L4A

$\mathbb{N} = \{1, 2, 3, \dots\}$ as produced via Peano Axioms

How do the set theorists generate $\mathbb{N}$?

Hint: Use ϕ

Is there any inherent/ essential/intrinsic difference between
{cat, hat, dog, blue} and $\{1, 2, 3, 4\}$?

I4B

ANS. One set has an implicit ordering induced by Peano axioms.

Def: $<$ is a (strict) ordering on S $(S \neq \phi)$

iff $\forall$ a, b, c in S

(i) Exactly one of

$a < b$

$b < a$

$a = b$ is true

(ii) If $a < b$ and $b < c$

then $a < c$

This is an axiomatic presentation

L4C

How could this notion be achieved set theoretically?
Note: it involves pairs of elements of S.

<u>1st step</u>

need $<$ to be a subset G of $S \times S = \{(s_1, s_2) : s_1, s_2 \in S\}$

For all a, b, c in S

(i) $(a, b) \in G$ iff $(b, a) \notin G$

(ii) if (a, b) and $(b, c) \in G$ then $(a, c) \in G$

£4D

<u>Step 2</u> Sets are Unordered.
Present A x B (and so S x S)
set theoretically:
(a, b) can be re-expressed
so that its order on
this surface does
not affect its meaning.
How? For example
consider

$\{\{a,1\},\{b,2\}\}$ instead of (a, b)

or even $\{a, \{a, b\}\}$
to denote a typical
element of A x B.

£4E

Most importantly, the set S can be reconstructed to exhibit its order. For example, ~~that~~ given $(S, <)$ and $s \in S$ let

$$\mathcal{L}_s = \{a \in S : a < s\}$$

Then let $\mathcal{S} = \{\mathcal{L}_s : s \in S\}$.

$(\mathcal{S}, \subset)$ represents $(S, <)$

Can you Find another?

L4F

Given $\{1, 2, 3, \dots\}$
what do we do next?

Ans: Fiddle with what we have.

What is the 2^{nd} successor
of 1? of 2? etc.
What is the k^{th} successor
of n?

These questions lead
to the discovery of the
operation of addition
on $\mathbb{N}$. Since this
operation is one-to-one it can (often) be
inverted. Hence

L4G

what is the $k^{\underline{th}}$ predecessor of n?

Currently, this question can be answered iff $k < n$.

We want to be able to answer it for all n and all k. How?

Necessarily we need numbers

L4H

equal to

1-1

1-2

1-3, etc.

We can get these axiomatically (and hence set theoretically)

We call them

0, -1, -2, etc

We get

$$\mathbb{Z} \equiv \mathbb{N} \cup \bigcup_{n \in \mathbb{N}} \{-n\}$$

$$\equiv \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$$

Addition and subtraction hold for pairs of elements of $\mathbb{Z}$

L4I

And we can extend our ordering to $\mathbb{Z}$:

$n < m$ iff $n - m < 0$.
iff $m - n > 0$

Now what? ~~Is~~ there any other number?

Could there be a numerical creature with step lengths so small he/she/it required 2 steps to go from 0 to 1?