Math 104
Topics Covered

How are sets defined?
What is a theorem?
What is a proof?
Why does proof by contrad often simplify a problem?

Construction of
the integers
(by set theory
or Peano axioms)
the rationals
Need to define cross products and Equiv Relation
basic thm about equiv rels

② To construct the reals:
Dedekind cuts and the
no-gaps thm

Or else:
use total ordering
upper bounds
lower bounds
LUB(S) sup(S)
GLB(S) inf(S)
Least Upper Bound Property
(Completeness Axiom)
Archimedean property
Reals via sequences of
decimals
limits of sequences

mappings (3)

Cardinalities of sets

$|A| \le |B|$

Schroeder - Bernstein

rationals are countable

countable union of countable sets is countable

reals are uncountable

$|S| < |2^S|$

reals are totally ordered

$b = LUB(S)$ iff b is an upper bound of S and $\forall\, b_0 < b$ b_0 is not an upper bound of S

seqs, subseqs

subseq limits

$\liminf_{n\to\infty} a_n$ $\limsup_{n\to\infty} a_n$

What can prevent a seq from converging?

Bdd monotonic ~~seqs~~ converge in $\mathbb{R}$.

Thm $\{a_n\}$ converges in $\mathbb{R}$ iff $\exists$ bdd monotonic $\{b_n\}$ s.t. $a_n - b_n \longrightarrow 0.$

Cauchy seq

$\{x_n\}$ conv in $\mathbb{R}$ iff $\{x_n\}$ is a Cauchy seq

5

Cauchy seqs are bdd

If a subseq of a Cauchy seq converges to x, so does the entire seq.

Series $s_n = \sum_{j=1}^{n} a_j$

Conv & Diverg of series

s_n conv $\Rightarrow a_n \to 0$

~~Term series~~

Comparison Test

Integral test

Ratio Test

Root Test

Cauchy Condensation Thm

Summary (6) by goals

Alternating Series

Power Series

Radius of Conv of pwr series

Metric Spaces (M, d)

metrics & examples

constructing metrics

Convergent seqs and

Cauchy seqs in M

Completeness of M

Every metric space

has a completion

Open sets

defn + properties

$D_r(x)$

7

A° interior, interior pts

pts of closure $\bar{A}$

A' limit pts (pts of accum / cluster pts)

$\bar{A} = A \cup A'$

∂A boundary of A

$\bar{A} = \bigcap_{\{F \subseteq M : A \subseteq F \text{ and } F \text{ closed}\}} F$

Relatively open sets

" closed sets

$f : \overset{d_1}{M_1} \longrightarrow \overset{d_2}{M_2}$

Which maps preserve

convg seqs $x_n \longrightarrow x^*$?

Def: f cont at x^* iff

$\forall \varepsilon > 0 \ \exists \delta > 0$:

$$f(D_\delta(x^*)) \subseteq D_\varepsilon(f(x^*))$$

Equivalently, f is cont at x^* iff $\forall (x_n) \subseteq M_1$:

$$x_n \to x^* \Rightarrow f(x_n) \to f(x^*)$$

~~[illegible]~~

f cont on M_1 iff

$f^{-1}(\mathcal{O})$ open in M_1 whenever $\mathcal{O}$ open in M_2

iff pre-images of closed sets are closed

(9)

When does F preserve
Cauchy seqs?
uniform cont

Connectedness

a separation
(a,b) connected
cont fcns preserve
connectedness

(EVTh)

If A connected and
$A \subseteq B \subseteq \bar{A}$ then B connected

If A_γ connected for $\gamma \in J$
and $A_\gamma \cap A_\beta \neq \phi$ for all
γ, β in J then $\bigcup_{\gamma \in J} A_\gamma$ conn

Cont fcns also preserve sequentially cpt sets

Since we may "always" assume we are dealing with a bounded metric (i.e. just replace $d(x,y)$ by $\min\{d(x,y),1\}$)

when is a metric space intrinsically large? small?

M is intrinsically small a finitary iff it is totally bounded.

This extends to compactness

Fact: K cpt, F closed $\Rightarrow K \cap F$ cpt

Let $K \subseteq \mathcal{M}^d$

TFAE

(i) K is cpt

(ii) K is totally bdd and complete

(iii) K is sequentially cpt (also limit pt cpt)

~~In~~ In $\mathbb{R}^k$ K is cpt iff K is closed + bdd

What about other metric spaces? Which sets are cpt?

$\mathcal{M}$ separable $\Rightarrow$ $\mathcal{M}$ has countable base

Cont fcns preserve cpt sets

cont f achieves its max & its min over that cpt set

f unif cont on cpt set if f cont there

Integration

Construction of R-S Integral

$\int_a^b f(x)\,d\alpha(x)$ exists if

(i) f cont (ii) f monot & $\alpha(\cdot)$ cont

~~(ii)~~

Right-hand and left-hand limits of F (and f')

Then if for $x \in [a,b]$

$$0 = \lim_{y \to x} (F(y) - F(x))(\alpha(y) - \alpha(x))$$

and $D_F = \{\text{disconts of } f \text{ on } [a,b]\}$

Then $\int_a^b f\,d\alpha$ exists iff

$m^*(D_F) = 0$

If $\alpha(\cdot)$ is strictly increasing and $\int_a^b f\,d\alpha$ exists
Then $m^*(D_F) = 0$

$\int_a^b \Phi(f(x))\,d\alpha(x)$

graph of f $\{(x, f(x))\}$

monotonic fcns have at most countably many disconts on $\mathbb{R}$

derivative

properties of deriv

$(af+bg)'$

$(fg)'$

$(f \circ g)'$

Fund Thm Calc

Taylor's Thm with integral and/or differential remainder

Taylor series expansion of f

15

Power series is diff inside its radius of conv

$f'(x_0) = 0$ at an extremum

mean value thm

$f(b) - f(a) = (b-a)f'(\xi)$

generalized mean val thm

cont of derivs

EVThm for deriv

L' ~~H~~ospital's Rule