# Sequences

seqs
limits
constructing the limit
via (i) monotonic seqs
(ii) monotonic subseqs
Cauchy seqs
subsequential limits

premise

Besides $\vec{a} = (a_0, a_1, a_2, \ldots)$ how else might we express elements of $R$?

$$s_0 = a_0$$

$$s_1 = a_0 + \frac{a_1}{2}$$

$$\vdots$$

$$s_n = \sum_{j=0}^{n} a_j 2^{-j}$$

The sequence $\{s_n\}$ represents

$\{z \in \mathbb{Z}_\infty : z < s_n$ for all $n$ suff large$\}$

It has limit $s_\infty$.

# Sequences

Def a seq $\{a_n\}_1^\infty$ is a map from the integers. a real-valued seq is a map into the reals from the integers.

## Examples

$$a_n = \frac{1}{n^2}$$

$$a_n = (-1)^n$$

$$a_n = \cos nx$$

$$a_n = n^{1/n}$$

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

# (49) Convergence of seqs

**Def:** $a_n \to a$ iff

$\forall \varepsilon > 0 \ \exists N_\varepsilon :$

$|a_n - a| < \varepsilon$ for all $n \geq N_\varepsilon$,

We write $a = \lim_{n\to\infty} a_n$.

**Def:** $a_n \to +\infty$ iff

$\forall b < \infty \ \exists N_b$ s.t.

$a_n \geq b$ for all $n \geq N_b$,

**Def:** $a_n \to -\infty$ iff ?

**Thm** (Limits are Unique)

If $a_n \to a$ and $a_n \to b$

then $a = b$.

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# Conv in $R_\infty$: an elegant reformulation

<u>Def</u>: Let $\{a_n\}$ be a seq of reals and $a_\infty \in \mathbb{R} \cup \{\pm\infty\}$.

$$\lim_{n\to\infty} a_n = a_\infty$$

iff

(i) $\forall$ real $b > a_\infty$ $\exists N_b < \infty$ : for all $n \geq N_b$, $a_n < b$

and (ii) $\forall$ real $b < a_\infty$ $\exists N_b < \infty$ : for all $n \geq N_b$, $a_n > b$

you prove

# Finding 50 Limits and Proving Convergence

**Example 1** $\lim_{n\to\infty} \frac{1}{n^2} = 0$

**Example 2** $\lim_{n\to\infty} \frac{3n+1}{7n-4} = ?$

**Example 3** $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = ?$

HW Spse $a_n \to a > 0$
Prove: $\sqrt{a_n} \to \sqrt{a}$

<u>Thm</u> Spse $a_n \to a$ and $b_n \to b$. Then

$a_n + b_n \to a + b$

(and $\lambda a_n \to \lambda a$)

and $a_n b_n \to a \cdot b$

<u>Thm</u> If $a_n \to a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$

<u>Cor</u> If $a_n \to a$ and $b_n \to b \neq 0$ then

$\frac{a_n}{b_n} \to \frac{a}{b}$

5th Method (5.3) of Proving Conv:

Monotone Sequences

Let $\{a_n\}$ be a seq of reals.

We say $\{a_n\}$ is monotonic

if either

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$$

or else

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots$$

In the first case $\{a_n\}$ is said to be (monotone) non-decreasing and in the second case it is (monotone) non-increasing.

Thm Spse $\{a_n\}$ is monotonic and bounded. Then it converges in $\mathbb{R}$.

**Thm** Let $\{a_n\}$ be monot in $\mathbb{R}_\infty$ in $\mathbb{R}$. Then

$$\lim_{n\to\infty} a_n = \begin{cases} \sup\{a_k\} & \text{if } a_1 \le a_2 \le a_3 \le \cdots \\ \inf\{a_k\} & \text{if } a_1 \ge a_2 \ge \cdots \end{cases}$$

**Pf:** W.l.o.g. spse $a_1 \le a_2 \le \cdots$

Take any $b < \sup\{a_k\}$

Then $\exists\, k_* < \infty$ s.t.

$b < a_{k_*}$. For all

$n \ge k_*$, $a_n \ge a_{k_*}$ so

$a_n > b$.

But also $a_n \le \sup\{a_k\}$

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**Thm** Let $\{a_n\}$ be any seq in $\mathbb{R}$

Then $\exists\ 1 \le n_1 < n_2 < \cdots$ s.t.

$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is monot.

(i.e. $\{a_n\}$ has a monotonic subsequence)

Pf: Either $a_1 \le a_n$ for infinitely many $n > 1$ or $a_1 \ge a_n$ for infinitely many $n > 1$.

Applying this question to each $a_j$:

Let $J = \{j \ge 1 : a_n \ge a_j$ for infinitely many $n > j\}$

**Case 1** $|J| = \infty$

Let $n_1$ = smallest $j \in J$

$\exists\ n_2 > n_1$ with $n_2 \in J, \ldots$ etc get $\{n_k\}$

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and $a_{n_1} \le a_{n_2} \le \cdots$

Case 2 $|J| < \infty$

one and

Take any $n_1 > \max_{j \in J} j$

By construction, ~~if~~

$n_2$ is large enough

$n_2 > n_1$ and $a_{n_2} < a_{n_1}$

By induction, having constructed $n_1 < n_2 < \cdots < n_k$

s.t. $a_{n_1} > a_{n_2} > \cdots > a_{n_k}$

$\exists\, n_{k+1} > n_k$ s.t.

$a_{n_{k+1}} < a_{n_k}$

and so the Thm holds