Metric Spaces I

$m1$

Def: Let M be non-empty set

A metric d on M is a map

$$d : M \times M \longrightarrow [0, \infty)$$

satisfying (for all x, y, z in M)

$$\text{(i)}\quad d(x, y) = d(y, x) \begin{cases} > 0 \text{ if } x \neq y \\ = 0 \text{ if } x = y \end{cases}$$

$$\text{(ii)}\quad d(x, y) \leq d(x, z) + d(z, y)$$

(ii) is called the triangle ineq

$M2$

Examples

① $d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$

② If $M = \mathbb{R}^k$ and $1 \leq p < \infty$

$$d_p(\vec{x},\vec{y}) = \left(\sum_{j=1}^{k} |x_j - y_j|^p\right)^{\frac{1}{p}}$$

$$d_\infty(\vec{x},\vec{y}) = \max_{1 \leq j \leq k} |x_j - y_j|$$

New Metrics from Old

Spse $d(x,y)$ is a metric on M and $f:[0,\infty)\to[0,\infty)$ is non-decreasing with $f(0)=0$ and $f(x)>0$ if $x>0$.

When will
$$g(x,y)=f(d(x,y))$$
be a metric?

We need to guarantee that the triangle ineq holds for g.

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Spse $f(a+b) \leq f(a) + f(b)$

for $a \geq 0, b \geq 0$ (Then f is <u>sub-additive</u>)

Then

$$g(x,y) = f(d(x,y))$$
$$\leq f(d(x,z) + d(z,y))$$
$$\leq f(d(x,z)) + f(d(z,y))$$
$$= g(x,z) + g(z,y)$$

If $f'(y) \geq 0$ and $f''(y) \leq 0$ and $f(0) = 0$ then f is sub-additive.

Ex: $f(y) = \sqrt{y}$

M5

If d_1 and d_2 are metrics on M then

$$d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$$

is a metric

Show by example

$$\min\{d_1(x,y), d_2(x,y)\}$$

may not be a metric.

but

$$\min\{d(x,y), 1\}$$

is a metric

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An Aside: Why do mathematicians call virtually every non-empty set a <u>space</u>?

<u>Answer</u>: ~~[illegible]~~ The <u>space</u> we inhabit can be represented by the <u>set</u> of ordered triples (x_1, x_2, x_3) of reals. Hence we think of any non-empty set as an abstract space.

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Convergence of a Sequence

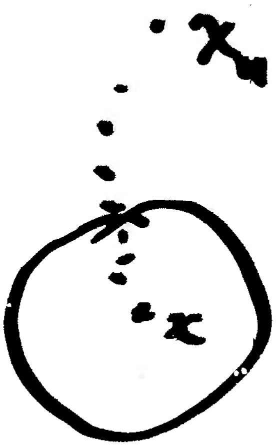
Let $x, x_n \in M$. Then

$x_n \to x$ in M in metric d

iff

$\forall \varepsilon > 0 \ \exists N_\varepsilon < \infty$

s.t.

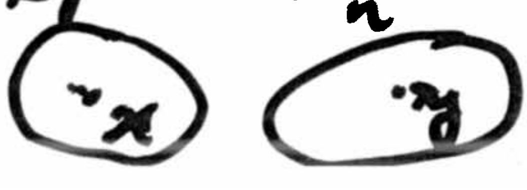


$d(x_n, x) < \varepsilon$

for all $n \geq N_\varepsilon$

Fact 1. Limits are unique

Pf: Suppose $x_n \to x$ and $x_n \to y$



How can x_n be in both circles

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If $d_0 = d(x, y) > 0$

$\exists N < \infty$ s.t. for all

$n \geq N$ $\quad d(x_n, x) < \frac{d_0}{2}$

and $d(x_n, y) < \frac{d_0}{2}$

Hence

$$d_0 = d(x, y) \leq d(x, x_n) + d(x_n, y)$$

$$< \frac{d_0}{2} + \frac{d_0}{2} = d_0$$

contrad

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<u>Def:</u> metrics d_1 and d_2 on M are <u>equivalent</u> iff they have the <u>same</u> <u>convergent sequences</u>

<u>Ex</u> Two metrics on $\mathbb{R}^k$

$$d_1(\vec{x},\vec{y}) = \sum_{j=1}^{k} |x_j - y_j|$$

$$d_\infty(\vec{x},\vec{y}) = \max_{1\le j\le k} |x_j - y_j|$$

<u>Ex</u>. On $\mathbb{R}^\infty$ let

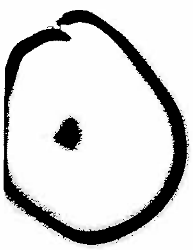
$$d_1(\vec{x},\vec{y}) = \min\left\{1, \sum_{j=1}^{\infty}|x_j - y_j|\right\}$$

$$d_\infty(\vec{x},\vec{y}) = \min\left\{1, \sup_{1\le j<\infty} |x_j - y_j|\right\}$$

Reformulating MM8 Convergence

We can use collections of points to determine whether a seq converges.

Let $D_r(x) = \{y \in M : d(x,y) < r\}$



Fact: $\lim_{n\to\infty} x_n = x$ iff

$\forall \varepsilon > 0 \ \exists N_\varepsilon < \infty : x_n \in D_\varepsilon(x)$ for all $n \geq N_\varepsilon$

$D_r(x)$ is called the open disk (ball, sphere) of radius r about x.

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Rudin calls its an r neighborhood of x, or just a neighborhood.

Are there more general families of sets of points in a metric space (M, d) that could be used to determine convergence of sequences regardless of the particular seq or its limit?

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For each point x of such a set $\mathcal{O}$, if $x_n \longrightarrow x$

there must be an $\varepsilon_0 > 0$ s.t.

for all $0 < \varepsilon < \varepsilon_0$

$D_\varepsilon(x) \subseteq \mathcal{O}$

so that $x_n \in \mathcal{O}$ for all

n suff large

<u>Defn</u> $\mathcal{O} \subseteq \mathfrak{M}$ is $(\mathfrak{M}, d)$ an <u>open set</u> in $\mathfrak{M}$ iff

$\forall x \in \mathcal{O} \;\; \exists \; \varepsilon > 0$ s.t.

$D_\varepsilon(x) \subseteq \mathcal{O}$.

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For all metric spaces (M, d), ϕ, M and $D_r(x)$ for $x \in M$ and $r > 0$ are always open sets

Properties of Open Sets

(i) If $\{O_\alpha : \alpha \in J\}$ are open in M so is $\bigcup_{\alpha \in J} O_\alpha$

(ii) If $O_1, \ldots, O_n$ are open in M so is $\bigcap_{j=1}^{n} O_j$

(Arbitrary unions of open sets are open. Intersections of finitely many open sets are open.)

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Defn x is an <u>interior point</u> of $A \subseteq \mathcal{M}$ iff
$\exists \varepsilon > 0 : D_\varepsilon(x) \subseteq A$

$A^\circ = \{$ all interior pts of $A\}$

<u>Fact:</u> A° is the largest open subset of A.

<u>Defn</u> x is a limit pt of $A \subseteq \mathcal{M}$
iff $\forall \varepsilon > 0 \quad |D_\varepsilon(x) \cap A| \geq 2$

iff $\forall \varepsilon > 0 \quad (D_\varepsilon(x) - \{x\}) \cap A \neq \phi$

<u>Prop</u> x is a limit pt of A iff $\exists$ seq of distinct pts of A that converges to x.

<u>Defn</u> $F \subseteq M$ is <u>closed</u> iff every limit pt of F belongs to F

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Examples of closed sets in $\mathbb{R}^1$ and $\mathbb{R}^2$

ϕ and m are always closed and F_j and F_α are closed in m space

(i) if $F_1, \ldots, F_n$ are closed, so is $\bigcup_{j=1}^{n} F_j$

(ii) if F_α is closed for $\alpha \in J$ so is $\bigcap_{\alpha \in J} F_\alpha$

<u>Thm</u> $F \subseteq M$ is closed iff F^c is open

Pf: ($\Longrightarrow$) Spse F is closed.

If F^c is not open $\exists\, x \in F^c$ s.t. $\forall\, \varepsilon > 0$ $\exists\, x_\varepsilon \neq x$:

$$x_\varepsilon \in D_\varepsilon(x) \cap F$$

Hence x is a limit pt of F

But F contains all its limit pts so $x \in F$, contrad.

Thus F^c is open.

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$(\Leftarrow)$ Spse F^c is open

If F is not closed $\exists$ seq of distinct pts x_n of F which converges to a pt $x \notin F$.

Since $x \in F^c$ and F^c is open

$\exists\ \varepsilon > 0$ s.t. $D_\varepsilon(x) \subseteq F^c$

Also $\exists\ N : x_n \in D_\varepsilon(x)$ for all $n \geq N$

But then $x_N \in F \cap F^c = \phi$

contrad. Hence F is closed

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Defn Let $E \subseteq \mathfrak{m}$. Let E' denote the limit pts of E. Then $\bar{E}$, the closure of E, is defined to be $E \cup E'$.

Fact 1 $\bar{E}$ is closed

Fact 2 If F is closed and $E \subseteq F$ then $\bar{E} \subseteq F$

Corollary $\bar{E} = \bigcap_{\{F \text{ closed}:\ E \subseteq F\}} F$

How should we define
∂E, the boundary of a
set E?

Certainly
$\partial E \subseteq \overline{E}$
since the boundary could
~~the~~ involve pts of E and
pts infinitessimally
close to pts of E.
However, $\partial E = \partial E^c$
so $\partial E \subseteq \overline{(E^c)}$

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Defn $\partial E \equiv \overline{E} \cap \overline{(E^c)}$

∂E is called the <u>boundary</u> of the set E.

Can $\partial E \supsetneq E$?

Fact: $\overline{E} = E \cup (\partial E)$

(m, d)

Spse $E \subseteq F \subseteq m$

Defn E is dense in F

iff $\overline{E} \supseteq F$

iff every pt of F is in E or is a limit pt of E

Thm E is dense in F iff $\forall x \in F \ \exists x_n \in E$ s.t. $x_n \to x$