# 1 Set Theory

To demystify mathematics consider

(i) What is a theorem?

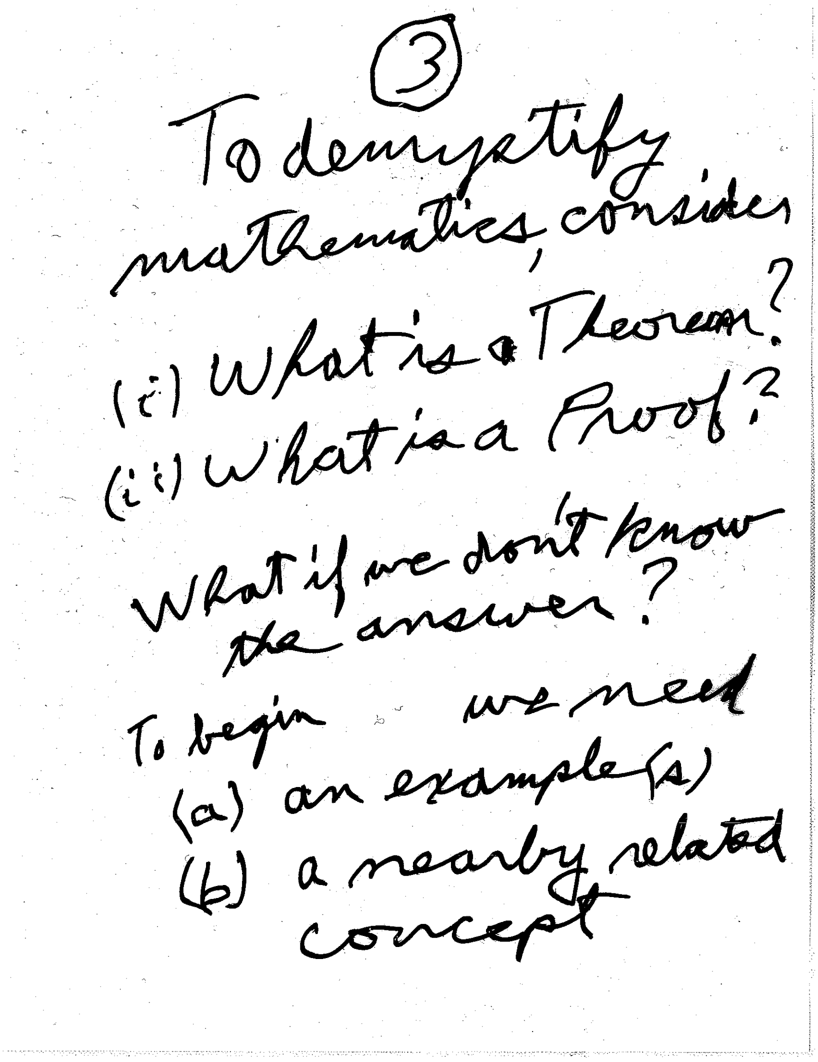
(ii) What is a proof?

What if we don't know the answer?

To begin we need

(a) an example(s)

(b) a nearly related concept



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To demystify mathematics, consider

(i) What is a Theorem?

(ii) What is a Proof?

What if we don't know the answer?

To begin we need

(a) an example(s)

(b) a nearby related concept

Related Concept: Greek Syllogism
<u>example:</u>

1. All men are mortal.

2. Socrates is a man.

3. Therefore, Socrates must die.

To analyze, recast in set theoretic terms via Venn Diagram.

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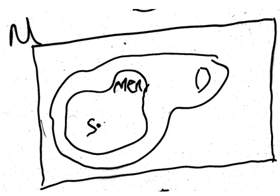
Related Concept:
Greek Syllogism

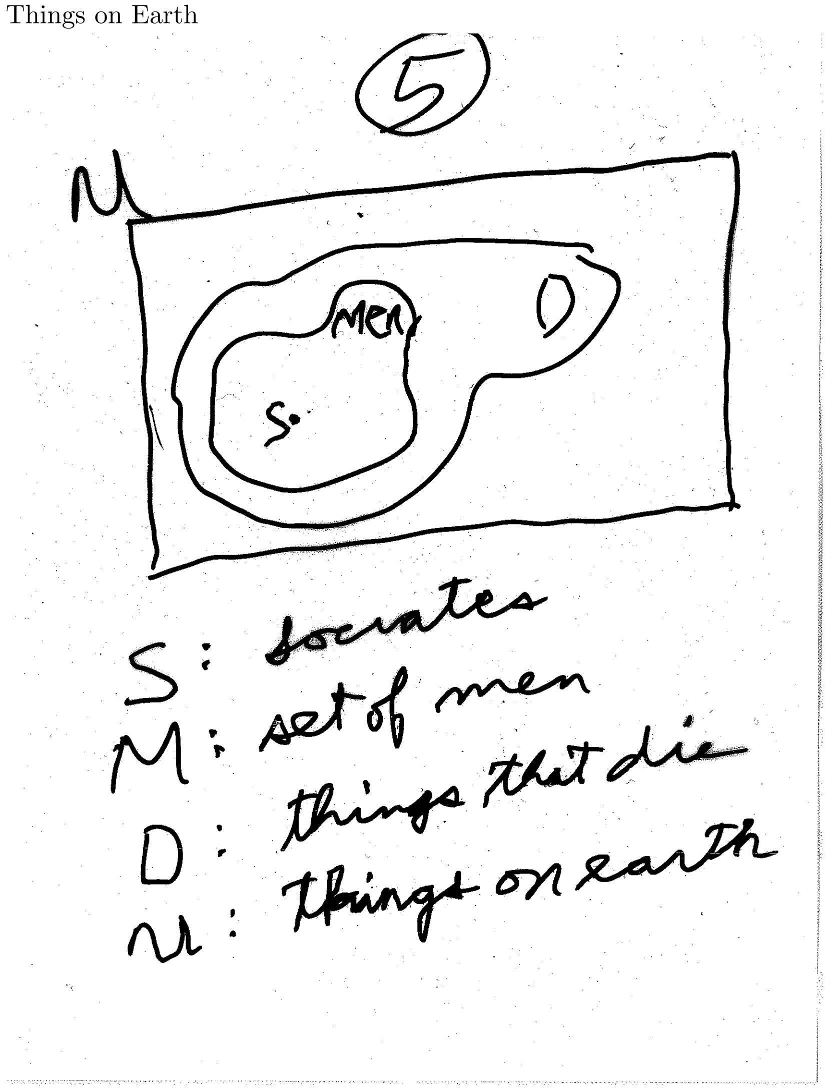
<u>Example:</u>

(1) All men are mortal
(2) Socrates is a man
$\therefore$ (3) Socrates must die

To analyze, recast
in set theoretic terms
via Venn diagram




$S$: Socrates
$M$: Set of Men
$D$: Things that will die
$\mathcal{U}$: Things on Earth



# 2 Generate $\mathbb{N}$

# 3 From $\mathbb{Z}$ to $\mathbb{R}$ via ordering

# 4 Sequence and Limits

Sequences
 Limits
 Constructing the limit via:

(i) Monotonic Sequences

(ii) Monotonic Sub-sequences

Cauchy Sequences
Subsequential Limits

Sequences

seqs
limits
constructing the limit
via (i) monotonic seqs
(ii) monotonic subseqs
Cauchy seqs
subsequential limits

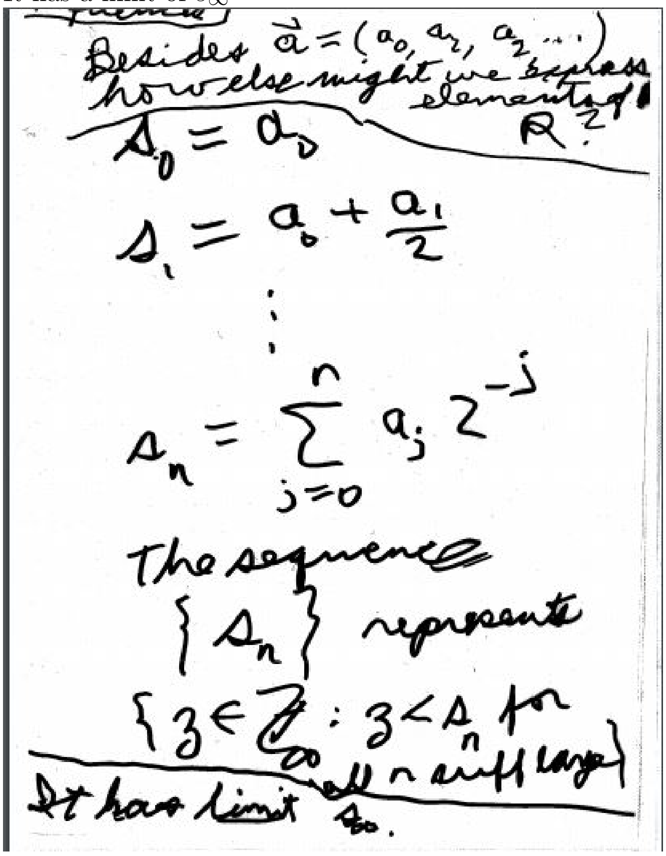
Besides $\vec{a} = (a_0, a_1, a_2, \ldots)$, how else might we harness elements of $\mathbb{R}^2$?

$$s_0 = a_0$$

$$s_1 = a_0 + \frac{a_1}{2}$$

$$s_n = \sum_{j=0}^{n} a_j 2^{-j}$$

The sequence $\{s_n\}$ represents $\{\ z \in \mathbb{Z}_\infty : z < s_n$ for all in sufficient large $\}$
It has a limit of $s_\infty$



Besides $\vec{a} = (a_0, a_1, a_2 \ldots)$ how else might we express elements of $\mathbb{R}^2$?

$$s_0 = a_0$$

$$s_1 = a_0 + \frac{a_1}{2}$$

$$\vdots$$

$$s_n = \sum_{j=0}^{n} a_j 2^{-j}$$

The sequence $\{s_n\}$ represents $\{z \in \mathbb{Z}_\infty : z < s_n$ for all $n$ suff large$\}$

It has limit $s_\infty$.

**Sequences**

Def: a sequence $\{ a_{n1}^{\infty} \}$ is a map from the integers. A real valued sequence is a map into the reale from the integers.

Examples:

$a_n = \frac{1}{n^2}$

$a_n = (-c)^n$

$a_n = \cos(nx)$

$a_n = n^{\frac{1}{n}}$

$a_n = (1 + \frac{1}{n})^n$

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Sequences

Def: a seq $\{a_n\}^{\infty}$ is a map from the integers. a real valued seq is a map into the reals from the integers.

Examples

$a_n = \frac{1}{n^2}$

$a_n = (-1)^n$

$a_n = \cos nx$

$a_n = n^{1/n}$

$a_n = (1 + \frac{1}{n})^n$

**Convergence of Sequences**

Def: $a_n \to a$ if and only if $\forall \epsilon > 0 \exists N$:
for all n $\geq N$, $|a_n - a| < \epsilon$
we write...

$$a = \lim_{n\to\infty} a_n$$

Def: $a_n \to +\infty$ if and only if $\forall b < \infty\ \epsilon N_b$ such that for all $n \geq, N_b, a_n \geq b$
Def: $a_n \to -\infty$ ?
Then (limits are unique)
if $a_n \to a$ and $a_n \to b$
then $a = b$

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(49) Convergence of Seqs

Def: $a_n \to a$ iff
$\forall \varepsilon > 0\ \exists N_\varepsilon$:
for all $n \geq N_\varepsilon$, $|a_n - a| < \varepsilon$
We write $a = \lim_{n\to\infty} a_n$.

Def: $a_n \to +\infty$ iff
$\forall\ b < \infty\ \exists N_b$ s.t.
for all $n \geq N_b$, $a_n \geq b$

Def: $a_n \to -\infty$ iff ?

Thm (Limits are Unique)
If $a_n \to a$ and $a_n \to b$
then $a = b$.

**Convergence in $\mathbb{R}_\infty$ An Elegant Reformation**

Def: Let $\{a_n\}$ be a sequence of reals and $a_\infty \epsilon \mathbb{R} \bigcup \{\pm\infty\}$ .
$\lim_{n\to\infty} a_n = a_\infty$ if and only if...

(i) $\forall$ real $b > a_\infty$

$\exists N_b < \infty$ : for all

$n \geq N_b$, $a_n < b$

and

(ii) $\forall$ real $b < a_\infty$

$\exists N_b < \infty$: for all

$n \geq N_b$, $a_n > b$

you prove

#9A)

Conv in $R_\infty$ : an Elegant Reformulation

Def: Let $\{a_n\}$ be a seq of reals and $a_\infty \in R \cup \{\pm\infty\}$.

$\lim_{n\to\infty} a_n = a_\infty$

iff

(i) $\forall$ real $b > a_\infty$
$\exists N_b < \infty$ : for all
$n \geqslant N_b$, $a_n < b$

and (ii) $\forall$ real $b < a_\infty$
$\exists N_b < \infty$ : for all
$n \geqslant N_b$, $a_n > b$

you prove

**Finding Limits and Proving Convergence**

Example 1: $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2: $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$

Example 3: $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$

Homework: Suppose $a_n \to a > 0$

Prove $\sqrt{a_n} \to \sqrt{a}$

Finding (50) Limits and Proving Convergence

Example 1 $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2 $\lim_{n\to\infty} \frac{3n+1}{7n-4} = ?$

Example 3 $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = ?$

HW Spse $a_n \to a > 0$

Prove: $\sqrt{a_n} \to \sqrt{a}$

# 5 Limit and Convergence

# 6 Infinite Series

A frog is 2 feet from a wall. He makes a succession of jumps toward it, always jumping half his remaining distance to the wall. Hence his finest jump is one foot. Now he is one foot from the wall. So

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Infinite Series

A frog is 2 feet from a wall. He makes a succession of jumps toward it, always jumping half his remaining distance to the wall.
Hence his first jump is one foot.
Now he is one foot from the wall. So

his second jump is $\frac{1}{2}$ feet. He is now $\frac{1}{2}$ feet from the wall so he jumps $\frac{1}{4}$ feet. He makes successive jumps of $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ After n jumps he has moved

$$s_n = \sum_{j=1}^{n} \frac{1}{2^{j-1}}$$

feet.

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his second jump is $\frac{1}{2}$ foot

He is now $\frac{1}{2}$ foot from the wall so he jumps $\frac{1}{4}$ foot.

He makes successive jumps of

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

After n jumps he has moved $s_n = \sum_{j=1}^{n} \frac{1}{2^{j-1}}$ feet and is now

$$\frac{1}{2^{n-1}}$$

feet from the wall so

$$s_n = 2 - \frac{1}{2^{n-1}}$$

If he keeps jumping forever, how far does he go? He moves

$$\sum_{j=1}^{n} \frac{1}{2^{j-1}}$$

feet. This number is at most 2 and yet it exceeds

$$2 - \frac{1}{2^n}$$

for all

$$n \geq 1$$

.

and is now $\frac{1}{2^{n-1}}$ feet from the wall

so $s_n = 2 - \frac{1}{2^{n-1}}$

If he keeps jumping forever, how far does he go?

He moves $\sum_{j=1}^{\infty} \frac{1}{2^{j-1}}$ feet

This number is at most 2 and yet it exceeds $2 - \frac{1}{2^n}$ for all $n \geq 1$.

Hence the sum of this infinites collection of numbers $1, \frac{1}{2}, \frac{1}{4}, ...$ must be two. How can we generalize this? <u>Generalization 1</u>: Genetic Series How large is $1 + r + r^2 + \ . \ . \ .$ If $|r| < |$? Solution: Let $Sn = 1 + r + ... + r^n$ For $r \leq 0$, $S1 \leq S2 \leq ...$ and we expect him $S_n = \sum_{j-0}^{\infty} r^j$

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Hence the sum of this infinite collection of numbers $\{1, \frac{1}{2}, \frac{1}{4}, \cdots\}$ must be two.

How can we generalize this?

<u>Generalization 1</u> Geometric Series

How large is $1 + r + r^2 + \cdots$ if $|r| < 1$?

<u>Soln</u>: Let $S_n = 1 + r + \cdots + r^n$

For $r \geq 0$, $S_1 \leq S_2 \leq \cdots$

and we expect $\lim_{n \to \infty} S_n = \sum_{j=0}^{\infty} r^j$

$S_n = 1 + r + ... + r^n$ is complex in that it has too many terms. how can we simplify it? We need to capitalize on the regularity of the expensive. Notice: $rS_n = r + r^2 + ... + r^{n+1}$, subtracting equals from equal $S_n - rS_n = 1 - r^{n+1}$ so for $r \neq 1, S_n = \frac{1-r^{n+1}}{1-r}$

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$S_n = 1 + r + \dots + r^n$

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Notice:

$rS_n = r + r^2 + \dots + r^{n+1}$

Subtracting equals from equal

$S_n - rS_n = 1 - r^{n+1}$

so for $r \neq 1$, $S_n = \frac{1-r^{n+1}}{1-r}$

$\to \frac{1}{1-r}$

Theorem 2 (The Integral Test)

Suppose $a_1 \geq a_2 \geq \cdots \geq 0$.
Extend $a_n$ to $a(x)$ where $a(x)$ is continuous, $a(n) = a_n$ and $a(x)$ decreases,
Then $\sum_n a_n < \infty$ if and only if $\int_1^\infty a(x)dx < \infty$

Proof

$$\int_1^\infty a(x)dx = \sum_{n=1}^\infty \int_n^{n+1} a(x)dx$$
$$\leq \sum_{n=1}^\infty \int_n^{n+1} a_n dx$$
$$= \sum_{n=1}^\infty a_n$$

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Thm 2 (The Integral Test)

Spse $a_1 \geq a_2 \geq \cdots \geq 0$

Extend $a_n$ to $a(x)$ where $a(x)$ is continuous, $a(n) = a_n$ and $a(x) \searrow$

Then $\sum_n a_n < \infty$ iff $\int_1^\infty a(x)\,dx < \infty$

Pf:

$$\int_1^\infty a(x)dx = \sum_{n=1}^\infty \int_n^{n+1} a(x)dx$$
$$\leq \sum_{n=1}^\infty \int_n^{n+1} a_n dx$$
$$= \sum_{n=1}^\infty a_n$$

Lower-bounding,

$$
\begin{aligned}
\int_1^n a(x)dx &= \sum_{n=2}^{\infty} \int_{n-1}^{n} a(x)dx \\
&\geq \sum_{n=2}^{\infty} \int_{n-1}^{\infty} a_n dx \\
&= \sum_{n=2}^{\infty} a_n
\end{aligned}
$$

Hence, $\sum_{n=2}^{\infty} a_n$ and $\int_1^{\infty} a(x)dx$
converge or diverge together.

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&= \sum_{n=2}^{\infty} a_n
\end{aligned}
$$

Hence $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} a(x)dx$

converge or diverge together

Are $\sum_{j=1}^{\infty} a_j$ and $\sum_{k=1}^{\infty} b_k$ equal, where

$$b_k = \sum_{n_k \le j \le n_{k+1}} a_j$$

and $n_o = 0 < 1 = n_1 < n_2 < \cdots$?

More generally, when do all re-orderings of the terms of a series produce the same sum?

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Are $\sum_{j=1}^{\infty} a_j$ and $\sum_{k=1}^{\infty} b_k$ equal, where $b_k = \sum_{n_k \le j < n_{k+1}} a_j$

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More generally, when do all re-orderings of the terms of a series produce the same sum?

Theorem: Let $a_j \geq 0$
and $F_1 \subseteq F_2 \subseteq \cdots$ with
$\cup_{n=1}^{\infty} F_n = \mathbb{N}$ and $F_n$ is finite.
Then

$$\sum_{j=1}^{\infty} a_j = \lim_{n \to \infty} \sum_{j \in F_n} a_j$$

Proof: Take away

$$s < S_\infty \equiv \sum_{j=1}^{\infty} a_j$$

$\exists N < \infty$ such that for all $n \geq N$, $a_1 + \cdots + a_n > s$

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Thm Let $a_j \geq 0$
and $F_1 \subseteq F_2 \subseteq \cdots$ with
$\bigcup_{n=1}^{\infty} F_n = \mathbb{N}$ and $F_n$ finite.

Then $\sum_{j=1}^{\infty} a_j = \lim_{n \to \infty} \sum_{j \in F_n} a_j$

Pf: Take any
$s < s_\infty \equiv \sum_{j=1}^{\infty} a_j$.
$\exists N < \infty$ s.t. for all
$n \geq N$, $a_1 + \cdots + a_n > s$

Since $\cup_{n=1}^{\infty} F_n = \mathbb{N}$,
$\exists N' \geq N$ such that
$\{1, 2, \cdots, N\} \subseteq F_{N'}$
Hence for all $n \geq N'$

$$\sum_{j \in F_n} a_j \geq \sum_{j=1}^{N} a_j > s$$

Moreover, since $F_n$ is finite,

$$\exists n* \geq max\{j \in F_n\}$$

Hence,$\sum_{j \in F_n} a_j \leq \sum_{j=1}^{n} a_j \leq s_\infty$ TEXT CUT OFF

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Since $\bigcup_{n=1}^{\infty} F_n = \mathbb{N}$,

$\exists N' \geq N$ s.t.

$\{1, 2, \cdots, N\} \subseteq F_{N'}$

Hence for all $n \geq N'$

$$\sum_{j \in F_n} a_j \geq \sum_{j=1}^{N} a_j > s$$

Moreover, since $F_n$ is finite, $\exists n^* \geq \max\{j \in F_n\}$

Hence $\sum_{j \in F_n} a_j \leq \sum_{j=1}^{n^*} a_j \leq s_\infty$

**Summation by Parts**

<u>Theorem</u> Let $A_o = 0$, $A_n = a_1 + \cdots + a_n$
Suppose $\{A_n : n \geq 1\}$ is bounded.
Let $b_1 \geq b_2 \geq \cdots$ with $b_n \to 0$.
Then $\sum_{j=1}^{\infty} a_j b_j$ converges.

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Summation by Parts

<u>Thm</u> Let $A_0 = 0$,
$A_n = a_1 + \cdots + a_n$. Spse
$\{A_n : n \geqslant 1\}$ is bounded.
Let $b_1 \geqslant b_2 \geqslant \cdots$ with
$b_n \to 0$.
Then $\sum_{j=1}^{\infty} a_j b_j$ conv

(for all $n \geq 1$)

Proof: Suppose $|A_n| < A^* < \infty$.
Fix any $\epsilon > 0$. Take any $\delta > 0$ to be chosen later $\exists N$ such that for $n \geq N$
$0 \leq b_n < \delta_\epsilon$
For $n \geq N$ and $p \geq 0$,

$$\sum_{j=n}^{n+p} A_j b_j = \sum_{j=n-1}^{n+p} (A_j - A_{j-1}) b_j$$

$$= \sum_{j=n}^{n+p} A_j b_j - \sum_{j=n-1}^{n+p-1} A_j b_{j+1}$$

28 30
(for all $n \geq 1$)

Pf: Spse $|A_n| \leq A^* < \infty$

Fix any $\varepsilon > 0$. Take any $\delta > 0$ to be chosen later

$\exists N$ s.t. for $n \geq N$

$0 \leq b_n < \delta\varepsilon$.

For $n \geq N$ and $p \geq 0$

$$\sum_{j=n}^{n+p} a_j b_j = \sum_{j=n}^{n+p} (A_j - A_{j-1}) b_j$$

$$= \sum_{j=n}^{n+p} A_j b_j - \sum_{j=n-1}^{n+p-1} A_j b_{j+1}$$

So

$$
|\sum_{j=n}^{n+p} a_j b_j|
$$

$$
= |A_{n+p} b_{n+p} - A_{n-1} b_n + \sum_{j=n}^{n+p-1} A_j (b_j - b_{j+1})|
$$

$$
\leq |A_{n+p}| b_{n+p} + |A_{n-1}| b_n + \sum_{j=n}^{n+p-1} |A_j| |b_j - b_{j+1}|
$$

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So

$$
\left|\sum_{j=n}^{n+p} a_j b_j\right|
$$

$$
= \left| A_{n+p} b_{n+p} - A_{n-1} b_n + \sum_{j=n}^{n+p-1} A_j (b_j - b_{j+1}) \right|
$$

$$
\leq |A_{n+p}| b_{n+p} + |A_{n-1}| b_n + \sum_{j=n}^{n+p-1} |A_j| |b_j - b_{j+1}|
$$

$$
\begin{aligned}
&\leq A^*\delta\epsilon + A^*\delta\epsilon + A^* \sum_{j=n}^{n+p-1} (b_j - b_{j+1}) \\
&\leq 2A^*\delta\epsilon + A^*(b_n - b_{n+p}) \\
&\leq 3A^*\delta\epsilon
\end{aligned}
$$

So let $\delta$ be any real number such that $0 < 3A^*\delta < 1$.
Hence $\sum_{j=1}^{n} a_j b_j$ satisfies the Cauchy criterion.
Therefore, it converges.

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$$
\leq A^*\delta\varepsilon + A^*\delta\varepsilon + A^* \sum_{j=n}^{n+p-1} (b_j - b_{j+1})
$$

$$
\leq 2A^*\delta\varepsilon + A^*(b_n - b_{n+p})
$$

$$
\leq 3A^*\delta\varepsilon
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So let $\delta$ be any real s.t. $0 < 3A^*\delta < 1$. Hence $\sum_{j=1}^{n} a_j b_j$ satisfies the Cauchy criterion. Therefore it converges.

3) Thinking grandly, maybe all of mathematics can be put on a set theoretic foundation.

Let's try to do so.

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Some Set Theory

A set can be defined by

(i) listing its elements

(ii) listing the properties that determine membership in the set.

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3 Thinking grandly maybe all of mathematics can be put on a set theoretic foundation

Let's try to do so,

---

Some Set Theory

a set can be defined by (i) listing its elements (ii) listing the properties that determine membership in the set

Examples

- $\{1, 2, 5\}$
- { cat, bat, dog }
- $\{\{1, 2\}, 5\}$
- { odd primes }
- { positive integers having no odd divisors }

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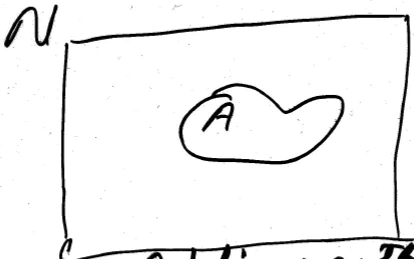
Ex $\{1, 2, 5\}$ {cat, bat, dog}

$\{\{1, 2\}, 5\}$

{odd primes}

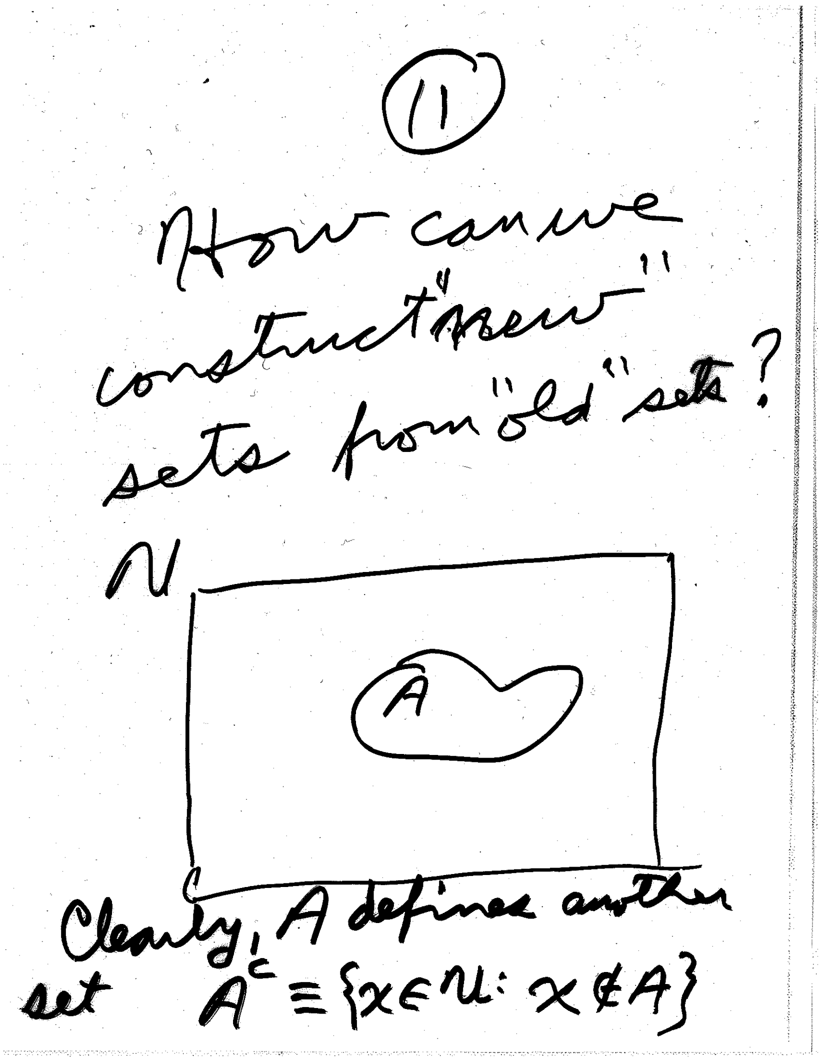
{positive integers having no odd divisors}

How can we construct "new" sets from "old" sets?




Clearly, $A$ defines another set

$$A^c \equiv \{x \in \mathbb{U} : x \notin A\}$$


(1)

How can we construct "new" sets from "old" sets?

Clearly, $A$ defines another set $A^c \equiv \{x \in \mathcal{U} : x \notin A\}$

So: What is a THEOREM?

It always has the form

$$\boxed{\text{If..., then...}}$$

Let

- $A \equiv \{x \in \mathbb{U} : x$ satisfies the conditions in the statement of the theorem $\}$
- $B \equiv \{x \in \mathbb{U} : x$ satisfies the conclusion of the theorem $\}$

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So: What is a THEOREM?

It always has the form:

$$\boxed{\text{If..., then...}}$$

Let $A \equiv \{x \in \mathcal{U} : x$ satisfies the conditions in the statement of the theorem$\}$

$B \equiv \{x \in \mathcal{U} : x$ satisfies the conclusion of the theorem$\}$

Hence, this theorem can be nested as nothing other than $A \subseteq B$.

Hence, a proof is just a logical demonstration:

For each $x \in A$, in fact $x \in B$ also.

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Hence, this theorem can be restated as nothing other than $A \subseteq B$,

Hence, a proof is just a logical demonstration:

For each $x \in A$, in fact $x \in B$ also

It is beyond the scope of this course to formalize how the statement $A \subseteq B$ may be proved. However, to illustrate what is required, it is sufficient to show:

For each $x \in A$, there exists sets

$$D_{x,1} \subseteq D_{x,2} \subseteq D_{x,3} \subseteq \cdots$$

such that $x \in D_{x,1}$ <u>and</u>

$$\bigcup_{j=1}^{\infty} D_{x,j} \subseteq B$$

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It is beyond the scope of this course to formalize how the statement $A \subseteq B$ may be proved. However, to illustrate what is required It suffices to show:

For each $x \in A$ there exist sets

$$D_{x,1} \subseteq D_{x,2} \subseteq D_{x,3} \subseteq \cdots$$

such that $x \in D_{x,1}$

<u>and</u>

$$\bigcup_{j=1}^{\infty} D_{x,j} \subseteq B$$

# 7 Metric Spaces Part 1

# 8 Metric Spaces Part 2