CURS2. EX.6.

Which is the limit of the error for computing $\sqrt{115}$ using Lagrange interpolation formula for $f(x) = \sqrt{x}$

$x_0 = 100$, $x_1 = 121$, $x_2 = 144$.? Find the aprox. value of $\sqrt{115}$.

$x_0 = 100 \to f(x_0) = \sqrt{100} = 10.$

$x_1 = 121 \to f(x_1) = \sqrt{121} = 11$

$x_2 = 144 \to f(x_2) = \sqrt{144} = 12$

x	100	121	144
$f(x)$	10	11	12

$x = 115.$

$$|R_m f(x)| \le \frac{|u(x)|}{(m+1)!} \underbrace{\overbrace{\|f^{(m+1)}\|_\infty}^{M^3 f}}_{\max\limits_{x \in [100, 144]} |f'''(x)|}$$

$u(x) = (x - x_0)(x - x_1)(x - x_2)$

$u(115) = (115 - 100)(115 - 121)(115 - 144) = 15 \cdot (-6) \cdot (-29) = 2610$

$f(x) = \sqrt{x}.$

$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \cdot x^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{2} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}-1} = -\frac{1}{4} \cdot x^{-\frac{3}{2}}$

$f'''(x) = +\frac{1}{4} \cdot \frac{3}{2} \cdot x^{-\frac{3}{2}-1} = \frac{3}{8} x^{-\frac{5}{2}}$

$M_3 f = \frac{3}{8} \cdot 100^{-\frac{5}{2}} = \frac{3}{8} 10^{2 \cdot (-\frac{5}{2})} = \frac{3}{8} 10^{-5}$

$\Rightarrow |(R_2 f)| \le \frac{|2610|}{6} \cdot \frac{3}{8} 10^{-5}$

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CURS 3 - EX 2.

Find $L_2 f$ for $f(x) = \sin \pi x$ and

$x_0 = 0$.

$x_1 = \frac{1}{6}$ in both forms

$x_2 = \frac{1}{2}$.

x	0	$\frac{1}{6}$	$\frac{1}{2}$
$f(x)$	0	$\frac{1}{2}$	1.

$f(0) = \sin 0 = 0$.

$f(\frac{1}{6}) = \sin \frac{\pi}{6} = \frac{1}{2}$

$f(\frac{1}{2}) = \sin \frac{\pi}{2} = 1$

a)

x	$f(x)$	$\mathcal{D}^1 f$	$\mathcal{D}^2 f$
$x_0 = 0$.	0	3	-9.
$x_1 = \frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{2}$.	
$x_2 = \frac{1}{2}$	1		

$$\mathcal{D} f(x_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$$

$$\mathcal{D}^1 f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\frac{1}{2} - 0}{\frac{1}{6} - 0} = \frac{1}{\cancel{2}_1} \cdot \frac{\cancel{6}^3}{1} = 3$$

$$\mathcal{D}^1 f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{{}^{2)}1 - \frac{1}{2}}{{}^{3)}\frac{1}{2} - \frac{1}{6}} = \frac{\frac{2-1}{2}}{\frac{3-1}{6}} = \frac{\frac{1}{2}}{\frac{\cancel{2}}{\cancel{6}_3}} = \frac{3}{2}$$

$$\mathcal{D}^2 f(x_0) = \frac{\frac{3}{2} - {}^{2)}3}{\frac{1}{6} - 0} = \frac{-\frac{3}{2}}{\frac{1}{6}} = -\frac{3}{\cancel{2}} \cdot \frac{\cancel{6}^3}{1} = -9$$

$$L_2 f(x) = f(x_0) + (x - x_0)\mathcal{D}^1 f(x_0) +$$

$$+ (x - x_0)(x - x_1)\mathcal{D}^2 f(x_0) =$$

$$= 0 + (x - 0) \cdot 3 + (x - 0)\left(x - \frac{1}{6}\right)(-9) =$$

$$= 0 + 3x - \left(x^2 - \frac{1}{6}x + \frac{1}{6}\right) 9 =$$

$$= 3x - 9x^2 + \frac{9}{6}x - \frac{9}{6} =$$

$$= 3x - 9x^2 + \frac{3}{2}x - \frac{3}{2} =$$

$$= 3x(1 - 3x) + \frac{3}{2}(x - 1)$$

b.) $$(L_2 f)(x) = \sum_{i=0}^{2} l_i(x) f(x_i) = \sum_{i=0}^{2} \frac{u_i(x)}{u_i(x_i)} f(x_i)$$

$$= \frac{\left(x - \frac{1}{6}\right)\left(x - \frac{1}{2}\right)}{\left(0 - \frac{1}{6}\right)\left(0 - \frac{1}{2}\right)} 0 + \frac{x\left(x - \frac{1}{2}\right)}{\left(\frac{1}{6} - 0\right)\left(\frac{1}{6} - \frac{1}{2}\right)} \cdot \frac{1}{2} + \frac{x\left(x - \frac{1}{6}\right)}{\left(\frac{1}{2} - 0\right)\left(\frac{1}{2} - \frac{1}{6}\right)} \cdot 1 =$$

= ~~......~~.

$$u(x) = \prod_{j=0}^{m} (x - x_j)$$

~~$u_i(x_i) = x_i$~~

$$l_i = \frac{u_i(x)}{u_i(x_i)}$$

$$u_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{m} (x - x_j) =$$

$$\Rightarrow u_0(x) = (\cancel{x} - x_1)(x - x_2) = \left(x - \frac{1}{6}\right)\left(x - \frac{1}{2}\right)$$

$$u_1(x) = (x - x_0)(x - x_2) = (x - 0)\left(x - \frac{1}{2}\right)$$

$$u_2(x) = (x - x_0)(x - x_1) = (x - 0)\left(x - \frac{1}{6}\right).$$

CURS 4 – Ex 5.

Find the hermite interpolation formula for the function $x \cdot e^x$ for which we know

$f(-1) = -0.3679$, $f(0) = 0$, $f'(0) = 1$, $f(1) = 2.7183$

$x_0 = -1$, $x_1 = 0$, $x_2 = 1$

Which is the limit of the error for approx $f(\frac{1}{2})$.

$f = H_m f + R_m f$

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi) \quad (H_m f)(x) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}$$

$x_0 = -1 \rightarrow f(-1) = -0.3679 \rightarrow r_0 = 0.$

$x_1 = 0 \rightarrow f(0) = 0 \quad f'(0) = 1 \rightarrow r_1 = 1$

$x_2 = 1 \rightarrow f(1) = 2.7183. \rightarrow r_0 = 0.$

$f(x) = x e^x$

$$u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k + 1}$$

$m = 3$. $\quad m = r_0 + r_1 + r_2 + m = 0 + 1 + 0 + 2 = 3.$

$$\cancel{u(x) = \frac{u(x)}{(x - x_0)^{r_k+1}} = \frac{(x-x_0)^1 (x-x_1)^2 (x-x_2)}{(x-x_0)^1}}$$

$$\cancel{(R_m f)(x) = \frac{\frac{(x+1)(x-0)(x-1)}{(x+1)}}{4!} M_4 f}$$

$$(R_m f)(x) = \frac{(x-x_0)^1 (x-x_1)^2 (x-x_2)^1}{(3+1)!} M_4 f =$$

$$= \frac{(x+1)(x-0)^2 (x-1)^1}{4!} M_4 f$$

2

$M_4 f \cdot f^{(4)}(x)$

$f(x) = x \cdot e^x$

$f'(x) = e^x + x \cdot e^x = e^x(x+1)$.

$f''(x) = e^x + e^x(x+1) = e^x + e^x x + e^x = e^x(x+2)$.

$f'''(x) = e^x(x+3)$

$f^{(4)}(x) = e^x(x+4)$.

$$\Rightarrow (R_3 f)\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}+1\right)\left(\frac{1}{2}-1\right)\cdot\left(\frac{1}{2}\right)^2}{16} \cdot M_4 f$$

CURS 4 – EX 7.

Consider the double nodes $x_0 = -1$ and $x_1 = 1$

and $f(-1) = 3$ $\quad f(1) = 1$

$f'(-1) = 10$ $\quad f'(1) = 2$

Find the Hermite interpolation polynomial, that approximates the funct. f, in both forms, using the classical formula and using divided differences

$x_0 = -1$ $\quad z_0 = -1$ $\quad z_1 = -1$

$x_1 = 1$ $\quad z_2 = 1$, $z_3 = 1$

$f(-1) = -3$

$f'(-1) = 10$

$f(1) = 1$

$f'(1) = 2$

a)

z	$f(z)$	$\mathcal{D}^1 f(z)$	$\mathcal{D}^2 f(z)$	$\mathcal{D}^3 f(z)$
$z_0 = -1$	-3	10	$\frac{2-10}{2} =$ [illegible]	$\frac{0-[illegible]}{2}$ [illegible]
$z_1 = -1$	-3	2	$\frac{2-2}{2} = 0$	
$z_2 = 1$	1	2		
$z_3 = 1$	1			

$n = n_0 + n_1 + m = 1 + 1 + 2 = 4$

$$(H_3f)(x) = f(z_0) + \sum_{i=1}^{3} (x-z_0)\dots(x-z_{i-1})(\mathcal{D}^i f(z_0))$$

$$= f(z_0) + (x-z_0)\mathcal{D}^1 f(z_0) + (x-z_0)(x-z_1)(\mathcal{D}^2 f)(z_0)$$

$$+ (x-z_0)(x-z_1)(x-z_2)(\mathcal{D}^3 f)(z_0)$$

$$\Rightarrow (H_3 f)(x) = f(-1) + (x+1)f'(-1) + (x+1)^2 \frac{f(1)-f(-1)-2f'(-1)}{4} +$$

$$+ (x+1)^2(x-1)\frac{2f'(1)-f(1)+f(-1)}{4} =$$

$$= -3 + 10(x+1) - 4(x+1)^2 + 2(x+1)^2(x-1) = \dots$$

CURS 6 EX. 14.

Approximate $\frac{\pi}{4}$ with repeated trapezium's formula, considering precision $\varepsilon = 10^{-2}$

$$\int_a^b f(x)\,dx = \frac{b-a}{2m}\left[f(a) + f(b) + 2\sum_{k=1}^{m-1} f(x_k)\right] +$$

$$+ R_m(f)$$

$$R_m f = -\frac{(b-a)^3}{12m^2} f''(\xi), \quad a < \xi < b.$$

$\Rightarrow$ we have $\frac{\pi}{4} = \operatorname{arctg}(1) = \int_0^1 \frac{dx}{1+x^2}$

$$f(x) = \frac{1}{1+x^2} \Rightarrow |R_m f| \leq \frac{(0-1)^3}{12m^2} M_2 f$$

$$M_2 f = \max_{x\in[0,1]} |f''(x)| = 2$$

$$|R_m f| \leq \frac{1}{12n^2} M_2 f$$

$$f'(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(x) = \frac{6x^2-2}{(1+x^2)^3} \qquad (*)$$

Info la C5 → Birkhoff. (*)

$f(0) \to I_0 = \{0\}$

$f'(1) \to I_1 = \{1\}$.

$f(2), f''(2) \Rightarrow I_2 = \{0, 2\}$.

$\to |R_n f| \le \frac{1}{\cancel{12}_{6} n^2} \cancel{2} < 10^{-2}$

$|R_m f| \le \frac{1}{6m^2} < 10^{-2}$

$\Rightarrow m^2 > \frac{10^2}{6} = 16.66$

$\Rightarrow m = 5$

<u>CURS 6.</u>

Aprox. the integral

$\int_0^2 \frac{1}{x+4}$ with precision $\varepsilon = 10^{-3}$ using Trapezium formula:

$$|R_n(f)| \le \frac{(b-a)^3}{12m^2} M_2 f = \frac{(2-0)^3}{12 \cdot m^2} M_2 f = \underbrace{\frac{2^3}{12n^2} M_2 f < \frac{1}{1000}}_{10^{-3}}$$

$M_n f$ - max din modul din —

$$M_2 f = \max_{a \le x \le b} |f''(x)|$$

$$\frac{2^3}{12m^2} M_2 f < \frac{1}{1000}. \Rightarrow (*)$$

$$M_2 f = \max_{x \in [0,2]} |f^{(2)}(x)| = \frac{2}{(0+4)^3} = \frac{2}{4^3}$$

$f'(x) = \frac{-1}{(x+2)^2}$

$f''(x) = \frac{2}{(x+4)^3}$

$$(*) = \frac{2^3}{\cancel{12}_{6} m^2} \cdot \frac{\cancel{2}^1}{4^3} < \frac{1}{1000} \Rightarrow \frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{\cancel{6}_3 \cdot m^2} \cdot \frac{1}{4 \cdot 4 \cdot \cancel{4}} < \frac{1}{1000} =$$

$$= \frac{1}{3n^2} \cdot \frac{1}{4^2} < \frac{1}{1000} \Rightarrow 1000 < 16 \cdot 3n^2 \Rightarrow m > \sqrt{\frac{1000}{16 \cdot 3}}$$

CURS 9 - EX 1.

Jacobi

Solve the following system using the Jacobi iterative method

Use $\varepsilon = 10^{-3}$ and $x^{(0)} = (0\ 0\ 0\ 0)$ → as the starting vector

$$\begin{cases} 7x_1 - 2x_2 + x_3 = 17 \\ x_1 - 9x_2 + 3x_3 - x_4 = 13 \\ 2x_1 + 10x_3 + x_4 = 15 \\ x_1 - x_2 + x_3 + 6x_4 = 10 \end{cases}$$

$$\begin{cases} x_1 = u_{12} x_2 + \dots + u_{1n} x_n + c_1; \\ x_2 = u_{21} x_1 + \dots + u_{2n} x_n + c_2; \\ \vdots \\ x_n = u_{n1} x_1 + \dots + u_{n\,n-1} x_{n-1} + c_n; \end{cases}$$

$$u_{ij} = -\frac{a_{ij}}{a_{ii}}; \quad c = \frac{b_i}{a_{ii}}; \quad i = \overline{1,n}$$

$$c_i = \frac{b_i}{a_{ii}}$$

rearanjăm ecuațiile.

$$x_1 = (17 + 2x_2 - x_3)/7$$

$$x_2 = (-13 + x_1 + 3x_3 - x_4)/9$$

$$x_3 = (15 - 2x_1 - x_4)/10$$

$$x_4 = (10 - x_1 + x_2 - x_3)/6$$

CURS 10 -> ex 5.

Use Newton's method to compute a root of $x^3 - x^2 - 1 = 0$ to an accuracy of 10^{-4}. Use $x_0 = 1$.

$\varepsilon = \frac{1}{10000} = 0.0001$

Newton's method

$f'(x) = 3x^2 - 2x = 0$

$$x_{i+1} = F_2^T(x_i) = x_i - \frac{f(x_i)}{f'(x_i)} \quad ; \; i = 0, 1, \ldots$$

$$x_{0+1} \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1 - 1 - 1}{3 \cdot 1 - 2 \cdot 1} = 1 - \frac{-1}{1} = 1 + 1 = 2$$

$x_0 = 1$

$x_1 = 2$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2^3 - 2^2 - 1}{3 \cdot 2^2 - 2 \cdot 2} = 2 - \frac{8 - 4 - 1}{3 \cdot 4 - 4} =$$

$$= 2 - \frac{3}{8} = 1.625.$$

$$\cancel{F_3^T(x_i) = x_i - \frac{f(x_i)}{f'(x_i)} - \frac{1}{2}\left[\frac{f(x_i)}{f'(x_i)}\right]^2 \frac{f''(x_i)}{f'(x_i)}}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.625 - \frac{1.625^3 - 1.625^2 - 1}{3 \cdot 1{,}625^2 - 2 \cdot 1{,}625} =$$

$$= 1{,}625 - \frac{4{,}2910 - 2{,}6406}{4{,}9218 - 3{,}25} =$$

$$= 1{,}625 - \frac{1.651}{4.671} = 1{,}625 - 0.353 = \underline{1.272}$$

$x_3 = 1.272$.

$x_3 - x_2 = -0.728 \qquad |-0.728| = 0 \leq \varepsilon$

CURS 5 - Ex 1.

Considering the following data:

	x_0	x_1	x_2
x	0	2	3
$f(x)$	0	10	12
$f'(x)$	5	3	7

$r_0 = 1$ $\quad m = 2$

$r_1 = 1$ $\quad n = 5$

$r_2 = 1$

	$f(x)$	$\mathcal{D}^1 f$	$\mathcal{D}^2 f$	$\mathcal{D}^3 f$
$z_0 = 0$	0	5	0	$\frac{-1}{2} = -\frac{1}{2}$
$z_1 = 0$	0	5	-1	$\frac{4/18}{3} = $ [illegible] 0
$z_2 = 2$	10	3	$-\frac{1}{4} = -1$	$\frac{6}{1} = 6$
$z_3 = 2$	10	2	$\frac{5}{1} = 5$	
$z_4 = 3$	12	7		
$z_5 = 3$	12			

$x_0 = z_0$

$x_0 = z_1$

$x_1 = z_2$

$x_1 = z_3$

$x_2 = z_4$

$x_2 = z_5$

$\mathcal{D}^4 f$	$\mathcal{D}^5 f$
$\frac{\frac{1}{2} \cdot \frac{1}{3} - \frac{1}{6}}{3}$	$\frac{6/2 - \frac{1}{6}}{3} = \frac{\frac{11}{6}}{3} =$
$\frac{6}{3} = 2$	

$$(N_5 f)(x) = f(x_0) + \sum_{i=1}^{5} (x - x_0) \ldots (x - x_{i-1}) (\mathcal{D}^i f)(z_0) =$$

$$= f(x_0) + (x - x_0)\mathcal{D}^1 f(x_0) + (x - x_0)(x - x_0)\mathcal{D}^2 f + (x - x_0)(x - x_1)(x - x_2)\mathcal{D}^3 f$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\mathcal{D}^4 f + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)\mathcal{D}^5 f =$$

$$= 0 + (x - 0) \cdot 5 + (x - 0)(x - 0) \cdot 0 + (x - 0)^2 (x - 2)\left(-\frac{1}{2}\right) +$$

$$(x - 0)^2 (x - 2)^2 \cdot \frac{1}{6} + (x - 0)^2 (x - 2)^2 (x - 3) \cdot \frac{11}{18} + \ldots$$

CURS 5 - EXS.

Let $f \in C^2[0,1]$

$x_0 = 0$ $\quad f(0) = 1$ $\quad I_0 = \{0\}$

$x_1 = 1$. $\quad f'(1) = \frac{1}{2}$ $\quad I_1 = \{1\}$

Find the corresponding interp. formula.

$m = \cancel{1} + \cancel{1} - \cancel{1} = 1.$

$$(B_1 f)(x) = \sum_{k=0}^{1} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k) =$$

$$= b_{00}(x) \cdot f^{(0)}(x_0) + b_{11}(x) f^{(1)}(x_1) =$$

$$= 1 \cdot 1 + x \cdot \frac{1}{2} = \underbrace{1 + x\frac{1}{2}}_{\text{Res.}}$$

$b_{00}(x) = ax + b. \quad \Rightarrow 1$

$$\begin{cases} b_{00}(x_0) = 1 \\ b'_{00}(x_1) = 0 \end{cases} \quad \begin{cases} b_{00}(0) = 1 \\ b'_{00}(1) = 0 \end{cases} \quad \begin{cases} b = 1 \\ a = 0. \end{cases}$$

$b_{11}(x) = a_2 x + b_2 \Rightarrow x$

$$\begin{cases} b_{11}(x_0) = 0 \\ b'_{11}(x_1) = 1 \end{cases} \quad \begin{cases} b_{11}(0) = 0 \\ b'_{11}(1) = 1 \end{cases} \quad \begin{cases} b = 0 \\ a = 1 \end{cases}$$