

# COURSE 1

## Outline of the course:

- Introductory notions. Finite and divided differences.
- Approximation of functions: interpolation of Lagrange, Hermite and Birkhoff type, Least squares approximation.
- Numerical integration. Newton-Cotes quadrature formulas. Repeated quadrature formulas. General quadrature formulas. Romberg's algorithm. Adaptive quadratures formulas. Gauss type quadrature formulas.
- Numerical methods for solving linear systems - direct methods (Gauss, Gauss-Jordan, LU-methods). Perturbations of a linear system.
- Numerical methods for solving linear systems - iterative methods (Jacobi, Gauss-Seidel, SOR).
- Methods for solving nonlinear equations in R: one-step methods (Newton (tangent) method) and multi-step methods (secant, bisection and false position methods). Lagrange, Hermite and Birkhoff inverse interpolation.
- Methods for solving nonlinear systems: successive approximation and Newton methods.
- Numerical methods for solving differential equations: Taylor interpolation, Euler and Runge-Kutta methods.
- Revision of the main types of problems.

## Evaluation methods

- Written exam: 70%
- Lab activities (evaluation and continuous observations during the semester): 30%
- Each lab will be evaluated.
- Each lab should be delivered in the same week or the next one.
- Respect delivery dates for each lab assignment. Each delay will be penalized: 1 point/one week of delay.
- A lab that will not be delivered will have grade 1.
- You may deliver 2 lab assignments during one lab.

## References:

1. O. Agratini, P. Blaga, I. Chiorean, Gh. Coman, R.T. Trîmbițaș, D.D. Stancu, *Analiză Numerică și Teoria Aproximării*, vol. (I,II,III), Presa Universitară Clujeană, 2001-2002;
2. R. L. Burden, J. D. Faires, *Numerical Analysis*, PWS Publishing Company, 2010.
3. I. Chiorean, T. Căținaș, R. Trîmbițaș, *Analiză Numerică*, Ed. Presa Univ. Clujeană, 2010.
- 4 R. Trîmbițaș, *Numerical Analysis*, Ed. Presa Univ. Clujeană, 2007.

## Chapter 1. Preliminary notions

We will study numerical methods and algorithms and analyze the error. In the most cases, finding an exact solution is not possible, so we approximate it, we find it numerically.

### 1.1. Preliminaries

**Definition 1** Let  $x^* \in \mathbb{R}$  be an unknown value of interest. An element  $\tilde{x} \in \mathbb{R}$  which approximates  $x^*$  is called **the approximation** or **the approximant** of  $x^*$ .

The expression  $\Delta x = x^* - \tilde{x}$  or  $\Delta x = \tilde{x} - x^*$  is called **the error**.

The value  $|\Delta x| = |x^* - \tilde{x}|$  is called **the absolute error** of approximation.

The value  $\delta x = \frac{|\Delta x|}{|x^*|} = \frac{|x^* - \tilde{x}|}{|x^*|}$ ,  $x^* \neq 0$  is called **the relative error** of approximation.

(An absolute error of, say, 1cm, may be good or poor, depending on the size of  $x^*$ : when  $x^*$  represents the length of a rectangular land (of hundreds of meters), an error of 1cm is acceptable, while when a carpenter is making furniture and cuts with an error of 1 cm may not be acceptable.)

The notions are correspondingly extended to normed space.

**Definition 2** If  $V$  is a  $K$ -linear space then a real functional  $p : V \rightarrow [0, \infty)$ , with the properties:

- 1)  $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ ,  $\forall v_1, v_2 \in V$ ,
- 2)  $p(\alpha v) = |\alpha|p(v)$ ,  $\forall \alpha \in K, v \in V$ ,
- 3)  $p(v) = 0 \implies v = 0$

is called **a norm** on  $V$ .

**Definition 3** Let  $K$  be a field and  $V$  be a given set. We say that  $V$  is a  **$K$ -linear space** (a linear space over  $K$ ) if there exist an internal operation:

$$" + " : V \times V \rightarrow V; \quad (v_1, v_2) \rightarrow v_1 + v_2,$$

and an external operation:

$$" \cdot " : K \times V \rightarrow V; \quad (\alpha, v) \rightarrow \alpha v$$

that satisfy the following conditions:

1)  $(V, +)$  is a commutative group

2)

- a)  $(\alpha + \beta)v = \alpha v + \beta v$ ,  $\forall \alpha, \beta \in K, \quad \forall v \in V$ ,
- b)  $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$ ,  $\forall \alpha \in K, \quad \forall v_1, v_2 \in V$ ,
- c)  $(\alpha\beta)v = \alpha(\beta v)$ ,  $\forall \alpha, \beta \in K, \quad \forall v \in V$ ,
- d)  $1 \cdot v = v$ ,  $\forall v \in V$ .

The elements of  $V$  are called *vectors* and those of  $K$  are called *scalars*.

**Definition 4** Let  $V$  and  $V'$  be two  $K$ -linear spaces. A function  $f : V \rightarrow V'$  is called *linear transformation* or *linear operator* if:

- 1)  $f(v_1 + v_2) = f(v_1) + f(v_2)$ ,  $\forall v_1, v_2 \in V$  (aditivity)
- 2)  $f(\alpha v) = \alpha f(v)$ ,  $\forall \alpha \in K, \quad \forall v \in V$  (homogeneity).

Or, shortly,

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2), \quad \forall \alpha, \beta \in K, \quad \forall v_1, v_2 \in V.$$

**Definition 5** Let  $V$  be a linear space on  $\mathbb{R}$  or  $\mathbb{C}$ . A linear operator  $P : V \rightarrow V$  is called **projector** if

$$P \circ P = P, \quad (\text{shortly, } P^2 = P); \quad (\text{idempotence}).$$

**Remark 6** 1) The identity operator  $I : V \rightarrow V$ ,  $I(v) = v$  and the null operator  $0 : V \rightarrow V$ ,  $0(v) = 0$  are projectors.

2)  $P$  is projector  $\Rightarrow P^C := I - P$ , **the complement of  $P$** , is projector.

## 1.2. Finite and divided differences

### Finite differences

Let  $M = \{a_i \mid a_i = a + ih, \text{ with } i = 0, \dots, m; a, h \in \mathbb{R}^*, m \in \mathbb{N}^*\}$  and  $\mathcal{F} = \{f \mid f : M \rightarrow \mathbb{R}\}$ .

**Definition 7** For  $f \in \mathcal{F}$ ,

$$(\Delta_h f)(a_i) = f(a_{i+1}) - f(a_i), \quad i < m$$

is called **the finite difference of the first order** of the function  $f$ , with step  $h$ , at point  $a_i$ .

**Theorem 8** The operator  $\Delta_h$  is a linear operator with respect to  $f$ .

**Proof.** If  $f, g : M \rightarrow \mathbb{R}$ ;  $A, B \in \mathbb{R}$  and  $i < m$ , we have

$$\begin{aligned} (\Delta_h(Af + Bg))(a_i) &= (Af + Bg)(a_{i+1}) - (Af + Bg)(a_i) \\ &= A[f(a_{i+1}) - f(a_i)] + B[g(a_{i+1}) - g(a_i)] \\ &= A(\Delta_h f)(a_i) + B(\Delta_h g)(a_i). \end{aligned} \quad (1)$$

■

**Definition 9** Let  $0 \leq i < m$ ,  $k \in \mathbb{N}$  and  $1 \leq k \leq m - i$

$$\begin{aligned} (\Delta_h^k f)(a_i) &= (\Delta_h(\Delta_h^{k-1} f))(a_i) \\ &= (\Delta_h^{k-1} f)(a_{i+1}) - \Delta_h^{k-1} f(a_i), \quad \text{with } \Delta_h^0 = I \text{ si } \Delta_h^1 = \Delta_h \end{aligned} \quad (2)$$

is called **the k-th order finite difference** of the function  $f$ , with step  $h$ , at point  $a_i$ .

**Theorem 10** If  $0 \leq i < m$ ;  $k, p \in \mathbb{N}$  and  $1 \leq p + k \leq m - i$ , then

$$(\Delta_h^p(\Delta_h^k f))(a_i) = \Delta_h^k(\Delta_h^p f)(a_i) = (\Delta_h^{p+k} f)(a_i). \quad (3)$$

**Finite differences table:** ( $f_i$  denotes  $f(a_i)$ )

$a$	$f$	$\Delta_h f$	$\Delta_h^2 f$	...	$\Delta_h^{m-1} f$	$\Delta_h^m f$
$a_0$	$f_0$	$\Delta_h f_0$	$\Delta_h^2 f_0$	...	$\Delta_h^{m-1} f_0$	$\Delta_h^m f_0$
$a_1$	$f_1$	$\Delta_h f_1$	$\Delta_h^2 f_1$	...	$\Delta_h^{m-1} f_1$	
	...					
$a_{m-3}$	$f_{m-3}$	$\Delta_h f_{m-3}$	$\Delta_h^2 f_{m-3}$			
$a_{m-2}$	$f_{m-2}$	$\Delta_h f_{m-2}$	$\Delta_h^2 f_{m-2}$			
$a_{m-1}$	$f_{m-1}$	$\Delta_h f_{m-1}$				
$a_m$	$f_m$					

where

$$\Delta_h^k f_i = \Delta_h^{k-1} f_{i+1} - \Delta_h^{k-1} f_i, \quad k = 1, \dots, m; \quad i = 0, 1, \dots, m - k.$$

### Examples.

1. Considering  $h = 0.25$ ,  $a = 1$ ,  $a_i = a + ih$ ,  $i = \overline{0, 4}$ , and  $f_0 = 0$ ,  $f_1 = 2$ ,  $f_2 = 6$ ,  $f_3 = 14$ ,  $f_4 = 17$  form the finite differences table.

Sol.: We get:

$a$	$f$	$\Delta_h f$	$\Delta_h^2 f$	$\Delta_h^3 f$	$\Delta_h^4 f$
1	0	2	2	2	-11
1.25	2	4	4	-9	
1.50	6	8	-5		
1.75	14	3			
2	17				

2. For  $f(x) = e^x$  find  $(\Delta_h^k f)(a_i)$ , with  $a_i = a + ih$ ,  $i \in \mathbb{N}$ .

### Divided differences

Let  $X = \{x_i \mid x_i \in \mathbb{R}, i = 0, 1, \dots, m, m \in \mathbb{N}^*\}$  and  $f : X \rightarrow \mathbb{R}$ .

**Definition 11** For  $r \in \mathbb{N}$ ,  $r < m$ ,

$$(\mathcal{D}f)(x_r) := [x_r, x_{r+1}; f] = \frac{f(x_{r+1}) - f(x_r)}{x_{r+1} - x_r}$$

is called **the first order divided difference** of the function  $f$ , regarding the points  $x_r$  and  $x_{r+1}$ .

**Theorem 12** The operator  $\mathcal{D}$  is linear with respect to  $f$ .

**Proof.**

$$\begin{aligned} (\mathcal{D}(\alpha f + \beta g))(x_r) &= \frac{(\alpha f + \beta g)(x_{r+1}) - (\alpha f + \beta g)(x_r)}{x_{r+1} - x_r} \\ &= \alpha(\mathcal{D}f)(x_r) + \beta(\mathcal{D}g)(x_r), \quad \text{for } \alpha, \beta \in \mathbb{R}. \end{aligned} \quad (4)$$

■

**Definition 13** Let  $r, k \in \mathbb{N}$ ,  $0 \leq r < m$  and  $1 \leq k \leq m - r$ ,  $m \in \mathbb{N}^*$ . The quantity

$$(\mathcal{D}^k f)(x_r) = \frac{(\mathcal{D}^{k-1} f)(x_{r+1}) - (\mathcal{D}^{k-1} f)(x_r)}{x_{r+k} - x_r}, \quad \text{with } \mathcal{D}^0 = 1, \mathcal{D}^1 = \mathcal{D}, \quad (5)$$

is called **the  $k$ -th order divided difference** of the function  $f$ , at  $x_r$ .

$(\mathcal{D}^k f)(x_r)$  is also denoted by  $[x_r, \dots, x_{r+k}; f]$ . Relation (5) can be written as

$$[x_r, \dots, x_{r+k}; f] = \frac{[x_{r+1}, \dots, x_{r+k}; f] - [x_r, \dots, x_{r+k-1}; f]}{x_{r+k} - x_r}. \quad (6)$$

**Remark 14** The operator  $\mathcal{D}^k$  is linear with respect to  $f$ .

For  $r = 0$  and  $k = m$  we have

$$(\mathcal{D}^m f)(x_0) = \sum_{i=0}^m \frac{f(x_i)}{(x_i - x_0) \dots (x_i - x_m)}. \quad (7)$$

**Theorem 15** If  $f, g : X \rightarrow \mathbb{R}$  then

$$[x_0, \dots, x_m; fg] = \sum_{k=0}^m [x_0, \dots, x_k; f][x_k, \dots, x_m; g].$$

**Proof.** The proof follows by complete induction with respect to  $m$ . ■

Table of divided differences:

$x$	$f$	$\mathcal{D}f$	$\mathcal{D}^2 f$	...	$\mathcal{D}^{m-1} f$	$\mathcal{D}^m f$
$x_0$	$f_0$	$\mathcal{D}f_0$	$\mathcal{D}^2 f_0$	...	$\mathcal{D}^{m-1} f_0$	$\mathcal{D}^m f_0$
$x_1$	$f_1$	$\mathcal{D}f_1$	$\mathcal{D}^2 f_1$		$\mathcal{D}^{m-1} f_1$	
$x_2$	$f_2$	$\mathcal{D}f_2$	$\mathcal{D}^2 f_2$			
...	...	...				
$x_{m-2}$	$f_{m-2}$	$\mathcal{D}f_{m-2}$	$\mathcal{D}^2 f_{m-2}$			
$x_{m-1}$	$f_{m-1}$	$\mathcal{D}f_{m-1}$				
$x_m$	$f_m$					

with  $f_i = f(x_i)$ ,  $i = 0, 1, \dots, m$ .

## Chapter 2. Polynomial interpolation

*Interpolation* is the science of "reading between the lines of a mathematical table" (E. Whittaker, G. Robinson)

Assume we know only some values  $f(x_i)$ ,  $i = 0, \dots, m$  of a function  $f$ .

$x$	$x_0$	$x_1$	$\dots$	$z$	$\dots$	$x_m$
$y = f(x)$	$y_0$	$y_1$	$\dots$	$?$	$\dots$	$y_m$

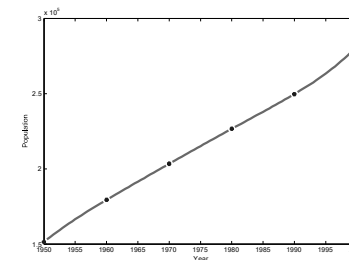
Is there a way to compute or approximate the function value  $f(z)$  for some given  $z$  without evaluating  $f$ ?

Applications:

1. approximating data at points where measurements are not available
2. sketching a function  $f$  which is expensive to evaluate if it is evaluated at some given points.

**Example 18** A census of the population of the United States is taken every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000.

1950	1960	1970	1980	1990	2000
151326	179323	203302	226542	249633	281422



*Question: these data could be used to provide a reasonable estimate of the population in 1975? Answer: population in 1975 is 215042.*

Predictions of this type can be obtained by using a function that fits the given data. This process is called **interpolation**. (If the desired

**Example 16** For  $x_0 = 0$ ,  $x_1 = 1$ ,  $x_2 = 2$ ,  $x_3 = 4$  and  $f_0 = 3$ ,  $f_1 = 4$ ,  $f_2 = 7$ ,  $f_3 = 19$  form the divided differences table.

$x$	$f$	$Df$	$D^2f$	$D^3f$
0	3	1	1	0
1	4	3	1	
2	7	6		
4	19			

**Example 17** Form the divided differences table for  $x_0 = 1$ ,  $x_1 = 2$ ,  $x_2 = 3$ ,  $x_3 = 5$ ,  $x_4 = 7$  and  $f_0 = 3$ ,  $f_1 = 5$ ,  $f_2 = 9$ ,  $f_3 = 11$ ,  $f_4 = 15$ .

value  $z$  is within the range of the interpolation points  $x_i$ , then we have *interpolation*; if  $z$  is outside the range, the process is called *extrapolation*.)

**Example 19** a) *Some values obtained by physical measurements: The temperature of the air outside a house during a day:*

$t$	8am	9am	11am	1pm	5pm
$T$ in $^{\circ}C$	12.1	13.6	15.9	18.5	16.1

*What temperature was at 10am?*

b) *Estimate  $\sin 1$  if we know  $\sin \frac{\pi}{6} = \frac{1}{2}$ ,  $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$  and  $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ .*

c) *Estimate  $\log_{10}(z)$  for values of the argument  $z$  that are intermediate between the tabulated values.*

One of the most useful classes of functions mapping the set of real numbers into itself is the algebraic polynomials.

Advantages:

- Easy to handle calculus with polynomials: the derivative and the primitive of a polynomial are easy to determine and are also polynomials. The evaluation of a polynomial can be made efficient (the Horner scheme).
- Properly chosen, they can approximate arbitrary well any continuous function:

(*Weierstrass Approximation Theorem*) Given any  $f \in C[a, b]$ ,  $\forall \varepsilon > 0$  (arbitr. small)  $\exists P(x)$  polynomial that is as “close” to  $f$  as desired:

$$|f(x) - P(x)| < \varepsilon, \quad \forall x \in [a, b].$$

Of course, the smaller  $\varepsilon$ , the greater the degree of  $P$  may become.

### 2.1. Taylor interpolation

**Theorem 20** (*Taylor theorem*) *Let  $f \in C^n[a, b]$ , such that there exists  $f^{(n+1)}$  on  $[a, b]$  and consider  $x_0 \in [a, b]$ . The Taylor polynomial is*

$$P_n(x) = \sum_{k=0}^n \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) \tag{8}$$

*and we have the approximation formula*

$$f(x) = P_n(x) + R_n(x),$$

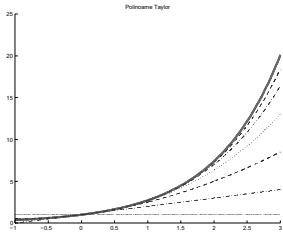
$R_n$  denoting the remainder (the error).

For  $\forall x \in [a, b]$  there exists a number  $\xi$  between  $x_0$  and  $x$  such that

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

**Remark 21** *Taylor polynomials agree as closely as possible with a given function around the specific point  $x_0$ , but not on the entire interval.*

**Example 22** *We calculate the first six Taylor polynomials about  $x_0 = 0$  for  $f(x) = e^x$ .*



Notice that even for the higher-degree polynomials, the error becomes progressively worse as we move away from  $x_0 = 0$ .

**Example 23** *Consider  $f(x) = \frac{1}{x}$  and  $x_0 = 1$ . Approximate the value of  $f(3)$  by Taylor polynomials of various degrees.*

$n$	0	1	2	3	4	5	6	7
$P_n(3)$	1	-1	3	-5	11	-21	43	-85

Taylor polynomial approximation is used when approximations are needed and only at numbers close to  $x_0$ . It is more efficient to use methods that include information at various points.

## 2.2. Lagrange interpolation

Let  $[a, b] \subset \mathbb{R}$ ,  $x_i \in [a, b]$ ,  $i = 0, 1, \dots, m$  such that  $x_i \neq x_j$  for  $i \neq j$  and consider  $f : [a, b] \rightarrow \mathbb{R}$ .

**The Lagrange interpolation problem** (LIP) consists in determining the polynomial  $P$  of the smallest degree for which

$$P(x_i) = f(x_i), \quad i = 0, 1, \dots, m$$

i.e., the polynomial of the smallest degree which passes through the distinct points  $(x_i, f(x_i))$ ,  $i = 0, 1, \dots, m$ .

**Definition 24** A solution of (LIP) is called **Lagrange interpolation polynomial**, denoted by  $L_m f$ .

**Remark 25** We have  $(L_m f)(x_i) = f(x_i)$ ,  $i = 0, 1, \dots, m$ .

$L_m f \in \mathbb{P}_m$  ( $\mathbb{P}_m$  is the space of polynomials of at most  $m$ -th degree).

The Lagrange interpolation polynomial is given by

$$(L_m f)(x) = \sum_{i=0}^m \ell_i(x) f(x_i), \quad (9)$$

where by  $\ell_i(x)$  denote **the Lagrange fundamental interpolation polynomials**.

We have

$$u(x) = \prod_{j=0}^m (x - x_j),$$

$$u_i(x) = \frac{u(x)}{x - x_i} = (x - x_0) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_m) = \prod_{\substack{j=0 \\ j \neq i}}^m (x - x_j)$$

$$\ell_i(x) = \frac{u_i(x)}{u_i'(x_i)} = \frac{(x - x_0) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_m)}{(x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_m)} = \prod_{\substack{j=0 \\ j \neq i}}^m \frac{x - x_j}{x_i - x_j} \quad (10)$$

for  $i = 0, 1, \dots, m$ .

**Proposition 26** We also have

$$\ell_i(x) = \frac{u(x)}{(x - x_i)u'(x_i)}, \quad i = 0, 1, \dots, m. \quad (11)$$

**Proof.** We have  $u_i(x) = \frac{u(x)}{x - x_i}$ , so  $u(x) = u_i(x)(x - x_i)$ . We get  $u'(x) = u_i(x) + (x - x_i)u_i'(x)$ , whence it follows  $u'(x_i) = u_i(x_i)$ . So, as

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)}$$

we get

$$\ell_i(x) = \frac{u_i(x)}{u_i'(x_i)} = \frac{u(x)}{(x - x_i)u'(x_i)}, \quad i = 0, 1, \dots, m. \quad (12)$$

■

**Theorem 27** The operator  $L_m$  is linear.

**Proof.**

$$\begin{aligned} L_m(\alpha f + \beta g)(x) &= \sum_{i=0}^m \ell_i(x)(\alpha f + \beta g)(x_i) = \sum_{i=0}^m [\ell_i(x)\alpha f(x_i) + \ell_i(x)\beta g(x_i)] \\ &= \alpha(L_m f)(x) + \beta(L_m g)(x), \end{aligned}$$

so

$$L_m(\alpha f + \beta g) = \alpha L_m f + \beta L_m g, \quad \forall f, g : [a, b] \rightarrow \mathbb{R} \text{ and } \alpha, \beta \in \mathbb{R}.$$

■