COURSE 5

Example 1 *Considering the the following data*

x	0	2	3
$f(x)$	0	10	12
$f\prime(x)$	5	3	7

find the corresponding Hermite interpolation polynomial.

2.4. Birkhoff interpolation

Let $x_k \in [a,b],\ \ k = 0,1,...,m,\ \ x_i \neq x_j$ for $i \neq j, r_k \in \mathbb{N}$ and $I_k \subset \{0,1,...,r_k\},\ k = 0,1,...,m,\ f:[a,b] \to \mathbb{R}$ s.t. $\exists f^{(j)}(x_k),\ k = 0,...,m,\ j \in I_k$, and denote $n = |I_0| + ... + |I_m| - 1$, where $|I_k|$ is the cardinal of the set I_k.

The Birkhoff interpolation problem (BIP) consists in determining the polynomial P of the smallest degree such that

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0,...,m;\ j \in I_k.$$

Remark 2 *If $I_k = \{0,1,...,r_k\},\ k = 0,...,m$, then (BIP) reduces to a (HIP). Birkhoff interpolation is also called* lacunary Hermite interpolation.

In order to check if (BIP) has solution, we consider the polynomial $P(x) = a_n x^n + ... + a_0$ and the $(n+1) \times (n+1)$ linear system

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0,...,m;\ j \in I_k, \tag{1}$$

having as unknowns the coefficients of the polynomial. If the determinant of the system (1) is nonzero than (BIP) has an unique solution.

Definition 3 *A solution of (BIP), if exists, is called* **Birkhoff interpolation polynomial**, *denoted by $B_n f$.*

Birkhoff interpolation polynomial is given by

$$(B_n f)(x) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k), \tag{2}$$

where $b_{kj}(x)$ denote the Birkhoff fundamental interpolation polynomials. They fulfill relations:

$$\begin{aligned} & b_{kj}^{(p)}(x_\nu) = 0,\ \nu \neq k, \quad p \in I_\nu \\ & b_{kj}^{(p)}(x_k) = \delta_{jp},\ p \in I_k, \quad \text{for } j \in I_k \text{ and } \nu, k = 0,1,...,m, \end{aligned} \tag{3}$$

with $\delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$

Remark 4 *Because of the gaps of the interpolation conditions, it is hard to find an explicit expression for $b_{kj},\ k = 0,...,m;\ j \in I_k$. They are found using relations (3).*

Birkhoff interpolation formula is

$$f = B_n f + R_n f,$$

where $R_n f$ denotes the remainder term.

Example 5 *Let $f \in C^2[0,1]$, the nodes $x_0 = 0,\ x_1 = 1$ and we suppose that we know $f(0) = 1$ and $f'(1) = \frac{1}{2}$. Find the corresponding interpolation formula.*

We have $m = 1,\ I_0 = \{0\},\ I_1 = \{1\}$, so $n = 1 + 1 - 1 = 1$.

We check if there exists a solution of the problem.

Consider $P(x) = a_1 x + a_0 \in \mathbb{P}_1$ and the system

$$\begin{cases} P(0) = f(0) \\ P'(1) = f'(1) \end{cases} \Longleftrightarrow \begin{cases} a_0 = f(0) \\ a_1 = f'(1) \end{cases}.$$

The determinat of the system is

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,$$

so the problem has an unique solution.

The Birkhoff polynomial is

$$(B_1 f)(x) = b_{00}(x) f(0) + b_{11}(x) f'(1) \in \mathbb{P}_1.$$

We have $b_{00}(x) = ax + b \in \mathbb{P}_1$ and

$$\begin{cases} b_{00}(x_0) = 1 \\ b'_{00}(x_1) = 0 \end{cases} \Longleftrightarrow \begin{cases} b_{00}(0) = 1 \\ b'_{00}(1) = 0 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ a = 0 \end{cases},$$

whence

$$b_{00}(x) = 1.$$

For $b_{11}(x) = cx + d \in \mathbb{P}_1$ *we have*

$$\begin{cases} b_{11}(x_0) = 0 \\ b_{11}'(x_1) = 1 \end{cases} \Longleftrightarrow \begin{cases} b_{11}(0) = 0 \\ b_{11}'(1) = 1 \end{cases} \Leftrightarrow \begin{cases} d = 0 \\ c = 1 \end{cases}$$

whence

$$b_{11}(x) = x.$$

So,

$$(B_1 f)(x) = f(0) + x f'(1) = 1 + \frac{1}{2}x.$$

Example 6 *Considering* $f'(0) = 1,\ f(1) = 2$ *and* $f'(2) = 1$. *Find the approximative value of* $f(\frac{1}{2})$.

2.5. Least squares approximation

- It is an extension of the interpolation problem.

- More desirable when the data are contaminated by errors.

- To estimate values of parameters of a mathematical model from measured data, which are subject to errors.

When we know $f(x_i)$, $i = 0, ..., m$, an interpolation method can be used to determine an approximation φ of the function f, such that

$$\varphi(x_i) = f(x_i),\ i = 0, ..., m.$$

If only approximations of $f(x_i)$ are available or the number of interp. conditions is too large, instead of requiring that the approx. function reproduces $f(x_i)$ exactly, we ask only that it fits the data "as closely as possible".

The least squares approximation φ is determined such that:

- in the discrete case:

$$\left(\sum_{i=0}^{m} [f(x_i) - \varphi(x_i)]^2 \right)^{1/2} \to \min,$$

- in the continuous case:

$$\left(\int_a^b [f(x) - \varphi(x)]^2 \, dx \right)^{1/2} \to \min,$$

Remark 7 *Notice that the interpolation is a particular case of the least squares approximation, with*

$$f(x_i) - \varphi(x_i) = 0, \quad i = 0, ..., m.$$

Linear least square. Consider the data

x	1	2	3	4	5
$f(x)$	1	1	2	2	4

The problem consists in finding a function φ that "best" represents the data.

Plot the data and try to recognize the shape of a "guess function φ" such that $f \approx \varphi$.

For this example, a resonable guess may be a linear one, $\varphi(x) = ax + b$. The problem: find a and b that makes φ the best function to fit the data. The least squares criterion consists in minimizing the sum

$$E(a, b) = \sum_{i=0}^{4} [f(x_i) - \varphi(x_i)]^2 = \sum_{i=0}^{4} [f(x_i) - (ax_i + b)]^2.$$

The minimum of the sum is obtained when

$$\frac{\partial E(a,b)}{\partial a} = 0$$
$$\frac{\partial E(a,b)}{\partial b} = 0.$$

We get

$$15a + b = 10$$
$$55a + 15b = 37$$

and further $\varphi(x) = 0.7x - 0.1$.

Consider a more general problem with the data from the table

x	x_0	x_1	...	x_m
$f(x)$	y_0	y_1	...	y_m

and the approximating linear function $\varphi(x) = ax + b$. We have to find a and b.

We have to minimize the sum

$$E(a,b) = \sum_{i=0}^{m} \left[f(x_i) - \varphi(x_i)\right]^2 = \sum_{i=0}^{m} \left[f(x_i) - (ax_i + b)\right]^2. \tag{4}$$

The minimum of the sum is obtained when

$$\frac{\partial E(a,b)}{\partial a} = 2\sum_{i=0}^{m} \left[f(x_i) - (ax_i + b)\right] \cdot x_i = 0$$

$$\frac{\partial E(a,b)}{\partial b} = 2\sum_{i=0}^{m} \left[f(x_i) - (ax_i + b)\right] = 0$$

further

$$\sum_{i=0}^{m} x_i f(x_i) = a\sum_{i=0}^{m} x_i^2 + b\sum_{i=0}^{m} x_i$$

$$\sum_{i=0}^{m} f(x_i) = a\sum_{i=0}^{m} x_i + (m+1)b.$$

These are called **normal equations**. The solution is

$$a = \frac{(m+1)\sum_{i=0}^{m} x_i f(x_i) - \sum_{i=0}^{m} x_i \sum_{i=0}^{m} f(x_i)}{(m+1)\sum_{i=0}^{m} x_i^2 - (\sum_{i=0}^{m} x_i)^2}$$

$$b = \frac{\sum_{i=0}^{m} x_i^2 \sum_{i=0}^{m} f(x_i) - \sum_{i=0}^{m} x_i f(x_i) \sum_{i=0}^{m} x_i}{(m+1)\sum_{i=0}^{m} x_i^2 - (\sum_{i=0}^{m} x_i)^2}. \tag{5}$$

Polynomial least squares. In many experimental results the data are not linear. Suppose that

$$\varphi(x) = \sum_{k=0}^{n} a_k x^k, \quad n < m$$

Find $a_i, i = 0, ..., n$, that minimize the sum

$$E(a_0, ..., a_n) = \sum_{i=0}^{m} \left[f(x_i) - \varphi(x_i)\right]^2 \tag{6}$$

$$= \sum_{i=0}^{m} \left[f(x_i) - \sum_{k=0}^{n} a_k x_i^k\right]^2.$$

The minimum is obtained when

$$\frac{\partial E(a_0, ..., a_n)}{\partial a_j} = 0, \quad j = 0, ...n,$$

which are **the normal equations** and have a unique solution.

General case. Solution of the least squares problem is

$$\varphi(x) = \sum_{i=1}^{n} a_i g_i(x),$$

where $\{g_i,\ i = 1, ..., n\}$ is a basis of the space and the coefficients a_i are obtained solving **the normal equations**:

$$\sum_{i=1}^{n} a_i \langle g_i, g_k \rangle = \langle f, g_k \rangle, \quad k = 1, ..., n.$$

In the discrete case

$$\langle f, g \rangle = \sum_{k=0}^{m} w(x_k) f(x_k) g(x_k)$$

and in the continuous case

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x),$$

where w is a weight function.

Example 8 *Having the data*

x	0	1	2	3
$f(x)$	-4	0	4	-2

find the corresponding least squares polynomial of first degree.

Sol. We have

$$E(a,b)=\sum_{i=0}^{3}\left[f\left(x_i\right)-\varphi\left(x_i\right)\right]^2=\sum_{i=0}^{3}\left[f\left(x_i\right)-\left(ax_i+b\right)\right]^2 \qquad (7)$$

and we have to find a and b from the system

$$\begin{cases} \frac{\partial E(a,b)}{\partial a}=2\sum\limits_{i=0}^{3}\left[f\left(x_i\right)-\left(ax_i+b\right)\right]\cdot x_i=0 \\ \frac{\partial E(a,b)}{\partial b}=2\sum\limits_{i=0}^{3}\left[f\left(x_i\right)-\left(ax_i+b\right)\right]=0 \end{cases}$$

$$\begin{cases} \sum\limits_{i=0}^{3}\left[f\left(x_i\right)-\left(ax_i+b\right)\right]\cdot x_i=0 \\ \sum\limits_{i=0}^{3}\left[f\left(x_i\right)-\left(ax_i+b\right)\right]=0 \end{cases}$$