Finding 50 Limits and

Proving Convergence

Example 1 $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2 $\lim_{n\to\infty} \frac{3n+1}{7n-4} = ?$

Example 3 $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = ?$

Example 4 Find $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

HW Spse $a_n \to a > 0$

Prove: $\sqrt{a_n} \to \sqrt{a}$

Properties of limits

Thm Spse $a_n \to a$ and $b_n \to b$. Then

$a_n + b_n \to a + b$

(and $\lambda a_n \to \lambda a$)

and $a_n b_n \to a \cdot b$

Thm If $a_n \to a \neq 0$,

then $\frac{1}{a_n} \to \frac{1}{a}$

Cor If $a_n \to a$

and $b_n \to b \neq 0$ then

$\frac{a_n}{b_n} \to \frac{a}{b}$

(50B)

Sample Proof:

<u>Prop</u> Spse $a_n \to a$, $b_n \to b$ $(a, b \in \mathbb{R})$ Then $a_n b_n \to ab$

<u>Pf</u>: Fix any $\varepsilon > 0$ and $\lambda > 0$ (λ to be specified later (depending on ε))

$\exists N < \infty$ s.t. for $n \geq N$ $|a_n - a| < \lambda\varepsilon$

$|b_n - b| < \lambda\varepsilon$

The problem posed by $a_n b_n$ is that two factors are varying with n

Ideally we'd like only one to vary at a time. But how?

50C

$|a_n b_n - ab|$

$= |a_n b_n - ab_n + a(b_n - b)|$

$\leq |b_n||a_n - a| + |a||b_n - b|$

$< |b_n| \lambda \varepsilon + |a| \lambda \varepsilon$ for $n \geq N$

since b_n conv in R to b

N can be chosen so large

that $|b_n| \leq 1 + |b| \equiv B$ for $n \geq N$

$\therefore$ for $n \geq N$, $|a_n b_n - ab| < (B + |a|) \lambda \varepsilon$

Let $\lambda = \dfrac{1}{B + |a| + 1}$

Then $|a_n b_n - ab| < \varepsilon$ for $n \geq N$

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Given $\{a_n\}$,

how might we construct its limit in R_∞ when it has one? in R when it has one?

<u>Def</u>: $\{a_n\}$ is monotonic in R iff $a_n \in R$ and either
(i) $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$
or (ii) $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$

<u>Def</u>: $\{a_n\}$ is bounded in R iff $\exists\ B \in R$ s.t. $|a_n| \leq B$ for all n

Thm Let $\{a_n\}$ be monot in $\mathbb{R}$. Then in $\mathbb{R}_\infty$

$$\lim_{n\to\infty} a_n = \begin{cases} \sup\{a_k\} \text{ if } a_1 \le a_2 \le a_3 \le \cdots \\ \inf\{a_k\} \text{ if } a_1 \ge a_2 \ge \cdots \end{cases}$$

The limit is in $\mathbb{R}$ iff $\{a_n\}$ is bounded.

Pf. W.l.o.g. spse $a_1 \le a_2 \le \cdots$

Take any $b < \sup\{a_k\}$

Then $\exists\, k_* < \infty$ s.t.

$b < a_{k_*}$. For all

$n \ge k_*$, $a_n \ge a_{k_*}$ so

$a_n > b$.

But also $a_n \le \sup\{a_k\}$

(52A)

What if $\{a_n\}$ is **not** monotonic?

Thm Let $\{a_n\}$ be seq in $\mathbb{R}$.
Then $\exists\ 1 \le n_1 < n_2 < \dots$ s.t. $a_{n_1}, a_{n_2}, \dots$ is monotonic

Pf: We need to consider the tail of a ~~seq~~. Let

$$J = \{j \ge 1 : a_j > a_{j+k} \text{ for all } k \ge 1\}$$

If J is unbounded we may write $J = \{n_j : j \ge 1\}$ where $1 \le n_1 < n_2 < \dots$

(52B)

By construction,

$a_{n_1} > a_{n_2} > \cdots$

On the other hand, if $|J| < \infty$, let $n_1 \geq 1$ exceed every $j \in J$. Having constructed $n_1 < n_2 < \cdots < n_k$ with $a_{n_1} \leq a_{n_2} \leq \cdots \leq a_{n_k}$ (since $n_k \notin J$) $\exists\, n_{k+1} > n_k$ s.t. $a_{n_k} \leq a_{n_{k+1}}$, and the desired conclusion follows by induction.

52C

(~~Bolzano~~ - Weierstrass)

<u>Corollary</u> If $\{a_n\}$ is a bounded seq in R, it has a convergent subsequence.

(Even if $\{a_n\}$ is <u>not</u> bounded, it has a subseq converging in R_∞.)

Cauchy Sequences

When can we guarantee that a seq conv?

Def: a seq of reals $\{x_n\}$ is said to be a *Cauchy seq* iff $\forall \varepsilon > 0 \ \exists N_\varepsilon$ such that for every $n, m \geq N_\varepsilon$

$$|x_n - x_m| < \varepsilon$$

Prop Spse $x_n \to x$. Then $\{x_n\}$ is a Cauchy seq.

(53A)

<u>Prop</u> Let $\{a_n\}$ be a Cauchy seq in $\mathbb{R}$. Then $\{a_n\}$ is <u>bdd</u>.

<u>Pf</u>: $\exists N < \infty$ s.t. for $j, k \geq N$ $|a_j - a_k| < 1$. Let

(*) $B = \max\{|a_1|, |a_2|, \cdots, |a_N|\} + 1$

For $1 \leq j \leq N$, $|a_j| < B$.

For $j > N$

$$|a_j| = |a_j - a_N + a_N| \leq |a_N| + |a_j - a_N| < |a_N| + 1 \leq B$$

Hence $\{a_j\}$ is <u>bounded</u>.

53B

<u>Thm</u> $\{a_n\}$ conv in $\mathbb{R}$ iff $\{a_n\}$ is Cauchy.

<u>Pf</u>: ($\Rightarrow$) ~~Spse~~ $a_n \to a$ in $\mathbb{R}$.
Take any $\varepsilon > 0$. $\exists N_\varepsilon \ni$
s.t. for $n \geq N_\varepsilon$, $|a_n - a| < \frac{\varepsilon}{2}$

For $j, k \geq N_\varepsilon$

$$
\begin{aligned}
|a_j - a_k| &= |a_j - a + a - a_k| \\
&\leq |a_j - a| + |a - a_k| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}
$$

(5.3C)

($\Leftarrow$) Spse $\{a_n\}$ is Cauchy.
Then $\exists\ B < \infty$ s.t. $|a_n| \le B$
for all n. Moreover
$\exists\ 1 \le n_1 < n_2 < \cdots$ s.t.
$\{a_{n_k} : k \ge 1\}$ is monotonic
w.l.o.g. spse $a_{n_1} \le a_{n_2} \le \cdots$
Let $L = \sup\{a_{n_k} : k \ge 1\}$
Then $\lim_{k\to\infty} a_{n_k} = L\ (\le B)$

Conjecture: $a_j \to L$

Pf: Take any $\varepsilon > 0$. $\exists K_\varepsilon < \infty$
s.t. for $k \ge K_\varepsilon$ $|a_{n_k} - L| < \varepsilon/2$
$\exists N_\varepsilon \ge K_\varepsilon$ s.t. $|a_i - a_j| < \varepsilon/2$
for $i, j \ge N_\varepsilon$. So for $j \ge N_\varepsilon$ and
$k_\varepsilon \ge N_\varepsilon$, $|a_j - L| \le \underbrace{|a_j - a_{n_k}|}_{<\varepsilon} + |a_{n_k} - L|$

Subseq Limits (54)

Given $\{a_n\}$, let

$\mathcal{L} = \{$ subsequential limits $\}$

Can we upper bound $\mathcal{L}$?

Let

$b_1 = \sup\{a_1, a_2, \dots\}$

$b_2 = \sup\{a_2, a_3, \dots\}$

$b_n = \sup\{a_n, a_{n+1}, \dots\}$

$b_1 \geq b_2 \geq \dots$

54A

Def: $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

$\equiv \lim_{n\to\infty} \left(\sup \{a_k : k \geq n\} \right)$

$\equiv \inf_n \left(\sup \{a_k : k \geq n\} \right)$

Clearly,

$\sup \mathcal{L} \leq \overline{\lim_{n\to\infty}} a_n$

Can you show they are equal?

What about $\inf \mathcal{L}$?