s_0 < s_1 < \cdots < s_N = b$ be a partition of $[a,b]$ and let $\epsilon > 0$ be given. If $a = t_0 < t_1 < \cdots < t_n = b$ is any partition with $\Delta t_i < \epsilon/N$ for every i, then the total length of the intervals $[t_{i-1}, t_i]$ which are not contained entirely within some (s_{p-1}, s_p) is less than ϵ.*

Proof Since there are just N points in the s-partition, there are no more than $2N$ intervals $[t_{i-1}, t_i]$ which contain points of the s-partition (it is $2N$ because s_t could be in two such intervals by being a common end point). Since each such interval has length strictly less than $\epsilon/2N$, the total length is $(2N)(\epsilon/2N) = \epsilon$.

Proof of Theorem 4 We give the proof of part 1. Part 2 is left for the reader (see Problem 13). Let ϵ be given and let $|f(x)| \leqslant M$ for all t in $[a, b]$. Choose piecewise constant functions $g(t) \leqslant f(t)$ and $h(t) \geqslant f(t)$ such that

$$\int_a^b h(t)\,dt - \int_a^b g(t)\,dt < \frac{\epsilon}{2}$$

(so both integrals are within $\epsilon/2$ of $\int_a^b f(t)\,dt$) which is possible since f is integrable.

Let $u_0 < u_1 < \cdots < u_r$ and $v_0 < v_1 < \cdots < v_s$ be adapted partitions for g and h. Let $s_0 < s_1 < \cdots < s_N$ be a partition adapted for both g and h obtained by taking all the u's and v's together. Let $g(x) = k_p$ on (s_{p-1}, s_p) and $h(x) = l_p$ on (s_{p-1}, s_p).

Choose N_1 so that $\Delta t_i < \dfrac{(\epsilon/2M)}{2N}$, if $n > N_1$; this is possible since the maximum of the Δt_i goes to zero as $n \to \infty$.

By the lemma, the total length of the intervals $[t_{i-1}, t_i]$ not contained in some (s_{p-1}, s_p) is less than $\epsilon/2M$.

Thus

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

$$= \sum_{\substack{i \text{ such that} \\ [t_{i-1}, t_i] \\ \text{lies in some} \\ (s_{p-1}, s_p)}} f(c_i)\,\Delta t_i + \sum_{\substack{\text{rest of} \\ \text{the } i\text{'s}}} f(c_i)\,\Delta t_i$$

$$\leqslant \sum_{p=1}^{N} l_p\,\Delta s_p + M \cdot \frac{\epsilon}{2M}$$

$$= \int_a^b h(t)\,dt + \frac{\epsilon}{2}$$

$$\leqslant \int_a^b f(t)\,dt + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \int_a^b f(t)\,dt + \epsilon$$

In a similar way we show that $S_n \geqslant \int_a^b f(t)\,dt - \epsilon$. Thus, if $n \geqslant N_1$, then $|S_n - \int_a^b f(t)\,dt| < \epsilon$, so our result is proved.

Solved Exercises

5. Let $S_n = \sum_{i=1}^{n} (1 + i/n)$. Prove that $S_n \to \frac{3}{2}$ as $n \to \infty$ (a) directly and (b) using Riemann sums.

6. Use Theorem 4 to demonstrate that $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. (You may assume that the limit of a sum is the sum of the limits.)

Exercises

10. (a) Prove that

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right) = \int_0^1 \frac{dx}{1+x} = \ln 2$$

(b) Evaluate the sum for $n = 10$ and compare with $\ln 2$.

11. Use Theorem 4 to demonstrate the following:

(a) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(b) If $f(x) \leqslant g(x)$ for all x in $[a, b]$, then $\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$.

12. Let f be a bounded function on $[a, b]$. Assume that for every $\epsilon > 0$ there is a $\delta > 0$ such that, if $a = t_0 < t_1 < \cdots < t_n = b$ is a partition with $\Delta t_i < \delta$, and c_i is any point in $[t_{i-1}, t_i]$, then

$$\left| \sum_{i=1}^{n} f(c_i)\,\Delta t_i - S \right| < \epsilon$$

Prove that f is integrable with integral S.

Problems for Chapter 13

1. Evaluate $\lim_{x \to \pi^2} (\cos\sqrt{x} + 1)/(x - \pi^2)$ by recognizing the limit to be a derivative.

2. Evaluate:

$$\lim_{x \to \pi^2} \left[\frac{\sin\sqrt{x}}{(\sqrt{x} - \pi)(\sqrt{x} + \pi)} + \tan\sqrt{x} \right]$$

3. Using Theorem 2 and the limit laws, prove that if f and g are continuous at x_0, then so are $f + g$, fg, and f/g (if $g(x_0) \neq 0$).

4. Prove the chain rule, $(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$, via limits as follows:

(a) Let $y = g(x)$ and $z = f(y)$, and write

$$\Delta y = g'(x_0)\Delta x + \rho(x)$$

Show that

$$\lim_{\Delta x \to 0} \frac{\rho(x)}{\Delta x} = 0$$

Also write

$$\Delta z = f'(y_0)\Delta y + \sigma(y) \qquad y_0 = g(x_0)$$

and show that

$$\lim_{\Delta y \to 0} \frac{\sigma(y)}{\Delta y} = 0$$

(b) Show that

$$\Delta z = f'(y_0)g'(x_0)\,\Delta x + f'(y_0)\rho(x) + \sigma(g(x))$$

(c) Note that $\sigma(g(x)) = 0$ if $\Delta y = 0$. Thus show that

$$\frac{o(g(x))}{\Delta x} = \begin{cases} \dfrac{o(g(x))}{\Delta y}\dfrac{\Delta y}{\Delta x} & \text{if } \Delta y \neq 0 \\ \\ 0 & \text{if } \Delta y = 0 \end{cases} \to 0$$

as $\Delta x \to 0$.

(d) Use parts (b) and (c) to show that $\lim\limits_{\Delta x \to 0} \Delta z/\Delta x = f'(y_0)g'(x_0)$.

5. Write down Riemann sums for the given functions. Sketch.
   (a) $f(x) = x/(x+1)$ for $1 \leqslant x \leqslant 6$ with $t_0 = 1, t_1 = 2, t_2 = 3, t_3 = 4, t_4 = 5, t_5 = 6; c_i = i - 1$ on $[i-1, i]$.
   (b) $f(x) = x + \sin[(\pi/2)x]$, $0 \leqslant x \leqslant 6$ with $t_i = i, i = 0, 1, 2, 3, 4, 5, 6$. Find Riemann sums S_6 with

   $c_i = i$ on $[i-1, i]$

   and

   $c_i = i - \frac{1}{2}$ on $[i-1, i]$

6. Write each of the following integrals as a limit.
   (a) $\int_1^3 [1/(x^2+1)]\, dx$; partition $[1,3]$ into n equal parts and use a suitable choice of c_i.
   (b) $\int_0^\pi (\cos\frac{1}{2}x + x)\, dx$; partition $[0,\pi]$ into n equal parts and use a suitable choice of c_i.

7. Let

$$S_n = \sum_{i=1}^{n} \left(\frac{i}{n} + \frac{i^2}{n^2}\right) \frac{1}{n}$$

Prove that $S_n \to \frac{5}{6}$ as $n \to \infty$ by using Riemann sums.

8. Expressing the following sums as Riemann sums, show that:

(a) $$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\sqrt{\frac{i}{n}} - \left(\frac{i}{n}\right)^{3/2}\right] \frac{1}{n} = \frac{4}{15}$$

(b) $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3n}{(2n+i)^2} = \frac{1}{2}$$

9. Write down a Riemann sum for $f(x) = x^3 + 2$ on $[-2,3]$ with $t_i = -2 + (i/2); i = 0, 1, 2, \ldots, 10$.

10. Write $\int_{-\pi/4}^{\pi/4} (1 + \tan x)\, dx$ as a limit. (Partition $[-\pi/4, \pi/4]$ into $2n$ equal parts and choose c_i appropriately.)

11. Use Theorem 4 to prove that $\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$.

12. Show that

$$f''(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + 2\,\Delta x) + f(x_0) - 2f(x_0 + \Delta x)}{(\Delta x)^2}$$

if f'' is continuous at x_0.

13. Prove part 2 of Theorem 4 using the following outline (demonstrate each of the statements). Let f be a bounded function on the interval $[p, q]$, and let ϵ be any positive number. Prove that there are real numbers m and M and numbers x_m and x_M in $[p, q]$ such that:

    1. $m \leqslant f(x) \leqslant M$ for all x in $[p, q]$
    2. $f(x_m) < m + \epsilon$ and $f(x_M) > M - \epsilon$

    [*Hint*: Let S be the set of real numbers z such that $f(x) \leqslant z \leqslant f(y)$ for some x and y in $[p, q]$. Prove that S is an interval by using the completeness axiom.]

14. Prove that $e = \lim_{h \to 0} (1 + h)^{1/h}$ using the following outline. Write down the equation $\ln'(1) = 1$ as a limit and substitute into $e = e^1$. Use the continuity of e^x and $e^{\ln y} = y$.

15. (a) A function f defined on a domain D is called *uniformly continuous* if for any $\epsilon > 0$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Show that a continuous function on $[a, b]$ is uniformly continuous. (You may wish to use the proof of Theorem 3, Chapter 11 for inspiration.)

    (b) Use (a) to show that a continuous function on $[a, b]$ is integrable.

16. (Cauchy Sequences.) A sequence $a_1, a_2, a_3, \ldots$ is called a *Cauchy sequence* if for every $\epsilon > 0$ there is a number N such that $|a_n - a_m| < \epsilon$ whenever $n \geqslant N$ and $m \geqslant N$. Prove that every convergent sequence is a Cauchy sequence.

17. Use the following outline to prove that every Cauchy sequence $a_1, a_2, a_3, \ldots$ converges to some real number.

    (a) Using the definition of a Cauchy sequence, with $\epsilon = 1$, prove that the sequence is bounded.

    (b) Let S be the set of real numbers x such that $a_n < x$ for infinitely many n. Prove that S is an interval of the form (l, ∞) or $[l, \infty)$. (Use the completeness axiom.)

    (c) Prove that $\lim_{n \to \infty} a_n = l$.