

# CISE301: Numerical Methods

## Topic 2:

# Solution of Nonlinear Equations

## Lectures 5-11:



# NUM

Read Chapters 5 and 6 of the textbook

# Lecture 5

## Solution of Nonlinear Equations ( Root Finding Problems )



- Definitions
- Classification of Methods
  - Analytical Solutions
  - Graphical Methods
  - Numerical Methods
    - Bracketing Methods
    - Open Methods
- Convergence Notations

Reading Assignment: Sections 5.1 and 5.2

# Root Finding Problems

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Many problems in Science and Engineering are expressed as:

Given a continuous function  $f(x)$ ,  
find the value  $r$  such that  $f(r) = 0$

These problems are called root finding problems.

# Roots of Equations

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A number *r* that satisfies an equation is called a root of the equation.

The equation :  $x^4 - 3x^3 - 7x^2 + 15x = -18$

has four roots :  $-2, 3, 3, \text{ and } -1$  .

i.e.,  $x^4 - 3x^3 - 7x^2 + 15x + 18 = (x + 2)(x - 3)^2(x + 1)$

The equation has two simple roots ( $-1$  and  $-2$ )  
and a repeated root ( $3$ ) with multiplicity  $= 2$ .

# Zeros of a Function

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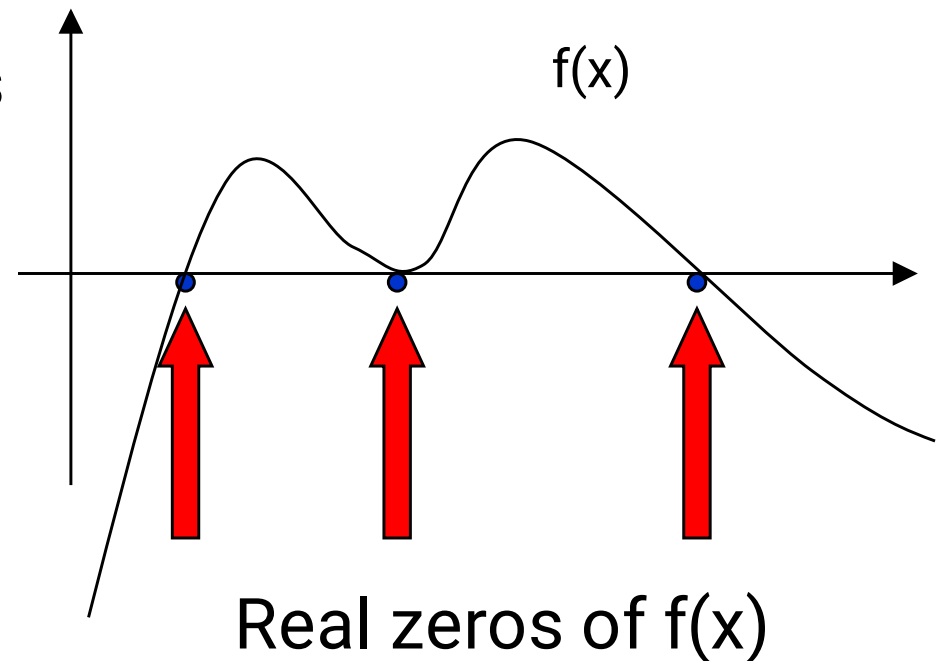
Let  $f(x)$  be a real-valued function of a real variable. Any number  $r$  for which  $f(r)=0$  is called a zero of the function.

*Examples:*

*2 and 3 are zeros of the function  $f(x) = (x-2)(x-3)$ .*

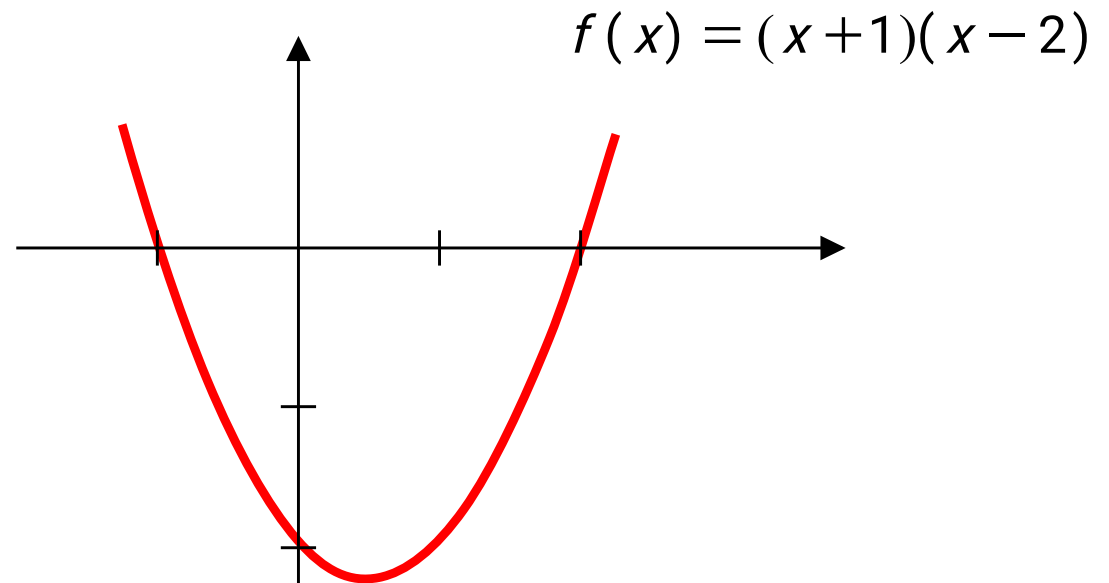
# Graphical Interpretation of Zeros

- The real zeros of a function  $f(x)$  are the values of  $x$  at which the graph of the function crosses (or touches) the  $x$ -axis.



# Simple Zeros

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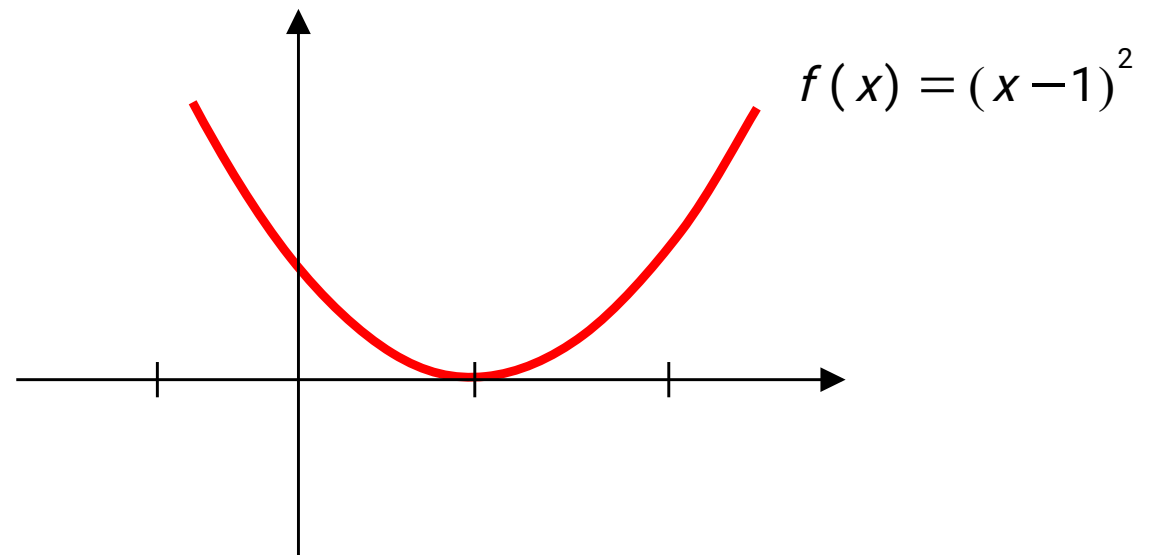


$$f(x) = (x+1)(x-2) = x^2 - x - 2$$

has two simple zeros (one at  $x = 2$  and one at  $x = -1$ )

# Multiple Zeros

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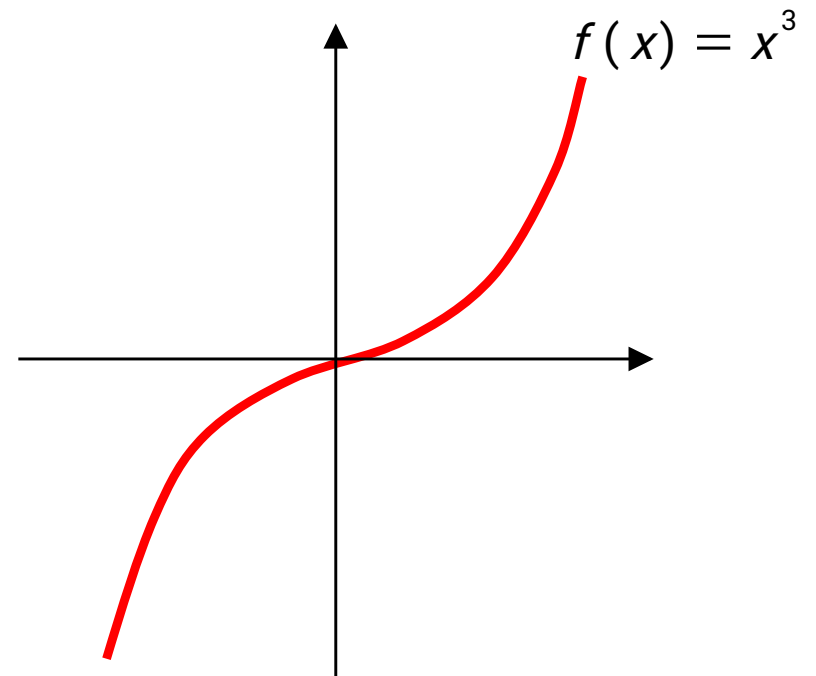
$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

has double zeros (zero with multiplicity  $y = 2$ ) at  $x = 1$



# Multiple Zeros

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$$f(x) = x^3$$

has a zero with multiplicity  $y = 3$  at  $x = 0$

# Facts

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- Any  $n^{\text{th}}$  order polynomial has exactly  $n$  zeros (counting real and complex zeros with their multiplicities).
- Any polynomial with an odd order has at least one real zero.
- If a function has a zero at  $x=r$  with multiplicity  $m$  then the function and its first  $(m-1)$  derivatives are zero at  $x=r$  and the  $m^{\text{th}}$  derivative at  $r$  is not zero.

# Roots of Equations & Zeros of Function

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Given the equation :

$$x^4 - 3x^3 - 7x^2 + 15x = -18$$

Move all terms to one side of the equation :

$$x^4 - 3x^3 - 7x^2 + 15x + 18 = 0$$

Define  $f(x)$  as :

$$f(x) = x^4 - 3x^3 - 7x^2 + 15x + 18$$

The zeros of  $f(x)$  are the same as the roots of the equation  $f(x) = 0$

(Which are  $-2$ ,  $3$ ,  $3$ , and  $-1$ )

# Solution Methods

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Several ways to solve nonlinear equations are possible:

- Analytical Solutions
  - Possible for special equations only
- Graphical Solutions
  - Useful for providing initial guesses for other methods
- Numerical Solutions
  - Open methods
  - Bracketing methods

# Analytical Methods

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Analytical Solutions are available for special equations only.

Analytical solution of :  $a x^2 + b x + c = 0$

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No analytical solution is available for :  $x - e^{-x} = 0$

# Graphical Methods

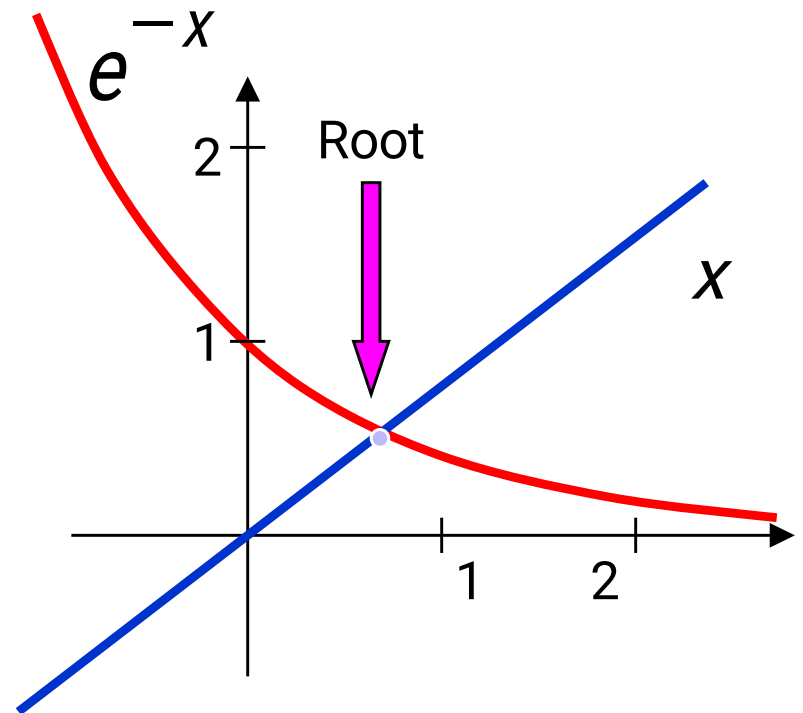
- Graphical methods are useful to provide an initial guess to be used by other methods.

*Solve*

$$x = e^{-x}$$

*The root  $\in [0,1]$*

*root  $\approx 0.6$*



# Numerical Methods

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Many methods are available to solve nonlinear equations:

- ❑ Bisection Method

- ❑ Newton's Method

- ❑ Secant Method

- False position Method

- Muller's Method

- Bairstow's Method

- Fixed point iterations

- .....

→ These will be  
covered in CISE301

# Bracketing Methods

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- ❑ In bracketing methods, the method starts with an interval that contains the root and a procedure is used to obtain a smaller interval containing the root.
  
- ❑ Examples of bracketing methods:
  - Bisection method
  - False position method



# Open Methods

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- ❑ In the open methods, the method starts with one or more initial guess points. In each iteration, a new guess of the root is obtained.
- ❑ Open methods are usually more efficient than bracketing methods.
- ❑ They may not converge to a root.

# Convergence Notation

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A sequence  $x_1, x_2, \dots, x_n, \dots$  is said to **converge** to  $x$  if to every  $\varepsilon > 0$  there exists  $N$  such that :

$$|x_n - x| < \varepsilon \quad \forall n > N$$

# Convergence Notation

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Let  $x_1, x_2, \dots$ , converge to  $x$ .

Linear Convergence : 
$$\frac{|x_{n+1} - x|}{|x_n - x|} \leq C$$

Quadratic Convergence : 
$$\frac{|x_{n+1} - x|}{|x_n - x|^2} \leq C$$

Convergence of order  $P$  : 
$$\frac{|x_{n+1} - x|}{|x_n - x|^p} \leq C$$

# Speed of Convergence

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- ❑ We can compare different methods in terms of their convergence rate.
- ❑ Quadratic convergence is faster than linear convergence.
- ❑ A method with convergence order  $q$  converges faster than a method with convergence order  $p$  if  $q > p$ .
- ❑ Methods of convergence order  $p > 1$  are said to have super linear convergence.

# Lectures 6-7

# Bisection Method



- The Bisection Algorithm
- Convergence Analysis of Bisection Method
- Examples

Reading Assignment: Sections 5.1 and 5.2

# Introduction

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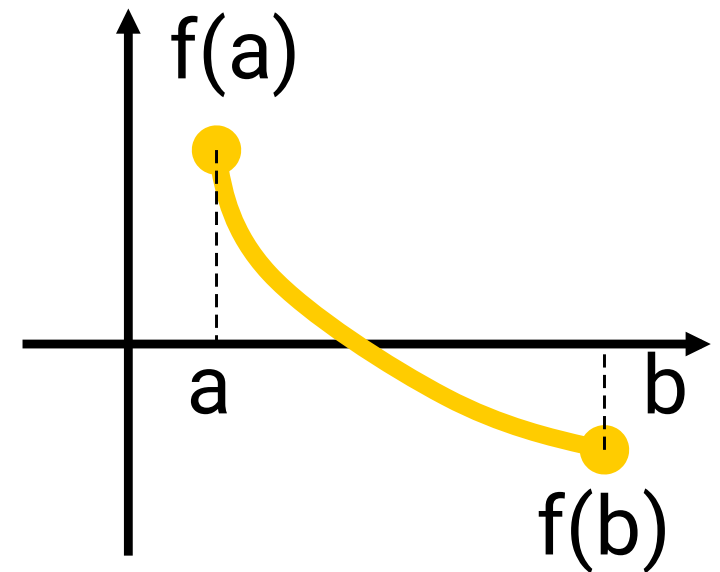
- ❑ The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- ❑ It is also called **interval halving** method.
- ❑ To use the Bisection method, one needs an initial interval that is known to contain a zero of the function.
- ❑ The method systematically reduces the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test, half of the interval is thrown away.
- ❑ The procedure is repeated until the desired interval size is obtained.

# Intermediate Value Theorem

□ Let  $f(x)$  be defined on the interval  $[a,b]$ .

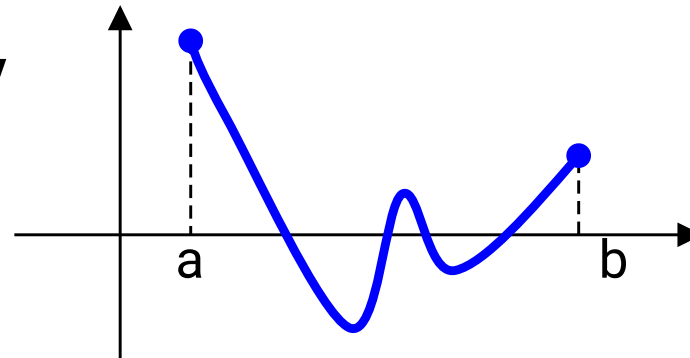
□ Intermediate value theorem:

if a function is continuous and  $f(a)$  and  $f(b)$  have different signs then the function has at least one zero in the interval  $[a,b]$ .

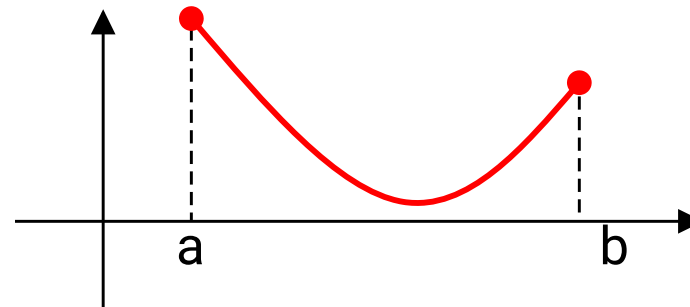


# Examples

- If  $f(a)$  and  $f(b)$  have the same sign, the function may have an even number of real zeros or no real zeros in the interval  $[a, b]$ .
- Bisection method can not be used in these cases.



The function has four real zeros

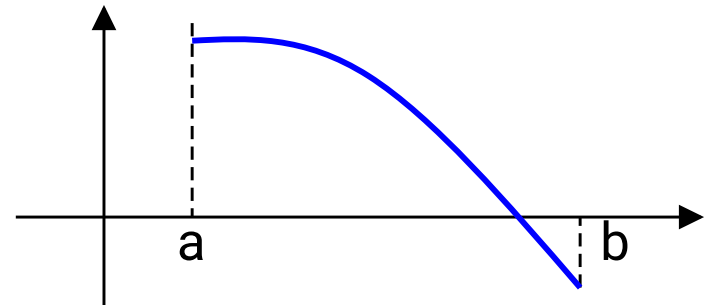


The function has no real zeros



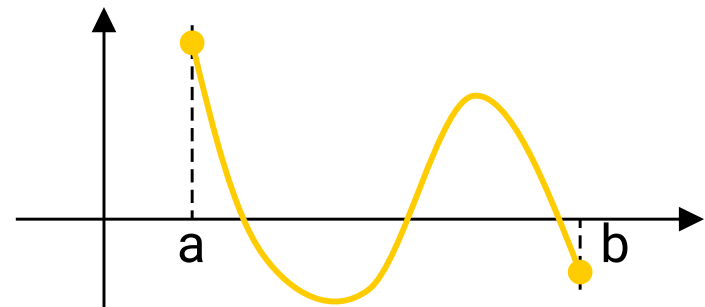
# Two More Examples

- If  $f(a)$  and  $f(b)$  have different signs, the function has at least one real zero.



The function has one real zero

- Bisection method can be used to find one of the zeros.



The function has three real zeros

# Bisection Method

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- If the function is continuous on  $[a,b]$  and  $f(a)$  and  $f(b)$  have different signs, Bisection method obtains a new interval that is half of the current interval and the sign of the function at the end points of the interval are different.
- This allows us to repeat the Bisection procedure to further reduce the size of the interval.

# Bisection Method

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## Assumptions:

Given an interval  $[a,b]$

$f(x)$  is continuous on  $[a,b]$

$f(a)$  and  $f(b)$  have opposite signs.

These assumptions ensure the existence of at least one zero in the interval  $[a,b]$  and the bisection method can be used to obtain a smaller interval that contains the zero.

# Bisection Algorithm

## Assumptions:

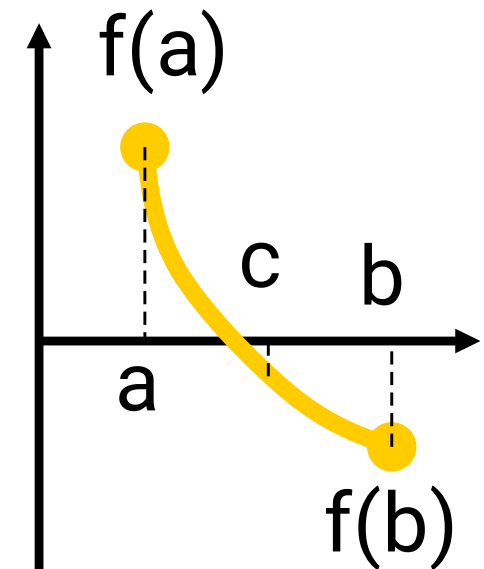
- $f(x)$  is continuous on  $[a, b]$
- $f(a) f(b) < 0$

## Algorithm:

### Loop

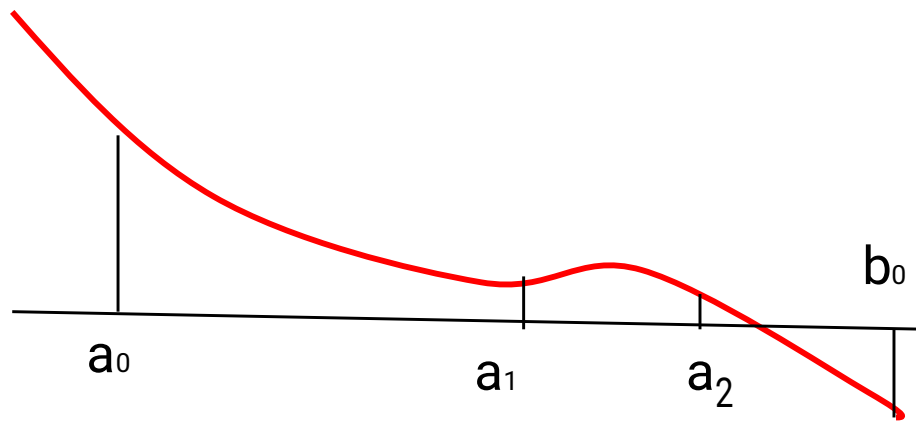
1. Compute the mid point  $c = (a+b)/2$
2. Evaluate  $f(c)$
3. If  $f(a) f(c) < 0$  then new interval  $[a, c]$   
If  $f(a) f(c) > 0$  then new interval  $[c, b]$

### End loop

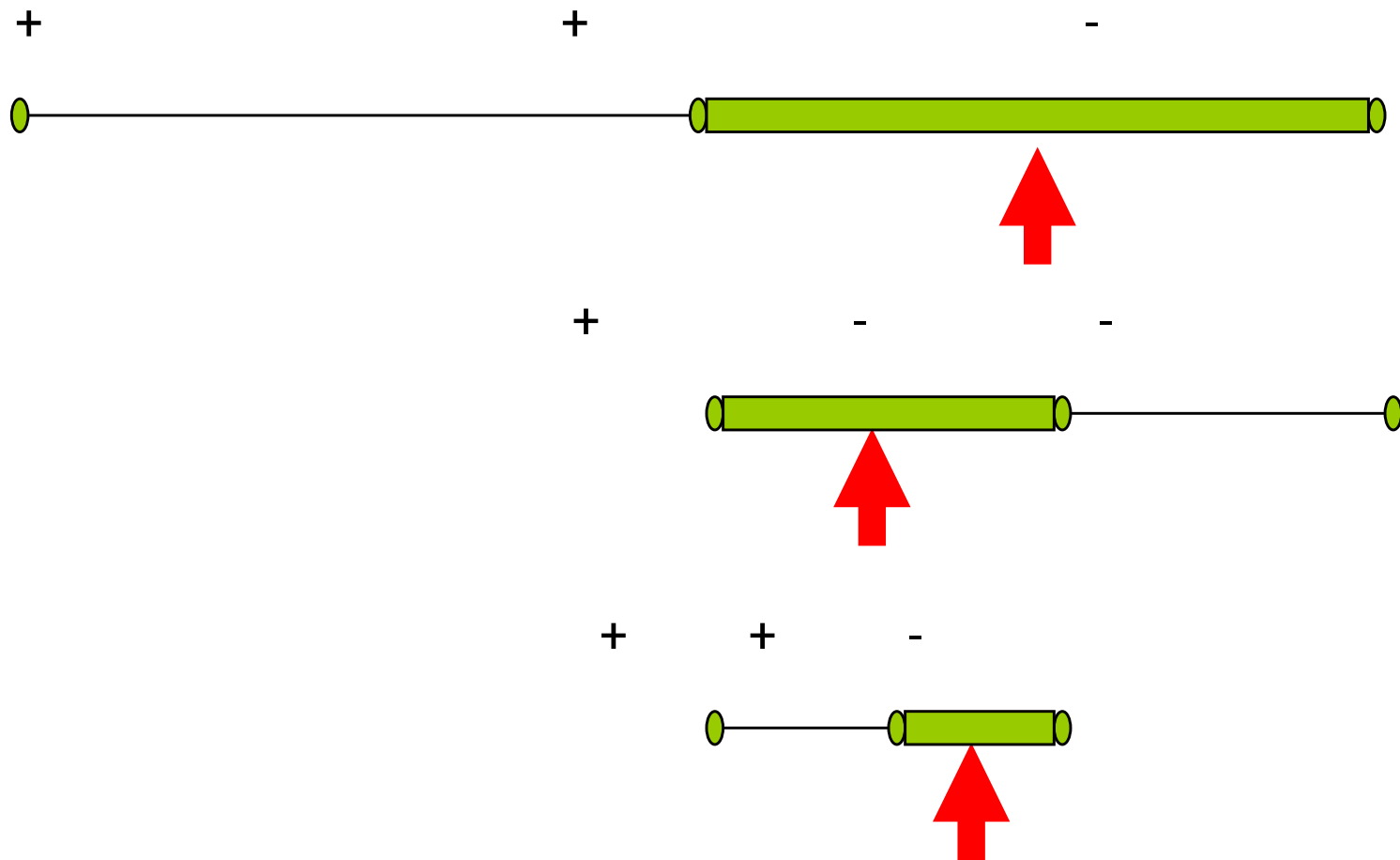


# Bisection Method

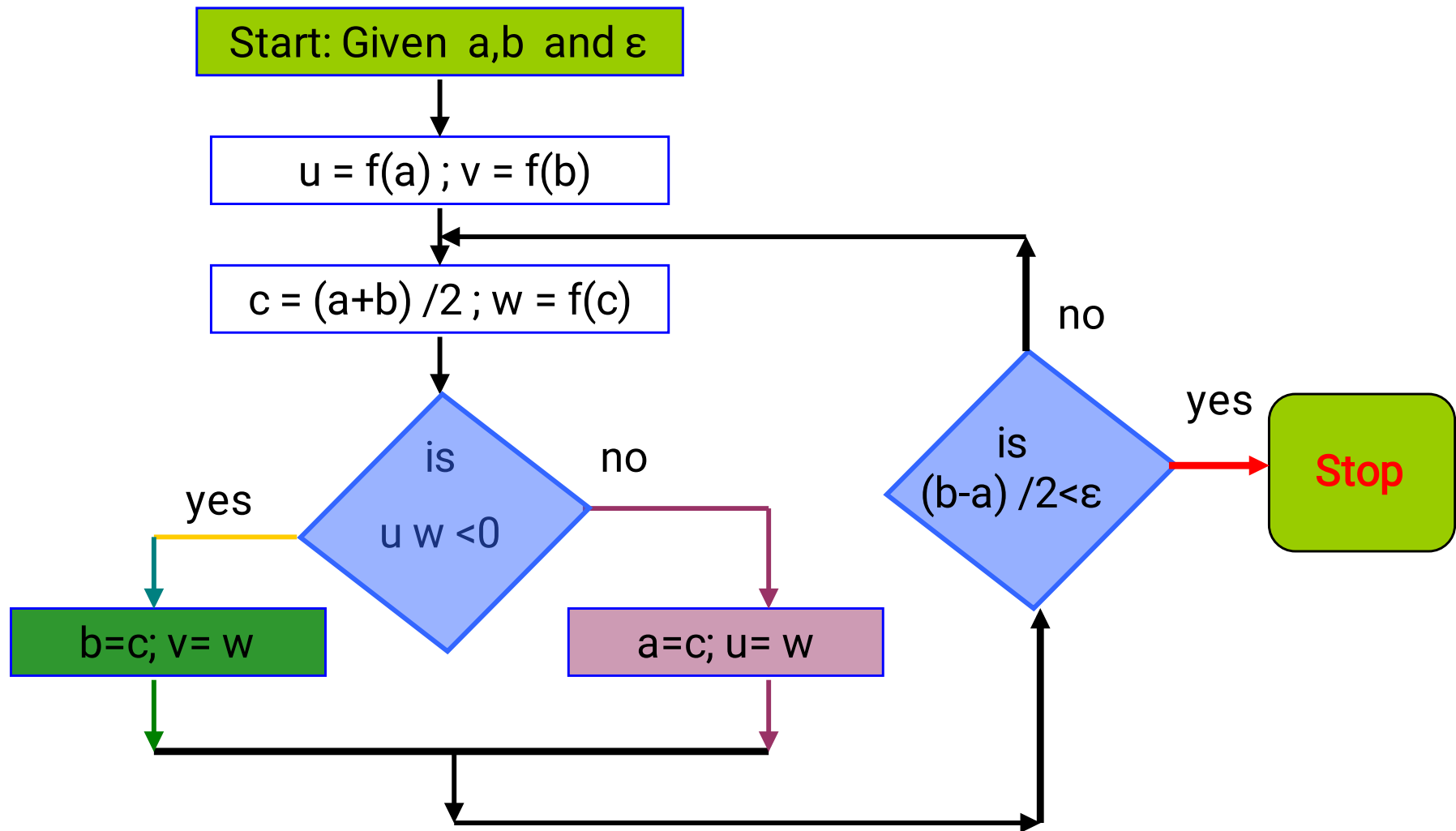
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# Example



# Flow Chart of Bisection Method



# Example

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Can you use Bisection method to find a zero of :

$$f(x) = x^3 - 3x + 1 \text{ in the interval } [0,2]?$$

**Answer:**

$f(x)$  is continuous on  $[0,2]$

$$\text{and } f(0) * f(2) = (1)(3) = 3 > 0$$

$\Rightarrow$  Assumption s are not satisfied

$\Rightarrow$  Bisection method can not be used



# Example

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Can you use Bisection method to find a zero of :

$$f(x) = x^3 - 3x + 1 \text{ in the interval } [0,1]?$$

**Answer:**

$f(x)$  is continuous on  $[0,1]$

$$\text{and } f(0) * f(1) = (1)(-1) = -1 < 0$$

$\Rightarrow$  Assumptions are satisfied

$\Rightarrow$  Bisection method can be used

# Best Estimate and Error Level

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Bisection method obtains an interval that is guaranteed to contain a zero of the function.

## Questions:

- What is the best estimate of the zero of  $f(x)$ ?
- What is the error level in the obtained estimate?

# Best Estimate and Error Level

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The best estimate of the zero of the function  $f(x)$  after the first iteration of the Bisection method is the mid point of the initial interval:

$$\text{Estimate of the zero : } r = \frac{b + a}{2}$$

$$\text{Error} \leq \frac{b - a}{2}$$

# Stopping Criteria

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Two common stopping criteria

1. Stop after a fixed number of iterations
2. Stop when the absolute error is less than a specified value

How are these criteria related?

# Stopping Criteria

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$c_n$  : is the midpoint of the interval at the  $n^{\text{th}}$  iteration  
(  $c_n$  is usually used as the estimate of the root).

$r$  : is the zero of the function.

After  $n$  iterations :

$$|error| = |r - c_n| \leq E_a^n = \frac{b - a}{2^n} = \frac{\Delta x^0}{2^n}$$

# Convergence Analysis

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Given  $f(x)$ ,  $a$ ,  $b$ , and  $\varepsilon$

How many iterations are needed such that  $|x - r| \leq \varepsilon$   
where  $r$  is the zero of  $f(x)$  and  $x$  is the  
bisection estimate (i.e.,  $x = c_k$ )?

$$n \geq \frac{\log(b - a) - \log(\varepsilon)}{\log(2)}$$

# Convergence Analysis – Alternative Form

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$$n \geq \frac{\log(b - a) - \log(\mathcal{E})}{\log(2)}$$

$$n \geq \log_2 \left( \frac{\text{width of initial interval}}{\text{desired error}} \right) = \log_2 \left( \frac{b - a}{\mathcal{E}} \right)$$

# Example

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$$a = 6, \quad b = 7, \quad \varepsilon = 0.0005$$

How many iterations are needed such that :  $|x - r| \leq \varepsilon$ ?

$$n \geq \frac{\log(b - a) - \log(\varepsilon)}{\log(2)} = \frac{\log(1) - \log(0.0005)}{\log(2)} = 10.9658$$

$$\Rightarrow n \geq 11$$



# Example

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- Use Bisection method to find a root of the equation  $x = \cos(x)$  with absolute error  $< 0.02$   
(assume the initial interval  $[0.5, 0.9]$ )

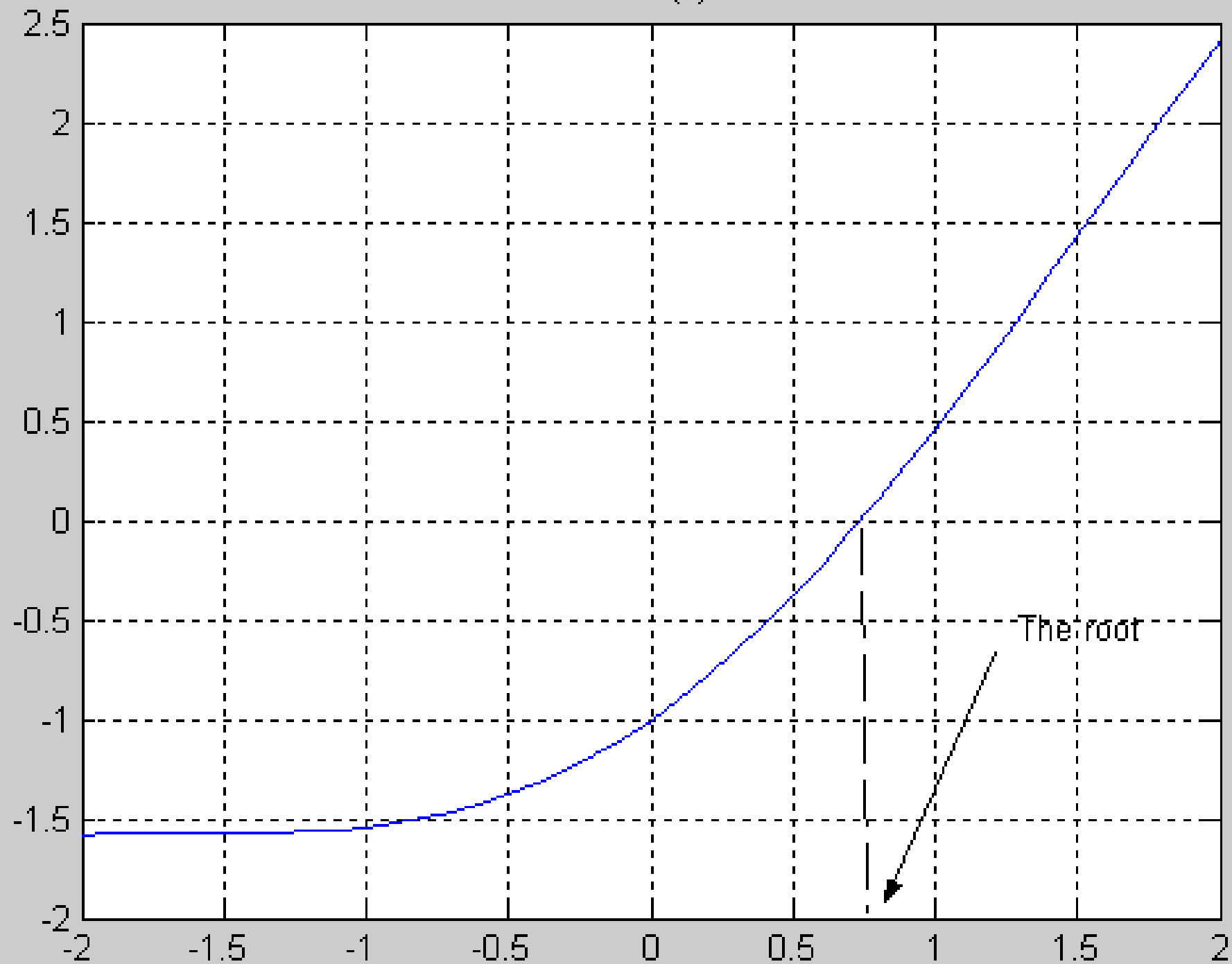
Question 1: What is  $f(x)$  ?

Question 2: Are the assumptions satisfied ?

Question 3: How many iterations are needed ?

Question 4: How to compute the new estimate ?

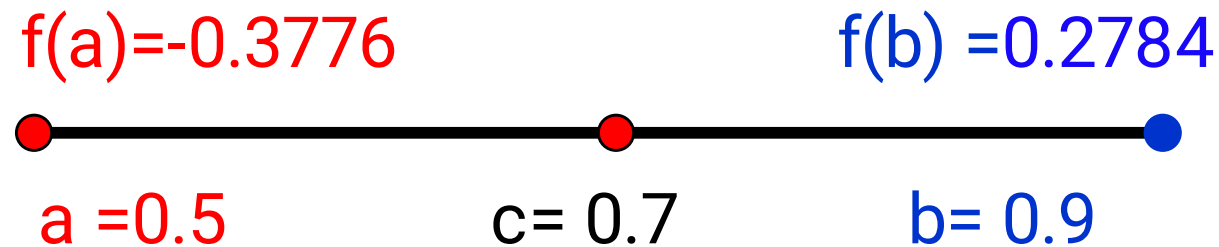
$$x - \cos(x)$$



# Bisection Method

## Initial Interval

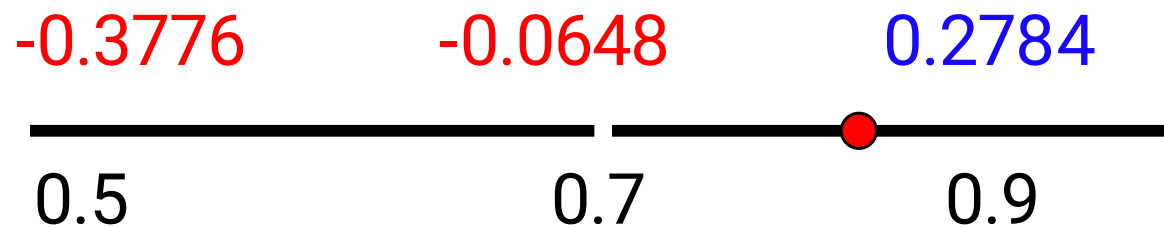
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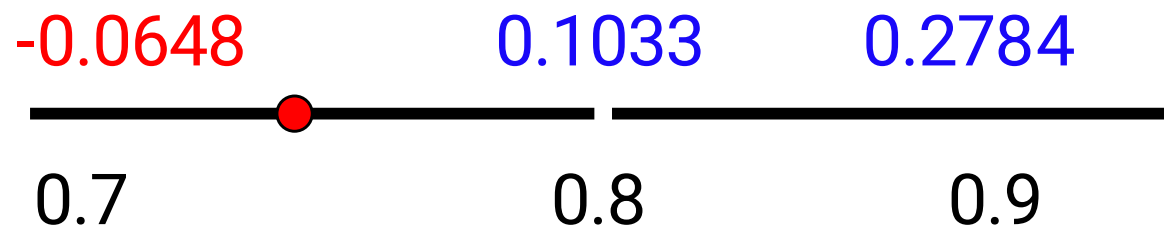
Error < 0.2

# Bisection Method

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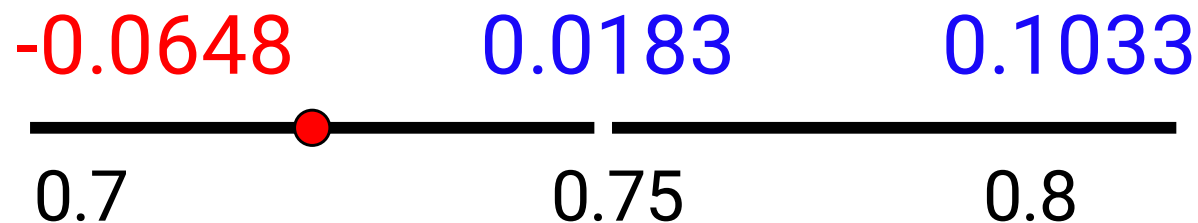
Error < 0.1



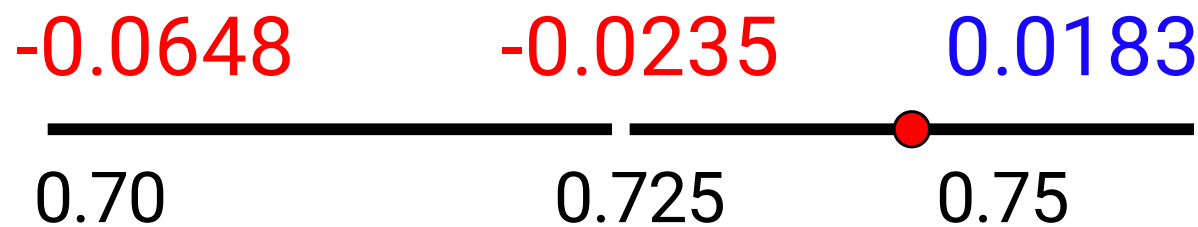
Error < 0.05

# Bisection Method

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Error < 0.025



Error < .0125

# Summary

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- Initial interval containing the root:  $[0.5, 0.9]$
  
- After 5 iterations:
  - Interval containing the root:  $[0.725, 0.75]$
  - Best estimate of the root is  $0.7375$
  - $|\text{Error}| < 0.0125$

# A Matlab Program of Bisection Method

```
a=.5; b=.9;
u=a-cos(a);
v=b-cos(b);
for i=1:5
    c=(a+b)/2
    fc=c-cos(c)
    if u*fc<0
        b=c; v=fc;
    else
        a=c; u=fc;
    end
end
```

```
c =
    0.7000
fc =
   -0.0648
c =
    0.8000
fc =
    0.1033
c =
    0.7500
fc =
    0.0183
c =
    0.7250
fc =
   -0.0235
```

# Example

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Find the root of:

$$f(x) = x^3 - 3x + 1 \text{ in the interval } : [0,1]$$

\*  $f(x)$  is continuous

\*  $f(0) = 1, \quad f(1) = -1 \Rightarrow f(a) f(b) < 0$

$\Rightarrow$  Bisection method can be used to find the root



# Example

Iteration	a	b	$c = \frac{a+b}{2}$	f(c)	$\frac{(b-a)}{2}$
1	0	1	0.5	-0.375	0.5
2	0	0.5	0.25	0.266	0.25
3	0.25	0.5	.375	-7.23E-3	0.125
4	0.25	0.375	0.3125	9.30E-2	0.0625
5	0.3125	0.375	0.34375	9.37E-3	0.03125

# Bisection Method

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## Advantages


- ❑ Simple and easy to implement
- ❑ One function evaluation per iteration
- ❑ The size of the interval containing the zero is reduced by 50% after each iteration
- ❑ The number of iterations can be determined a priori
- ❑ No knowledge of the derivative is needed
- ❑ The function does not have to be differentiable

## Disadvantage

- ❑ Slow to converge
- ❑ Good intermediate approximations may be discarded

# Lecture 8-9

# Newton-Raphson Method



- Assumptions
- Interpretation
- Examples
- Convergence Analysis

# Newton-Raphson Method

(Also known as Newton's Method)

Given an initial guess of the root  $x_0$  Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.

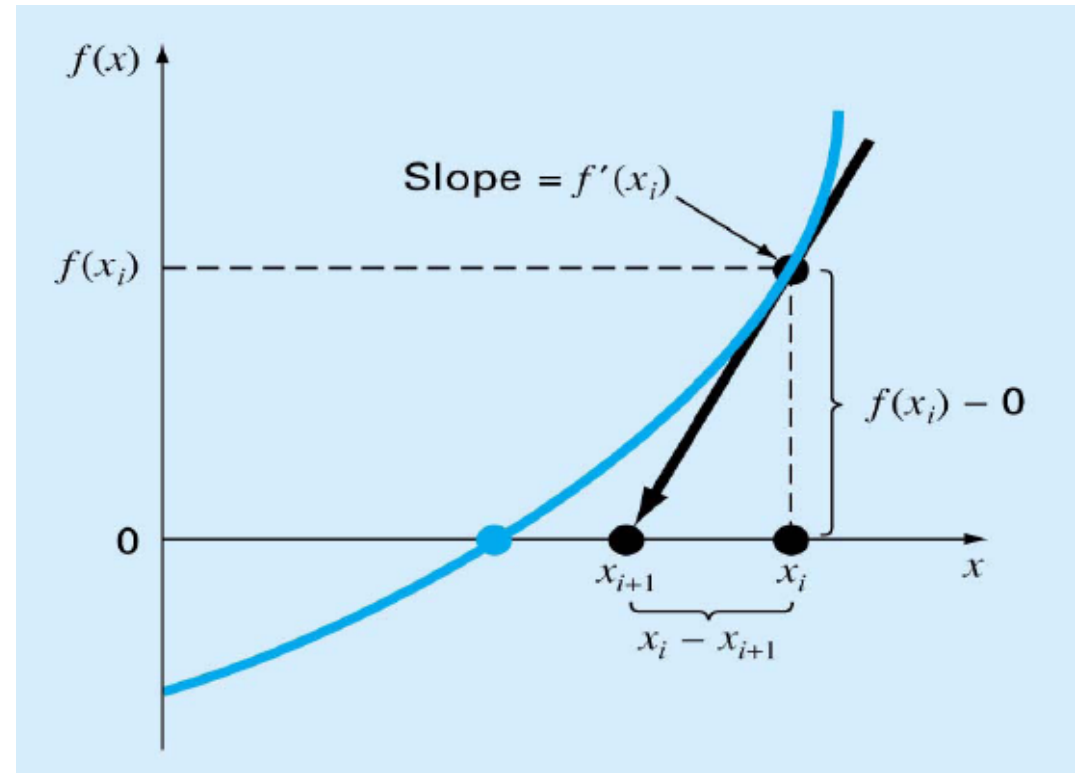
## Assumptions:

- $f(x)$  is continuous and the first derivative is known
- An initial guess  $x_0$  such that  $f'(x_0) \neq 0$  is given

# Newton Raphson Method

## - Graphical Depiction -

- If the initial guess at the root is  $x_i$ , then a tangent to the function of  $x_i$  that is  $f'(x_i)$  is extrapolated down to the  $x$ -axis to provide an estimate of the root at  $x_{i+1}$ .



# Derivation of Newton's Method

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*Given* :  $x_i$  an initial guess of the root of  $f(x) = 0$

*Question* : How do we obtain a better estimate  $x_{i+1}$ ?

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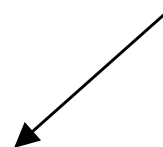
Taylor Theorem :  $f(x+h) \approx f(x) + f'(x)h$

Find  $h$  such that  $f(x+h) = 0$ .

$$\Rightarrow h \approx -\frac{f(x)}{f'(x)}$$

Newton – Raphson Formula

A new guess of the root :  $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$



# Newton's Method

Given  $f(x)$ ,  $f'(x)$ ,  $x_0$

Assumption  $f'(x_0) \neq 0$

for  $i = 0:n$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

end

C    FORTRAN    PROGRAM

$F(X) = X^{**3} - 3 * X^{**2} + 1$

$FP(X) = 3 * X^{**2} - 6 * X$

$X = 4$

DO 10 I = 1, 5

$X = X - F(X) / FP(X)$

PRINT \*, X

10 CONTINUE

STOP

END

# Newton's Method

Given  $f(x)$ ,  $f'(x)$ ,  $x_0$

Assumption  $f'(x_0) \neq 0$

```
for i = 0 : n
```

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

```
end
```

F.m

```
function [F] = F(X)
    F = X^3 - 3*X^2 + 1
```

FP.m

```
function [FP] = FP(X)
    FP = 3*X^2 - 6*X
```

```
% MATLAB PROGRAM
X = 4
for i = 1:5
    X = X - F(X) / FP(X)
end
```



# Example

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Find a zero of the function  $f(x) = x^3 - 2x^2 + x - 3$ ,  $x_0 = 4$

$$f'(x) = 3x^2 - 4x + 1$$

Iteration 1: 
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 - \frac{33}{33} = 3$$

Iteration 2: 
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{9}{16} = 2.4375$$

Iteration 3: 
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.4375 - \frac{2.0369}{9.0742} = 2.2130$$

# Example

k (Iteration)	$x_k$	$f(x_k)$	$f'(x_k)$	$x_{k+1}$	$ x_{k+1} - x_k $
0	4	33	33	3	1
1	3	9	16	2.4375	0.5625
2	2.4375	2.0369	9.0742	2.2130	0.2245
3	2.2130	0.2564	6.8404	2.1756	0.0384
4	2.1756	0.0065	6.4969	2.1746	0.0010

# Convergence Analysis

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Theorem :

Let  $f(x)$ ,  $f'(x)$  and  $f''(x)$  be continuous at  $x \approx r$   
where  $f(r) = 0$ . If  $f'(r) \neq 0$  then there exists  $\delta > 0$

such that  $|x_0 - r| \leq \delta \Rightarrow \frac{|x_{k+1} - r|}{|x_k - r|^2} \leq C$

$$C = \frac{1}{2} \frac{\max_{|x_0 - r| \leq \delta} |f''(x)|}{\min_{|x_0 - r| \leq \delta} |f'(x)|}$$

# Convergence Analysis

## Remarks

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When the guess is close enough to a **simple** root of the function then Newton's method is guaranteed to converge quadratically.

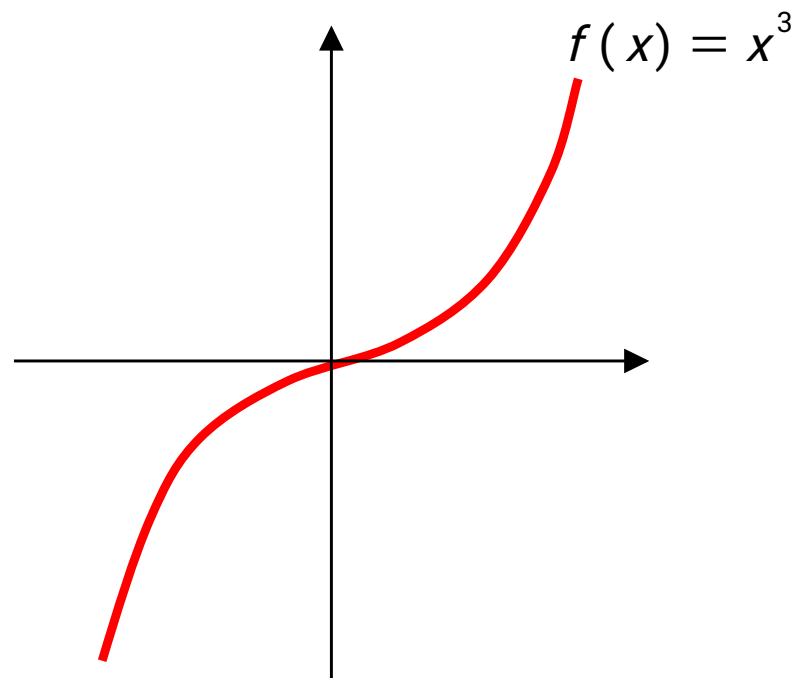
Quadratic convergence means that the number of correct digits is nearly doubled at each iteration.

# Problems with Newton's Method

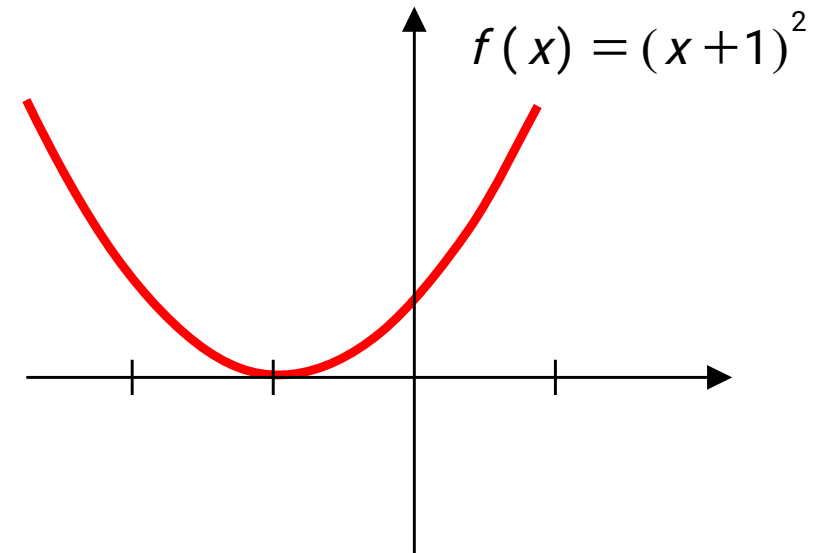
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- If the initial guess of the root is far from the root the method may not converge.
- Newton's method converges linearly near multiple zeros  $\{ f(r) = f'(r) = 0 \}$ . In such a case, modified algorithms can be used to regain the quadratic convergence.

# Multiple Roots



$f(x)$  has three  
zeros at  $x = 0$

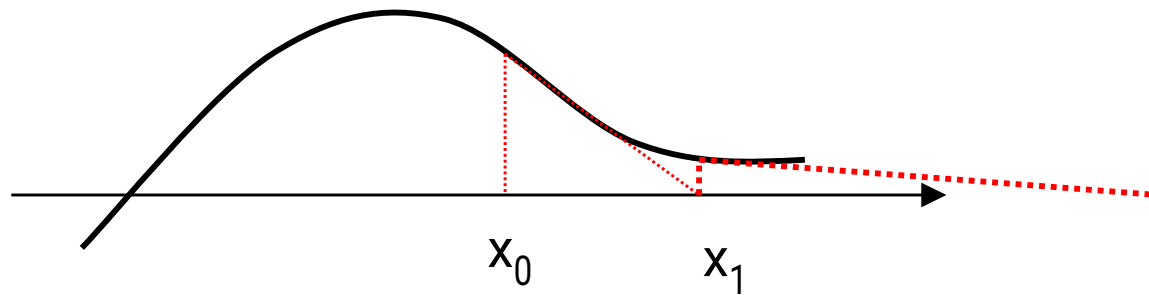


$f(x)$  has two  
zeros at  $x = -1$

# Problems with Newton's Method

## - Runaway -

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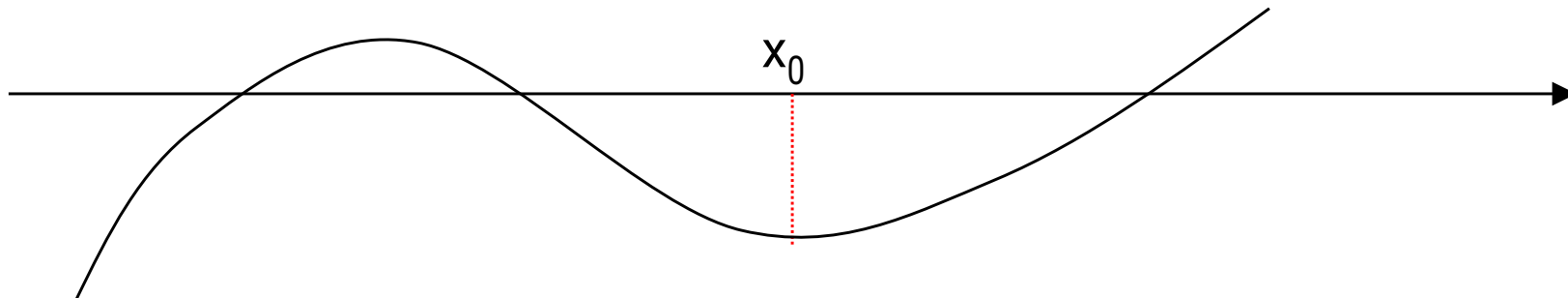


The estimates of the root is going away from the root.

# Problems with Newton's Method

## - Flat Spot -

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The value of  $f'(x)$  is zero, the algorithm fails.

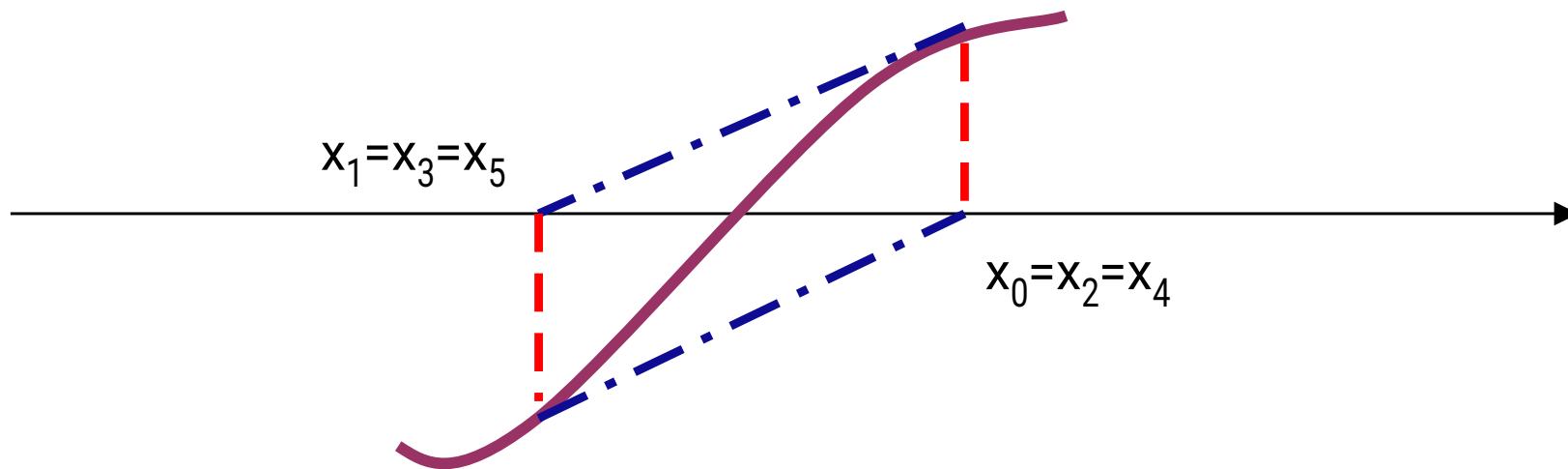
If  $f'(x)$  is very small then  $x_1$  will be very far from  $x_0$ .



# Problems with Newton's Method

## - Cycle -

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The algorithm cycles between two values  $x_0$  and  $x_1$

# Newton's Method for Systems of Non Linear Equations

*Given* :  $X_0$  an initial guess of the root of  $F(x) = 0$

*Newton's Iteration*

$$X_{k+1} = X_k - [F'(X_k)]^{-1} F(X_k)$$

$$F(X) = \begin{bmatrix} f_1(x_1, x_2, \dots) \\ f_2(x_1, x_2, \dots) \\ \vdots \end{bmatrix}, \quad F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \vdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \vdots \end{bmatrix}$$

# Example

□ Solve the following system of equations:

$$y + x^2 - 0.5 - x = 0$$

$$x^2 - 5xy - y = 0$$

Initial guess  $x = 1, y = 0$

$$F = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix}, \quad F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -5x - 1 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Solution Using Newton's Method

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Iteration 1:

$$F = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad F' = \begin{bmatrix} 2x-1 & 1 \\ 2x-5y & -5x-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}^{-1} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix}$$

Iteration 2:

$$F = \begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix}, \quad F' = \begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}^{-1} \begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 1.2332 \\ 0.2126 \end{bmatrix}$$

# Example

Try this

□ Solve the following system of equations:

$$y + x^2 - 1 - x = 0$$

$$x^2 - 2y^2 - y = 0$$

Initial guess  $x = 0, y = 0$

$$F = \begin{bmatrix} y + x^2 - 1 - x \\ x^2 - 2y^2 - y \end{bmatrix}, \quad F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x & -4y - 1 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# Example

## Solution

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<i>Iteration</i>	0	1	2	3	4	5
$X_k$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -0.6 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} -0.5287 \\ 0.1969 \end{bmatrix}$	$\begin{bmatrix} -0.5257 \\ 0.1980 \end{bmatrix}$	$\begin{bmatrix} -0.5257 \\ 0.1980 \end{bmatrix}$

# Lectures 10

# Secant Method

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- ❑ Secant Method
- ❑ Examples
- ❑ Convergence Analysis

# Newton's Method (Review)

*Assumption s:  $f(x)$ ,  $f'(x)$ ,  $x_0$  are available ,  
 $f'(x_0) \neq 0$*

*Newton 's Method new estimate :*

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

*Problem :*

*$f'(x_i)$  is not available,  
or difficult to obtain analytically.*



# Secant Method

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

if  $x_i$  and  $x_{i-1}$  are two initial points :

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}} = x_i - f(x_i) \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# Secant Method

Assumption s :

Two initial points  $x_i$  and  $x_{i-1}$

*such that*  $f(x_i) \neq f(x_{i-1})$

New estimate (Secant Method) :

$$x_{i+1} = x_i - f(x_i) \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# Secant Method

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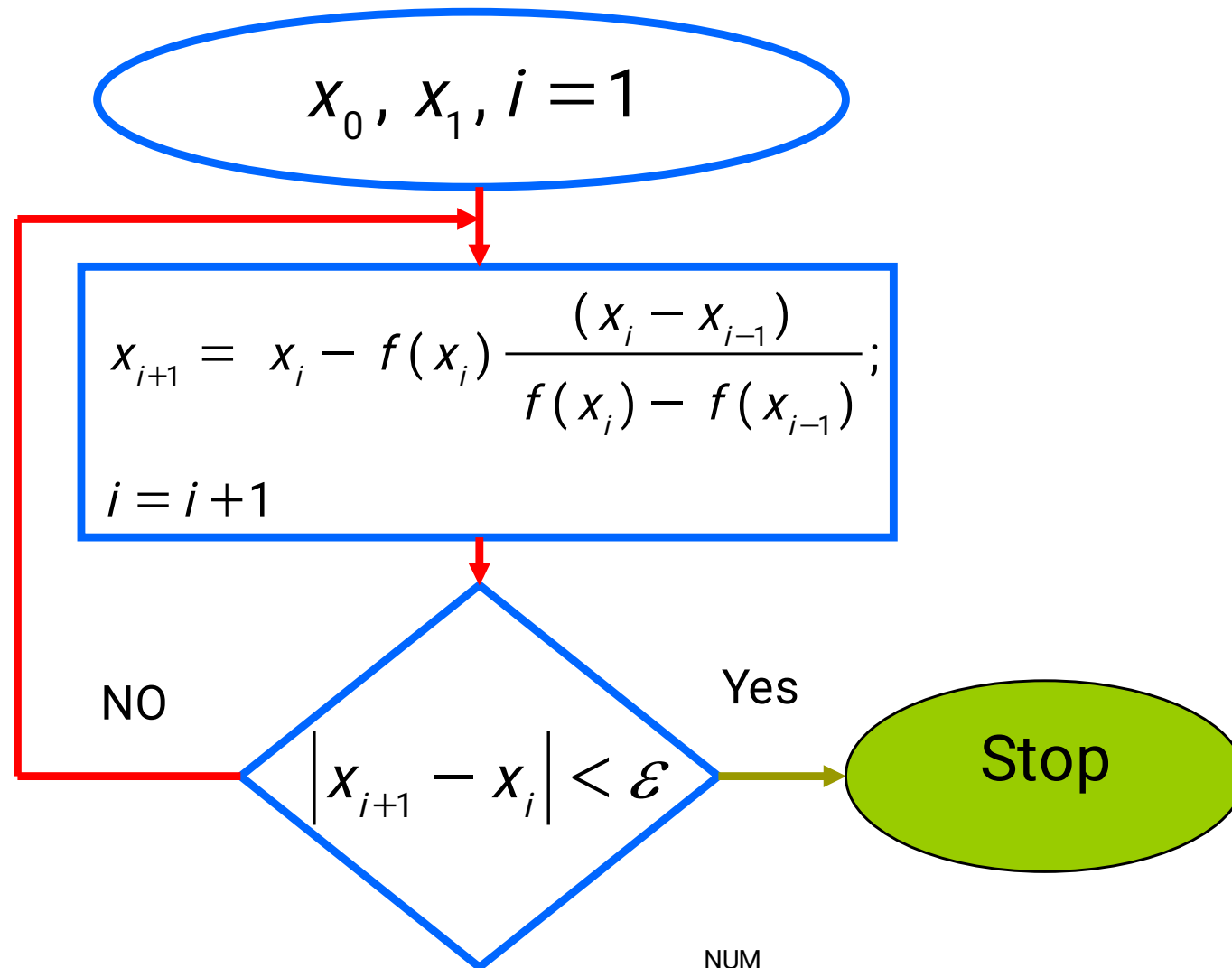
$$f(x) = x^2 - 2x + 0.5$$

$$x_0 = 0$$

$$x_1 = 1$$

$$x_{i+1} = x_i - f(x_i) \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# Secant Method - Flowchart



# Modified Secant Method

In this modified Secant method, only one initial guess is needed :

$$f'(x_i) \approx \frac{f(x_i + \delta_i x_i) - f(x_i)}{\delta_i x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\frac{f(x_i + \delta_i x_i) - f(x_i)}{\delta_i x_i}} = x_i - \frac{\delta_i x_i f(x_i)}{f(x_i + \delta_i x_i) - f(x_i)}$$

Problem : How to select  $\delta_i$  ?

If not selected properly, the method may diverge .

# Example

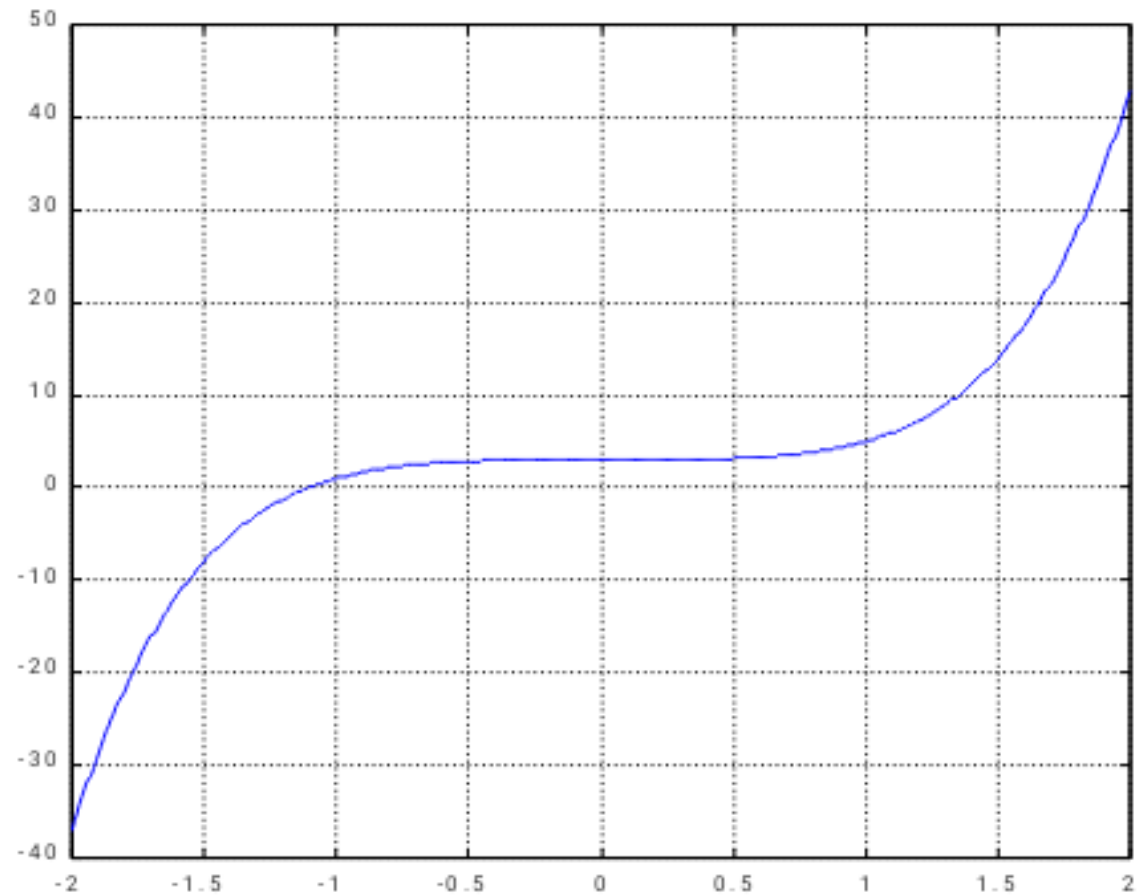
Find the roots of :

$$f(x) = x^5 + x^3 + 3$$

Initial points

$$x_0 = -1 \text{ and } x_1 = -1.1$$

with error  $< 0.001$



# Example

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$x(i)$	$f(x(i))$	$x(i+1)$	$ x(i+1)-x(i) $
-1.0000	1.0000	-1.1000	0.1000
-1.1000	0.0585	-1.1062	0.0062
-1.1062	0.0102	-1.1052	0.0009
-1.1052	0.0001	-1.1052	0.0000

# Convergence Analysis

- The rate of convergence of the Secant method is super linear:

$$\frac{|x_{i+1} - r|}{|x_i - r|^\alpha} \leq C, \quad \alpha \approx 1.62$$

$r$  : root       $x_i$  : estimate of the root at the  $i^{\text{th}}$  iteration.

- It is better than Bisection method but not as good as Newton's method.



# Lectures 11

## Comparison of Root Finding Methods



- Advantages/disadvantages
- Examples

# Summary

Method	Pros	Cons
Bisection	<ul style="list-style-type: none"><li>- Easy, Reliable, Convergent</li><li>- One function evaluation per iteration</li><li>- No knowledge of derivative is needed</li></ul>	<ul style="list-style-type: none"><li>- Slow</li><li>- Needs an interval <math>[a,b]</math> containing the root, i.e., <math>f(a)f(b)&lt;0</math></li></ul>
Newton	<ul style="list-style-type: none"><li>- Fast (if near the root)</li><li>- Two function evaluations per iteration</li></ul>	<ul style="list-style-type: none"><li>- May diverge</li><li>- Needs derivative and an initial guess <math>x_0</math> such that <math>f'(x_0)</math> is nonzero</li></ul>
Secant	<ul style="list-style-type: none"><li>- Fast (slower than Newton)</li><li>- One function evaluation per iteration</li><li>- No knowledge of derivative is needed</li></ul>	<ul style="list-style-type: none"><li>- May diverge</li><li>- Needs two initial points guess <math>x_0, x_1</math> such that <math>f(x_0) - f(x_1)</math> is nonzero</li></ul>

# Example

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Use Secant method to find the root of :

$$f(x) = x^6 - x - 1$$

Two initial points  $x_0 = 1$  and  $x_1 = 1.5$

$$x_{i+1} = x_i - f(x_i) \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# Solution

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k	$x_k$	$f(x_k)$
0	1.0000	-1.0000
1	1.5000	8.8906
2	1.0506	-0.7062
3	1.0836	-0.4645
4	1.1472	0.1321
5	1.1331	-0.0165
6	1.1347	-0.0005

# Example

---

Use Newton' s Method to find a root of :

$$f(x) = x^3 - x - 1$$

Use the initial point :  $x_0 = 1$ .

Stop after three iterations , or

if  $|x_{k+1} - x_k| < 0.001$  , or

if  $|f(x_k)| < 0.0001$  .

# Five Iterations of the Solution

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□	k	$x_k$	$f(x_k)$	$f'(x_k)$	ERROR
□	<hr/>				
□	0	1.0000	-1.0000	2.0000	
□	1	1.5000	0.8750	5.7500	0.1522
□	2	1.3478	0.1007	4.4499	0.0226
□	3	1.3252	0.0021	4.2685	0.0005
□	4	1.3247	0.0000	4.2646	0.0000
□	5	1.3247	0.0000	4.2646	0.0000

# Example

---

Use Newton' s Method to find a root of :

$$f(x) = e^{-x} - x$$

Use the initial point :  $x_0 = 1$ .

Stop after three iterations , or

$$\text{if } |x_{k+1} - x_k| < 0.001, \text{ or}$$

$$\text{if } |f(x_k)| < 0.0001 .$$

# Example

---

Use Newton's Method to find a root of :

$$f(x) = e^{-x} - x, \quad f'(x) = -e^{-x} - 1$$

$x_k$	$f(x_k)$	$f'(x_k)$	$\frac{f(x_k)}{f'(x_k)}$
1.0000	- 0.6321	- 1.3679	0.4621
0.5379	0.0461	- 1.5840	- 0.0291
0.5670	0.0002	- 1.5672	- 0.0002
0.5671	0.0000	- 1.5671	- 0.0000



# Example

---

Estimates of the root of:  $x - \cos(x) = 0$ .

0.6000000000000000

Initial guess

0.74401731944598

1 correct digit

0.73909047688624

4 correct digits

0.73908513322147

10 correct digits

0.73908513321516

14 correct digits

# Example

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In estimating the root of:  $x - \cos(x) = 0$ , to get more than 13 correct digits:

- 4 iterations of Newton ( $x_0 = 0.8$ )
- 43 iterations of Bisection method (initial interval  $[0.6, 0.8]$ )
- 5 iterations of Secant method ( $x_0 = 0.6, x_1 = 0.8$ )