COURSE 2

2.2. Lagrange interpolation

Let $[a,b]\subset\mathbb{R},\ x_i\in[a,b],\ i=0,1,...,m$ such that $x_i\neq x_j$ for $i\neq j$ and consider $f:[a,b]\to\mathbb{R}$.

The Lagrange interpolation polynomial is given by

$$(L_m f)(x)=\sum_{i=0}^{m}\ell_i(x)f(x_i). \tag{1}$$

We have

$$u(x)=\prod_{j=0}^{m}(x-x_j),\qquad u_i(x)=\frac{u(x)}{x-x_i}$$

and

$$\ell_i(x)=\frac{u_i(x)}{u_i(x_i)},\quad i=0,1,...,m. \tag{2}$$

Example 1 *1) Consider the nodes x_0,x_1 and a function f to be interpolated.*

We have $m=1$,

$$\begin{aligned}u(x)&=(x-x_0)(x-x_1)\\u_0(x)&=x-x_1\\u_1(x)&=x-x_0\end{aligned}$$

$$\begin{aligned}(L_1f)(x)&=l_0(x)f(x_0)+l_1(x)f(x_1)\\&=\frac{x-x_1}{x_0-x_1}f(x_0)+\frac{x-x_0}{x_1-x_0}f(x_1),\end{aligned}$$

which is the line passing through the given points $(x_0,f(x_0))$ and $(x_1,f(x_1))$.

2) Find the Lagrange polynomial that interpolates the data in the following table and find the approximative value of $f(-0.5)$.

x	-1	0	3
$f(x)$	8	-2	4

Sol. We have $m=2$. The Lagrange polynomial is

$$(L_2f)(x)=l_0(x)f(x_0)+l_1(x)f(x_1)+l_2(x)f(x_2).$$

$u(x)=(x+1)(x-0)(x-3)$ *and it follows*

$$\begin{aligned}l_0(x)&=\frac{(x-0)(x-3)}{(-1-0)(-1-3)}=\frac{1}{4}x(x-3)\\l_1(x)&=\frac{(x+1)(x-3)}{(0+1)(0-3)}=-\frac{1}{3}(x+1)(x-3)\\l_2(x)&=\frac{(x+1)(x-0)}{(3+1)(3-0)}=\frac{1}{12}x(x+1),\end{aligned}$$

The polynomial is

$$(L_2f)(x)=2x(x-3)+\frac{2}{3}(x+1)(x-3)+\frac{1}{3}x(x+1).$$

and $(L_2f)(-0.5)=2.25$.

Remark 2 *Disadvantages of the form (1) of Lagrange polynomial: requires many computations and if we add or substract a point we have to start with a complete new set of computations.*

Some calculations allow us to reduce the number of operations:

$$(L_mf)(x)=\frac{(L_mf)(x)}{1}=\frac{\sum\limits_{i=0}^{m}l_i(x)f(x_i)}{\sum\limits_{i=0}^{m}l_i(x)}.$$

Dividing the numerator and the denominator by

$$u(x)=\prod_{i=1}^{m}(x-x_i)$$

and denoting

$$A_i=\frac{1}{\prod\limits_{j=0,j\neq i}^{m}(x_i-x_j)}=\frac{1}{u_i(x_i)}$$

one obtains

$$(L_mf)(x)=\frac{\sum\limits_{i=0}^{m}\frac{A_if(x_i)}{x-x_i}}{\sum\limits_{i=0}^{m}\frac{A_i}{x-x_i}}, \tag{3}$$

called **the barycentric form** *of Lagrange interpolation polynomial.*

Remark 3 *Formula (3) needs half of the number of arithmetic operations needed for (1) and it is easier to add or substract a point.*

The Lagrange polynomial generates **the Lagrange interpolation formula**

$$f = L_m f + R_m f,$$

where $R_m f$ denotes **the remainder** (**the error**).

Theorem 4 *Let* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$. *If* $f \in C^m[\alpha, \beta]$ *and* $f^{(m)}$ *is derivable on* (α, β) *then* $\forall x \in (\alpha, \beta)$, *there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi). \tag{4}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix}.$$

From hypothesis it follows that $F \in C^m[\alpha, \beta]$ and there exists $F^{(m+1)}$ on (α, β).

We have

$$F(x) = 0, \ F(x_i) = 0, \qquad i = 0, 1, ..., m,$$

as

$$u(x_i) = \prod_{j=0}^{m} (x_i - x_j) = 0$$

and

$$(R_m f)(x_i) = f(x_i) - (L_m f)(x_i) = f(x_i) - f(x_i) = 0,$$

so F has $m+2$ distinct zeros in (α, β). Applying successively the Rolle theorem it follows that: F has $m+2$ zeros in $(\alpha, \beta) \Rightarrow F'$ has at least $m+1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(m+1)}$ has at least one zero in (α, β)

So $F^{(m+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(m+1)}(\xi) = 0$.

We have

$$F^{(m+1)}(z) = \begin{vmatrix} u^{(m+1)}(z) & (R_m f)^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix},$$

with

$$u(z) = \prod_{i=0}^{m} (z - z_i) \Rightarrow u^{(m+1)}(z) = (m+1)!,$$

and

$$\begin{aligned} (R_m f)^{(m+1)}(z) &= (f - (L_m f))^{(m+1)}(z) \\ &= f^{(m+1)}(z) - (L_m f)^{(m+1)}(z) = f^{(m+1)}(z) \end{aligned}$$

(as, $L_m f \in \mathbb{P}_m$).

We have $F^{(m+1)}(\xi) = 0$, for $\xi \in (\alpha, \beta)$, so

$$F^{(m+1)}(\xi) = \begin{vmatrix} (m+1)! & f^{(m+1)}(\xi) \\ u(x) & (R_m f)(x) \end{vmatrix} = 0,$$

i.e., $(m+1)!(R_m f)(x) = u(x) f^{(m+1)}(\xi)$, whence $(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+}$

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Corollary 5 *If* $f \in C^{m+1}[a, b]$ *then*

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty, \qquad x \in [a, b]$$

where $\|\cdot\|_\infty$ *denotes the uniform norm, and* $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$.

Example 6 *Which is the limit of the error for computing* $\sqrt{115}$ *using Lagrange interpolation formula for* $f(x) = \sqrt{x}$ *and* $x_0 = 100$, $x_1 = 121$ *and* $x_2 = 144$? *Find the approximative value of* $\sqrt{115}$.

Example 7 *If we know that* $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699$, *find* $\lg 76$. *Study the approximation error.*

The Aitken's algorithm

Let $[a,b] \subset \mathbb{R}$, $x_i \in [a,b]$, $i = 0,1,...,m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a,b] \to \mathbb{R}$.

A practical method for computing the Lagrange polynomial is **the Aitken's algorithm.** This consists in generating the table:

$$\begin{array}{c|c|c|cccc}
x_0 & f_{00} & & & & & \\
x_1 & f_{10} & f_{11} & & & & \\
x_2 & f_{20} & f_{21} & f_{22} & & & \\
x_3 & f_{30} & f_{31} & f_{32} & f_{33} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
x_m & f_{m0} & f_{m1} & f_{m2} & f_{m3} & \dots & f_{mm}
\end{array}$$

where

$$f_{i0} = f(x_i), \quad i = 0,1,...,m,$$

and

$$f_{i,j+1} = \frac{1}{x_i - x_j}\begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}, \qquad i = 0,1,...,m; j = 0,...,i-1.$$

For example,

$$\begin{aligned}
f_{11} &= \frac{1}{x_1 - x_0}\begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} \\
&= \frac{1}{x_1 - x_0}[f_{00}(x_1 - x) - f_{10}(x_0 - x)] \\
&= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) = (L_1 f)(x),
\end{aligned}$$

so f_{11} is the value in x of Lagrange polynomial for the nodes x_0, x_1. We have

$$f_{ii} = (L_i f)(x),$$

$L_i f$ being Lagrange polynomial for the nodes $x_0, x_1, ..., x_i$.

So $f_{11}, f_{22}, ..., f_{ii}, ..., f_{mm}$ is a sequence of approximations of f.

If the interpolation procedure is convergent then the sequence is also convergent, i.e., $\lim\limits_{m\to\infty} f_{mm} = f(x)$. By Cauchy convergence criterion it follows

$$\lim_{i\to\infty} |f_{ii} - f_{i-1,i-1}| = 0.$$

This could be used as a stopping criterion, i.e.,

$$\left|f_{ii} - f_{i-1,i-1}\right| \leq \varepsilon, \quad \varepsilon = \text{error}.$$

Recommendation is to sort the nodes $x_0, x_1, ..., x_m$ with respect to the distance to x, such that

$$|x_i - x| \leq \left|x_j - x\right| \text{ if } i < j, \quad i, \quad j = 1,...,m.$$

Example 8 *Approximate* $\sqrt{115}$ *with precision* $\varepsilon = 10^{-3}$, *using Aitken's algorithm.*