SE301: Numerical Methods

Topic 5:

Interpolation

Lectures 20-22:

Lecture 20
Introduction to Interpolation

- Introduction
- Interpolation Problem
- Existence and Uniqueness
- Linear and Quadratic Interpolation
- Newton’s Divided Difference Method
- Properties of Divided Differences

Introduction

Interpolation was used for long time to provide an estimate of a tabulated function at values that are not available in the table.

What is sin (0.15)?

x	sin(x)
0	0.0000
0.1	0.0998
0.2	0.1987
0.3	0.2955
0.4	0.3894

Using **Linear Interpolation** sin (0.15) ≈ **0.1493**

True value (4 decimal digits) sin (0.15) = **0.1494**

The Interpolation Problem

Given a set of n+1 points,

$$(x_0,\ f(x_0)),\ (x_1,\ f(x_1)),\ \ldots,\ (x_n,\ f(x_n))$$

Find an nth order polynomial $f_n(x)$
that passes through all points, such that:

$$\boxed{f_n(x_i) = f(x_i) \quad for \quad i = 0, 1, 2, \ldots, n}$$

Example

An experiment is used to determine the viscosity of water as a function of temperature. The following table is generated:

Problem: Estimate the viscosity when the temperature is 8 degrees.

Temperature (degree)	Viscosity
0	1.792
5	1.519
10	1.308
15	1.140

Interpolation Problem

Find a polynomial that fits the data points exactly.

$$V(T) = \sum_{k=0}^{n} a_k T^k$$

$$V_i = V(T_i)$$

V : Viscosity

T : Temperature

a_k : Polynomial coefficients

Linear Interpolation: V(T)= 1.73 − 0.0422 T

V(8)= 1.3924

Existence and Uniqueness

Given a set of n+1 points:

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$$

Assumption: $x_0, x_1, \ldots, x_n$ are distinct

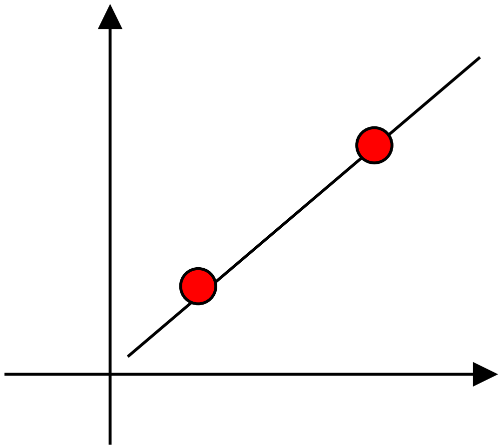
Theorem:

There is a unique polynomial $f_n(x)$ of order ≤ n such that:

$$f_n(x_i) = f(x_i) \qquad for \qquad i = 0, 1, \ldots, n$$

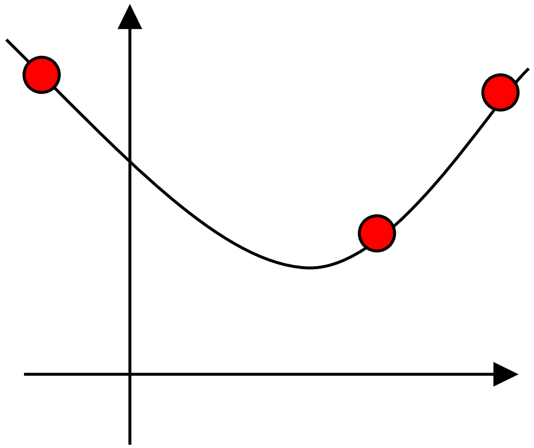
Examples of Polynomial Interpolation

Linear Interpolation



- Given any two points, there is one polynomial of order ≤ 1 that passes through the two points.

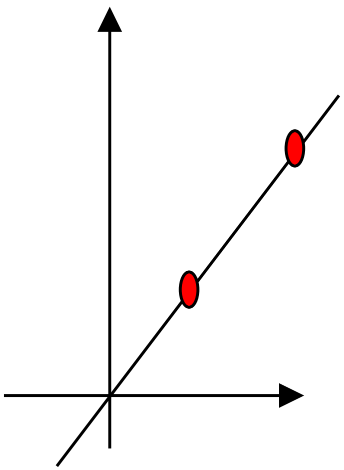
Quadratic Interpolation



Given any three points there is one polynomial of order ≤ 2 that passes through the three points.

Linear Interpolation

Given any two points, $(x_0,\ f(x_0)),\ (x_1,\ f(x_1))$



The line that interpolates the two points is:

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Example :

Find a polynomial that interpolates (1,2) and (2,4).

$$f_1(x) = 2 + \frac{4-2}{2-1}(x-1) = 2x$$

Quadratic Interpolation

- Given any three points: $(x_0,\ f(x_0)),\ (x_1,\ f(x_1)),\ and\ \ (x_2,\ f(x_2))$
- The polynomial that interpolates the three points is:

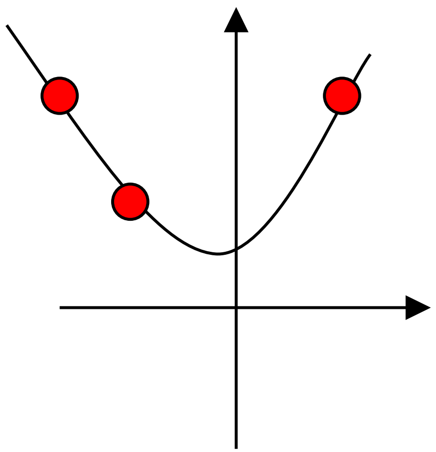
$$\boxed{f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)}$$

where :

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_0, x_1, x_2] = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



General nth Order Interpolation

Given any n+1 points: $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$

The polynomial that interpolates all points is:

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \ldots + b_n(x - x_0)\ldots(x - x_{n-1})$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1]$$

....

$$b_n = f[x_0, x_1, \ldots, x_n]$$

Divided Differences

$$f[x_k] = f(x_k)$$ Zeroth order DD

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$ First order DD

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$ Second order DD

…………

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}$$

Divided Difference Table

x	F[ ]	F[,]	F[, ,]	F[, , ,]
x_0	F[x0]	$F[x_0,x_1]$	$F[x_0,x_1,x_2]$	$F[x_0,x_1,x_2,x_3]$
x_1	$F[x_1]$	$F[x_1,x_2]$	$F[x_1,x_2,x_3]$	
x_2	$F[x_2]$	$F[x_2,x_3]$		
x_3	$F[x_3]$			

$$f_n(x) = \sum_{i=0}^{n} \left\{ F[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j) \right\}$$

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	$f(x_i)$
0	-5
1	-3
-1	-15

Entries of the divided difference table are obtained from the data table using simple operations.

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	$f(x_i)$
0	-5
1	-3
-1	-15

The first two column of the

table are the data columns.

Third column: First order differences.

Fourth column: Second order differences.

Divided Difference Table

x	F[]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{-3-(-5)}{1-0} = 2$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{-15-(-3)}{-1-1} = 6$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{6-(2)}{-1-(0)} = -4$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	y_i
0	-5
1	-3
-1	-15

$$f_2(x) = -5 + 2(x-0) - 4(x-0)(x-1)$$

$f_2(x)$= $F[x_0]+F[x_0,x_1]\,(x-x_0)+F[x_0,x_1,x_2]\,(x-x_0)(x-x_1)$

Two Examples

Obtain the interpolating polynomials for the two examples:

x	y
1	0
2	3
3	8

x	y
2	3
1	0
3	8

What do you observe?

Two Examples

x	Y		
1	0	3	1
2	3	5	
3	8		

$$P_2(x) = 0 + 3(x-1) + 1(x-1)(x-2)$$
$$= x^2 - 1$$

x	Y		
2	3	3	1
1	0	4	
3	8		

$$P_2(x) = 3 + 3(x-2) + 1(x-2)(x-1)$$
$$= x^2 - 1$$

Ordering the points should not affect the interpolating polynomial.

Properties of Divided Difference

Ordering the points should not affect the divided difference:

$$f[x_0, x_1, x_2] = f[x_1, x_2, x_0] = f[x_2, x_1, x_0]$$

Example

- Find a polynomial to interpolate the data.

x	f(x)
2	3
4	5
5	1
6	6
7	9

Example

x	f(x)	f[,]	f[, ,]	f[, , ,]	f[, , , ,]
2	3	1	-1.6667	1.5417	-0.6750
4	5	-4	4.5	-1.8333	
5	1	5	-1		
6	6	3			
7	9				

$$f_4 = 3 + 1(x-2) - 1.6667\,(x-2)(x-4) + 1.5417\,(x-2)(x-4)(x-5)$$
$$-0.6750\,(x-2)(x-4)(x-5)(x-6)$$

Summary

Interpolating Condition : $f(x_i) = f_n(x_i) \quad for \quad i = 0, 1, 2, ..., n$

* The interpolating Polynomial is unique.

* Different methods can be used to obtain it

  - Newton Divided Difference [Section 18.1]

  - Lagrange Interpolation [Section 18. 2]

  - Other methods

Ordering the points should not affect the interpolating polynomial .

Lecture 21
Lagrange Interpolation

The Interpolation Problem

Given a set of n+1 points:

$$(x_0,\ f(x_0)),\ (x_1,\ f(x_1)),\ \ldots.,\ (x_n,\ f(x_n))$$

Find an n[th] order polynomial: $f_n(x)$

that passes through all points, such that:

$$\boxed{f_n(x_i) = f(x_i) \quad for \ \ i = 0, 1, 2, \ldots, \ n}$$

Lagrange Interpolation

Problem:

Given

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

Find the polynomial of least order $f_n(x)$ such that:

$$f_n(x_i) = f(x_i) \qquad for \qquad i = 0,1,\ldots,\ n$$

Lagrange Interpolation Formula:

$$f_n(x) = \sum_{i=0}^{n} f(x_i)\ \ell_i(x)$$

$$\ell_i(x) = \prod_{j=0,\ j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

Lagrange Interpolation

$\ell_i(x)$ are called the cardinals.

The cardinals are n^{th} order polynomials:

$$\ell_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Lagrange Interpolation Example

x	1/3	1/4	1
y	2	-1	7

$$P_2(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x)$$

$$\ell_0(x) = \frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)} = \frac{(x-1/4)}{(1/3-1/4)}\frac{(x-1)}{(1/3-1)}$$

$$\ell_1(x) = \frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)} = \frac{(x-1/3)}{(1/4-1/3)}\frac{(x-1)}{(1/4-1)}$$

$$\ell_2(x) = \frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)} = \frac{(x-1/3)}{(1-1/3)}\frac{(x-1/4)}{(1-1/4)}$$

$$P_2(x) = 2\{-18(x-1/4)(x-1)\} - 1\{16(x-1/3)(x-1)\}$$

$$+7\{2(x-1/3)(x-1/4)\}$$

Example

Find a polynomial to interpolate:

Both Newton's interpolation method and Lagrange interpolation method must give the same answer.

x	y
0	1
1	3
2	2
3	5
4	4

Newton's Interpolation Method

0	1	2	-3/2	7/6	-5/8
1	3	-1	2	-4/3	
2	2	3	-2		
3	5	-1			
4	4				

Interpolating Polynomial

$$f_4(x) = 1 + 2(x) - \frac{3}{2}x(x-1) + \frac{7}{6}x(x-1)(x-2)$$

$$-\frac{5}{8}x(x-1)(x-2)(x-3)$$

$$f_4(x) = 1 + \frac{115}{12}x - \frac{95}{8}x^2 + \frac{59}{12}x^3 - \frac{5}{8}x^4$$

Interpolating Polynomial Using Lagrange Interpolation Method

$$f_4(x) = \sum_{i=0}^{4} f(x_i)\ \ell_i = \ell_0 + 3\ell_1 + 2\ell_2 + 5\ell_3 + 4\ell_4$$

$$\ell_0 = \frac{(x-1)}{(0-1)}\frac{(x-2)}{(0-2)}\frac{(x-3)}{(0-3)}\frac{(x-4)}{(0-4)} = \frac{(x-1)(x-2)(x-3)(x-4)}{24}$$

$$\ell_1 = \frac{(x-0)}{(1-0)}\frac{(x-2)}{(1-2)}\frac{(x-3)}{(1-3)}\frac{(x-4)}{(1-4)} = \frac{x(x-2)(x-3)(x-4)}{-6}$$

$$\ell_2 = \frac{(x-0)}{(2-0)}\frac{(x-1)}{(2-1)}\frac{(x-3)}{(2-3)}\frac{(x-4)}{(2-4)} = \frac{x(x-1)(x-3)(x-4)}{4}$$

$$\ell_3 = \frac{(x-0)}{(3-0)}\frac{(x-1)}{(3-1)}\frac{(x-2)}{(3-2)}\frac{(x-4)}{(3-4)} = \frac{x(x-1)(x-2)(x-4)}{-6}$$

$$\ell_4 = \frac{(x-0)}{(4-0)}\frac{(x-1)}{(4-1)}\frac{(x-2)}{(4-2)}\frac{(x-3)}{(4-3)} = \frac{x(x-1)(x-2)(x-3)}{24}$$

Lecture 22

Inverse Interpolation

Error in Polynomial Interpolation

Inverse Interpolation

Problem : Given a table of values

Find x such that : $f(x) = y_k$, where y_k is given

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

<u>One approach</u>:

Use polynomial interpolation to obtain $f_n(x)$ to interpolate the data then use Newton's method to find a solution to x

$$f_n(x) = y_k$$

Inverse Interpolation

Inverse interpolation:

1. Exchange the roles of x and y.

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

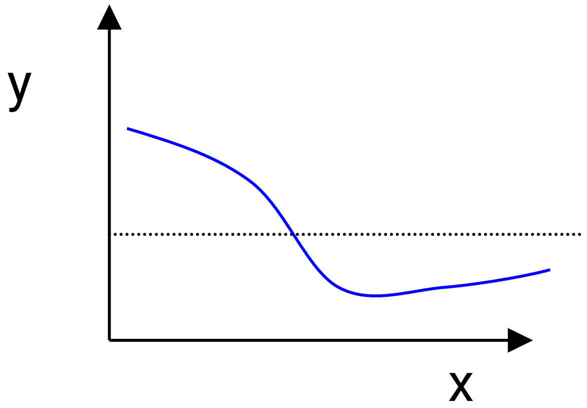
y_i	y_0	y_1	....	y_n
x_i	x_0	x_1	....	x_n

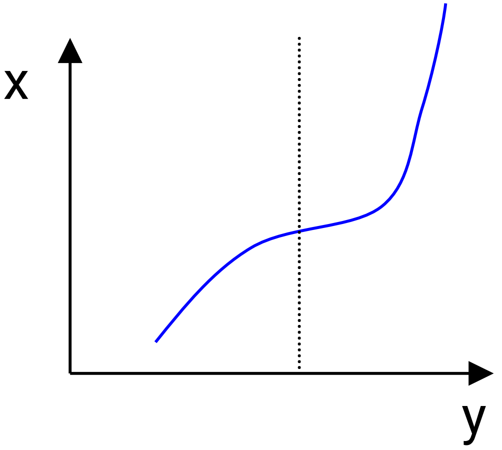
2. Perform polynomial Interpolation on the new table.

3. Evaluate

$$x = f_n(y_k)$$

Inverse Interpolation





Inverse Interpolation

<u>**Question:**</u>

What is the limitation of inverse interpolation?

- The original function has an inverse.
- $y_1, y_2, \ldots, y_n$ must be distinct.

Inverse Interpolation

Example

x	1	2	3
y	3.2	2.0	1.6

Problem :

Given the table. Find x such that $f(x) = 2.5$

3.2	1	-.8333	1.0417
2.0	2	-2.5	
1.6	3		

$$x = f_2(y) = 1 - 0.8333\,(y - 3.2) + 1.0417\,(y - 3.2)(y - 2)$$

$$x = f_2(2.5) = 1 - 0.8333\,(-0.7) + 1.0417\,(-0.7)(0.5) = 1.2187$$

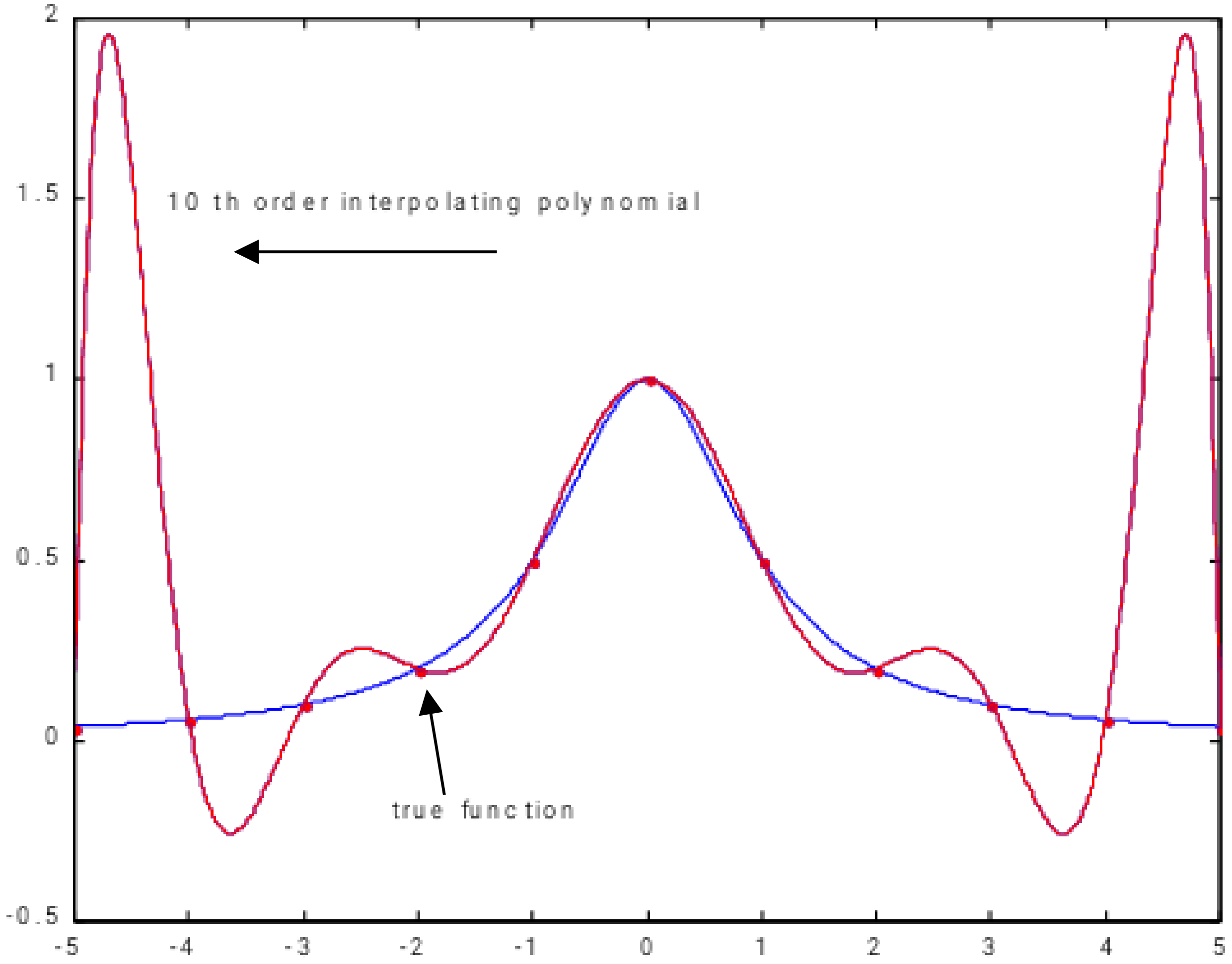
Errors in polynomial Interpolation

- Polynomial interpolation may lead to large errors (especially for high order polynomials).

BE CAREFUL

- When an n^{th} order interpolating polynomial is used, the error is related to the $(n+1)^{th}$ order derivative.

10^{th} Order Polynomial Interpolation



Errors in polynomial Interpolation

Theorem

Let $f(x)$ be a function such that :

$f^{(n+1)}(x)$ is continuous on [a, b], and $\left| f^{(n+1)}(x) \right| \leq M$.

Let *P(x)* be any polynomial of degree $\leq$ n

that interpolates f at $n+1$ equally spaced points

in [a, b] (including the end points). Then :

$$\left| f(x) - P(x) \right| \leq \frac{M}{4(n+1)} \left(\frac{b-a}{n} \right)^{n+1}$$

Example

$f(x) = \sin(x)$

We want to use 9^{th} order polynomial to interpolate f(x) (using 10 equally spaced points) in the interval [0,1.6875].

$$\left| f^{(n+1)} \right| \leq 1 \quad for \quad n > 0$$

$$M = 1, \quad n = 9$$

$$\left| f(x)\text{-}P(x) \right| \leq \frac{M}{4(n+1)} \left(\frac{b-a}{n} \right)^{n+1}$$

$$\left| f(x)\text{-}P(x) \right| \leq \frac{1}{4(10)} \left(\frac{1.6875}{9} \right)^{10} = 1.34 \times 10^{-9}$$

Summary

- The interpolating polynomial is unique.
- Different methods can be used to obtain it.
  - Newton's divided difference
  - Lagrange interpolation
  - Others
- Polynomial interpolation can be sensitive to data.
- BE CAREFUL when high order polynomials are used.