MM 1

Metric Spaces I.

and Constr. of a metric

How might we extend our mathematical investigations beyond the real numbers?

Reexamining what has already been useful, suppose we consider ordered pairs of reals

$$\mathbb{R}^2 \equiv \{(x, y) : x, y \in \mathbb{R}\}$$

Since sequences

m12

of reals were of vital significance, consider

$\{(x_n, y_n) : n \geq 1\}$ for $x_n, y_n \in \mathbb{R}$.

What would it mean to say that such a sequence converged in $\mathbb{R}^2$?

Clearly,

$$(x_n, y_n) \to (x_\infty, y_\infty)$$

iff

$x_n \to x_\infty$ and $y_n \to y_\infty$

So the notion of convergence in $\mathbb{R}^2$ seems clear, and it extends naturally to $\mathbb{R}^k$.

But how far is a particular term in the sequence from its limit? In order to be able to answer such a question, we require a notion of distance between pairs

MM4

of points in the underlying set, be it $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^k$, or just some abstract set S.

To achieve broadest applicability we seek the most general possible notion of a distance function.

What properties must a distance function have?

$\boxed{M5}$

Given a non-empty set M, if $d(\cdot,\cdot)$ is a distance function on M then

(i) $d : M \times M \to [0,\infty)$

(ii) $d(x,y) = 0$ iff $x = y$

symmetry (iii) $d(x.y) = d(y,x)$

Is this sufficient?

Suppose $d(x,y)$ denotes the distance along the shortest path from x to y. Then

$\boxed{m6}$

for any $z \in m$

the path

$x \to z$ followed by

$z \to y$

can be no shorter than the shortest path from x to y

Hence we require

(iv) $d(x,y) \leq d(x,z) + d(z,y)$

This is called the triangle inequality.

Def:

Any $d : m \times m \to [0, \infty)$ satisfying conditions (i)–(iv) will be called a **metric** on m.

How could we construct a metric?

Suppose m is a finite set of points. Thinking of these as cities in a given state/region,

$\boxed{\text{M8}}$

Call them $C_1, C_2, \ldots, C_L$

Suppose some pairs (i, j) of cities C_i and C_j are adjacent in that they are linked by a non-stop road of some positive, finite, known distance. g_{ij}

When is there a path by car between every two cities?

$\boxed{\text{M 9}}$

Ans.

Let $A = (a_{ij})$ be an $L \times L$ incidence matrix where $a_{ii} \equiv 1$ and for $i \neq j$

$$a_{ij} = \begin{cases} 1 \text{ if } C_i \text{ and } C_j \text{ are adjacent} \\ 0 \text{ otherwise} \end{cases}$$

Let $A^k \equiv (a_{ij;k})$

$a_{ij;k} > 0$ iff there is a path along adjacent cities between C_i and C_j taking at most k steps.

Hence transportation by car is feasible between any two cities iff $a_{ij}^{L-1} > 0$ for all $1 \leq i, j \leq L-1$.

The distance $d(c_i, c_j)$ would then be defined as the sum of the lengths producing the shortest path.