From $\mathbb{Z}$ to

R (reals) via

Orderings

pp 1-19

From $\mathbb{Z}$ to R via Ordering

Having constructed

$$\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$$

we can put a strict ordering on it so that for all $n \in \mathbb{Z}$, $n < n+1$.

With this ordering

$0 \to 1$ in one step

$0 \to 2$ in two successive steps

Similarly for

$n \to n+1$

$n \to n+2$

Suppose we want to propose/construct the existence of a set

$$\mathbb{Z}_{(2)} \not\supseteq \mathbb{Z}$$

which requires two successive steps of some kind to go from

$$0 \longrightarrow 1$$

and $n \longrightarrow n+1$.

What is required?

ZR 3

We need to go from

$0 \longrightarrow x$ in one step

and from $x \rightarrow 1$ in the next

Similarly we go from

$n \longrightarrow n+x$

and $n+x$ to $n+1$.

We need a symbol to specify x. Let $x = \frac{1}{2}$.

Then $\mathbb{Z}_{(2)} = \mathbb{Z} \cup \{n+\frac{1}{2} : n \in \mathbb{Z}$

We can put an

ZR 4

ordering $<_{(2)}$ on $\mathbb{Z}_{(2)}$

that <u>extends</u> the ordering

$<$ on $\mathbb{Z}$.

<u>Problem</u>: Let $(S_i, <_i)$ be two ordered sets such that $S_1 \subseteq S_2$.

<u>Define</u> what it means to say that the ordering $<_2$ extends $<_1$.

ZR ⑤

For $\mathbb{Z}_{(2)}$ there is a unique ordering $<_{(2)}$ such that

$$n <_{(2)} n + \tfrac{1}{2}$$

and $n + \tfrac{1}{2} <_{(2)} n + 1$

Similarly, one can generate the ordered set $(\mathbb{Z}_{(4)}, <_4)$ where $\mathbb{Z}_{(2)} \subsetneq \mathbb{Z}_{(4)}$ and $<_4$ extends $<_2$

ZR 6

Continuing in this way we generate

$$\left(\mathbb{Z}_{(2^k)}, <_{2^k}\right) \text{ for all } k \in \mathbb{N}$$

Then we let

$$\mathbb{Z}_{(\infty)} = \bigcup_{k=1}^{\infty} \mathbb{Z}_{(2^k)}$$

<u>Problem:</u> How do we put an ordering $<_{\infty}$ on $\mathbb{Z}_{(\infty)}$ that extends every $<_{2^k}$?

We can represent

$\mathbb{Z}_{(\infty)}$ in terms of integers j, k, n such that

$$\mathbb{Z}_{(\infty)} = \mathbb{Z} \cup \left\{ n + \frac{j}{2^k} : k \geq 1 \text{ and } 1 \leq j < 2^k \right\}$$

What does the ordering look like?

Suppose we want to construct a version of $\mathbb{Z}_{(\infty)}$ that exhibits the ordering.

For $x \in Z_{(\infty)}$ let

$$\mathcal{L}_x \equiv \{ y \in Z_{(\infty)} : y <_\infty x \}$$

Now $\mathcal{L}_x$ denotes the dyadic rational x.

Notice that for any x, y in $Z_{(\infty)}$,

$$\mathcal{L}_x \cup \mathcal{L}_y = \mathcal{L}_w \text{ for } w = x \text{ or } w = y.$$

ZR 9

Hence for any $n \geq 1$

and $x_1, \cdots, x_n \in \mathbb{Z}_\infty$

$$\bigcup_{j=1}^{n} \mathcal{L}_{x_j} = \mathcal{L}_w$$

for $w \in \bigcup_{j=1}^{n} \{x_j\}$.

Hence, finite unions of dyadic rationals are dyadic rationals.

Suppose we ~~define~~ want to think of arbitrary unions of $\{\mathcal{L}_x : x \in \mathbb{Z}_{(\infty)}\}$ as bona fide numbers.

ZR 10

Let

$$\mathbb{R}_{(\infty)} \equiv \left\{ \bigcup_{x \in A} \mathcal{L}_x : A \subseteq \mathbb{Z}_{(\infty)} \right\}$$

Letting $A = \emptyset$, $\emptyset \in \mathbb{R}_{(\infty)}$

Letting $A = \mathbb{N}$, $\mathbb{Z}_{(\infty)} \in \mathbb{R}_{(\infty)}$

Let

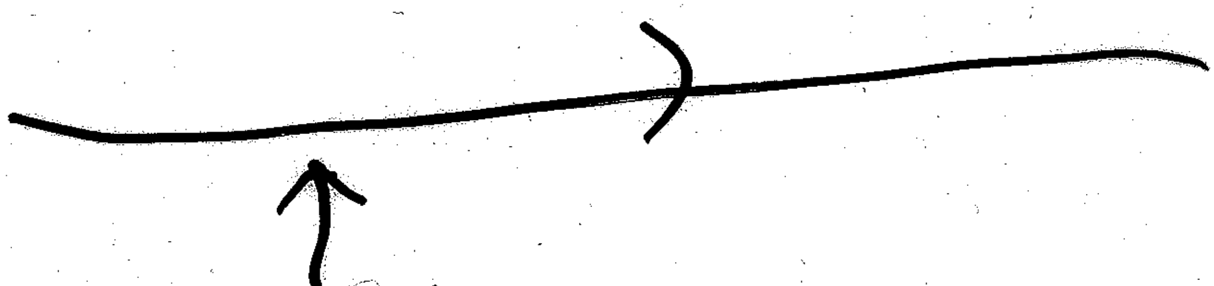
$$\mathbb{R} \equiv \left\{ B \in \mathbb{R}_{\infty} : \begin{array}{c} B \neq \emptyset \\ \text{and} \\ B \neq \mathbb{Z}_{(\infty)} \end{array} \right\}$$

$\mathbb{R}$ will represent the <u>real</u> <u>numbers</u>

ZR 11

What do the elements of R look like?

On the number line they look like



This set of dyadic rationals

How can we characterize them?

ZR 12

Thm $B \in R$ iff

(i) $\phi \subsetneq B \subsetneq \mathbb{Z}_{(\infty)}$

(ii) If $b \in B$ and $a \in \mathbb{Z}_{(\infty)}$ with $a <_\infty b$ then $a \in B$

(iii) If $b \in B$ $\exists\, b' \in B$ s.t. $b <_\infty b'$

ZR (13)

($\Rightarrow$)

Pf: Spse $B \in \mathcal{R}$.

$\exists$ non-empty $A \subseteq \mathbb{Z}_{(\infty)}$

s.t. $B = \bigcup_{a \in A} \mathcal{L}_a$

We are given that

$B \neq \emptyset$ & $B \neq \mathbb{Z}_{(\infty)}$.

(ii) Spse $b \in B$.

and $b_0 \leq_\infty b$ with $b_0 \in \mathbb{Z}_{(\infty)}$

$\exists a \in A : b \in \mathcal{L}_a$

Hence $b_0 \in \mathcal{L}_a$

so $b \in B$

(iii) holds for $\longrightarrow \mathcal{L}_a$ so [illegible]

ZR (14)

($\Leftarrow$) Conversely,

Spse B satisfies (i), (ii), (iii)

Let $A = B$

$\phi \subsetneqq A \subsetneqq \mathbb{Z}_{(\infty)}$

Claim $B = \bigcup_{a \in A} L_a$

Clearly $\bigcup_{a \in A} L_a \subseteq B$. (Why?)

Take any $b \in B$.

$\exists\, b' >_\infty b$ s.t. $b' \in B$

$\therefore b \in L_{b'}$ so

$B \subseteq \bigcup_{a \in A} L_a$ qed