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To demystify mathematics, consider

(i) What is a Theorem?

(ii) What is a Proof?

What if we don't know the answer?

To begin we need

(a) an example(s)

(b) a nearby related concept

Related Concept:
Greek Syllogism

<u>Example</u>:

(1) All men are mortal
(2) Socrates is a man
∴ (3) Socrates must die

To analyze, recast in set theoretic terms via Venn diagram

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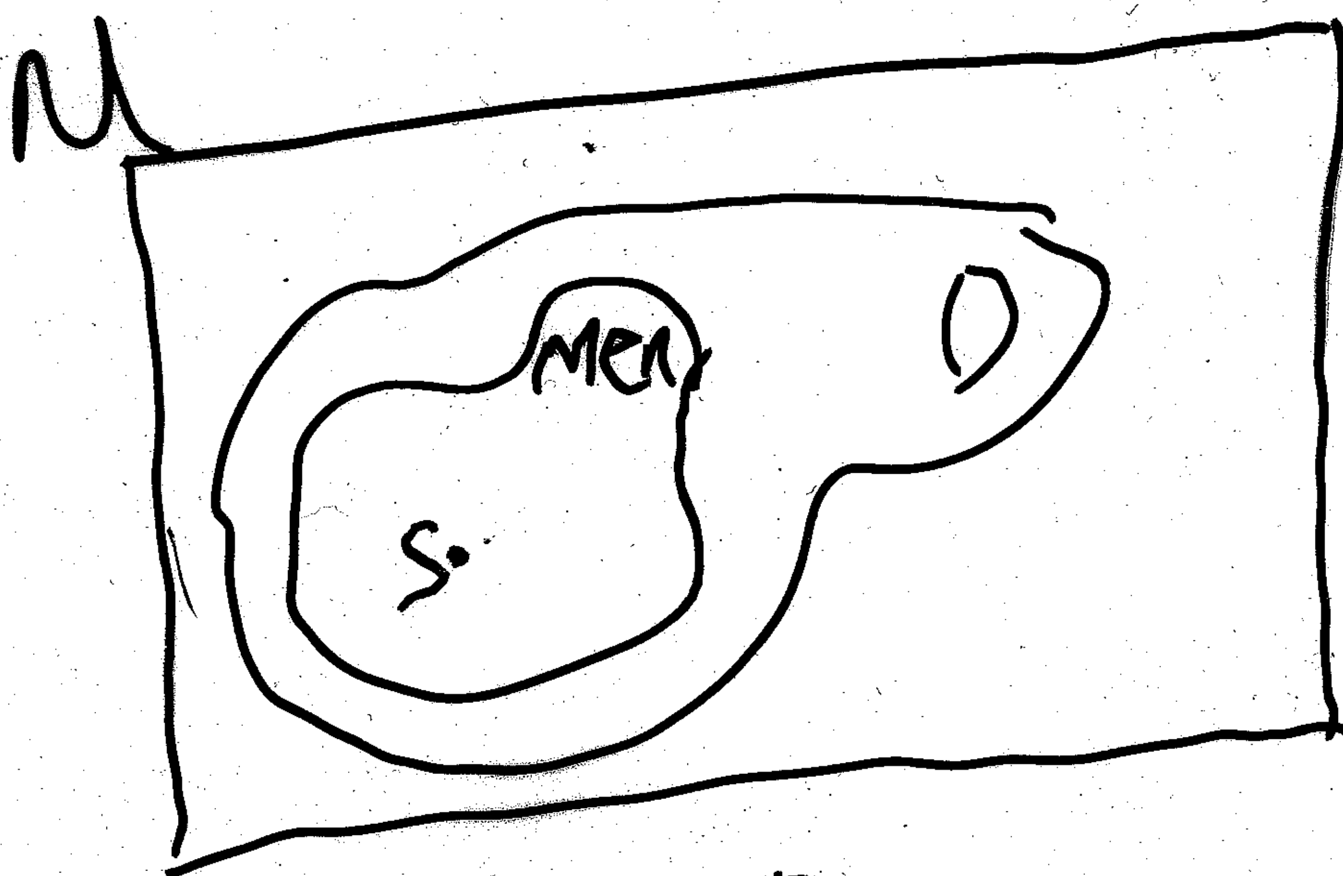

S : Socrates
M : set of men
D : things that die
U : things on earth

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Notice:

proper noun Socrates became an element (point)

common noun men became a set (collection of pts)

the property of being mortal became a set

the universe of all things under consideration became the Universe of Possibilities

The set theoretic representation of the syllogism:

(1) $M \subseteq D$

(2) $S \in M$

$\therefore$ (3) $S \in D$

Rearranging, we obtain a more logical ordering of the facts (doing so by successive inclusion):

$S \in M,\ M \subseteq D;\ \therefore S \in D.$

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What has the syllogism taught us?

1. Issues of fact (truth or falsehood) can be put in a set-theoretic context.
2. The simplest deductive argument has the form if $x \in A$ and $A \subseteq B$ then $x \in B$.

So: What is a THEOREM?

It always has the form:

$$\boxed{\text{If} \ldots, \text{then} \ldots}$$

Let $A \equiv \{x \in \mathcal{U} : x \text{ satisfies the conditions in the statement of the } \cancel{\text{theorem}}\}$

$B \equiv \{x \in \mathcal{U} : x \text{ satisfies the conclusion of the theorem}\}$

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Hence, this theorem can be restated as nothing other than $A \subseteq B$; proof

Hence, a 'logical proof' is just a demonstration:

For each $x \in A$, in fact $x \in B$ also

It is beyond the scope of this course to formalize how the statement $A \subseteq B$ may be proved. However, to illustrate what is required:

It suffices to show:

for each $x \in A$ there exist sets

$$D_{x,1} \subseteq D_{x,2} \subseteq D_{x,3} \subseteq \cdots$$

such that $x \in D_{x,1}$

and

$$\bigcup_{j=1}^{\infty} D_{x,j} \subseteq B$$

(9)

(3) Thinking grandly maybe all of mathematics can be put on a set theoretic foundation

Let's try to do so.

Some Set Theory

A set can be defined by (i) listing its elements (ii) listing the properties that determine membership in the set

(10)

Ex $\{1, 2, 5\}$ $\{\text{cat}, \text{hat}, \text{dog}\}$

$\{\{1, 2\}, 5\}$

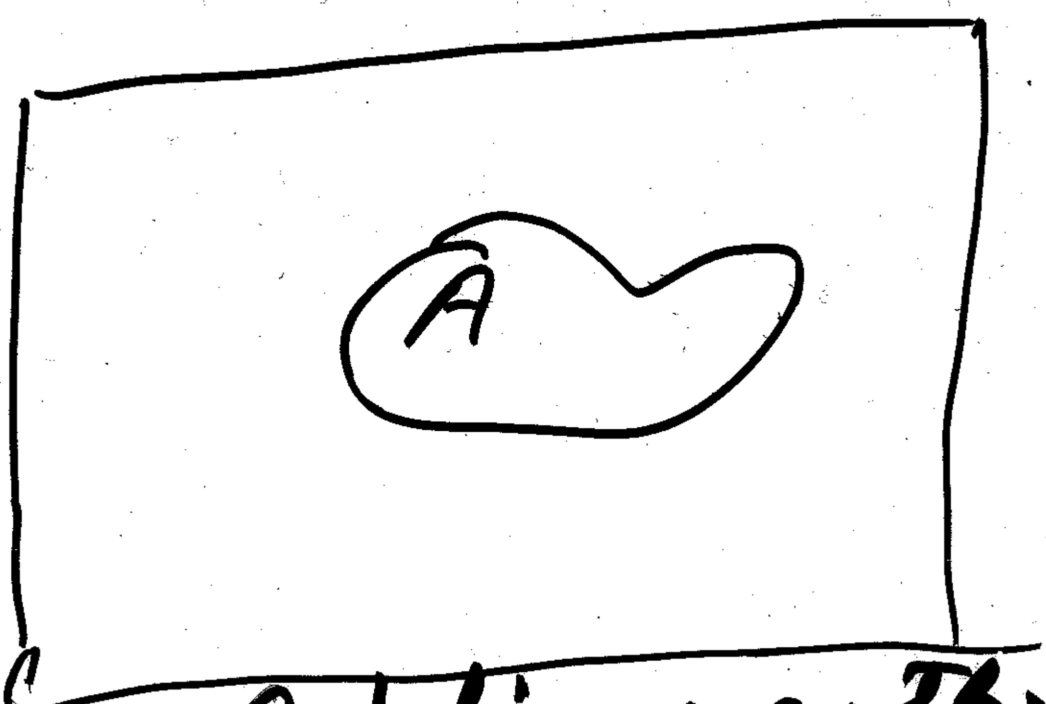
$\{\text{odd primes}\}$

$\{\text{positive integers having no odd divisors}\}$

(11)

How can we construct "new" sets from "old" sets?

$\mathcal{U}$



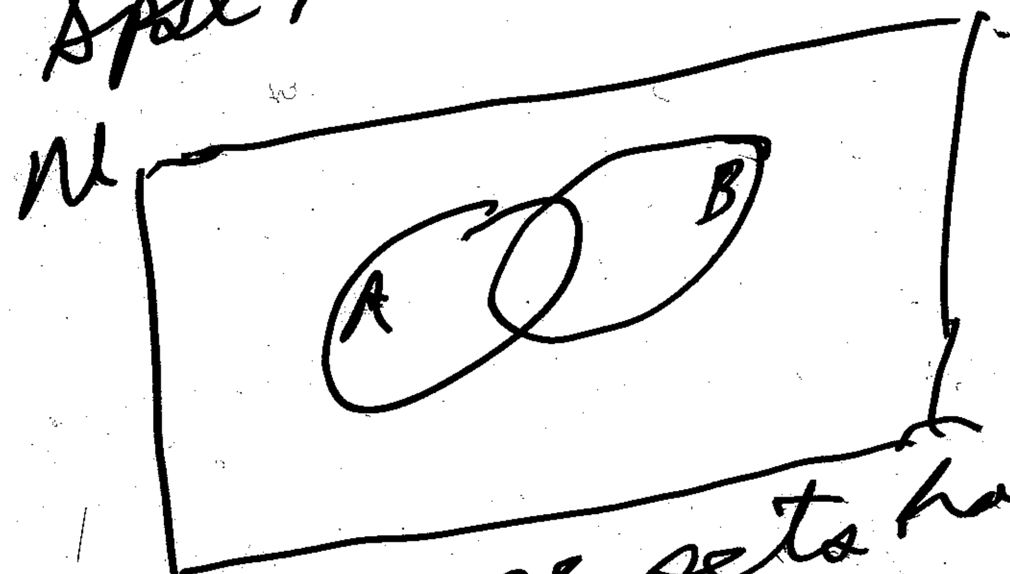
Clearly, A defines another set $A^c \equiv \{x \in \mathcal{U} : x \notin A\}$

(12)

A^c ; complement of A

also written $A^c \equiv \mathcal{U} \setminus A$

Spse we have two sets A, B



Now more sets have been generated.

union $A \cup B \equiv \{x \in \mathcal{U} : x \in A \text{ or } x \in B \text{ or both}\}$

intersection $A \cap B \equiv \{x \in \mathcal{U} : x \in A \text{ and } x \in B\}$

Spse we require that these 3 operations ^{c}, $\cup$, $\cap$ always produce sets.

Then

$$A \cap A^c \equiv \{x \in U : x \in A \text{ and } x \notin A\}$$

must be a set in U.

It is called <u>the</u> <u>empty</u> <u>set</u> ϕ. The set with no elements.

13A

1) $B \cap \left(\bigcup_{\alpha \in J} A_\alpha \right) \stackrel{?}{=}$

2) $B \cup \left(\bigcap_{\alpha \in J} A_\alpha \right) \stackrel{?}{=}$

3) $\left(\bigcap_{\alpha \in J} A_\alpha \right)^c \stackrel{?}{=}$

4) $\left(\bigcup_{\alpha \in J} A_\alpha \right)^c \stackrel{?}{=}$

What do we do next?

Begin constructing sets.

{apple}, {pear}, {grape}

{eagle}, {bear, deer},

{pencil, paper}

{sun, moon, orion}, ...

These examples motivate the need to

(22A)

What next?

~~Spse~~ we try to generate all (?) sets from ~~the simple~~ simple to complex

Examples

{apple} {pear}

{cat} {dog, hat}

{atoms in your body}

What ~~makes~~ these sets most similar?

22B

Notice:

{apple} and {pear}

are merely relabelings of one another.

As are

{dog, hat} and {cat, owl}

What is a relabeling of a set?

Perhaps sets A and B have the same 'size' if they are relabelings of one another.

We seem to be dancing around the concept of number.