Infinite Series

A frog is 2 feet from a wall. He makes a succession of jumps toward it, always jumping half his remaining distance to the wall.

Hence his first jump is one foot.

Now he is one foot from the wall. So

his second jump

is $\frac{1}{2}$ foot

He is now $\frac{1}{2}$ foot from

the wall so he jumps

$\frac{1}{4}$ foot.

He makes successive

jumps of

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

After n jumps he has

moved $s_n = \sum_{j=1}^{n} \frac{1}{2^{j-1}}$ feet and is now $\frac{1}{2^{n-1}}$ feet from the wall

so $s_n = 2 - \frac{1}{2^{n-1}}$

If he keeps jumping forever, how far does he go?

He moves $\displaystyle\sum_{j=1}^{\infty} \frac{1}{2^{j-1}}$ feet

This number is at most 2 and yet it exceeds $2 - \frac{1}{2^n}$ for all $n \geq 1$.

Hence the sum of this infinite collection of numbers $\{1, \frac{1}{2}, \frac{1}{4}, \cdots\}$ must be two.

How can we generalize this?

<u>Generalization</u>! Geometric Series

How large is $1+r+r^2+\cdots$ if $|r|<1$?

<u>Soln</u>: Let $S_n = 1+r+\cdots+r^n$

For $r \geq 0$, $S_1 \leq S_2 \leq \cdots$ and we expect $\lim_{n\to\infty} S_n = \sum_{j=0}^{\infty} r^j$

$$S_n = 1 + r + \cdots + r^n$$

is complex in that it has too many terms. How can we simplify it? We need to capitalize on the regularity of the expression.

Notice:

$$rS_n = r + r^2 + \cdots + r^{n+1}$$

Subtracting equals from equals

$$S_n - rS_n = 1 - r^{n+1}$$

so for $r \neq 1$, $S_n = \dfrac{1-r^{n+1}}{1-r}$

$\to \dfrac{1}{1-r}$

Generalization 2

Given a sequence of reals

$a_1, a_2, \ldots$, how (and when) can we assign a unique meaning to

$$\sum_{j=1}^{\infty} a_j \ ?$$

Soln Let $s_n = \sum_{j=1}^{n} a_j$

If the sequence of partial sums $s_n \longrightarrow L$ we say $\sum_{j=1}^{\infty} a_j = L$

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If the seq of partial sums s_n ~~either~~ has finite limit L we say the series <u>converges</u> (to L). Otherwise either $\{s_n\}$ has no limit or $|s_n| \to \infty$. In these cases we say the series <u>diverges</u>.

The most definitive statement that can be make concerning whether a series conv or div is based on the following so-called <u>Cauchy criterion</u>.

<u>Theorem</u> $\sum_{j=1}^{\infty} a_j$ converges iff $\forall\ \varepsilon > 0\ \exists\ N < \infty$ s.t. for all $n \geq N$ and $0 \leq k < \infty$

$$|a_n + a_{n+1} + \cdots + a_{n+k}| < \varepsilon$$

Pf: Let $s_n = a_1 + \cdots + a_n$

The series converges iff $\{s_n\}$ is a Cauchy seq

iff $\forall \varepsilon > 0 \ \exists N < \infty$

s.t. for all $n \geq N$, $k \geq 0$,

$$\left| s_{n+k} - s_{n-1} \right| < \varepsilon$$

Equivalently, iff $\forall n \geq N, k \geq 0$

$$\left| a_n + a_{n+1} + \cdots + a_{n+k} \right| < \varepsilon.$$

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Corollary 1 If $\sum_1^\infty a_j$ conv

then $\lim_{n\to\infty} a_n = 0$

Pf: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1})$

$= 0$ if $\{s_n\}$ is Cauchy

Corollary 2 If $\limsup_{n\to\infty} |a_n|$

> 0 then $\sum_1^\infty a_j$ diverges

Comparison Test

<u>Thm</u> Spse $|a_n| \le b_n$

Let $S_n = \sum_{j=1}^{n} a_j$, $T_n = \sum_{j=1}^{n} b_j$

and $\bar{S}_n = \sum_{j=1}^{n} |a_j|$

Then

(i) If $\lim_{n\to\infty} T_n < \infty$ then

S_n and $\bar{S}_n$ converge

(ii) If $\bar{S}_n$ diverges

then T_n diverges

<u>Def</u>: When $\lim_{n\to\infty} \bar{S}_n < \infty$ $\sum^{\infty} a_j$ is said to converge absolutely

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Pf: Spse T_n conv.

Fix any $\varepsilon > 0$

$\exists N$: $|T_{n+k} - T_n| < \varepsilon$

for all $n \geq N$ and $k > 0$

For such $n \geq N$ and $k > 0$

$$\left|S_{n+k} - S_n\right| = \left|\sum_{n<j\leq n+k} a_j\right|$$

$$\leq \sum_{n<j\leq n+k} |a_j|$$

$$= \bar{S}_{n+k} - \bar{S}_n$$

$$\leq \sum_{n<j\leq n+k} b_j = T_{n+k} - T_n$$

$< \varepsilon$ Hence S_n and $\bar{S}_n$ are Cauchy seq.

Therefore S_n and $\overline{S}_n$

converge.

(ii) Suppose $\overline{S}_n$ diverges.

Then $\infty = \lim_{n\to\infty} \overline{S}_n \leq \lim_{n\to\infty} T_n$

so $\{T_n\}$ diverges.

For series $s_n = \sum_{j=1}^{n} a_j$

whose terms decrease

"like" a geometric series

there are a couple of

methods of testing for

convergence – divergence.

The simpler is the ratio test

<u>Thm</u> Let $r^* = \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}$

and $r_* = \underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}$

(i) If $r^* < 1$ then $\sum_{n=1}^{\infty} a_n$ <u>conv</u> <u>absol.</u>

(ii) If $r_* > 1$ then $\sum_{n=1}^{\infty} a_n$ <u>div</u>

(iii) If $r_* \leq 1 \leq r^*$ the test in <u>not</u> conclusive

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Pf: Spse $r^* < 1$

Take any $r^* < \lambda < 1$

$\exists N$: for $n \geq N$, $\left|\frac{a_{n+1}}{a_n}\right| \leq \lambda$

$$\therefore \sum_{n=N+1}^{\infty} |a_n|$$

$$= \sum_{k=1}^{\infty} |a_{N+k}|$$

$$= \sum_{k=1}^{\infty} |a_N| \prod_{j=1}^{k} \left|\frac{a_{N+j}}{a_{N+j-1}}\right|$$

$$\leq |a_N| \sum_{k=1}^{\infty} \lambda^k < \infty$$

as series converges

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(iii) Spse $r_* > 1$

Take $1 < \lambda < r_*$

$\exists N : n \geq N$

$\Longrightarrow \left|\frac{a_{n+1}}{a_n}\right| \geq \lambda$

We assume this ~~means~~

$|a_n| > 0$ for $n \geq N$.

Hence $|a_{N+k}| \geq \lambda^k |a_N|$

for $k \geq 1$ so that

$|a_{N+k}| \to \infty$ as $N+k$ goes to infinity

Thus $\sum a_n$ diverges

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Case 1

(iii) $a_n = n \quad r_* = r^* = 1$

$$\sum a_n = \infty$$

Case 2

$a_n = \frac{1}{n^2} \quad r_* = r^* = 1$

yet $\sum a_n < \infty$

Pf: $$\sum_{n=1}^{\infty} \frac{1}{n^2} \le \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

Telescoping series

$$= \lim_{N\to\infty} 2\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{N\to\infty} 2\left(1 - \frac{1}{N+1}\right)$$

$$= 2 < \infty$$

Hence a series for which $r_* \le 1 \le r^*$ can either conv or div

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A refinement of the ratio test, the Root test does not require that every successive ratio is "well-behaved", but merely that some ~~prepondere~~ preponderance are.

Thm Let $r = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$

(i) If $r < 1$ the series conv absolutely

(ii) If $r > 1$ the series diverges

(iii) If $r = 1$ the test is inconclusive.

Pf: (i) Spse $r < 1$ and take any $r < \lambda < 1$

$\exists N : n \geq N \Rightarrow |a_n|^{\frac{1}{n}} \leq \lambda$

Hence $|a_n| \leq \lambda^n$

By the comparison test

$\sum a_n$ conv absol.

(ii) Spse $r > 1$ and take $1 < \lambda < r$

For ~~any~~ $N < \infty$

$\exists n > N : |a_n|^{\frac{1}{n}} \geq \lambda$

so $|a_n| \geq \lambda^n \longrightarrow \infty$

Hence $\limsup_{n \to \infty} |a_n| = \infty$

$\therefore \sum_n a_n$ diverges

(iii) If $r = 1$, test fails

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Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Let $R = \dfrac{1}{\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}}}$

Then

(i) series conv absol for $0 \le |x| < R$

(ii) series div for $|x| > R$

(iii) at $x = R$ we may have conv or div

$0 \le R \le \infty$. R is called the radius of conv of the power series.

$\Rightarrow$

Pf: By Root Test,

$\sum_n a_n x^n$ conv absol

if $\overline{\lim_{n\to\infty}} |a_n x^n|^{\frac{1}{n}} < 1$

iff $|x| \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} < 1$

iff $|x| < \dfrac{1}{\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}}} \equiv R$

If $\overline{\lim_{n\to\infty}} |a_n x^n|^{\frac{1}{n}} > 1$ series div

so series div if $|x| > R$

When $|x| = R$ anything can happen.

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For series $\sum a_n$

s.t. $a_1 \geq a_2 \geq \cdots \geq 0$

we can obtain two separate equiv conditions characterizing conv or div

Thm 1 (Cauchy Condensation Thm)

If $a_1 \geq a_2 \geq \cdots \geq 0$ then

$$\frac{1}{2}\sum_{k=1}^{\infty} 2^k a_{2^k} \leq \sum_{n=1}^{\infty} a_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k}$$

Hence $\sum a_n$ and $\sum 2^k a_{2^k}$ converge or diverge together

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Pf: $\sum_{k=1}^{\infty} a_k = \sum_{n=0}^{\infty} \left(\sum_{2^n \le k < 2^{n+1}} a_k \right)$

$$\frac{1}{2}\left(2^{n+1} a_{2^{n+1}}\right) \le \sum_{2^n \le k < 2^{n+1}} a_k \le 2^n a_{2^n}$$

Summing over $n \geq 0$ gives

$$\frac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2^n} \le \sum_{k=1}^{\infty} a_k \le \sum_{n=0}^{\infty} 2^n a_{2^n}$$

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application

$$\sum \frac{1}{n} = \infty \text{ iff } \sum_{n=0}^{\infty} 2^n \frac{1}{2^n} = \sum_{n=0}^{\infty} 1 = \infty$$

Since the RHS is infinite, so is the LHS

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Thm 2 (The Integral Test)

Spse $a_1 \geq a_2 \geq \cdots \geq 0$

Extend a_n to $a(x)$ where

$a(x)$ is continuous, and $a(x) \searrow$

$a(n) = a_n$

Then $\sum_n a_n < \infty$ iff $\int_1^\infty a(x)\,dx < \infty$

Pf:

$$\int_1^\infty a(x)\,dx = \sum_{n=1}^\infty \int_n^{n+1} a(x)\,dx$$

$$\leq \sum_{n=1}^\infty \int_n^{n+1} a_n\,dx$$

$$= \sum_{n=1}^\infty a_n$$

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Lower-bounding,

$$\int_1^\infty a(x)\,dx = \sum_{n=2}^{\infty} \int_{n-1}^{n} a(x)\,dx$$

$$\geq \sum_{n=2}^{\infty} \int_{n-1}^{n} a_n\,dx$$

$$= \sum_{n=2}^{\infty} a_n$$

Hence $\sum_{n=1}^{\infty} a_n$ and $\int_1^\infty a(x)\,dx$

converge or diverge together

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Are $\sum_{j=1}^{\infty} a_j$ and

$\sum_{k=1}^{\infty} b_k$ equal,

where $b_k = \sum_{n_k \le j < n_{k+1}} a_j$

and $n_0 = 0 < 1 = n_1 < n_2 < \cdots$?

More generally, when do all re-orderings of the terms of a series produce the same sum?

Thm Let $a_j \geq 0$ and $F_1 \subseteq F_2 \subseteq \cdots$ with $\bigcup_{n=1}^{\infty} F_n = \mathbb{N}$ and F_n finite.

Then $\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j\in F_n} a_j$

Pf: Take any $s < s_\infty \equiv \sum_{j=1}^{\infty} a_j$.

$\exists N < \infty$ s.t. for all $n \geq N$, $a_1 + \cdots + a_n > s$

Since $\bigcup_{n=1}^{\infty} F_n = \mathbb{N}$,

$\exists\, N' \geq N$ s.t.

$\{1, 2, \cdots, N\} \subseteq F_{N'}$

Hence for all $n \geq N'$

$$\sum_{j \in F_n} a_j \geqslant \sum_{j=1}^{N} a_j > s$$

Moreover, since F_n is finite, $\exists\, n^* \geqslant \max\{j \in F_n\}$

Hence $\sum_{j \in F_n} a_j \leq \sum_{j=1}^{n^*} a_j \leq s$ and ∞

Summation by Parts

<u>Thm</u> Let $A_0 = 0$,

$A_n = a_1 + \cdots + a_n$. Spse

$\{A_n : n \geq 1\}$ is bounded.

Let $b_1 \geq b_2 \geq \cdots$ with

$b_n \longrightarrow 0$.

Then $\sum_{j=1}^{\infty} a_j b_j$ conv

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(for all $n \geq 1$)

<u>Pf:</u> Suppose $|A_n| \leq A^* < \infty$

Fix any $\varepsilon > 0$. Take any $\delta > 0$ to be chosen later

$\exists N$ s.t. for $n \geq N$

$0 \leq b_n < \delta\varepsilon$.

For $n \geq N$ and $p \geq 0$

$$\sum_{j=n}^{n+p} a_j b_j = \sum_{j=n}^{n+p} (A_j - A_{j-1}) b_j$$

$$= \sum_{j=n}^{n+p} A_j b_j - \sum_{j=n-1}^{n+p-1} A_j b_{j+1}$$

So

$$\left|\sum_{j=n}^{n+p} a_j b_j\right|$$

$$= \left| A_{n+p} b_{n+p} - A_{n-1} b_n + \sum_{j=n}^{n+p-1} A_j (b_j - b_{j+1}) \right|$$

$$\leq |A_{n+p}| b_{n+p} + |A_{n-1}| b_n + \sum_{j=n}^{n+p-1} |A_j| |b_j - b_{j+1}|$$

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$$\leq A^* \delta \varepsilon + A^* \delta \varepsilon$$

$$+ A^* \sum_{j=n}^{n+p-1} (b_j - b_{j+1})$$

$$\leq 2A^* \delta \varepsilon + A^* (b_n - b_{n+p})$$

$$\leq 3A^* \delta \varepsilon$$

So let δ be any real s.t.

$0 < 3A^* \delta < 1$.

Hence $\sum_{j=1}^{n} a_j b_j$ satisfies the Cauchy criterion.

Therefore it converges.

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Corollary (Alternating Series)

Let $b_1 \geq b_2 \geq \cdots , b_n \to 0$

Then $\sum_{j=1}^{\infty} (-1)^{j-1} b_j$ conv

Pf: $A_n = \sum_{j=1}^{n} (-1)^{j-1}$ is bdd

Hence $A_n = \sum_{j=1}^{n} (-1)^{j-1} b_j$

conv in R,

$0 \leq A_2 \leq A_4 \leq \ldots$

SO

$\exists \; 0 \le \Delta_\infty < \infty$ s.t.

$\Delta_n \to \Delta_\infty$. Moreover,

(*) $\Delta_{2n} \le \Delta_\infty \le \Delta_{2n-1}$

since $\Delta_1 \ge \Delta_3 \ge \cdots$

Consequently

$|\Delta_\infty - \Delta_n| \le b_{n+1}$.

Power Series

Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$

Then $\exists$ $0 \le R \le \infty$ s.t.

$\sum_{j=0}^{\infty} a_j x^j$ converges absolutely

for $0 \le |x| < R$.

and diverges for

$|x| > R$

Pf: Want

$|a_j x^j| \le \lambda^j < 1$ for conv

So, taking j^{th} roots,

Need $|x|\,|a_j|^{\frac{1}{j}} \le \lambda < 1$

Need $|x| \limsup_{j\to\infty} |a_j|^{\frac{1}{j}} < 1$

$$|x| < \frac{1}{\overline{\lim}_{j\to\infty} |a_j|^{\frac{1}{j}}} \equiv R$$

If $|x| > R$, $\overline{\lim}_{j\to\infty} |x^j a_j| = \infty$

Multip of Power Series

Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$

and $g(x) = \sum_{k=0}^{\infty} b_k x^k$

Spse f has rad of conv R
and g has rad of conv $R' \geq R$

Let $h(x) = f(x)\, g(x)$ for $|x| \leq R$ with $|x| < R'$

Then $\exists\, c_n$ s.t.

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

has rad of conv at least R.

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

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P1: Let $F(x) = \sum_{j=0}^{\infty} |a_j x^j|$

$$G(x) = \sum_{k=0}^{\infty} |b_k x^k|$$

Fix any $|x| < R$.

There exists $j_* < \infty$ and $0 < \lambda < 1$ s.t.

$$|a_j x^j| < \lambda^j$$

and $|b_j x^j| < \lambda^j$ for all $j \geq j_*$

Let $f_n(x) = \sum_{j=0}^{2n} a_j x^j$

$$g_n(x) = \sum_{k=0}^{2n} b_k x^k$$

Then

$$h_n(x) \equiv f_n(x)\, g_n(x) \longrightarrow h(x)$$

Notice that

$$h_n(x) = \sum_{j=0}^{2n} a_j x^j \sum_{k=0}^{n} b_k x^k$$

$$= \sum_{r=0}^{2n} \left(\sum_{j=0}^{r} a_j b_{r-j} \right) x^r$$

$$+ \sum_{\{0 \leq j,k \leq 2n \,:\, j+k > 2n\}} a_j b_k x^{j+k}$$

$$= \tilde{h}_n(x) + \gamma_n(x)$$

For $n \geqslant j_*$,

$$|\gamma_n(x)| \leq \sum_{j=0}^{n} |a_j x^j| \sum_{k=n+1}^{\infty} |b_k x^k|$$

$$+ \sum_{j=n+1}^{\infty} |a_j x^j| \sum_{k=0}^{n} |b_k x^k|$$

$$\leq F(x) \sum_{k=n+1}^{\infty} \lambda^k$$

$$+ G(x) \sum_{j=n+1}^{\infty} \lambda^j$$

$$= \frac{(F(x)+G(x))\lambda^{n+1}}{1-\lambda} \to 0$$

as $n \to \infty$

Therefore

$$h_n(x) - \tilde{h}_n(x) \to 0$$

so

$$h(x) = \lim_{n\to\infty} \tilde{h}_n(x)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) x^n$$