COURSE 11

Hermite inverse interpolation

Let $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$. Consider the equation

$$f(x) = 0, \quad x \in \Omega. \tag{1}$$

Assume that α is a solution of equation $f(x) = 0$ and $V(\alpha)$ is a neighborhood of α. If $y_k = f(x_k)$, where $x_k \in V(\alpha)$, $k = 0, ..., m$, are approximations of α, $r_k \in \mathbb{N}$, then if there exist $g^{(j)}(y_k) = (f^{-1})^{(j)}(y_k), j = 0, ..., r_k$, one considers the Hermite type interpolation problem.

Theorem 1 *Let α be a solution of equation $f(x) = 0$, $V(\alpha)$ a neighborhood of α and $x_0, x_1 ..., x_m \in V(\alpha)$. For $n = r_0 + ... + r_m + m$, where r_k represents the multiplicity order of the nodes x_k, $k = 0, ..., m$, if $f \in C^{n+1}(V(\alpha))$ and $f'(x) \neq 0$ for $x \in V(\alpha)$, we have the following Hermite approximation method for α:*

$$\begin{aligned} \alpha \approx F_n^H(x_0, ..., x_m) = & \\ = (H_n g)(0) = & \sum_{k=0}^{m} \sum_{j=0}^{r_k} \sum_{\nu=0}^{r_k - j} \frac{(-1)^{j+\nu}}{j!\nu!} f_k^{j+\nu} v_k(0) \left(\frac{1}{v_k(y)}\right)_{y=f_k}^{(\nu)} g^{(j)}(f_k), \end{aligned} \tag{2}$$

where $f_k = f(x_k), k = 0, ..., m, g = f^{-1}$, and

$$v_k(y) = (y - f_0)^{r_0+1} ... (y - f_{k-1})^{r_{k-1}+1} (y - f_{k+1})^{r_{k+1}+1} ... (y - f_m)^{r_m+1}.$$

For $g = f^{-1}$ the corresponding Hermite polynomial is

$$(H_n g)(y) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(y) g^{(j)}(y_k),$$

and it satisfies the conditions:

$$(H_n g)^{(j)}(y_k) = g^{(j)}(y_k), \ \ j = 0, ..., r_k; \quad k = 0, ..., m,$$

where h_{kj} are the fundamental interpolating polynomials, i.e.,

$$h_{kj}^{(p)}(y_\nu) = 0, \ k \neq \nu, \ p = 0, ..., r_\nu$$
$$h_{kj}^{(p)}(y_k) = \delta_{pj}, \ p = 0, ..., r_k$$

and the corresponding interpolation formula is

$$g = H_n g + R_n g,$$

where $R_n g$ is the remainder term.

Taking into account that

$$\alpha = g(0) \approx (H_n g)(0),$$

defines a new approximation to α we have that

$$F_n^H(x_0, ..., x_m) = (H_n g)(0)$$

is an approximation method for α.

Regarding the order of Hermite-type inverse interpolation method F_n^H, we have two results, first for the case of equal information (the same multiplicity order for all the nodes x_k, $k = 0, ..., m$) and then for different multiplicities.

Theorem 2 *(Equal information) The order $ord(F_n^H)$ is the unique positive root of the equation:*

$$t^{m+1} - (r + 1) \sum_{j=0}^{m} t^j = 0,$$

where r is the multiplicity order of the points x_k, $\forall k = 0, ..., m$.

Theorem 3 *(Unequal information) The order of F_n^H is the unique positive and real root of the equation:*

$$t^{m+1} - (r_m + 1)t^m - (r_{m-1} + 1)t^{m-1} - .... - (r_1 + 1)t - (r_0 + 1) = 0,$$

where $r_0, ..., r_m$ are real numbers, permutation of the multiplicity orders of the nodes x_k, $k = 0, ..., m$ satisfying the conditions:

$$r_0 + r_1 + ... + r_m > 1 \tag{3}$$

and

$$r_m \geq r_{m-1} \geq ... \geq r_1 \geq r_0. \qquad (4)$$

Remark 4 *The order of the Taylor-type inverse interpolation method, can be expressed as the solution of equation*

$$t-(r_0+1)=0,$$

where r_0 is the multiplicity order of the node x_0.

Particular cases.

1) For nodes x_0, x_1; with $r_0=0$, $r_1=1$, we have the following approximation method:

$$F_2^H(x_0,x_1)=x_1-\left[\frac{f(x_1)}{f(x_0)-f(x_1)}\right]^2(x_1-x_0)-\frac{f(x_1)}{f(x_0)-f(x_1)}\frac{f(x_0)}{f'(x_1)}.$$

The *order* of this method is the solution of the equation:

$$t^2-r_1t-r_0=0,$$

so,

$$t^2-2t-1=0,$$

and $p=1+\sqrt{2}$.

2) For nodes x_0, x_1, x_2; with $r_0=r_1=0$; $r_2=1$, the method is:

$$F_4^L(x_0,x_1,x_2)=$$
$$=\frac{f(x_2)^2}{f(x_1)-f(x_0)}\left[\frac{x_0f(x_1)}{[f(x_0)-f(x_2)]^2}-\frac{x_1f(x_0)}{[f(x_1)-f(x_2)]^2}\right]$$
$$+\frac{f(x_0)f(x_1)}{[f(x_2)-f(x_0)][f(x_2)-f(x_1)]}\left[1+\frac{f(x_2)}{[f(x_2)-f(x_0)][f(x_2)-f(x_1)]}\right]\left[x_2-\frac{f(x_2)}{f'(x_2)}\right].$$

The *order* of this method is the solution of the equation:

$$t^3-2t^2-t-1=0,$$

so $p=2.548$.

Birkhoff inverse interpolation

Assume that α is a solution of equation $f(x)=0$ and $V(\alpha)$ is a neighborhood of α. If $y_k=f(x_k)$, where $x_k\in V(\alpha)$, $k=0,...,m$, are approximations of α, $r_k\in N$ and $I_k\subset\{0,...,r_k\}$, then if there exist $g^{(j)}(y_k)=(f^{-1})^{(j)}(y_k), j\in I_k$, one considers the Birkhoff type interpolation problem.

The Birkhoff polynomial

$$(B_ng)(y)=\sum_{k=0}^{m}\sum_{j\in I_k}b_{kj}(y)g^{(j)}(y_k),$$

satisfies the conditions:

$$(B_ng)^{(j)}(y_k)=g^{(j)}(y_k),\ \ j\in I_k,\ \ k=0,...,m,$$

where b_{kj} are the fundamental interpolating polynomials, i.e.,

$$b_{kj}^{(p)}(y_\nu)=0,\ k\neq\nu,\ \ p\in I_\nu$$
$$b_{kj}^{(p)}(y_k)=\delta_{pj},\ p\in I_k$$

and the corresponding interpolation formula is

$$g=B_ng+R_ng,$$

where R_ng is the remainder term.

Taking into account that

$$\alpha=g(0)\approx(B_ng)(0),$$

defines a new approximation to α we have that

$$F_n^B(x_0,...,x_m)=(B_ng)(0)$$

is an approximation method for α.

Particular case.

1) Let $x_0, x_1 \in V(\alpha)$, $I_0 = \{0\}$, $I_1 = \{1\}$, so $n = 1$ and $y_0 = f(x_0)$, $y_1 = f(y_1)$.

Taking

$$F_1^B(x_0, x_1) = (B_1 g)(0),$$

we obtain the method defined by

$$F_1^B(x_0, x_1) = x_0 - \frac{f(x_0)}{f'(x_1)}.$$

9. Numerical methods for solving differential equations

We consider a Cauchy problem:

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned} \tag{5}$$

with f defined on $D = \{(x, y) \in \mathbb{R}^2 \mid |x - x_0| \le a,\ |y - y_0| \le b\}$, $a, b \in \mathbb{R}_+$, continuous and derivable.

9.1. Taylor interpolation method

Let $f \in C^p(D)$ and y be a solution of the problem (5). We attach Taylor interpolation formula to y, with respect to x_0:

$$y = T_p y + R_p y,$$

where

$$(T_p y)(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + ... + \frac{(x - x_0)^p}{p!} y^{(p)}(x_0),$$

and the remainder term:

$$(R_p y)(x) = \frac{(x - x_0)^{p+1}}{(p+1)!} y^{(p+1)}(\xi), \quad \xi \text{ between } x_0 \text{ and } x. \tag{6}$$

We know only $y(x_0) = y_0$ and $y'(x_0) = f(x_0, y_0)$ in this polynomial, so we have to compute $y^{(k)}(x_0)$, $k = 2, ..., p$. Using equation (5) we get

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y', ...$$

and so on. Taking the values of these derivatives in x_0, the approximation of y is completely determined.

Denoting $y^{(k)} = f^{(k-1)}$, Taylor polynomial can be written as

$$\begin{aligned} (T_p y)(x) =& y(x_0) + \frac{(x - x_0)}{1!} f(x_0, y(x_0)) + \frac{(x - x_0)^2}{2!} f'(x_0, y(x_0)) \\ &+ ... + \frac{(x - x_0)^p}{p!} f^{(p-1)}(x_0, y(x_0)). \end{aligned} \tag{7}$$

If $\left| y^{(p+1)}(x) \right| \le M_{p+1}$, we get the error delimitation:

$$y(x) - (T_p y)(x) \le \frac{|x - x_0|^{p+1}}{(p+1)!} M_{p+1},\ x \in I. \tag{8}$$

So, we've proved the following theorem:

Theorem 5 *If $f \in C^p(D)$ then the solution y of the Cauchy problem (5) can be approximated by Taylor polynomial (7) with delimitation of the error given in (8).*

Remark 6 *Disadvantage: for large values of p, the derivatives $f^{(k)}$, $k = 1, ..., p$, are more and more complicated to compute. In practical applications, it is used for small values of p.*

For the equidistant points $x_i = x_0 + ih$, with $y_i = y(x_i)$, $i = 0, ..., N$; $N \in \mathbb{N}$; $h = \frac{b-a}{N}$, **Taylor interpolation method of order** n can be written as

$$y_{i+1} = y_i + h T_n(x_i, y_i), \tag{9}$$

with

$$T_n(x_i, y_i) = f(x_i, y_i) + \frac{h}{2!} f'(x_i, y_i) + ... + \frac{h^{n-1}}{n!} f^{(n-1)}(x_i, y_i). \tag{10}$$

9.2. Euler's method

Consider the equation (5) and an equidistant partition of the interval $[a,b]$: $x_i = x_0 + ih$, $h = \frac{b-a}{N}$; $i = 0, 1, ..., N$; $N \in \mathbb{N}^*$.

Remark 7 *The method (9), for $n = 1$, i.e.,*

$$y_{i+1} = y_i + hf(x_i, y_i),\ \ i = 0, 1, ..., N$$

is called **Euler's method.**

Geometrical interpretation: by Euler's method, the graph of the solution y is approximated by the polygonal line with vertices (x_k, y_k), $k = 0, 1, ..., N$, hence this method is also called *the method of polygonal lines.*

Solution. We have $h = \frac{1}{10}$, $f(x,y) = 2x - y$ and we get

$$\begin{aligned} y(0.1) &\approx y_1 = y_0 + 0.1f(0,-1) = -0.9 \\ y(0.2) &\approx y_2 = y_1 + 0.1f(0.1,-0.9) = -0.79 \\ &\ldots \\ y_{10} &= 0.348678. \end{aligned}$$

Algorithm for Euler's method:

$h \leftarrow \frac{b-a}{N}$

$\alpha \leftarrow y_0$

for $i = 0, 1, ..., N-1$

$\quad y_{i+1} \leftarrow y_i + hf(x_i, y_i)$

end

Example 8 *Approximate the solution of the Cauchy problem:*

$$\begin{aligned} y'(x) &= 2x - y \\ y(0) &= -1 \end{aligned}$$

on the equidistant nodes $x_i = a + ih$, $i = 0, ..., N; h = \frac{b-a}{N}$, with $a = 0$, $b = 1$, $N = 10$, using Euler's method.

9.3. Runge-Kutta methods

One way of finding these methods is based on Taylor interpolation. There are determined the values a_1, α_1 and β_1 such that $a_1 f(x + \alpha_1, y +$ to approximate Taylor polynomial of second order, given by (10):

$$T_2(x,y) = f(x,y) + \frac{h}{2} f'(x,y), \quad h = \frac{b-a}{N}, N\text{-given.}$$

We have

$$f'(x,y) = \frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y) \cdot y'(x)$$

so

$$T_2(x,y) = f(x,y) + \frac{h}{2}\frac{\partial f}{\partial x}(x,y) + \frac{h}{2}\frac{\partial f}{\partial y}(x,y) y'(x).$$

Expanding $f(x + \alpha_1, y + \beta_1)$ using Taylor series we get

$$\begin{aligned} a_1 f(x + \alpha_1, y + \beta_1) = {} & a_1 f(x,y) + a_1\alpha_1 \frac{\partial f}{\partial x}(x,y) + a_1\beta_1 \frac{\partial f}{\partial y}(x,y) \\ & + a_1 (R_1 f)(x + \alpha_1, y + \beta_1). \end{aligned}$$

Indentifying the coefficients of the terms $f(x,y)$, $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$, we obtain

$$\begin{aligned} a_1 &= 1 \\ \alpha_1 &= \frac{h}{2} \\ \beta_1 &= \frac{h}{2}y'(x) = \frac{h}{2}f(x,y). \end{aligned}$$

In (9), for $n=2$ we have

$$y_{i+1} = y_i + hT_2(x_i, y_i), \tag{11}$$

and replacing T_2 by $a_1 f(x+\alpha_1, y+\beta_1)$, it is obtained **"the midpoint method"**:

$$\begin{aligned} y_0 &= \alpha \\ y_{i+1} &= y_i + hf\left(x_i + \frac{h}{2}, y + \frac{h}{2}f(x_i, y_i)\right), \quad i = 0, ..., N-1 \end{aligned}$$

Runge-Kutta methods of second order:

Consider $y_0 = \alpha$.

1) Midpoint method

$$y_{i+1} = y_i + hf\left(x_i + \frac{h}{2}, y + \frac{h}{2}f(x_i, y_i)\right), \quad i = 0, ..., N-1$$

2) **Modified Euler method**:

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_{i+1}, y_i + hf(x_i, y_i)\right)\right], \quad i = 0, ..., N-1$$

3) **Heun method**:

$$y_{i+1} = y_i + \frac{h}{4}\left[f(x_i, y_i) + 3f\left(x_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(x_i, y_i)\right)\right], \quad i = 0, ..., N-1.$$

Runge-Kutta method of fourth order (one of the most used in practice):

$$\begin{aligned} y_0 &= \alpha \\ k_1 &= hf(x_i, y_i) \\ k_2 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right) \\ k_3 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_2\right) \\ k_4 &= hf\left(x_{i+1}, y_i + k_3\right) \\ y_{i+1} &= y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad i = 0, ..., N-1. \end{aligned}$$

Algorithm for Runge-Kutta method of 4-th order:

$h \leftarrow \frac{b-a}{N}$; $y_0 \leftarrow \alpha$

for $i = 0, ..., N$

$$\begin{aligned} k_1 &= hf(x_i, y_i) \\ k_2 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right) \\ k_3 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_2\right) \\ k_4 &= hf\left(x_{i+1}, y_i + k_3\right) \\ y_{i+1} &= y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

end

For the data from Example 8 we have the following graph. (The exact solution is $y(x) = e^{-x} + 2x - 2$.)