x

1 Set Theory

To demystify mathematics consider

(i) What is a theorem?

(ii) What is a proof?

What if we don't know the answer?
To begin we need

(a) an example(s)

(b) a nearly related concept

3

To demystify mathematics, consider

(i) What is a Theorem?

(ii) What is a Proof?

What if we don't know the answer?

To begin we need

(a) an example(s)

(b) a nearby related concept

Related Concept: Greek Syllogism
<u>example:</u>

1. All men are mortal.
2. Socrates is a man.
3. Therefore, Socrates must die.

To analyze, recast in set theoretic terms via Venn Diagram.

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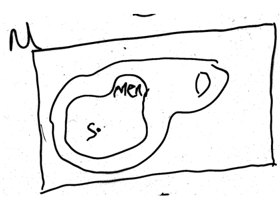
Related Concept:
Greek Syllogism

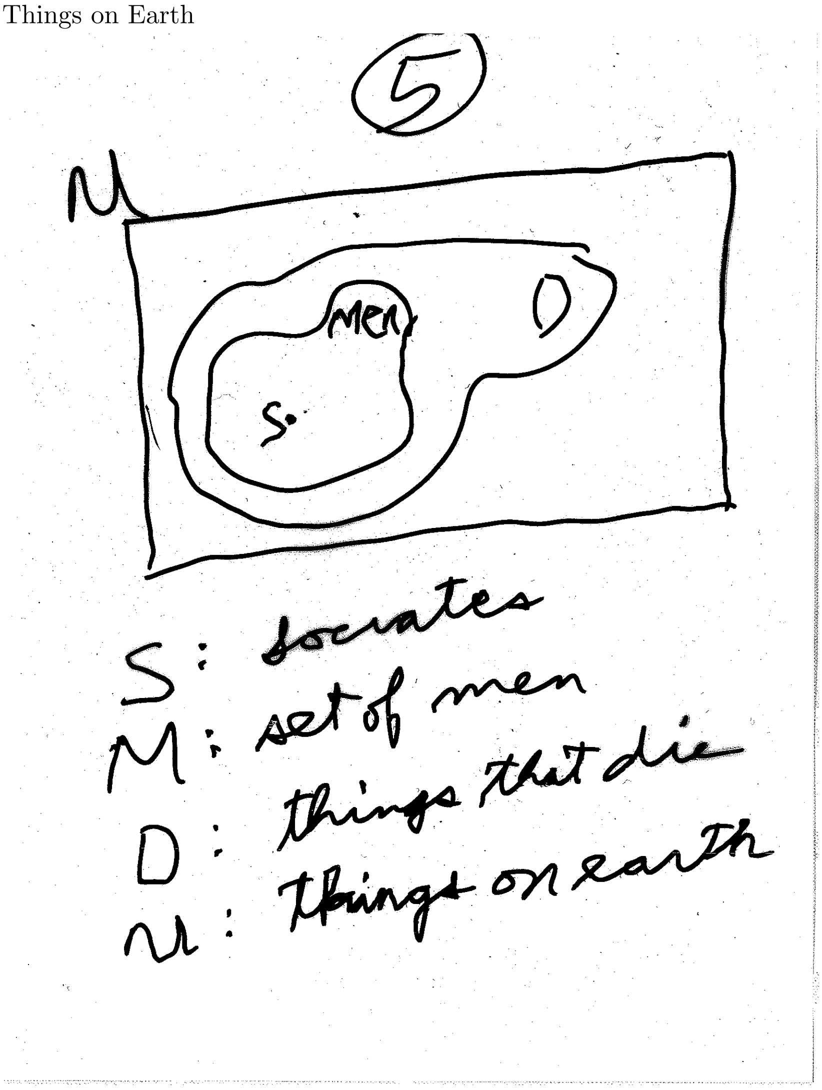
Example:

(1) All men are mortal
(2) Socrates is a man
$\therefore$ (3) Socrates must die

To analyze, recast
in set theoretic terms
via Venn diagram



S: Socrates
M: Set of Men
D: Things that will die
$\mathcal{U}$: Things on Earth



¿¿¿¿¿¿¿ origin/master

Notice: proper noun Socrates became an $\frac{\textit{element}}{(\textit{point})}$

common noun <u>men</u> became a <u>set</u> (collection of pts)

<u>the property</u> of being mortal became a <u>set</u> the universe of all things under consider became the Universe of Possibilities

6

Notice:

proper noun Socrates
became an element
(point)

common noun men
became a set
(collection of pts)

the property of being
mortal became a set

the universe of all
things under consider
became the Universe
of Possibilities

The set theoretic representation of the syllogism:
(1)$M \leq D$
(2)SεM
(3)SεD
Rearranging, we obtain a more logical ordering of the facts (doing so by successive inclusion):
SεM, $M \leq D$; SεD

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The set theoretic representation of the syllogism:

(1) $M \subseteq D$
(2) $S \in M$
$\therefore$ (3) $S \in D$

Rearranging, we obtain a more logical ordering of the facts (doing so by successive inclusion):

$S \in M, M \subseteq D; \therefore S \in D$

What has the syllogism taught us?

1. Games of fact (Truth on falsehood) can be put in a set theoretic context.

2. The simplest deductive argument has the form if $X \varepsilon$A and $A \leq B$ then $X \varepsilon$ B

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What has the syllogism taught us?

1 Issues of fact (truth or falsehood) can be put in a set-theoretic context.

2 The simplest deductive argument has the form if $x \in A$ and $A \subseteq B$ then $x \in B$

21. It is beyond the scope of this course to formulize how the statement A = B may be proved. However, to illustrate what is required it suffices to show these exist sets $D_{x,1}/subseteqD_{x2}, \subseteq D_{x3}, CsuchthatxED_x, and\infty, D_x,' s \subseteq B$

9. Thinking grandly, maybe all of mathematics can be put on a set theoretic foundation. Let's try to do so. ——————————————— Some Set Theory

A set can be defined by (i) listing its elements (ii) listing the properties that determine membershing in the set.

Ex $1, 2, 5 cat, hat, dog 1, 2, 5 odd primes positive integers have no odd divisions$

11. How can we construct ”new: sets from ”old” sets?
Clearly, A defines another set Ac = x E u: x /notin A

13

Spse we require that these 3 operations c, $\cup$, $\cap$ always produce sets.

Then

$A \cap A^c \equiv \{x \in U : x \in A$ and $x \notin A\}$ must be a set in U.

It is called <u>the empty set</u> ϕ. The set with no elements.

13. Suppose we require that these 3 operations $^C, \bigcup, \bigcap$ always produce sets. Then $A \bigcap A^C \equiv \{x \in u : x \in A$ and $x \notin$ must be a set in U. It is called <u>the empty set</u> $\emptyset$. The set with no elements.

13A ?

1) $B \cap \left(\bigcup_{\alpha \in J} A_\alpha \right) =$

2) $B \cup \left(\bigcap_{\alpha \in J} A_\alpha \right) \stackrel{?}{=}$

3) $\left(\bigcap_{\alpha \in J} A_\alpha \right)^c \stackrel{?}{=}$

4) $\left(\bigcup_{\alpha \in J} A_\alpha \right)^c \stackrel{?}{=}$

13A.

1.
$$B \bigcap (\bigcup_{\alpha \in J} A_\alpha) \stackrel{?}{=}$$

2.
$$B \bigcup (\bigcap_{\alpha \in J} A_\alpha) \stackrel{?}{=}$$

3.
$$(\bigcap_{\alpha \in J} A_\alpha)^c \stackrel{?}{=}$$

4.
$$(\bigcup_{\alpha \in J} A_\alpha)^c \stackrel{?}{=}$$

22

What do we do next?
Begin constructing sets,

{ apple }, { pear }, { grape }
{ eagle }, { bear, deer },
{ pencil, paper }
{ sun, moon, orion }...
These examples motivate the need to

22. What do we do next? Begin constructing sets, { apple } , { pen }, { grape }, { eagle }, { bear, deer }, { pencil, paper }, { sun, moon, onion. } These examples motivate the need to

22A

What next?
Spse we try
to generate all(?) ~~the simple~~
sets from
simple to complex

Examples
{apple} {pear}
{cat} {dog, hat}
{atoms in your body}
What makes these sets
most similar?

22A. What next? Suppose we try to generate all sets from simple to complex Examples { apple } { pear } { cat } { dog,hat } { atoms in your body } What makes these sets most similar?

(22B)

Notice:

{apple} and {pear}

are merely relabelings
of one another.

As are

{dog, hat} and {cat oval}

What is a relabeling
of a set?

Perhaps Sets A and B
have the same 'size'
if they are relabelings
of one another.

We seem to be
dancing around the concept
of number.

22B. Notice { apple } and { pear } are merely relating of one another as one { dog, hat } and { cat, oval } What is a relabeling of a set? Perhaps sets A and B have the same 'size' if they are relabeling of one another. We seem to be dancing around the concept of number.

2 Generate $\mathbb{N}$

L4A Generate $\mathbb{N}$

$\mathbb{N}$= { 1, 2, 3,.. } as produced via Peono Axions. How do the set theorists generate $\mathbb{N}$ Hint: Use ϕ Is there any inherent/essential/intrinsic difference between {cat, hat, dog, blue} and {1, 2, 3, 4}?

L4A

$\mathbb{N} = \{1, 2, 3, \cdots\}$ as produced via Peano Axioms

How do the set theorists generate $\mathbb{N}$?

Hint: Use ϕ

Is there any inherent/essential/intrinsic difference between $\{$cat, hat, dog, blue$\}$ and $\{1, 2, 3, 4\}$?

L4B Generate $\mathbb{N}$

Ans.One set has an implicit ordering induced by Peono Axions. Def: $<$is a (strict) ordering on S (S $\neq \emptyset$) if $\forall$ a, b, c in S

(i) Exactly one of

a $<$ b

b $<$ a

a $=$ b is true

(ii) If a $<$ b and b $<$ c

then a $<$ c This is an axiomatic presentation

L4B

Ans. One set has an implicit ordering induced by Peano Axioms.

Def: $<$ is a (strict) ordering on S $(S \neq \phi)$ iff $\forall$ a, b, c in S

(i) Exactly one of

a $<$ b

b $<$ a

a $=$ b is true

(ii) If $a < b$ and $b < c$ then $a < c$

This is an axiomatic presentation

L4C Generate ℕ

How could this notion be achieved set theoretically? Note: It involves pairs of elements of S.

1st step Need $<$ to be a subset 6 of SxS = (S 1 S 2) For all a, b, c in S

(i) (a, b) ϵ 6 if (b, a) 6 $\nexists$

(ii) if (a, b) and (b, c) ϵ C then (a, c) ϵ C

L4C

How could this notion be achieved set theoretically? Note: It involves pairs of elements of S.

1st step

Need $<$ to be a subset G of $S \times S = \{(s_1, s_2) : s_1, s_2 \in S\}$

For all a, b, c in S

(i) $(a, b) \in G$ iff $(b, a) \notin G$

(ii) if (a, b) and $(b, c) \in G$ then $(a, c) \in G$

L4D Generate ℕ

Step 2 Sets are Undefined Present $A \times B$ (and so $S \times S$) set theoretically: (a, B) can be re-exposed so that its order on this surface does not affect its meaning How? for example consider { { a, 1 }, { b, 2 } } instead of (a, b) or even { a ${}_2$ {a, b} } to denote a typical element of AxB

L4D

<u>Step 2</u> Sets are Unordered,
Present $A \times B$ (and so $S \times S$)
set theoretically:
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How? For example
consider
$\{\{a,1\},\{b,2\}\}$ instead
of (a, b)

or even $\{a, \{a,b\}\}$
to denote a typical
element of $A \times B$.

L4E Generate $\mathbb{N}$ Most importantly the set S can be reconstructed to exhibit its order For example give $(S, <)$ and $s \in S$ let $Ls = \{ a \in S; a < s \}$ Then let $S = \{ Ls : s \in S \}$ (S, c) represents $(S, <)$ Can you find another?

L4E

Most importantly, the set S can be reconstructed to exhibit its order. For example, ~~let~~ given $(S, <)$ and $s \in S$ let

$$\mathcal{L}_s = \{ a \in S : a < s \}$$

Then let $\mathcal{S} = \{ \mathcal{L}_s : s \in S \}$. $(\mathcal{S}, \subset)$ represents $(S, <)$

Can you find another?

L4F Generate $\mathbb{N}$

Given { 1, 2, 3,... } what do we do next? Ans: Fiddle with what we have. What is the 2nd successor of 1? 2? 3? etc. What is the 6th successor of n? These questions lead to the discovery of the operation of addition on $\mathbb{N}$. Since this operation is one-to-one it can (often) be inverted. Hence

L4F

Given $\{1, 2, 3, \dots\}$ what do we do next?

Ans: Fiddle with what we have.

What is the 2nd successor of 1? of 2? etc.

What is the kth successor of n?

These questions lead to the discovery of the operation of addition on $\mathbb{N}$. Since this operation is one-to-one it can (often) be inverted. Hence

L4G Generate ℕ

What is the 6th predecessor of n? Currently, this question can be answered if $k < n$. We want to be able to answer it for all n and How? Necessarily we need numbers

L4G

what is the k^{th} predecessor of n?

Currently, this question can be answered iff $k < n$.

We want to be able to answer it for all n and all k.

How?

Necessarily we need numbers

3 From $\mathbb{Z}$ to $\mathbb{R}$ via ordering

From Z to R via Orderings

$\mathbb{Z}$ = ...,-1,0,1,...
We can put a strict ordering on it so that for all $n \in \mathbb{Z}$, $n < n + 1$.

With this ordering
$0 \to 1$ in one step
$0 \to 2$ in two successive step

Simplify for
$n \to n + 1$
$n \to n + 2$

ZR (i)

From Z to R via Orderings

Having constructed

$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$

we can put a strict ordering on it so that for all $n \in \mathbb{Z}$, $n < n+1$.

With this ordering
$0 \longrightarrow 1$ in one step
$0 \longrightarrow 2$ in two successive steps

Similarly for
$n \longrightarrow n+1$
$n \longrightarrow n+2$

Suppose we want to propose/construct the existence of a set
$\mathbb{Z}_{(2)} \neq \mathbb{Z}$
which requires two successive steps of some kind to go from
$0 \to 1$
and $n \to n+1$.
What is required?

ZK(2)

Suppose we want
to propose/construct
the existence of a set
$\mathbb{Z}_{(2)} \neq \mathbb{Z}$
which requires two
successive steps
of some kind to go from
$0 \to 1$
and $n \to n+1$.
What is required?

We need to go from
$0 \to x$ in one step and from $x \to 1$ in the next
Similarly we go from $n \to n + x$ and $n + x$ to $n + 1$.
We need a symbol to specify x. Let $x = \frac{1}{2}$.
Then $\mathbb{Z}_{(2)} = \mathbb{Z} \bigcup \{ \in + \frac{1}{2} : n \in Z$
We can put an...

ZR 3

We need to go from

$0 \longrightarrow x$ in one step

and from $x \rightarrow 1$ in the next

Similarly we go from

$n \longrightarrow n+x$

and $n+x$ to $n+1$.

We need a symbol
to specify x. Let $x = \frac{1}{2}$.

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We can put an

{

...ordering $<_{(2)}$ on $Z_{(2)}$ that <u>extends</u> the ordering $<$ on Z.
<u>Problem:</u> Let $(\mathcal{S};, <;)$ be two ordered sets such that $\mathcal{S}, \subseteq \mathcal{S}_2$.
<u>Define</u> what it means to say that the ordering $<_2$ extends $<,$.

ZR 4

ordering $<_{(2)}$ on $Z_{(2)}$

that <u>extends</u> the ordering

$<$ on Z.

<u>Problem</u>: Let $(\mathcal{S}_i, <_i)$

be two ordered sets

such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

<u>Define</u> what it means

to say that the ordering

$<_2$ extends $<_1$.

For $Z_{(2)}$ there is a unique ordering $<_{(2)}$ such that
$n <_{(2)} n + \frac{1}{2}$ and $n + \frac{1}{2} <_{(2)} n + 1$
Similarly, we can generate the ordered set $(Z_{(4)}, <_4)$ where $Z_{(2)}$ sign $Z_{(4)}$ and $<_4$ extends $<_2$

ZR 5

For $\mathbb{Z}_{(2)}$ there is a unique ordering $<_{(2)}$ such that

$$n <_{(2)} n+\frac{1}{2}$$

and $n+\frac{1}{2} <_{(2)} n+1$

Similarly, one can generate the ordered set $(\mathbb{Z}_{(4)}, <_4)$ where $\mathbb{Z}_{(2)} \subsetneq \mathbb{Z}_{(4)}$ and $<_4$ extends $<_2$

ZR6
Continuing in this way we generate
$(\mathbb{Z}_{2^k}, < 2_{2^k})$ For all $k \in \mathbb{N}$
Then we get

$$\mathbb{Z}_\infty = \bigcup_{k=1}^{\infty} Z_{(2k)}$$

Problem: How do we put an ordering $<_\infty$ on $Z_{(\infty)}$
That extends every $<_{2k}$?

ZR 6

Continuing in this way we generate

$$\left(\mathbb{Z}_{(2^k)}, <_{2^k}\right) \text{ for all } k \in \mathbb{N}$$

Then we let

$$\mathbb{Z}_{(\infty)} = \bigcup_{k=1}^{\infty} \mathbb{Z}_{(2^k)}$$

<u>Problem:</u> How do we put an ordering $<_\infty$ on $\mathbb{Z}_{(\infty)}$ that extends every $<_{2^k}$?

ZR7

May 23, 2016

We can represent $\mathbb{Z}$ in terms of integers j, k, n and that $\mathbb{Z}_\infty = \mathbb{Z} \bigcup \left\{ n + \frac{j}{2k} = k \geqslant 1 and \lneqq j < z^k \right\}$
What does the ordering look like?
Suppose we want to construct a version of $\mathbb{Z}_{(\infty)}$ that exhibits.

$ZR(Z)$

We can represent $\mathbb{Z}_{(\infty)}$ in terms of integers j, k, n such that

$$\mathbb{Z}_{(\infty)} = \mathbb{Z} \cup \left\{ n + \frac{j}{2^k} : k \geqslant 1 \text{ and } 1 \leqslant j < 2^{k} \right\}$$

What does the ordering look like?

Suppose we want to construct a version of $\mathbb{Z}_{(\infty)}$ that exhibits

ZR8

The ordering. for $x \in \mathbb{Z}_\infty$ let

$L_x = \{y \in \mathbb{Z} : y <_\infty x\}$

Now L_x denotes the dyadic rational x.

notice that for any

x, y in $\mathbb{Z}_\infty$,

$L_x \bigcup L_y = L_w$ for

$w = x$ or $w = 0$

ZR8

the ordering.

For $x \in \mathbb{Z}_{(\infty)}$ let

$\mathcal{L}_x \equiv \{y \in \mathbb{Z}_{(\infty)} : y <_\infty x\}$

Now $\mathcal{L}_x$ denotes the dyadic rational x.

Notice that for any

x, y in $\mathbb{Z}_{(\infty)}$,

$\mathcal{L}_x \cup \mathcal{L}_y = \mathcal{L}_w$ for

$w = x$ or $w = y$.

ZR9

Hence for any $n \geqslant 1$ and $x, ..., x_n \in \mathbb{Z}_\infty$

$\bigcup_{j=1}^{n} L_{x_j} = L_w$

For w $\in \bigcup_{j=1}^{n} \{x_j\}$

Hence, finite unions of dyadic rationals one dyadic rationals suppose we want to think of arbitrary unions of $Lx = x \in \mathbb{Z}_\infty$ as bone fide numbers.

ZR9

Hence for any $n \geqslant 1$ and $x_1, \dots, x_n \in \mathbb{Z}_\infty$

$$\bigcup_{j=1}^{n} \mathcal{L}_{x_j} = \mathcal{L}_w$$

for $w \in \bigcup_{j=1}^{n} \{x_j\}$.

Hence, finite unions of dyadic rationals are dyadic rationals. Suppose we want to think of arbitrary unions of $\{\mathcal{L}_x : x \in \mathbb{Z}_{(\infty)}\}$ as bona fide numbers.

ZR10
LET
$R_\infty = \left\{\bigcup_{X\in A} L_x = a \subseteq \mathbb{Z}_\infty\right\}$
Letting $A = \emptyset, \emptyset \in R_\infty$
Letting $A = \mathbb{N}, \mathbb{Z}_\infty \in R_\infty$
Let $R = \left\{B \in R_\infty : B \neq \emptyset \text{ and } B \neq \mathbb{Z}_{(\infty)}\right\}$

ZR 10

Let

$$R_{(\infty)} \equiv \left\{ \bigcup_{x \in A} \mathcal{L}_x : A \subseteq \mathbb{Z}_{(\infty)} \right\}$$

Letting $A = \Phi$, $\Phi \in R_{(\infty)}$

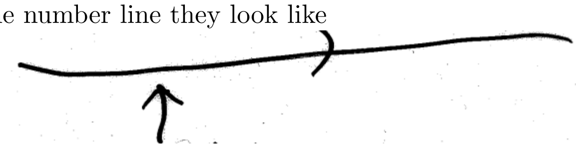
Letting $A = \mathbb{N}$, $\mathbb{Z}_{(\infty)} \in R_{(\infty)}$

Let

$$R \equiv \left\{ B \in R_\infty : \begin{array}{l} B \neq \Phi \\ \text{and} \\ B \neq \mathbb{Z}_{(\infty)} \end{array} \right\}$$

R will represent the <u>real</u> <u>numbers</u>

R will represent real numbers What do the elements of $\mathbb{R}$ look like?
On the number line they look like

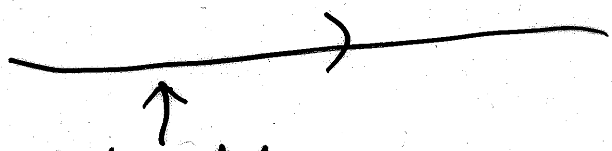


This set of dynamic rationals
How can we characterize them?

ZR 11

What do the elements of R look like?

On the number line they look like



This set of dyadic rationals

How can we characterize them?

Therom $B \in \mathbb{R}$ iff

(i) $\phi \subsetneq B \subsetneq \mathbb{Z}_{(\infty)}$

(ii) If $b \in B$ and $a \in \mathbb{Z}_{(\infty)}$ with $a <_\infty b$ then $a \in B$

(iii) If $b \in B$
$\exists b' \in B$ s.t
$b <_\infty b'$

ZR 12

Thm $B \in R$ iff

(i) $\phi \subsetneq B \subsetneq \mathbb{Z}_{(\infty)}$

(ii) If $b \in B$ and $a \in \mathbb{Z}_{(\infty)}$ with $a <_\infty b$ then $a \in B$

(iii) If $b \in B$
$\exists b' \in B$ s.t.
$b <_\infty b'$

$\underline{Pf}$:$^{(\Rightarrow)}$ Spse $B \in \mathbb{R}$
$\exists$ non-entry $A \subseteq \mathbb{Z}_{(\infty)}$
s.t

$$B = \bigcup_{a \in A} \mathcal{L}_a$$

We are given that
$B \neq \emptyset$ and $B \neq \mathbb{Z}_{(\infty)}$

(ii) Spse $b \in B$
and $b_o <_\infty b$ with $b_o \in \mathbb{Z}_{(\infty)}$
$\exists a \in A : b \in \mathcal{L}_a$
hence $b_o \in \mathcal{L}_a$
so $b_o \in B$

(iiii) holds for ——-> $\mathcal{L}_a$ so (iii) holds.

ZR(13)

$\underline{Pf}$:$^{(\Rightarrow)}$ Spse $B \in R$.
$\exists$ non-entity $A \subseteq \mathbb{Z}_{(\infty)}$
s.t. $B = \bigcup_{a \in A} \mathcal{L}_a$

We are given that
$B \neq \phi$ & $B \neq \mathbb{Z}_{(\infty)}$.

(ii) Spse $b \in B$.
and $b_o \leq_\infty b$ with $b_o \in \mathbb{Z}_{(\infty)}$
$\exists a \in A : b \in \mathcal{L}_a$
Hence $b_o \in \mathcal{L}_a$
so $b_o \in B$

(iii) holds for $\longrightarrow \mathcal{L}_a$ so (iii) holds

(⇐)Conversely,
Spse B satisfies (i),(ii),(iii)
Let $A = B$
$\emptyset \subsetneq A \subsetneq \mathbb{Z}_{(\infty)}$
Claim

$$B = \bigcup_{a \in A} \mathcal{L}_a$$

Clearly $\bigcup \mathcal{L}_a \subseteq B$ (Why?)
Take away $b \in B$.
$\exists b' >_\infty b$ s.t $b' \in B$
$b \in \mathcal{L}_b{}'$ so

$$B \subseteq \bigcup_{a \in A} \mathcal{L}_a$$

qed

ZR (14)

(⇐) Conversely,

Spse B satisfies (i),(ii),(iii)

Let $A = B$

$\phi \subsetneq A \subsetneq \mathbb{Z}_{(\infty)}$

Claim $B = \bigcup_{a \in A} \mathcal{L}_a$

Clearly $\bigcup_{a \in A} \mathcal{L}_a \subseteq B$. (Why?)

Take any $b \in B$.

$\exists b' >_\infty b$ s.t. $b' \in B$

$\therefore b \in \mathcal{L}_{b'}$ so

$B \subseteq \bigcup_{a \in A} \mathcal{L}_a$ qed

4 Sequence and Limits

¡¡¡¡¡¡¡ HEAD =======

Sequences ¿¿¿¿¿¿¿ 02ab88f2602e0e82d1db1124244f955fe13fd437

Limits

Constructing the limit via:

Prop let $\{a_n\}$ be a Cauchy sequence in R. Then $\{a_n\}$ is <u>bdd</u>. $P1 : \exists N < \infty$ such that for j,k $\geq N$ $| a_j - a_k |< 1, let\ B = max\{|a_1|, |a_2|, ..., |a_N|\} + 1$ for $1 \leq j \leq N, |a_j| < B$. for $j > N$ $|a_j| = |a_j - a_N + a_N|$ $|a_N| + |a_j - a_N|$ $< |a_N| + 1 \leq B$ Hence $\{a_j\}$ is <u>bounded</u>

(5 3 A)

<u>Prop</u> Let $\{a_n\}$ be a Cauchy seq in R. Then $\{a_n\}$ is <u>bdd</u>.

<u>P1</u>: $\exists N < \infty$ s.t. for $j,k \geq N$ $|a_j - a_k| < 1$. Let

(*) $B = \max\{|a_1|, |a_2|, \dots, |a_N|\} + 1$

For $1 \leq j \leq N$, $|a_j| < B$.

For $j > N$

$$|a_j| = |a_j - a_N + a_N|$$
$$\leq |a_N| + |a_j - a_N|$$
$$< |a_N| + 1 \leq B$$

Hence $\{a_j\}$ is <u>bounded</u>.

Then a_n con in R if a_n is Cauchy P1 | $\Rightarrow$ | space $a_n \longrightarrow$ a in R. Take any $\varepsilon > 0. \exists N\varepsilon < \frac{\varepsilon}{2}$ $For j, k >, N\varepsilon$ $|a_j - a_k| = |a_j - a + a - a_k| \le |a_j - a| + |a - a_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

53B

Thm $\{a_n\}$ conv in R iff $\{a_n\}$ is Cauchy.

Pf: $(\Rightarrow)$ ~~Spse~~ $a_n \to a$ in R. Take any $\varepsilon > 0$. $\exists N_\varepsilon \in \mathbb{N}$ s.t. for $n \geq N_\varepsilon$, $|a_n - a| < \frac{\varepsilon}{2}$

For $j, k \geq N_\varepsilon$

$$
\begin{aligned}
|a_j - a_k| &= |a_j - a + a - a_k| \\
&\leq |a_j - a| + |a - a_k| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}
$$

(⇐) space a_n is Cauchy. Then $\exists B < \infty$ such that $|a_n| \le B$ for all n. more over $\exists 1 \le n_1 < n_2 ...$ such as $a_{n_k} :\ge 1$ is montonic

w.l.o.g. space $a_n, \le a_{n_2} \le ...$ let $L = raysa_{n_k} : k \ge 1$ Then lim $a_{n_k} L(\le B)k \longrightarrow \infty$ Conjecture:$a_j \to L$

P1: Take away $\varepsilon > O.\exists_\varepsilon < \infty$ s.t. for $k \ge k_\varepsilon |a_{n_k} - L| < \frac{\varepsilon}{2}$ for $i, J \ge N_\varepsilon$. do for $j \ge N_\varepsilon$ and $k \ge N_\varepsilon, |a_j - 4 \le |a_j - an_k - 4$

(53c)

(⟸) Spse $\{a_n\}$ is Cauchy.
Then $\exists B < \infty$ s.t. $|a_n| \le B$
for all n. moreover
$\exists 1 \le n_1 < n_2 < \cdots$ s.t.
$\{a_{n_k} : k \ge 1\}$ is monotonic
w.l.o.g. spse $a_{n_1} \le a_{n_2} \le \cdots$
Let $L = \sup\{a_{n_k} : k \ge 1\}$
Then $\lim_{k \to \infty} a_{n_k} = L \ (\le B)$
Conjecture: $a_j \to L$
Pf: Take any $\varepsilon > 0$. $\exists K_\varepsilon < \infty$
s.t. for $k \ge K_\varepsilon$ $|a_{n_k} - L| < \varepsilon/2$
$\exists N_\varepsilon \ge K_\varepsilon$ s.t. $|a_i - a_j| < \varepsilon/2$
for $i, j \ge N_\varepsilon$. So for $j \ge N_\varepsilon$ and
$k \ge N_\varepsilon$, $|a_j - L| \le \underbrace{|a_j - a_{n_k}|}_{<\varepsilon} + |a_{n_k} - L|$

=======

1. Monotonic Sequences
2. Monotonic Sub-sequences ¿¿¿¿¿¿¿ 02ab88f2602e0e82d1db1124244f955fe13fd437

Cauchy Sequences
Subsequential Limits

Sequences

seqs
limits
constructing the limit
via (i) monotonic seqs
(ii) monotonic subseqs
Cauchy seqs
subsequential limits

Besides $\vec{a} = (a_0, a_1, a_2, ...)$, how else might we harness elements of $\mathbb{R}^2$?

$$s_0 = a_0$$

$$s_1 = a_0 + \frac{a_1}{2}$$

¡¡¡¡¡¡ HEAD

5 Metric Spaces Part 2

=======

$$s_n = \sum_{j=0}^{n} a_j 2^{-j}$$

¿¿¿¿¿¿¿ origin/master

¡¡¡¡¡¡ HEAD Def: Let m be non-empty set a metric d on m is a map
$d : m \times m \to$ [0,∞)
satisfying (for all x, y, z in m)

(i) $d(x, y) = d(y, x) \begin{cases} > 0 \text{ if } x \neq y \\ = 0 \text{ if } x = y \end{cases}$

(ii) $d(x, y) \leq d(x, z) + d(z, y)$

(iii) is called the triangle ineq

~~II~~ Metric Spaces ~~II~~

m |

Def: Let m be non-empty set a metric d on m is a map

$d : m \times m \to [0, \infty)$

satisfying (for all x, y, z in m)

(i) $d(x, y) = d(y, x) \begin{cases} >0 \text{ if } x \neq y \\ =0 \text{ if } x = y \end{cases}$

(ii) $d(x, y) \leq d(x, z) + d(z, y)$

(ii) is called the triangle ineq

Examples

1. $d(x,y) = \begin{cases} 1 \ if x \neq y \\ 0 \ if x = y \end{cases}$

2. If $m = R^k$ and $1 \leq P \leq \infty$

$d_p(\vec{x},\vec{y}) = (\sum_{j=1}^{k} (x_j - y_j)^p)^{\frac{1}{p}}$

$d_\infty(\vec{x},\vec{y}) = max_{1 \leq j \leq k} |x_j - y_j|$

MZ

Examples

(1) $d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$

(2) If $m = \mathbb{R}^k$ and $1 \leq p \leq \infty$

$d_p(\vec{x},\vec{y}) = \left(\sum_{j=1}^{k} |x_j - y_j|^p\right)^{\frac{1}{p}}$

$d_\infty(\vec{x},\vec{y}) = \max_{1 \leq j \leq k} |x_j - y_j|$

New Metrics From Old

Suppose $d(x, y)$ is a metric
on m and f:[0,∞)$\rightarrow$ [0,∞) is non-decreasing with
$f(0)$=0 and $f(x)$ >0 if x >0.
When will
$g(x, y) = f(d(x, y))$ be a metric?
We need to guarantee that the triangle inequality holds for g.

m3

New Metrics from Old

Spse $d(x,y)$ is a metric
on m and $F: [0,\infty) \rightarrow [0,\infty)$
is non-decreasing with
$f(0) = 0$ and $f(x) > 0$ if $x > 0$.
When will
$g(x,y) = f(d(x,y))$
be a metric?

We need to guarantee
that the triangle ineq
holds for g.

Suppose $f(a+b) \leq f(a) + f(b)$
for $a \geq 0$, $b \geq 0$ (Then f is sub-additive)
then
$g(x,y) = f(d(x,y))$
$\leq f(d(x,z) + d(z,y))$
$\leq f(d(x,z)) + f(d(z,y))$
$= g(x,z) + g(z,y)$
If $f'(y) \geq 0$ and $f''(y) \leq 0$ and $f(0) = 0$ then f is sub-additive
ex. $f(y) = \sqrt{y}$

m 4

Spse $f(a+b) \leq f(a) + f(b)$
for $a \geq 0$, $b \geq 0$ (Then f is sub-additive)
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If $f'(y) \geq 0$ and $f''(y) \leq 0$
and $f(0) = 0$ then f is
sub-additive.
Ex: $f(y) = \sqrt{y}$

If d_1 and d_2 are metrics on m then $d(x, y) = \max\{d_1(x, y), d_2(x, y)\}$ may not be a metric, but $\min\{d(x, y), 1\}$ is a metric

m5

If d_1 and d_2 are metrics on m then

$d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$

is a metric

Show by example

$\min\{d_1(x,y), d_2(x,y)\}$

may not be a metric.

but

$\min\{d(x,y), 1\}$

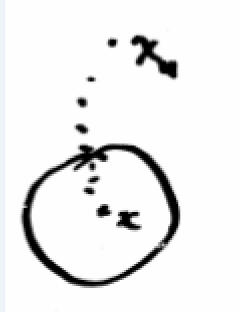
is a metric

An Aside: why do mathematicians call virtually every non-empty set a <u>*space*</u>?
<u>*Answer*</u>:The <u>*space*</u> we inhabit can be represented by the <u>*set*</u> of ordered triples (x_1, x_2, x_3) of reals. Hence we think of a way non-empty set as an abstract space.

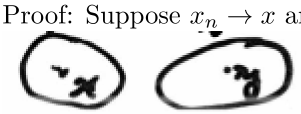
M 5A

An Aside: Why do mathematicians call virtually every non-empty set a <u>space</u>?
<u>Answer</u>: The space we inhabit can be represented by the <u>set</u> of ordered triples (x_1, x_2, x_3) of reals. Hence we think of any non-empty set as an abstract space.

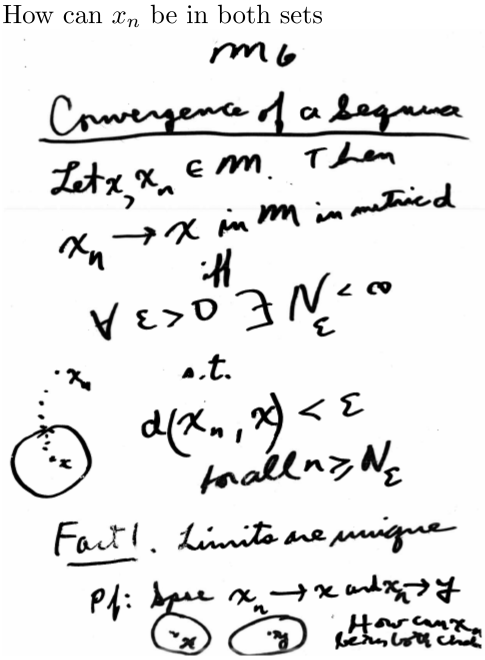
$\underline{Convergence of a Sequence}$
Let $x, x_n \in m$ Then $x_n \to x$ in m in metric d
if
$\forall\ \epsilon \geq 0\ \exists N_\epsilon < \infty$
such that
$d(x_n, x) < \epsilon$
for all $n \geq N_\epsilon$



$\underline{Fact 1}$. Limits are unique
Proof: Suppose $x_n \to x$ and $x_n \to y$



How can x_n be in both sets


rm6
Convergence of a Sequence
Let $x, x_n \in m$. Then
$x_n \to x$ in m in metric d
iff
$\forall\ \varepsilon > 0\ \exists N_\varepsilon < \infty$
s.t.
$d(x_n, x) < \varepsilon$
for all $n \geq N_\varepsilon$
Fact 1. Limits are unique
Pf: Suppose $x_n \to x$ and $x_n \to y$
How can x_n be in both circles

Subsequent Limits

Given $\{ a_n \}$, let $\mathcal{L}=$ { susequential limits }

Can we upper bound $\mathcal{L}$

Let $b_1=$ suppose $\{a_1, a_2, ...\}$

$b_2=$ suppose $\{a_2, a_3, ...\}$

$b_n=$ suppose $\{a_n, a_{n+1}, ...\}$

$b_1 \geq b_2 \geq ...$

Subseq Limits

54

Given $\{a_n\}$, let

$\mathcal{L} = \{$ subsequential limits $\}$

Can we upper bound $\mathcal{L}$?

Let

$b_1 = \sup\{a_1, a_2, \cdots\}$

$b_2 = \sup\{a_2, a_3, \cdots\}$

$b_n = \sup\{a_n, a_{n+1} \cdots\}$

$b_1 \geq b_2 \geq \cdots$

Definition:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

$$\equiv \lim_{n\to\infty}[\sup(a_k : k \geq n)] = \inf_n[\sup(a_k - k > n)]$$

Clearly,

$$\sup \mathcal{L} \leq \lim_{n\to\infty} a_n$$

Can you show they are equal?
What about inf $\mathcal{L}$?

54A

Def: $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

$\equiv \lim_{n\to\infty} \left(\sup\{a_k : k \geq n\}\right)$

$\equiv \inf_n \left(\sup\{a_k : k \geq$

Clearly,

$\sup \mathcal{L} \leq \lim_{n\to\infty} a_n$

Can you show they are equal?
What about inf $\mathcal{L}$?

======= ¿¿¿¿¿¿¿ origin/master
The sequence $\{s_n\}$ represents $\{\ z \in \mathbb{Z}_\infty : z < s_n$ for all in sufficient large $\}$
It has a limit of s_∞

Besides $\vec{a} = (a_0, a_1, a_2 \dots)$ how else might we express elements of $\mathbb{R}$?

$$s_0 = a_0$$

$$s_1 = a_0 + \frac{a_1}{2}$$

$$\vdots$$

$$s_n = \sum_{j=0}^{n} a_j 2^{-j}$$

The sequence $\{s_n\}$ represents

$\{z \in \mathbb{Z}_\infty : z < s_n$ for all n suff large$\}$

It has limit s_∞.

Sequences

Def: a sequence $\{ {a_n}_1^\infty \}$ is a map from the integers. A real valued sequence is a map into the reale from the integers.

Examples:

$a_n = \frac{1}{n^2}$

$a_n = (-c)^n$

$a_n = \cos(nx)$

$a_n = n^{\frac{1}{n}}$

$a_n = (1 + \frac{1}{n})^n$

---

(48)

Sequences

Def: a seq $\{a_n\}_1^\infty$ is a map from the integers. a real valued seq is a map into the reals from the integers.

Examples

$a_n = \frac{1}{n^2}$

$a_n = (-1)^n$

$a_n = \cos nx$

$a_n = n^{1/n}$

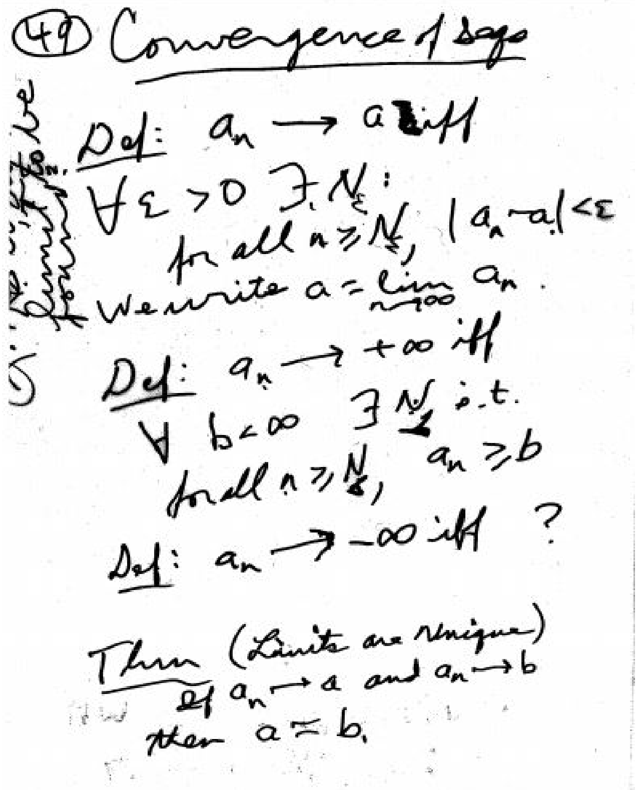
$a_n = (1 + \frac{1}{n})^n$

Convergence of Sequences

Def: $a_n \to a$ if and only if $\forall \epsilon > 0 \exists N$:
for all n $\geq N$, $|a_n - a| < \epsilon$
we write...

$$a = \lim_{n \to \infty} a_n$$

Def: $a_n \to +\infty$ if and only if $\forall b < \infty\ \epsilon N_b$ such that for all $n \geq, N_b, a_n \geq b$
Def: $a_n \to -\infty$?
Then (limits are unique)
if $a_n \to a$ and $a_n \to b$
then $a = b$



(49) Convergence of Seqs

Def: $a_n \to a$ iff
$\forall \varepsilon > 0\ \exists N_\varepsilon$:
for all $n \geq N_\varepsilon$, $|a_n - a| < \varepsilon$
We write $a = \lim_{n \to \infty} a_n$.

Def: $a_n \to +\infty$ iff
$\forall\ b < \infty\ \exists N_b$ s.t.
for all $n \geq N_b$, $a_n \geq b$

Def: $a_n \to -\infty$ iff ?

Thm (Limits are Unique)
If $a_n \to a$ and $a_n \to b$
then $a = b$.

Convergence in $\mathbb{R}_\infty$ An Elegant Reformation

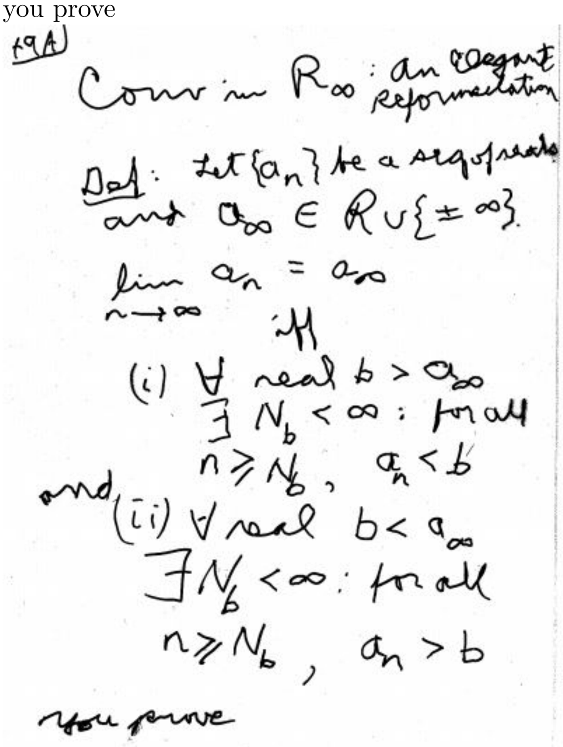
Def: Let $\{a_n\}$ be a sequence of reals and $a_\infty \epsilon \mathbb{R} \bigcup \{\pm\infty\}$.
$\lim_{n\to\infty} a_n = a_\infty$ if and only if...

(i) $\forall$ real $b > a_\infty$
   $\exists N_b < \infty$: for all
   $n \geq N_b$, $a_n < b$

and

(ii) $\forall$ real $b < a_\infty$
   $\exists N_b < \infty$: for all
   $n \geq N_b$, $a_n > b$

you prove



19A)

Conv in R_∞ : an elegant reformulation

Def: Let $\{a_n\}$ be a seq of reals and $a_\infty \in R \cup \{\pm\infty\}$.

$\lim_{n\to\infty} a_n = a_\infty$

iff

(i) $\forall$ real $b > a_\infty$
$\exists N_b < \infty$: for all
$n \geqslant N_b$, $a_n < b$

and (ii) $\forall$ real $b < a_\infty$
$\exists N_b < \infty$: for all
$n \geqslant N_b$, $a_n > b$

you prove

Finding Limits and Proving Convergence

Example 1: $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2: $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$

Example 3: $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

Homework: Suppose $a_n \to a > 0$

Prove $\sqrt{a_n} \to \sqrt{a}$

Finding 50 Limits and Proving Convergence

Example 1 $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2 $\lim_{n\to\infty} \frac{3n+1}{7n-4} = ?$

Example 3 $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = ?$

HW Suppose $a_n \to a > 0$

Prove: $\sqrt{a_n} \to \sqrt{a}$

Theorem Suppose $a_n \to a$ and $b_n \to b$. Then $a_n + b_n \to a + b$ (and $\lambda a_n \to \lambda a$)and $a_n b_n \to a * b$

Theorem if $a_n \to a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$

corollary if $a_n \to a$ and $b_n \to b \neq 0$ then $\frac{a_n}{b_n} \to \frac{a}{b}$

(57)

Thm Spse $a_n \to a$ and $b_n \to b$. Then $a_n + b_n \to a + b$ (and $\lambda a_n \to \lambda a$) and $a_n b_n \to a \cdot b$

Thm If $a_n \to a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$

Cor ~~then~~ If $a_n \to a$ and $b_n \to b \neq 0$ then $\frac{a_n}{b_n} \to \frac{a}{b}$

A method of proving corner:

monotone sequences

Lat $\{a_n\}$ be a sequence of reals. We say $\{a_n\}$ is monotonic if either $a_1 \leq a_2 ... \leq a_a \leq ...$ or else $a_1 \geq a_2 ... \geq a_a \geq ...$ In the first case $\{a_n\}$ is said to be (monotone) non-decreasing and in the second case it is (monotone) non-increasing. Theorem suppose a_n is monotonic and bowled. Then it conveys in R.

A Method (53) of Proving Conv:

Monotone Sequences

Let $\{a_n\}$ be a seq of reals. We say $\{a_n\}$ is monotonic ~~iff~~ either

$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$

or else

$a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$

In the first case $\{a_n\}$ is said to be (monotone) non-decreasing and in the second case it is (monotone) non-increasing.

Thm Spse $\{a_n\}$ is monotonic and bounded. Then it converges in $\mathbb{R}$.

Theorem Let $\{a_n\}$ be monotonic in R. Then in R_∞

$\lim_{n\to\infty} a_n = \begin{cases} sup\ \{a_k\} & if a_1 \le a_2 \le a_3 \le ... \\ inf\ \{a_k\} & if a_1 \ge a_2 \ge ... \end{cases}$

Pf: W l.o.g suppose $a_1 \le a_2 \le$ take away $b < supposea_k$ Then $\exists k < \infty s.t. b < \{a_k\}$. For all $n \ge k, a_n \ge a_n so\ a_n > b$. But also $a_n/leqsuppose\ \{a_k\}$

(52)

Thm Let $\{a_n\}$ be monot in R. Then in R_∞

$$\lim_{n\to\infty} a_n = \begin{cases} \sup\{a_k\} \text{ if } a_1 \le a_2 \le a_3 \le \dots \\ \inf\{a_k\} \text{ if } a_1 \ge a_2 \ge \dots \end{cases}$$

Pf. Wlog spse $a_1 \le a_2 \le \cdots$

Take any $b < \sup\{a_k\}$

Then $\exists\ k_* < \infty$ s.t. $b < a_{k_*}$. For all $n \ge k_*$, $a_n \ge a_{k_*}$ so $a_n > b$.

But also $a_n \le \sup\{a_k\}$

Theorem Let $\{a_n\}$ be any sequence in $\mathbb{R}$ then $\exists 1 \leq n_1 < n_2 < ...s.t. a_{n_1}, a_{n_2}, a_{n_3}, ...$ is monotonic (i.e. a_n has a monotonic sub sequence)
Pf: Either $a_1 \leq a_n$ for infinity many $n > 1$. Applying to each aj Let J $= \{j \geq 1 : a_n \geq a_j$ for infinity many $n > j\}$ Cosel $|J| = \infty$ Let $n_1 =$ smallest $j \in J$ $\exists n_2 > n_1$ with $n_2/inJ, ...$ ete get n_k

(52A)

Thm Let $\{a_n\}$ be any seq in $\mathbb{R}$
Then $\exists\ 1 \leq n_1 < n_2 < \cdots$ s.t.
$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is monot.
(i.e. $\{a_n\}$ has a monotonic subsequence)

Pf: ~~Let~~ Either $a_1 \leq a_n$ for infinitely many $n > 1$ or $a_1 \geq a_n$ for infinitely many $n > 1$.
Applying this question to each a_j:
Let $J = \{ j \geq 1 : a_n \geq a_j$ for infinitely many $n > j \}$

Case1 $|J| = \infty$
Let $n_1 =$ smallest $j \in J$
$\exists\ n_2 > n_1$ with $n_2 \in J, \ldots$ etc get $\{n_k\}$

and $a_{n_1} \leq a_{n_2} \leq \dots$

<u>Case 2</u> $|J| < \infty$ take away $n_1 >$ one and max. $j \leftarrow J$

By construction, if n_2 is large enough $n_2 > n_1$ and $a_{n_2} < a_{n_1}$ By induction, having constructed $n_1 < n_2 < \dots < n_k$ p.t. $a_{n_1} > a_{n_2} > \dots > a_{n_k}$ $\exists n_{k+1} > n_k$ s.t. $a_{n_{k+1}} < a_{n_k}$

and so the theorem holds.

<u>52B</u>

and $a_{n_1} \leq a_{n_2} \leq \dots$

<u>Case 2</u> $|J| < \infty$

Take any $n_1 >$ one and max. $j \in J$

By construction, if n_2 is large enough

$n_2 > n_1$ and $a_{n_2} < a_{n_1}$

By induction, having constructed $n_1 < n_2 < \dots < n_k$

p.t. $a_{n_1} > a_{n_2} > \dots > a_{n_k}$

$\exists$ $n_{k+1} > n_k$ s.t.

$a_{n_{k+1}} < a_{n_k}$.

and so the Thm holds

6 Limit and Convergence

z

Finding Limits and Proving Convergence

Example 1 $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2 $\lim_{n\to\infty} \frac{3n+1}{7n-4} = ?$

Example 3 $\lim_{n\to\infty} (1 + \frac{1}{n})^n = ?$

Example 4 Find $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$

HW suppose $a_n \to a > o$

Prove: $\sqrt{a_n} \to \sqrt{a}$

Finding (50) Limits and Proving Convergence

Example 1 $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2 $\lim_{n\to\infty} \frac{3n+1}{7n-4} = ?$

Example 3 $\lim_{n\to\infty} (1 + \frac{1}{n})^n = ?$

Example 4 Find $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$

HW Spse $a_n \to a > 0$

Prove: $\sqrt{a_n} \to \sqrt{a}$

Properties of Limits

Theorem Suppose $a_n \to a$ and
$b_n \to b$. Then
$a_n + b_n \to a + b$
(and $\lambda a_n \to \lambda a$)
and $a_n b_n \to a \cdot b$

Theorem of $a_n \to a \neq o$,
then $\frac{1}{a_n} \to \frac{1}{a}$

Corol if $a_n \to a$
and $b_n \to b \neq 0$ then
$\frac{a_n}{b_n} \to \frac{a}{b}$

Properties of limits

Thm Spse $a_n \to a$ and
$b_n \to b$. Then
$a_n + b_n \to a + b$
(and $\lambda a_n \to \lambda a$)
and $a_n b_n \to a \cdot b$

Thm If $a_n \to a \neq 0$,
then $\frac{1}{a_n} \to \frac{1}{a}$

Corol ~~then~~ If $a_n \to a$
and $b_n \to b \neq 0$ then
$\frac{a_n}{b_n} \to \frac{a}{b}$

Therm Let $\{a_n\}$ be monotonic in $\mathbb{R}$. Then in $\mathbb{R}_\infty$ $\lim_{n\to\infty}$

$$a_n = \begin{cases} \sup\{a_k\} & \text{if } a \leq a_2 \leq a_3 \leq ... \\ \inf\{a_k\} & \text{if } a \geq a_2 \geq ... \end{cases}$$

The limit is in $\mathbb{R}$ if $\{a_n\}$ is bounded

$\underline{Pf}$: WL.o.g spse $a_1 \le a_2 \le ...$ Take any b < sup $\{a_k\}$ Then
$\exists\, k_* < \infty$ s.t. $b < a_{k*}$ for all $n \ge k_*, \{a_n\} \ge a_{k*}$ so $a_n > b$. But also $a_n \le a_n$

52

__Thm__ Let $\{a_n\}$ be monot in $\mathbb{R}_\infty$ in $\mathbb{R}$. Then

$$\lim_{n\to\infty} a_n = \begin{cases} \sup\{a_k\} \text{ if } a_1 \le a_2 \le a_3 \le \cdots \\ \inf\{a_k\} \text{ if } a_1 \ge a_2 \ge \cdots \end{cases}$$

The limit is in $\mathbb{R}$ iff $\{a_n\}$ is bounded.

__Pf__: Wl.o.g spse $a_1 \le a_2 \le \cdots$
Take any $b < \sup\{a_k\}$
Then $\exists\ k_* < \infty$ s.t.
$b < a_{k*}$. For all
$n \ge k_*$, $a_n \ge a_{k*}$ so
$a_n > b$.
But also $a_n \le \sup\{a_k\}$

What if $\{a_n\}$ is *not* monotonic? *Therom* Let $\{a_n\}$ be seq in $\mathbb{R}$. Then $\exists\ 1 \geq$ n, $< n_2 < \ldots$ s.t. a_{n_1},a_{n_2}, ... is monotonic. *Pf*: We need to consider the tail of a seq. Let

$$J = \{j \leq | : a_j > a_{jtk}\ forall\ K \leq 1\}$$

If J is unbounded we may write J = $\{n_j \quad J \leq 1$ where $1 \leq n_1 < n_2 < \ldots$

(52A)

What if $\{a_n\}$ is *not* monotonic?

Thm Let $\{a_n\}$ be seq in $\mathbb{R}$.
Then $\exists\ 1 \leq n_1 < n_2 < \cdots$ s.t. $a_{n_1}, a_{n_2}, \ldots$ is monotonic

Pf: We need to consider the Tail of a seq. Let
$J = \{j \geq 1 : a_j > a_{j+k}$ forall $k \geq 1\}$
If J is unbounded we may write $J = \{n_j : j \geq 1\}$ where $1 \leq n_1 < n_2 < \cdots$

By construction, $a_{n_1} > a_{n_2} > ...$ on the other hand, if $|J| < \infty$, let $n_1 \geq 1$ exceed every j $\in$ J. Having constructed $n_1 < n_2 < ... < n_k$ with $a_{n_1} \leq a_{n_2} ... \leq a_{n_K}$ (since n_k and J) $\exists$ n_k+1 $> n_k$ s.t. a_{n_k}+1, and the desired construction follows by

52B)

By construction, $a_{n_1} > a_{n_2} > \cdots$ On the other hand, if $|J| < \infty$, let $n_1 \geq 1$ exceed every $j \in J$. Having constructed $n_1 < n_2 < \cdots < n_k$ with $a_{n_1} \leq a_{n_2} \leq \cdots \leq a_{n_k}$ (since $n_k \notin J$) $\exists\, n_{k+1} > n_k$ s.t. $a_{n_k} \leq a_{n_{k+1}}$, and the desired conclusion follows by induction.

induction.

(Balzono weierstras) *Corllary* if $|a_n|$ is a bounded seq in $\mathbb{R}$, it has a covergent subsequence. (Even if $\{a_n\}$ is *not* bounded, it has a subseq converging in $\mathbb{R}_\infty$.)

52C

(Bolzano-Weierstrass)

Corollary If $\{a_n\}$ is a bounded seq in R, it has a convergent subsequence.

(Even if $\{a_n\}$ is not bounded, it has a subseq converging in R_∞.)

CauchySequence When can we gaurentee that a seq converges? *Def*: a seq of vals $\{x_n\}$ is said to be a *Cauchyseq* is $\forall \epsilon > 0 \exists N_\epsilon$ such that for every $n, m \geq N_\epsilon$

$$|x_n - x_m| < \epsilon$$

Prop Spse $x_n \to x$ Then $\{x_n\}$ is a Cauchy seq.

Cauchy Sequences

When can we guarantee that a seq conv?

Def: a seq of reals $\{x_n\}$ is said to be a Cauchy seq iff $\forall \varepsilon > 0$ $\exists$ N_ε such that for every $n, m \geq N_\varepsilon$

$$|x_n - x_m| < \varepsilon$$

Prop Spse $x_n \to x$ Then $\{x_n\}$ is a Cauchy seq.

7 Infinite Series

A frog is 2 feet from a wall. He makes a succession of jumps toward it, always jumping half his remaining distance to the wall. Hence his finest jump is one foot. Now he is one foot from the wall. So

2S1

Infinite Series

A frog is 2 feet from a wall. He makes a succession of jumps toward it, always jumping half his remaining distance to the wall.
Hence his first jump is one foot.
Now he is one foot from the wall. So

his second jump is $\frac{1}{2}$ feet. He is now $\frac{1}{2}$ feet from the wall so he jumps $\frac{1}{4}$ feet. He makes successive jumps of $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ After n jumps he has moved

$$s_n = \sum_{j=1}^{n} \frac{1}{2^{j-1}}$$

feet.

Is2

his second jump
is $\frac{1}{2}$ foot
He is now $\frac{1}{2}$ foot from
the wall so he jumps
$\frac{1}{4}$ foot.
He makes successive
jumps of
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
After n jumps he has
moved $s_n = \sum_{j=1}^{n} \frac{1}{2^{j-1}}$ feet

and is now

$$\frac{1}{2^{n-1}}$$

feet from the wall so

$$s_n = 2 - \frac{1}{2^{n-1}}$$

If he keeps jumping forever, how far does he go? He moves

$$\sum_{j=1}^{n} \frac{1}{2^{j-1}}$$

feet. This number is at most 2 and yet it exceeds

$$2 - \frac{1}{2^n}$$

for all

$$n \geq 1$$

.

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and is now $\frac{1}{2^{n-1}}$ feet from the wall

so $s_n = 2 - \frac{1}{2^{n-1}}$

If he keeps jumping forever, how far does he go?

He moves $\sum_{j=1}^{\infty} \frac{1}{2^{j-1}}$ feet

This number is at most 2 and yet it exceeds $2 - \frac{1}{2^n}$ for all $n \geq 1$.

Hence the sum of this infinites collection of numbers $1, \frac{1}{2}, \frac{1}{4}, ...$ must be two. How can we generalize this? <u>Generalization 1</u>: Genetic Series How large is $1 + r + r^2 + \ . \ . \ .$ If $|r| < |$? Solution: Let $Sn = 1 + r + ... + r^n$ For $r \leq 0$, $S1 \leq S2 \leq ...$ and we expect him $S_n = \sum_{j-0}^{\infty} r^j$

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Hence the sum of this infinite collection of numbers $\{1, \frac{1}{2}, \frac{1}{4}, \cdots\}$ must be two.

How can we generalize this?

<u>Generalization 1</u> Geometric Series

How large is $1+r+r^2+\cdots$ if $|r|<1$?

<u>Soln</u>: Let $S_n = 1+r+\cdots+r^n$

For $r \geq 0$, $S_1 \leq S_2 \leq \cdots$

and we expect $\lim_{n\to\infty} S_n = \sum_{j=0}^{\infty} r^j$

$S_n = 1 + r + ... + r^n$ is complex in that it has too many terms. how can we simplify it? We need to capitalize on the regularity of the expensive. Notice: $rS_n = r + r^2 + ... + r^{n+1}$, subtracting equals from equal $S_n - rS_n = 1 - r^{n+1}$ so for $r \neq 1, S_n = \frac{1-r^{n+1}}{1-r}$

I & S

$S_n = 1 + r + \cdots + r^n$

is complex in that it has too many terms. How can we simplify it? We need to capitalize on the regularity of the expression. Notice:

$rS_n = r + r^2 + \cdots + r^{n+1}$

Subtracting equals from equal

$S_n - rS_n = 1 - r^{n+1}$

so for $r \neq 1$, $S_n = \frac{1-r^{n+1}}{1-r}$

$\rightarrow \frac{1}{1-r}$

(ie) Suppose $r_* > 1$
Take $1 < \lambda < r_*$
$\exists N : N > N$
$=> |\frac{^a n+1}{an}| = \lambda$
We assume this means
$|a_n| > O$ for $n \geq N$
ntense $|^a N + K| \geq \lambda^k$
for $k \geq 1$ so that
$|^a N + K| - > \infty$ and $N + K -$
goes to infinity
this Σan diverges

22/16

(ii) Spse $r_* > 1$

Take $1 < \lambda < r_*$

$\exists N : n \geq N$

$\implies \left|\frac{a_{n+1}}{a_n}\right| \geq \lambda$

We assume ~~this means~~

$|a_n| > 0$ for $n \geq N$.

Hence $|a_{N+k}| \geq \lambda^k |a_N|$

for $k \geq 1$ so that

$|a_{N+k}| \to \infty$ as $N+k$

goes to infinity

Thus $\sum a_n$ diverges

Infinite Series 17

Case 1

$$(iii) a_n = n \; r_{what} = r^{what} = 1$$
$$\Sigma a_n = \infty$$

Case 2

$$a_n = \frac{1}{n^2} \; r_* = r^* = 1$$
$$\text{yet } \Sigma a_n < \infty$$
$$\text{Pf: } = \infty \Sigma n = 1 \frac{1}{n2} \leq \infty \Sigma n = 1 \frac{2}{n(n+1)}$$

$= lim 2 N n = 1 \Sigma (\frac{1}{n} - \frac{1}{n+1})$
$= lim \underset{N \Rightarrow \infty}{lim} 2 \, (1 - \frac{1}{N+1})$
$= 2 < \infty$
Hence a series for which $r_* \leq 1 \leq r^*$ is either common div?

IS 17

Case 1

$$(iii) \quad a_n = n \qquad r_* = r^* = 1$$
$$\sum a_n = \infty$$

Case 2

$$a_n = \frac{1}{n^2} \qquad r_* = r^* = 1$$
$$\text{yet } \sum a_n < \infty$$
$$\text{Pf: } \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

Telescoping series

$$= \lim_{N \to \infty} 2 \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$
$$= \lim_{N \to \infty} 2 \left(1 - \frac{1}{N+1} \right)$$
$$= 2 < \infty$$

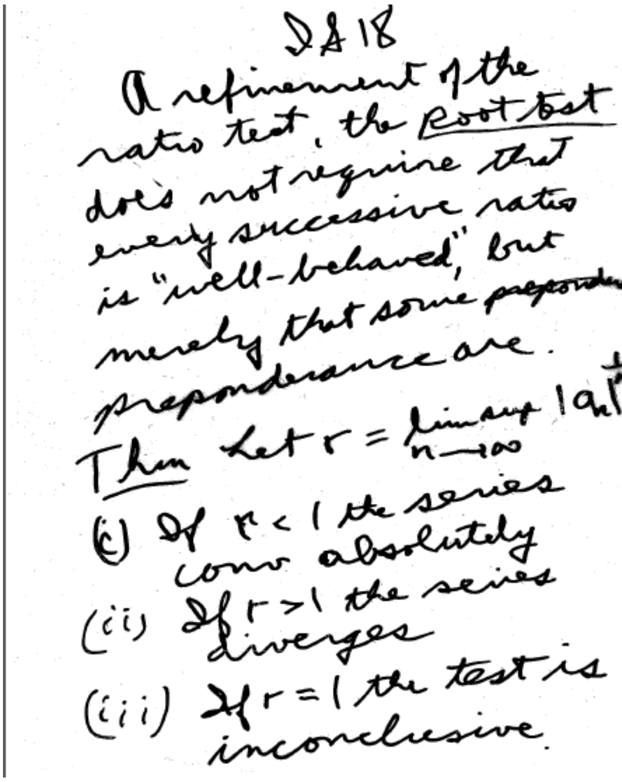
Hence a series for which $r_* \leq 1 \leq r^*$ can either conv or div

A refinement of the ratio teat, the Root Test does not require that evenly excessive ratio is "well-behaved", but merely that some p-what preponderance are. <u>Then</u> let r = $\underset{N \Rightarrow \infty}{lim} |a_n|^{\frac{1}{n}}$

(i) if $r < 1$ the something corns absolutely

(ii) if $r > 1$ the something diverges

(iii) if $r = 1$ the test is inconclusive



IS 18

A refinement of the ratio test, the <u>Root test</u> does not require that every successive ratio is "well-behaved", but merely that some ~~preponder~~ preponderance are.

<u>Then</u> let $r = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}$

(i) If $r < 1$ the series conv absolutely

(ii) If $r > 1$ the series diverges

(iii) If $r = 1$ the test is inconclusive.

Infinite Series 19

Pf: (i) Suppose $r < 1$ and take any $r < \lambda < 1$
$\exists N : n \geq N => |a_n|^{\frac{1}{n}} \leq \lambda$
ntense $|a_n| \leq what^n$
By the comparison test Σa_n comes ahead.
(ii) Suppose $r > 1$ and the $1 < \lambda < r$
For any $N < \infty$
$\exists n > N : |a_n|^{\frac{1}{n}} \geq what$
ntence lim says $|a_n| = \infty$
$n- > \infty$
$\Sigma a_n diveges$
(iii) if $r = 1$, test fail

IS 19

Pf: (i) Spse $r < 1$ and take any $r < \lambda < 1$
$\exists N : n \geq N \Rightarrow |a_n|^{\frac{1}{n}} \leq \lambda$
Hence $|a_n| \leq \lambda^n$
By the comparison test
Σa_n conv absol.
(ii) Spse $r > 1$ and take $1 < \lambda < r$
For any $N < \infty$
$\exists\, n > N : |a_n|^{\frac{1}{n}} \geq \lambda$
so $|a_n| \geq \lambda^n \longrightarrow \infty$
Hence $\limsup_{n \to \infty} |a_n| = \infty$
$\therefore \sum_n a_n$ diverges
(iii) If $r = 1$, test fail

Infinite Series 20

Power Series

Let $f(x) = \infty\Sigma\ n = 0\ a_n x^n$

Let $R = \frac{1}{lim}|a_n|^{\frac{1}{n}}$

Then

(i) series comes absol for $o \leq |x| < R$

(ii) series div for $x > R$

(iii) at $x = R$ we may have common div

$O \leq R \leq \infty$. R is called the radius of com of the power series.

IS 20.

Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Let $R = \dfrac{1}{\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}}$

Then

(i) series conv absol for $0 \leq |x| < R$

(ii) series div for $|x| > R$

(iii) at $x = R$ we may have conv or div

$0 \leq R \leq \infty$. R is called the radius of conv of the power series.

<u>PF:</u> By Root Test,

$\Sigma a_n x^n lonesomething$

if $\underset{N \Rightarrow \infty}{\bar{lim}} |a_n x^n|^{\frac{1}{n}} < 1$

if $|x| \underset{N \Rightarrow \infty}{\bar{lim}} |a_n|^{\frac{1}{n}}$

if $|x| < \frac{1}{lim|a_n|}^{\frac{1}{n}}$

if $|x|$ $limn- > \infty$ $|a_n|^{\frac{1}{n}}$

if $|x| < \frac{1}{\bar{lim}}|a_n|^{\frac{1}{n}} = R$

If $\bar{lim}$ $|a_n x^n|\frac{1}{n} > 1$ soive div

And series div if $|x| > R$

When $|x| = R$ anything can happen

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<u>Pf</u>: By Root Test,

$\sum_n a_n x^n$ conv absol

iff $\overline{\lim}_{n\to\infty} |a_n x^n|^{\frac{1}{n}} < 1$

iff $|x| \overline{\lim}_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$

iff $|x| < \frac{1}{\overline{\lim}_{n\to\infty} |a_n|^{\frac{1}{n}}} \equiv R$

If $\overline{\lim}_{n\to\infty} |a_n x^n|^{\frac{1}{n}} > 1$ series div

so series div if $|x| > R$

When $|x| = R$ anything can happen.

Infinite Series 22

For Series Σ $a_n st.$ $a, \ge a_z \ge .. \ge 0$

We can obtain two separate equal conditions chastening comes or div

<u>Thml</u> (Cauchy Condoumsatism The)If $a_1 \ge a_2 \ge ... \ge 0$ then

$$\frac{1}{2} \sum_{k=1}^{\infty} 2^{k} a_{2k} \le \sum_{n=1}^{\infty} a_n \le \sum_{k=0}^{\infty} 2^{k} a_{2k}$$

ntense Σa_n and $\Sigma 2^k$ a_{2k}

consomething n divege something

IS 22

For series $\sum a_n$

s.t. $a_1 \geq a_2 \geq \dots \geq 0$

we can obtain two separate equiv conditions characterizing conv or div

<u>Thm1</u> (Cauchy Condensation Thm)

If $a_1 \geq a_2 \geq \dots \geq 0$ then

$$\frac{1}{2}\sum_{k=1}^{\infty} 2^k a_{2^k} \leq \sum_{n=1}^{\infty} a_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k}$$

Hence $\sum a_n$ and $\sum 2^k a_{2^k}$ converge or diverge together

Infinite Series 23

$$\underline{\text{Pfi}} \sum_{k=1}^{\infty} a_k = \sum_{\infty=0}^{\infty} \Big(\sum_{2^n \le k < 2^{n+s}} a_k \Big)$$

$$\tfrac{1}{2}(2^{n+1} a_{2n+1}) \le \sum_{2^n \le k < 2^{n+1}} a_k \le 2^n a_{2n}$$

Summing over $n \ge 0$ given

$$\tfrac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2n} \le \sum_{k=1}^{\infty} a_k \sum_{n=0}^{\infty} 2^n a_{2n}$$

Application

$$\Sigma \tfrac{1}{n} = \infty \; iH \sum_{n=0}^{\infty} a_k \le \sum_{n=0}^{\infty} 1 = \infty$$

$$\underline{\text{Pf:}} \sum_{k=1}^{\infty} a_k = \sum_{n=0}^{\infty} \Big(\sum_{2^n \le k < 2^{n+1}} a_k \Big)$$

$$\frac{1}{2}\left(2^{n+1} a_{2^{n+1}}\right) \le \sum_{2^n \le k < 2^{n+1}} a_k \le 2^n a_{2^n}$$

Summing over $n \ge 0$ gives

$$\frac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2^n} \le \sum_{k=1}^{\infty} a_k \le \sum_{n=0}^{\infty} 2^n a_{2^n}$$

Application

$$\sum \frac{1}{n} = \infty \text{ iff } \sum_{n=0}^{\infty} 2^n \frac{1}{2^n} = \sum_{n=0}^{\infty} 1 = \infty$$

Since the RHS is infinite, so is the LHS

Since the RHS is infinite, so is the LHS

Theorem 2 (The Integral Test)

Suppose $a_1 \geq a_2 \geq \cdots \geq 0$.
Extend a_n to $a(x)$ where $a(x)$ is continuous, $a(n) = a_n$ and $a(x)$ decreases,
Then $\sum_n a_n < \infty$ if and only if $\int_1^\infty a(x)dx < \infty$

Proof

$$\int_1^\infty a(x)dx = \sum_{n=1}^\infty \int_n^{n+1} a(x)dx$$
$$\leq \sum_{n=1}^\infty \int_n^{n+1} a_n dx$$
$$= \sum_{n=1}^\infty a_n$$

Oct 24

Thm 2 (The Integral Test)

Spse $a_1 \geq a_2 \geq \cdots \geq 0$

Extend a_n to $a(x)$ where $a(x)$ is continuous, $a(n) = a_n$ and $a(x) \searrow$

Then $\sum_n a_n < \infty$ iff $\int_1^\infty a(x)\,dx < \infty$

Pf:

$$\int_1^\infty a(x)dx = \sum_{n=1}^\infty \int_n^{n+1} a(x)dx$$
$$\leq \sum_{n=1}^\infty \int_n^{n+1} a_n dx$$
$$= \sum_{n=1}^\infty a_n$$

Lower-bounding,

$$
\begin{aligned}
\int_1^n a(x)dx &= \sum_{n=2}^{\infty} \int_{n-1}^{n} a(x)dx \\
&\geq \sum_{n=2}^{\infty} \int_{n-1}^{\infty} a_n dx \\
&= \sum_{n=2}^{\infty} a_n
\end{aligned}
$$

Hence, $\sum_{n=2}^{\infty} a_n$ and $\int_1^{\infty} a(x)dx$
converge or diverge together.

Lower-bounding,

$$
\begin{aligned}
\int_1^{\infty} a(x) dx &= \sum_{n=2}^{\infty} \int_{n-1}^{n} a(x) dx \\
&\geq \sum_{n=2}^{\infty} \int_{n-1}^{n} a_n dx \\
&= \sum_{n=2}^{\infty} a_n
\end{aligned}
$$

Hence $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} a(x)dx$
converge or diverge together

Are $\sum_{j=1}^{\infty} a_j$ and $\sum_{k=1}^{\infty} b_k$ equal, where

$$b_k = \sum_{n_k \le j \le n_{k+1}} a_j$$

and $n_o = 0 < 1 = n_1 < n_2 < \cdots$?

More generally, when do all re-orderings of the terms of a series produce the same sum?

Q8 26

Are $\sum_{j=1}^{\infty} a_j$ and $\sum_{k=1}^{\infty} b_k$ aqual,

where $b_k = \sum_{n_k \le j < n_{k+1}} a_j$

and $n_o = 0 < 1 = n_1 < n_2 < \cdots$?

More generally, when do all re-orderings of the terms of a series produce the same sum?

Theorem: Let $a_j \geq 0$
and $F_1 \subseteq F_2 \subseteq \cdots$ with
$\cup_{n=1}^{\infty} F_n = \mathbb{N}$ and F_n is finite.
Then

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j\in F_n} a_j$$

Proof: Take away

$$s < S_\infty \equiv \sum_{j=1}^{\infty} a_j$$

$\exists N < \infty$ such that for all $n \geq N$, $a_1 + \cdots + a_n > s$

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Thm Let $a_j \geqslant 0$

and $F_1 \subseteq F_2 \subseteq \cdots$ with

$\bigcup_{n=1}^{\infty} F_n = \mathbb{N}$ and F_n finite.

Then $\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j\in F_n} a_j$

Pf: Take any

$s < s_\infty \equiv \sum_{j=1}^{\infty} a_j$.

$\exists N < \infty$ s.t. for all

$n \geqslant N$, $a_1 + \cdots + a_n > s$

Since $\cup_{n=1}^{\infty} F_n = \mathbb{N}$,
$\exists N' \geq N$ such that
$\{1, 2, \cdots, N\} \subseteq F_{N'}$
Hence for all $n \geq N'$

$$\sum_{j \in F_n} a_j \geq \sum_{j=1}^{N} a_j > s$$

Moreover, since F_n is finite,

$$\exists n* \geq max\{j \in F_n\}$$

Hence,$\sum_{j \in F_n} a_j \leq \sum_{j=1}^{n} a_j \leq s_\infty$ TEXT CUT OFF

Q8 28

Since $\bigcup_{n=1}^{\infty} F_n = \mathbb{N}$,

$\exists N' \geq N$ s.t.

$\{1, 2, \cdots, N\} \subseteq F_{N'}$

Hence for all $n \geq N'$

$$\sum_{j \in F_n} a_j \geq \sum_{j=1}^{N} a_j > s$$

Moreover, since F_n is finite, $\exists n^* \geq max\{j \in F_n\}$

Hence $\sum_{j \in F_n} a_j \leq \sum_{j=1}^{n^*} a_j \leq s_\infty$

Summation by Parts

Theorem Let $A_o = 0$, $A_n = a_1 + \cdots + a_n$
Suppose $\{A_n : n \geq 1\}$ is bounded.
Let $b_1 \geq b_2 \geq \cdots$ with $b_n \to 0$.
Then $\sum_{j=1}^{\infty} a_j b_j$ converges.

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Summation by Parts

Thm Let $A_0 = 0$,

$A_n = a_1 + \cdots + a_n$. Spse

$\{A_n : n \geq 1\}$ is bounded.

Let $b_1 \geq b_2 \geq \cdots$ with

$b_n \to 0$.

Then $\sum_{j=1}^{\infty} a_j b_j$ conv

(for all $n \geq 1$)

Proof: Suppose $|A_n| < A^* < \infty$.
Fix any $\epsilon > 0$. Take any $\delta > 0$ to be chosen later $\exists N$ such that for $n \geq N$
$0 \leq b_n < \delta_\epsilon$
For $n \geq N$ and $p \geq 0$,

$$\sum_{j=n}^{n+p} A_j b_j = \sum_{j=n-1}^{n+p} (A_j - A_{j-1}) b_j$$

$$= \sum_{j=n}^{n+p} A_j b_j - \sum_{j=n-1}^{n+p-1} A_j b_{j+1}$$

28 30
(for all $n \geq 1$)

Pf: Spse $|A_n| \leq A^* < \infty$

Fix any $\varepsilon > 0$. Take any $\delta > 0$ to be chosen later $\exists N$ s.t. for $n \geq N$

$0 \leq b_n < \delta \varepsilon$.

For $n \geq N$ and $p \geq 0$

$$\sum_{j=n}^{n+p} a_j b_j = \sum_{j=n}^{n+p} (A_j - A_{j-1}) b_j$$

$$= \sum_{j=n}^{n+p} A_j b_j - \sum_{j=n-1}^{n+p-1} A_j b_{j+1}$$

So

$$|\sum_{j=n}^{n+p} a_j b_j|$$

$$= |A_{n+p} b_{n+p} - A_{n-1} b_n + \sum_{j=n}^{n+p-1} A_j (b_j - b_{j+1})|$$

$$\leq |A_{n+p}| b_{n+p} + |A_{n-1}| b_n + \sum_{j=n}^{n+p-1} |A_j| |b_j - b_{j+1}|$$

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So

$$\left|\sum_{j=n}^{n+p} a_j b_j\right|$$

$$= \left|A_{n+p} b_{n+p} - A_{n-1} b_n + \sum_{j=n}^{n+p-1} A_j (b_j - b_{j+1})\right|$$

$$\leq |A_{n+p}| b_{n+p} + |A_{n-1}| b_n + \sum_{j=n}^{n+p-1} |A_j| |b_j - b_{j+1}|$$

$$
\begin{aligned}
&\le A^*\delta\epsilon + A^*\delta\epsilon + A^* \sum_{j=n}^{n+p-1} (b_j - b_{j+1}) \\
&\le 2A^*\delta\epsilon + A^*(b_n - b_{n+p}) \\
&\le 3A^*\delta\epsilon
\end{aligned}
$$

So let δ be any real number such that $0 < 3A^*\delta < 1$.
Hence $\sum_{j=1}^{n} a_j b_j$ satisfies the Cauchy criterion.
Therefore, it converges.

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$$
\le A^*\delta\varepsilon + A^*\delta\varepsilon + A^* \sum_{j=n}^{n+p-1} (b_j - b_{j+1})
$$

$$
\le 2A^*\delta\varepsilon + A^*(b_n - b_{n+p})
$$

$$
\le 3A^*\delta\varepsilon
$$

So let δ be any real st. $0 < 3A^*\delta < 1$.
Hence $\sum_{j=1}^{n} a_j b_j$ satisfies the Cauchy criterion.
Therefore it converges.

3) Thinking grandly, maybe all of mathematics can be put on a set theoretic foundation.

Let's try to do so.

<u>Some Set Theory</u>

A set can be defined by

(i) listing its elements

(ii) listing the properties that determine membership in the set.

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3 Thinking grandly maybe all of mathematics can be put on a set theoretic foundation

Let's try to do so,

Some Set Theory

A set can be defined by (i) listing its elements

(ii) listing the properties that determine membership in the set

Examples

- $\{1, 2, 5\}$
- { cat, bat, dog }
- $\{\{1, 2\}, 5\}$
- { odd primes }
- { positive integers having no odd divisors }

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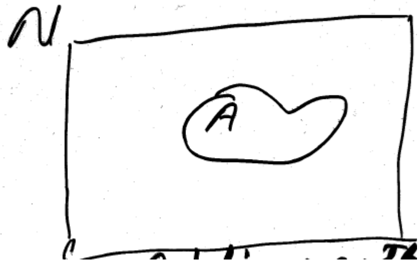
Ex $\{1, 2, 5\}$ {cat, bat, dog}

$\{\{1,2\}, 5\}$

{odd primes}

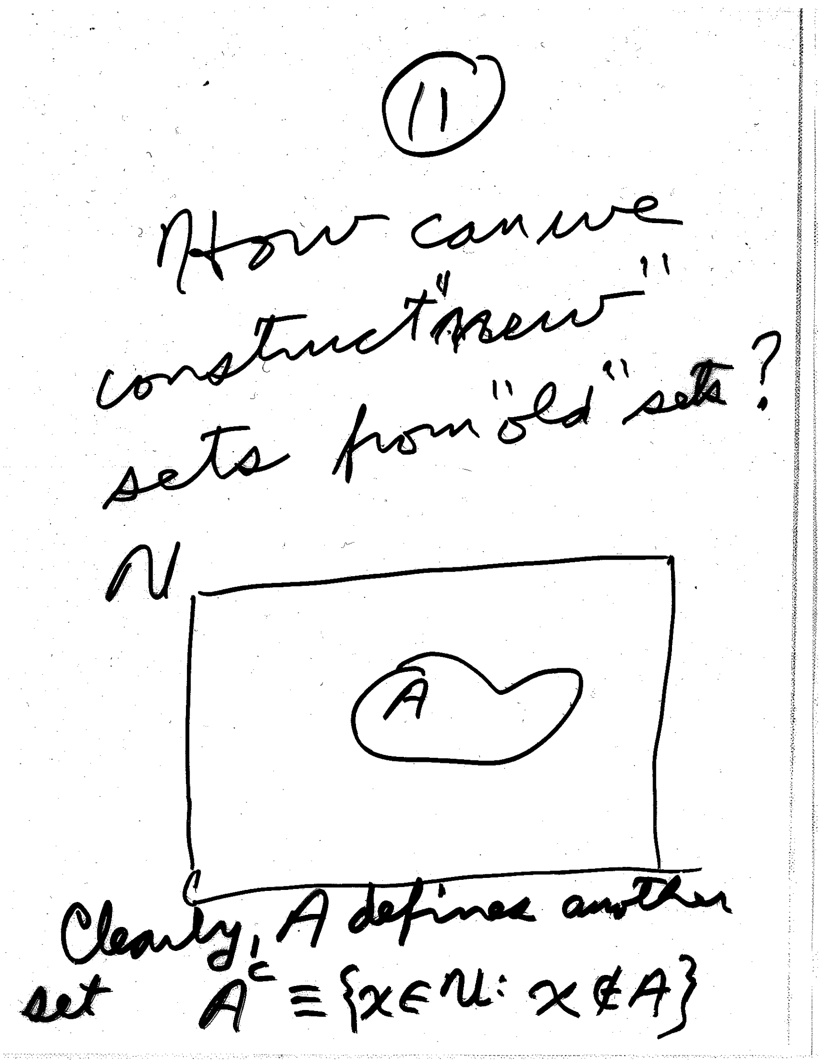
{positive integers having no odd divisors}

How can we construct "new" sets from "old" sets?



Clearly, A defines another set

$$A^c \equiv \{x \in \mathbb{U} : x \notin A\}$$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How can we construct "new" sets from "old" sets?

Clearly, A defines another set $A^c \equiv \{x \in \mathbb{U} : x \notin A\}$

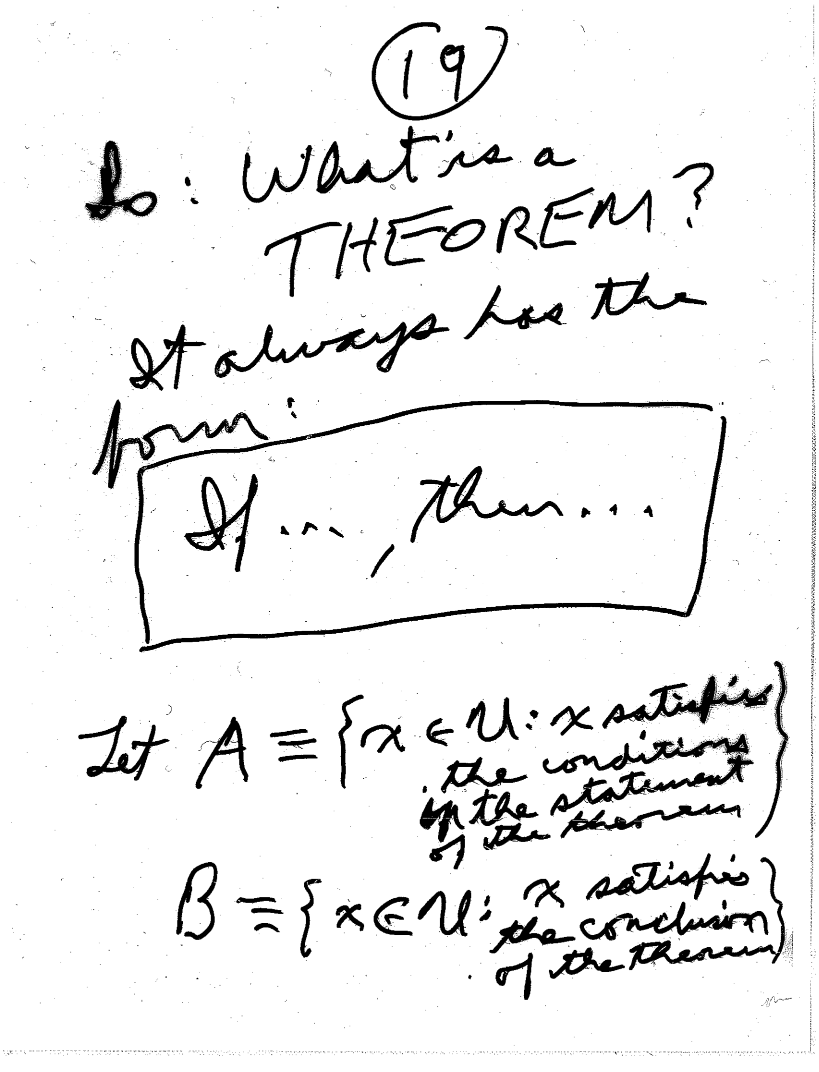
So: What is a THEOREM?

It always has the form

$$\boxed{\text{If..., then...}}$$

Let

- $A \equiv \{x \in \mathbb{U} : x$ satisfies the conditions in the statement of the theorem $\}$
- $B \equiv \{x \in \mathbb{U} : x$ satisfies the conclusion of the theorem $\}$


19

So: What is a THEOREM?

It always has the form:

If..., then...

Let $A \equiv \{x \in \mathcal{U} : x$ satisfies the conditions in the statement of the theorem$\}$

$B \equiv \{x \in \mathcal{U} : x$ satisfies the conclusion of the theorem$\}$

Hence, this theorem can be nested as nothing other than $A \subseteq B$.

Hence, a proof is just a logical demonstration:

For each $x \in A$, in fact $x \in B$ also.

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Hence, this theorem can be restated as nothing other than $A \subseteq B$,

Hence, a proof is just a logical demonstration:

For each $x \in A$, in fact $x \in B$ also

It is beyond the scope of this course to formalize how the statement $A \subseteq B$ may be proved. However, to illustrate what is required, it is sufficient to show:

For each $x \in A$, there exists sets

$$D_{x,1} \subseteq D_{x,2} \subseteq D_{x,3} \subseteq \cdots$$

such that $x \in D_{x,1}$ and

$$\bigcup_{j=1}^{\infty} D_{x,j} \subseteq B$$

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It is beyond the scope of this course to formalize how the statement $A \subseteq B$ may be proved. However, to illustrate what is required: It suffices to show:

For each $x \in A$ there exist sets

$$D_{x,1} \subseteq D_{x,2} \subseteq D_{x,3} \subseteq \cdots$$

such that $x \in D_{x,1}$

and $$\bigcup_{j=1}^{\infty} D_{x,j} \subseteq B$$

Corollary

(estimating series)

Let $b_1 \geq b_2 \geq ..., b_n \to 0$

then

$$\sum_{j=1}^{\infty}(-1)^{j-1}b_j$$

converges

Proof:

$$A_n = \sum_{J=1}^{n}(-1)^{j-1}$$

is less

Hence

$$s_n = \sum_{J=1}^{n}(-1)^{j-1}b_j$$

converges in R,

$$0 \leq s_2 \leq s_4 \leq ..,$$

so

D8 33

Corollary

(alternating Series)

Let $b_1 \geq b_2 \geq \cdots, b_n \to 0$

Then $\sum_{j=1}^{\infty}(-1)^{j-1} b_j$ conv

Pf: $A_n = \sum_{j=1}^{n}(-1)^{j-1}$ is bdd

Hence $s_n = \sum_{j=1}^{n}(-1)^{j-1} b_j$

conv in R,

$0 \leq s_2 \leq s_4 \leq \cdots,$

so

$\exists d \leq s_\infty \leq \infty$ such that
$s_n \to s_\infty$, moreover,
$*s_{2n} \leq s_\infty \leq s_{2n-1}$
since $s_1 \geq s_3 \geq ...$
consequently
$|s_\infty - s_n| \leq b_{n+1}$

d 8 34

$\exists\ 0 \leq s_\infty < \infty$ s.t.
$s_n \to s_\infty$. moreover,

$*$ $s_{2n} \leq s_\infty \leq s_{2n-1}$
since $s_1 \geq s_3 \geq \cdots$
Consequently
$|s_\infty - s_n| \leq b_{n+1}$.

Power Series

let

$$f(x) = \sum_{j=0}^{\infty} a_j x^j$$

Then $\exists 0 \leq R \leq \infty$ such that

$$\sum_{j=0}^{\infty} a_j x^j$$

converges absolutely for $0 \leq |x| < R$. and diverges for $|x| > R$

§8 35

Power Series

Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$

Then $\exists$ $0 \leq R \leq \infty$ s.t.

$\sum_{j=0}^{\infty} a_j x^j$ converges absolutely for $0 \leq |x| < R$.

and diverges for

$|x| > R$

Proof:Want $|a_j x^j| \leq \lambda^j < 1$ for converges
So, taking j^{th} roots, need $|x||a_j|^{\frac{1}{j}} \leq \lambda < 1$
Need

$$|x| \limsup_{j\to\infty} |a_j|^{\frac{1}{j}} < 1$$

$$|x| < \frac{1}{\bar{\lim}_{j\to\infty}|a_j|^{\frac{1}{j}}} \equiv R$$

Proof

$$|x| > R, \bar{\lim}_{J\to\infty}|x^j a_j| = \infty$$

2.2.36

Pf: Want

$|a_j x^j| \leq \lambda^j < 1$ for conv

So, taking j^{th} roots,

Need $|x| |a_j|^{\frac{1}{j}} \leq \lambda < 1$

Need $|x| \limsup_{j\to\infty} |a_j|^{\frac{1}{j}} < 1$

$$|x| < \frac{1}{\bar{\lim}_{j\to\infty} |a_j|^{\frac{1}{j}}} \equiv R$$

If $|x| > R$, $\bar{\lim}_{j\to\infty} |x^j a_j| = \infty$

8 Metric Spaces Part 1

How might we extend our mathematical investigations beyond the real numbers?
Reexamining what has already been useful, suppose we consider ordered pairs of reals $\mathbb{R}^2 \equiv \{(x, y) : x, y\epsilon\mathbb{R}\}$ since sequences

$\boxed{\text{MM1}}$ Metric Spaces I.

and Constr. of a metric

How might we extend our mathematical investigations beyond the real numbers?
Reexamining what has already been useful, suppose we consider ordered pairs of reals
$\mathbb{R}^2 \equiv \{(x, y) : x, y \in \mathbb{R}\}$
since sequences

of reals were of vital significance, consider $\{(x_n, y_n) : n \geq 1\} m x_n, y_n \epsilon \mathbb{R}$. What would it mean to say that such a sequence converged in $\mathbb{R}^2$? Clearly, $(x_n, y_n) \to (x_\infty, y_\infty)$ iff $x_n \to x_\infty$ and $y_n \to y_\infty$

m12

of reals were of vital significance, consider

$\{(x_n, y_n) : n \geq 1\}$ for $x_n, y_n \in \mathbb{R}$.

What would it mean to say that such a sequence converged in $\mathbb{R}^2$?

Clearly,

$(x_n, y_n) \to (x_\infty, y_\infty)$

iff

$x_n \to x_\infty$ and $y_n \to y_\infty$

So the notion of convergence in $\mathbb{R}^2$ seems clear, and it extends naturally to $\mathbb{R}^k$ But how far is a particular term in the sequence from its limit? In order to be able to answer such a question, we require a notion of distance between pairs

$\boxed{M3}$

So the notion of convergence in $\mathbb{R}^2$ seems clear, and it extends naturally to $\mathbb{R}^k$. But how far is a particular term in the sequence from its limit? In order to be able to answer such a question, we require a notion of distance between pairs

of points in the underlying set, be it $\mathbb{R}$, $\mathbb{R}^2$,$\mathbb{R}^k$, or just some abstract sets. To achieve broadest applicability we seek the most general possible notion of a distance function. What properties must a distance function have?

m4

of points in the underlying set, be it $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^k$, or just some abstract sets. To achieve broadest applicability we seek the most general possible notion of a distance function. What properties must a distance function have?

Given a non-empty set M, if $d(\cdot,\cdot)$ is a distance function on M then
(i) $d : M \times M \to (0, \infty)$
(ii) $d(x, y) = 0$ if $x = y$
(iii) $d(x, y) = d(y, x)$
Is this sufficient?
Suppose $d(x, y)$ denotes the distance along the shortest path from x to y. Then

$\boxed{m5}$

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symmetry (iii) $d(x. y) = d(y, x)$

Is this sufficient?

Suppose $d(x, y)$ denotes the distance along the shortest path from x to y. Then

Multiplication of Power Series

Let

$$f(x) = \sum_{j=0}^{\infty} a_j x^j$$

and

$$g(x) = \sum_{k=0}^{\infty} b_k x^k$$

Suppose f has radius of convergence R and g has radius of convergence $R^1 \geq R$

Let $h(x) = f(x)\ g(x)$ for $|x| \leq R$ with $|x| < R^1$

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Multip of Power Series

Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$

and $g(x) = \sum_{k=0}^{\infty} b_k x^k$

Spse f has rad of conv R

and g has rad of conv $R' \geq R$

Let $h(x) = f(x)\, g(x)$ for $|x| \leq R$ with $|x| < R'$

Then $\exists$ C_n such that

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

has radius of convergence at least R, where

$$C_n = \sum_{k=0}^{n} a_k b_{n-k}$$

I838

Then $\exists$ C_n s.t.

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

has rad of conv at least R.

where

$$C_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Pf: Let

$$F(x) = \sum_{j=0}^{\infty} |a_j x^j|$$

$$G(x) = \sum_{k=0}^{\infty} |b_k x^k|$$

Fix any $|x| < R$.
There exists $j_x < \infty$ and $0 < \lambda < 1$ such that $|a_j x^j| < \lambda^j$ and $|b_j x^j| < \lambda^j$ for all $j \geq j$
Let

$$f_n(x) = \sum_{j=0}^{2n} a j x^j$$

$$g_n(x) = \sum_{k=0}^{2n} b_k x^k$$

§ 8.39

Pf: Let $F(x) = \sum_{j=0}^{\infty} |a_j x^j|$

$G(x) = \sum_{k=0}^{\infty} |b_k x^k|$

Fix any $|x| < R$.
There exists $j_x < \infty$ and $0 < \lambda < 1$ s.t. $|a_j x^j| < \lambda^j$ and $|b_j x^j| < \lambda^j$ for all $j \geq j_x$

Let $f_n(x) = \sum_{j=0}^{2n} a_j x^j$

$g_n(x) = \sum_{k=0}^{2n} b_k x^k$

9 Metric Spaces Part 2

Def: Let m be non-empty set a metric d on m is a map
$d : m \times m \to [0,\infty)$
satisfying (for all x, y, z in m)
(i) $d(x, y) = d(y, x) \begin{cases} > 0 \text{ if } x \neq y \\ = 0 \text{ if } x = y \end{cases}$
(ii) $d(x, y) \leq d(x, z) + d(z, y)$
(iii) is called the triangle ineq

Metric Spaces II

m1

Def: Let m be non-empty set a metric d on m is a map
$d : m \times m \to [0, \infty)$
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(ii) $d(x, y) \leq d(x, z) + d(z, y)$
(ii) is called the triangle ineq

Examples

1. $d(x,y) = \begin{cases} 1\ if x \neq y \\ 0\ if x = y \end{cases}$

2. If $m = R^k$ and $1 \leq P \leq \infty$

$d_p(\vec{x},\vec{y}) = (\sum_{j=1}^{k}(x_j - y_j)^p)^{\frac{1}{p}}$

$d_\infty(\vec{x},\vec{y}) = max_{1\leq j\leq k}|x_j - y_j|$

MZ

Examples

① $d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$

② If $m = \mathbb{R}^k$ and $1 \leq P \leq \infty$

$d_p(\vec{x},\vec{y}) = \left(\sum_{j=1}^{k} (x_j - y_j)^p\right)^{\frac{1}{p}}$

$d_\infty(\vec{x},\vec{y}) = \max_{1 \leq j \leq k} |x_j - y_j|$

New Metrics From Old

Suppose $d(x, y)$ is a metric
on m and f:[0,∞)$\rightarrow$ [0,∞) is non-decreasing with
f(0)=0 and $f(x)$ >0 if x >0.
When will
$g(x, y) = f(d(x, y))$ be a metric?
We need to guarantee that the triangle inequality holds for g.

m3

New Metrics from Old

Spse $d(x,y)$ is a metric
on m and $f:[0,\infty)\to[0,\infty)$
is non-decreasing with
$f(0)=0$ and $f(x)>0$ if $x>0$.
When will
$g(x,y) = f(d(x,y))$
be a metric?

We need to guarantee
that the triangle ineq
holds for g.

Suppose $f(a+b) \leq f(a) + f(b)$
for $a \geq 0$, $b \geq 0$ (Then f is sub-additive)
then
$g(x,y) = f(d(x,y))$
$\leq f(d(x,z) + d(z,y))$
$\leq f(d(x,z)) + f(d(z,y))$
$= g(x,z) + g(z,y)$
If $f'(y) \geq 0$ and $f''(y) \leq 0$ and $f(0) = 0$ then f is sub-additive
ex. $f(y) = \sqrt{y}$

Pm 4

Spse $f(a+b) \leq f(a) + f(b)$
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$= g(x,z) + g(z,y)$
If $f'(y) \geq 0$ and $f''(y) \leq 0$
and $f(0) = 0$ then f is
sub-additive. Ex:
$f(y) = \sqrt{y}$

If d_1 and d_2 are metrics on m then
$d(x, y) = \max\{d_1(x, y), d_2(x, y)\}$ may not be a metric, but
$\min\{d(x, y), 1\}$ is a metric

m5

If d_1 and d_2 are metrics
on m then
$d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$
is a metric
Show by example
$\min\{d_1(x,y), d_2(x,y)\}$
may not be a metric.
but
$\min\{d(x,y), 1\}$
is a metric

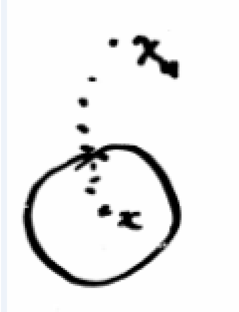
An Aside: why do mathematicians call virtually every non-empty set a <u>*space*</u>?
<u>*Answer*</u>:The <u>*space*</u> we inhabit can be represented by the <u>*set*</u> of ordered triples (x_1, x_2, x_3) of reals. Hence we think of a way non-empty set as an abstract space.

M 5A

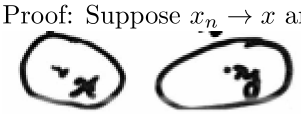
An Aside: Why do mathematicians call virtually every non-empty set a <u>space</u>?
<u>Answer</u>: The space we inhabit can be represented by the <u>set</u> of ordered triples (x_1, x_2, x_3) of reals. Hence we think of any non-empty set as an abstract space.

Convergence of a Sequence

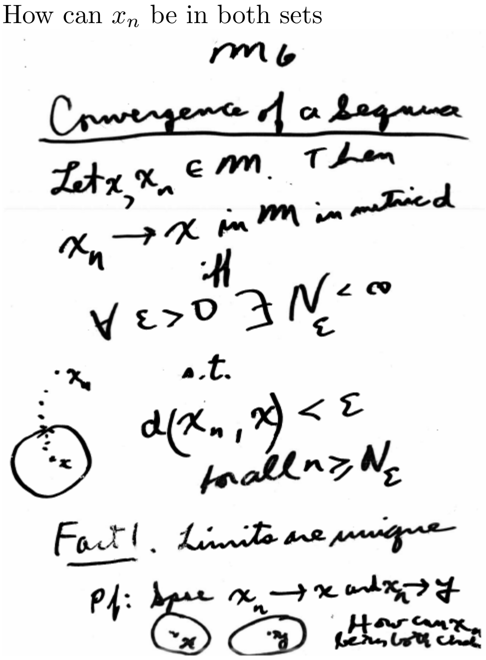
Let $x, x_n \in m$ Then $x_n \to x$ in m in metric d
if
$\forall\ \epsilon \geq 0\ \exists N_\epsilon < \infty$
such that
$d(x_n, x) < \epsilon$
for all $n \geq N_\epsilon$



Fact1. Limits are unique
Proof: Suppose $x_n \to x$ and $x_n \to y$



How can x_n be in both sets


rm6

Convergence of a Sequence

Let $x, x_n \in m$. Then
$x_n \to x$ in m in metric d
iff
$\forall\ \varepsilon > 0\ \exists N_\varepsilon < \infty$
s.t.
$d(x_n, x) < \varepsilon$
for all $n \geq N_\varepsilon$

Fact 1. Limits are unique

Pf: Suppose $x_n \to x$ and $x_n \to y$ How can x_n be in both circles

Subsequent Limits

Given { a_n }, let $\mathcal{L}$= { susequential limits }
Can we upper bound $\mathcal{L}$
Let b_1= suppose $\{a_1, a_2, ...\}$
b_2= suppose $\{a_2, a_3, ...\}$
b_n= suppose $\{a_n, a_{n+1}, ...\}$
$b_1 \geq b_2 \geq ...$

Subseq Limits

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Given $\{a_n\}$, let
$\mathcal{L} = \{$ subsequential limits $\}$

Can we upper bound $\mathcal{L}$?

Let
$b_1 = \sup\{a_1, a_2, \cdots\}$

$b_2 = \sup\{a_2, a_3, \cdots\}$

$b_n = \sup\{a_n, a_{n+1} \cdots\}$
$b_1 \geq b_2 \geq \cdots$

Definition:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

$$\equiv \lim_{n\to\infty}[\sup(a_k : k \geq n)] = \inf_n[\sup(a_k - k > n)]$$

Clearly,

$$\sup \mathcal{L} \leq \lim_{n\to\infty} a_n$$

Can you show they are equal?
What about inf $\mathcal{L}$?

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Def: $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

$\equiv \lim_{n\to\infty} \left(\sup\{a_k : k \geq n\}\right)$

$= \inf_n \left(\sup \{a_k : k \geq\right.$

Clearly,

$\sup \mathcal{L} \leq \lim_{n\to\infty} a_n$

Can you show they are equal?
What about inf $\mathcal{L}$?

Reformulation M8 Convergence

We can use collections of points to determine whether a sequence converges.
Let $D_r(x) = \{y \in M = d(x,y) < r\}$

Fact:

$$\lim_{n\to\infty} x_n = x$$

if and only if $\forall \varepsilon > 0 \exists N_\varepsilon < \infty : x_n \in D_\varepsilon(x)$
for all n $\geqslant N_\varepsilon$ $D_r(x)$ is called the open disk (ball, sphere) of radius r about x.

Reformulating M8 Convergence
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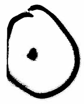

Fact: $\lim_{n\to\infty} x_n = x$ iff
$\forall \varepsilon > 0\ \exists N_\varepsilon < \infty : x_n \in D_\varepsilon(x)$ for all $n \geqslant N_\varepsilon$

$D_r(x)$ is called the open disk (ball, sphere) of radius r about x.

Rudin calls it an r neighborhood of x, or just a neighborhood. Are there more general families of sets of points in a metric space (m, d) that could be used to determine convergence of sequences regardless of the particular sequence or its limit?

mm4

Rudin calls its an r neighborhood of x, or just a neighborhood.

Are there more general families of sets of points in a metric space (m, d) that could be used to determine convergence of sequences regardless of the particular seq or its limit?

For each point x of such a set $\mathcal{O}$, if $x_n \to x$ there must be an $\varepsilon_o > 0$ such that for all $0 < \varepsilon < \varepsilon_o$
$D_\varepsilon(x) \subseteq \mathcal{O}$ so that $X_n \in \mathcal{O}$ for all n sufficiently large.
Definition $\mathcal{O} \subseteq \mathcal{M}$ is (m, d) an open set in $\mathcal{M}$ if and only if $\forall x \in \mathcal{O} \exists E > 0$ such that $D_\varepsilon(X) \subseteq \mathcal{O}$

$\mathcal{M}$ 10

For each point x of such a set $\mathcal{O}$, if $x_n \to x$ there must be an $\varepsilon_0 > 0$ s.t. for all $0 < \varepsilon < \varepsilon_0$

$D_\varepsilon(x) \subseteq \mathcal{O}$

so that $x_n \in \mathcal{O}$ for all n suff large

Defn $\mathcal{O} \subseteq \mathcal{M}$ is (m,d) an open set in $\mathcal{M}$ iff $\forall x \in \mathcal{O}\ \exists\, \varepsilon > 0$ s.t.

$D_\varepsilon(x) \subseteq \mathcal{O}$.

For all metric spaces (m, d), $\varnothing, \mathcal{M}$ and $D_r(x)$ for $x \in \mathcal{M}$ and $r \geq O$ are always open sets.

Properties of Open Sets

(i) Definition $\{\mathcal{O}_\alpha : \alpha \in J\}$ are open in $\mathcal{M}$ so is

$$\bigcup_{\alpha \in J} \mathcal{O}_\alpha$$

(ii) Definition $\mathcal{O}, \cdots, \mathcal{O}_n$ are open in $\mathcal{M}$ so is

$$\bigcap_{j=1}^{n} \mathcal{O}_j$$

(Arbitrary unions of open sets are open.
Intersections of finitely many open sets are open.)

m 11

For all metric spaces (m, d), ϕ, m and $D_r(x)$ for $x \in m$ and $r \geq 0$ are always open sets

Properties of Open Sets

(i) If $\{O_\alpha : \alpha \in J\}$ are open in m so is $\bigcup_{\alpha \in J} O_\alpha$

(ii) If $O_1, \ldots, O_n$ are open in m so is $\bigcap_{j=1}^{n} O_j$

(Arbitrary unions of open sets are open. Intersections of finitely many open sets are open.)

Definition x is an <u>interior point</u> of A $\subseteq M$ if an only if $\exists\varepsilon > 0 : D_\varepsilon(x) \subseteq A$

$A^O = \{$All interior points of A $\}$ <u>Fact: </u>A^O is the largest open subset of A.

<u>Definition</u> x is a limit point of $A \subseteq M$ if and only if $\forall\varepsilon > 0 |D_\varepsilon(x) \bigcap A| \geq 2$ if an only if $\forall\varepsilon > 0 (D_\varepsilon(x)\backslash\{x\}) \bigcap A \neq \emptyset$

m112

<u>Defn</u> x is an <u>interior point</u> of $A \subseteq m$ iff $\exists\, \varepsilon > 0 :\ D_\varepsilon(x) \subseteq A$

$A^0 = \{$ All interior pts of $A\}$

<u>Fact:</u> A^0 is the largest <u>open subset</u> of A.

<u>Defn</u> x is a limit pt of $A \subseteq m$ iff $\forall\, \varepsilon > 0\ |D_\varepsilon(x) \cap A| \geq 2$

iff $\forall\, \varepsilon > 0\ (D_\varepsilon(x) - \{x\}) \cap A \neq \phi$

Proposition x is a limit point of A if and only if $\exists$ sequence of distinct points of A that converges to x.
Definition $F \subseteq M$ is closed if and only if every limit point of F belongs to F.

M 13

Prop x is a limit pt of A iff $\exists$ seq of distinct pts of A that converges to x.

Defn $F \subseteq M$ is closed iff every limit pt of F belongs to F

Examples of closed sets in $\mathbb{R}^1$ and $\mathbb{R}^2$

$\emptyset$ and M are always closed space F_j and F_α are closed in M.

i if $F_1, \cdots, F_n$ are closed, so is

$$\bigcup_{j=1}^{n} F_j$$

ii if F_α is closed for $\alpha \in J$ so is

$$\bigcap_{\alpha \in J} F_\alpha$$

m 14

(I) Examples of closed sets in $\mathbb{R}^1$ and $\mathbb{R}^2$

ϕ and M are always closed space F_j and F_α are closed in M

(i) if $F_1, \cdots, F_n$ are closed, so is $\bigcup_{j=1}^{n} F_j$

(ii) if F_α is closed for $\alpha \in J$. so is $\bigcap_{\alpha \in J} F_\alpha$

How should we define ∂E, the boundary of a set E? Certainly $\partial E \subseteq \overline{E}$ since the boundary could involve points of E and points infinitesimally close to points of E. However, $\partial E = \partial E^c$ so $\partial E \subseteq \overline{(E^c)}$

M 18

How should we define ∂E, the boundary of a set E?

Certainly $\partial E \subseteq \overline{E}$ since the boundary could involve pts of E and pts infinitessimally close to pts of E.

However, $\partial E = \partial E^c$ so $\partial E \subseteq \overline{(E^c)}$

Definition $\partial E = \overline{E} \bigcap \overline{(E^c)}$
∂E is called the boundary of the set E.
Can $\partial E \underset{\neq}{\supseteq} E$?
Fact: $\overline{E} = E \bigcup (\partial E)$

Suppose $E \subseteq F \subseteq M(m,d)$
Definition E is dense in F
if and only if $\overline{E} \supseteq F$
if and only if every point of F in $E \bigcap$ is a limit point of E
if an only if $\forall x \in F \exists x_n \in$ such that $x_n \to x$
Then E is dense in F

m 19

Defn $\partial E = \overline{E} \cap \overline{(E^c)}$
∂E is called the boundary of the set E.

Can $\partial E \underset{\neq}{\supseteq} E$?

Fact: $\overline{E} = E \cup (\partial E)$

Spse $E \subseteq F \subseteq M$ (m,d)
Defn E is dense in F
iff $\overline{E} \supseteq F$
iff every pt of F is in $E \cap$ is a limit pt of E
iff $\forall x \in F \; \exists x_n \in E$ s.t. $x_n \to x$

Then E is dense in F

$m15$

<u>Theorem</u> $F \subseteq M$ is closed if and only if F^c is open
<u>Pf</u> : $(\Rightarrow)$ Suppose F is closed if F^c is not open $\exists x \in F^c$
suppose that $\forall \in > o \; \exists x_{\in} \neq x$:

$$x_{\in} \in D_{\in}(x) \bigcap F$$

Hence x is a limit point of F But F contains all its limit points so $x \in F$ continued. Thus F^c is open

m15

<u>Thm</u> $F \subseteq M$ is closed
iff F^c is open
Pf: $(\Longrightarrow)$ Spse F is closed
If F^c is not open $\exists x \in F^c$
s.t. $\forall \varepsilon > 0 \; \exists x_\varepsilon \neq x$:

$$x_\varepsilon \in D_\varepsilon(x) \cap F$$

Hence x is a limit pt of F
But F contains all its limit
pts so $x \in F$, contrad
Thus F^c is open.

$m16$

($\Leftarrow$) suppose F^c is open if F is not closed $\exists$ sequence of distinct points x_n of F which conveys to a point $x \notin F$. Since $x \in F^c$ and F^c is open
$\exists \in > 0$ such that $D_\in$ (x) $\subset F^c$ also $\exists N : x_n \in D_e(x)$ for all $n > N$ But then $x_v \in F \bigcap F^c = \emptyset$ continued. Hence F is closed.

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m 16

($\Leftarrow$) Spse F^c is open
If F is not closed $\exists$
seq of distinct pts x_n of F
which converges to a
pt $x \notin F$.
Since $x \in F^c$ and
F^c is open
$\exists\ \varepsilon > 0$ s.t. $D_\varepsilon(x) \subseteq F^c$
Also $\exists N$: $x_n \in D_\varepsilon(x)$
for all $n \geq N$
But then $x_N \in F \cap F^c = \emptyset$
contrad. Hence F is closed

$m17$

Define let E $\leq$ m. Let E' denote the limit points of E then $\overline{E}$, the closure of E, is defined to be E$\bigcup$E'

- Fact 1 $\overline{E}$ is closed
- Fact 2 If F is closed and E $\leq$ F then $\overline{E} \leq$ F

Corollary $\overline{E} = \bigcap$ F $/Fclosed : E \leq F/$

m17

Defn Let $E \subseteq m$. Let E' denote the limit pts of E Then $\overline{E}$, the closure of E, is defined to be $E \cup E'$.

Fact 1 $\overline{E}$ is closed

Fact 2 If F is closed and $E \subseteq F$ then $\overline{E} \subseteq F$

Corollary $\overline{E} = \bigcap_{\{F \text{ closed}:\, E \subseteq F\}} F$