1. Solve, $y'' + y = t$, $y(0) = 1$, $y'(0) = -2$.

Taking the laplace transform of both sides of the differential equation and using the given conditions.

We have,

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t\}.$$

$$\Rightarrow s^2 y - s y(0) - y'(0) + y = \frac{1}{s^2}$$

$$\Rightarrow s^2 y - s + 2 + y = \frac{1}{s^2}$$

$$\Rightarrow y(s^2+1) - s + 2 = \frac{1}{s^2}$$

$$\Rightarrow y(s^2+1) = \frac{1}{s^2} + s - 2$$

$$\Rightarrow y = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}.$$

* Using pertial fractions

$$\frac{1}{s^2(s^2+1)} \equiv \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1} \quad \text{--- (i)}$$

$$\Rightarrow 1 \equiv (As+B)(s^2+1) + (Cs+D)(s^2)$$

$$\Rightarrow 1 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$\Rightarrow 1 = s^3(A+C) + s^2(B+D) + As + B$$

Now equating co-efficients,

$$A = 0$$
$$B = 1.$$

another,

$$A+C=0$$
$$0+C=0$$
$$\Rightarrow C=0.$$

Others

$$B+D=0$$
$$\Rightarrow 1+D=0$$
$$\therefore D=-1.$$

Putting the all values in equation (i)

$$= \frac{0+1}{s^2} + \frac{0-1}{s^2+1}$$

$$= \frac{1}{s^2} - \frac{1}{s^2+1}.$$

$$\therefore y = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s-2}{s^2+1}$$

$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

and $Y = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right\}$

$= t + \cos t - 3\sin t.$

(Ans).

(2) $Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$

We have,

$$\mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 4\mathcal{L}\{e^{2t}\}$$

$$\Rightarrow \{s^2 y - sY(0) - Y'(0)\} - 3\{sy - Y(0)\} + 2y = \frac{4}{s}$$

$$\Rightarrow \{s^2 y + 3s - 5\} - 3\{sy + 3\} + 2y = \frac{4}{s-2}.$$

$$\Rightarrow y(s^2 - 3s + 2) + 3s - 14 = \frac{4}{s-2}$$

$$\Rightarrow y(s^2 - 3s + 2) = \frac{4}{s-2} + 14 - 3s$$

$$\Rightarrow y = \frac{4}{(s-2)(s^2-3s+2)} + \frac{14-3s}{s^2-3s+2}.$$

$$\Rightarrow y = \frac{4 + 14s - 3s^2 - 28 + 6s}{(s-2)(s^2-3s+2)}$$

$$\Rightarrow y = \frac{-3s^2 + 20s - 24}{(s-2)(s^2-3s+2)}.$$

$$\Rightarrow y = \frac{-3s^2 + 20s - 24}{(s-2)\{s^2 - 4s + 4 + s - 2\}}$$

$$\Rightarrow y = \frac{-3s^2+20s-24}{(s-2)\{(s-2)^2+(s-2)\}}$$

$$= \frac{-3s^2+20s-24}{(s-2)(s-2)\{(s-2)+1\}}$$

$$= \frac{-3s^2+20s-24}{(s-1)(s-2)^2}$$

Using partial fractions:

$$\Rightarrow \frac{-3s^2+20s-24}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2} \quad \text{——— (i)}$$

$$\Rightarrow -3s^2+20s-24 = A(s-2)^2 + B(s-2)(s-1) + C(s-1)$$

Now putting the values, $s = 1, 2$.

When, $s = 1$,

$$\Rightarrow -3+20-24 = A(1-2)^2 + 0 + 0$$

$$\Rightarrow A = -7.$$

When, $s=2$.

$$\Rightarrow -12+40-24 = 0+0+C(2-1)$$

$$\Rightarrow C = 4.$$

Now Equating co-efficient.

$$\Rightarrow -3s^2+20s-24 = A(s-2)^2 + B(s-2)(s-1) + C(s-1)$$

$$\Rightarrow -3s^2+20s-24 = A(s^2-4s+4) + B(s^2-3s+2) + C(s-1).$$

* co-efficient of s^2.

$$A+B=-3$$

$$B=-3+7$$

$$B=4$$

Putting the all vales in Equ. (i)

$$y = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

Thus, $Y = \mathcal{L}^{-1}\left\{\frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right\}$

$$= -7e^{t} + 4e^{2t} + 4te^{2t}$$

3. $Y'' + 2Y' + 5Y = e^{-t}\sin t,$

$$Y(0) = 0$$

$$Y'(0) = 1.$$

We have,

$$\mathcal{L}\{Y''\} + 2\mathcal{L}\{Y'\} + 5\mathcal{L}\{Y\} = \mathcal{L}\{e^{-t}\sin t\}$$

$$\Rightarrow \{s^2y - sY(0) - Y(0)\} + 2\{sy - Y(0)\} + 5y$$

$$= \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}.$$

$$\Rightarrow \{s^2 y - s.(0) - 1\} + 2\{sy - 0\} + 5y = \frac{1}{s^2+2s+2}$$

$$\Rightarrow (s^2+2s+5)y - 1 = \frac{1}{s^2+2s+2}$$

$$\Rightarrow (s^2+2s+5)y = \frac{1}{s^2+2s+2} + 1$$

$$\Rightarrow y = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$\Rightarrow y = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}.$$

Using ~~fractio~~ partial Fractions.

$$\Rightarrow \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \equiv \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5} \quad \text{---} ($$

$$\Rightarrow s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

$$\Rightarrow s^2+2s+3 = As^3+2As^2+5As+Bs^2+2Bs+5B+Cs^3+2Cs^2+2Cs+Ds^2+2Ds+2D$$

Equating co-efficient:

$$A+C=0 \text{ ——— (ii)}$$
$$2A+B+2C+D=1 \text{ ——— (iii)}$$
$$5A+2B+2C+2D=2 \text{ ——— (iv)}$$
$$5B+2D=3. \text{ ——— (v)}$$

Now working Equ (iii)

$$2A+2C+B+D=1$$
$$\Rightarrow 2(A+C)+B+D=1$$
$$\Rightarrow 2\times 0+B+D=1.$$
$$\Rightarrow 1 = B+D \text{ ——— (vi)}$$

Now working (v) and (vi)

Scanned with CamScanner

$$\Rightarrow 5B + 2D = 3 \quad \text{--- (v)}$$
$$2B + 2D = 2 \quad \text{--- (vi)} \times 2$$

$$3B = 1$$

$$\therefore B = \frac{1}{3},$$

Now

$$B + D = 1.$$

$$D = 1 - \frac{1}{3} = \frac{2}{3}$$

Scanned with CamScanner

Now → (iv)

$5A+2B+2D+2C=2$

$\Rightarrow 5A+2C+2=2$

$\Rightarrow 5A+2C=0$ —— (viii)

When $2\times$(ii) − viii

$\Rightarrow$ $5A+2C=0$

$2A+2C=0$

$3A=0$

$\Rightarrow A=0$,

Now, $A+C=0$

$\Rightarrow C=0$,

putting all values equ: (i)

$$= \frac{A\cdot 0 + \frac{1}{3}}{s^2+2s+2} + \frac{0+\frac{2}{3}}{s^2+2s+5}$$

$$y \Rightarrow \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)}$$

$$Y = \mathcal{L}^{-1}\left\{ \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{3\{(s+1)^2+1\}} + \frac{2}{3\{(s+1)^2+2^2\}} \right\}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{1}{3} e^{-t} \sin 2t$$

$$\Rightarrow \frac{1}{3} \{ e^{-t} (\sin t + \sin 2t)$$

Scanned with CamScanner

(4)

$$Y''' - 3Y'' + 3Y' - Y = t^2 e^t$$

$Y(0) = 1$

$Y'(0) = 0$

$Y''(0) = -2.$

We have,

$$\mathcal{L}\{Y'''\} - 3\mathcal{L}\{Y''\} + 3\mathcal{L}\{Y'\} - \mathcal{L}\{Y\} = \mathcal{L}\{t^2 e^t\}.$$

$$\Rightarrow \{s^3 y - s^2 Y(0) - s Y'(0) - Y''(0)\} - 3\{s^2 y - s Y(0) - Y'(0)\} + 3\{s y - Y(0)\} - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow (s^3 - 3s^2 + 3s - 1) y - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

$$\Rightarrow y = \frac{s^2-3s+1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{s^2-2s+1-s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{(s-1)^2-(s-1)-1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$y = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$Y = e^t - te^t - \frac{t^2e^t}{2} + \frac{t^5e^t}{60}.$$

Scanned with CamScanner

⑥ $Y'' + 9Y = \cos 2t$

$Y'(0) = 1$

$Y(\pi/2) = -1$

Since $Y'(0)$ is not known, let

$Y'(0) = c$

$\mathcal{L}\{Y''\} + 9\mathcal{L}\{Y\} = \mathcal{L}\{\cos 2t\}$

$\Rightarrow s^2 y - sY(0) - Y'(0) + 9y = \frac{s}{s^2+4}$

$\Rightarrow (s^2+9)y - s - c = \frac{s}{s^2+4}$

$\Rightarrow y = \frac{s+c}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)}$

Using Partial fraction.

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow s = (As+B)(s^2+4) + (Cs+D)(s^2+9)$$

$$\Rightarrow s = As^3 + 4As + Bs^2 + 4B + Cs^3$$
$$+ 9Cs + Ds^2 + 9D.$$

Equating co-efficient :-

$A+C=0$ —— (ii)

$4A+9C=1$ —— (iii)

$B+D=0$ —— (iv)

$9D+4B=0$ —— (v)

*

(iii) - (ii)×4

$4A+9C=1$

$4A+4C=0$

$5C=1$

$C=\frac{1}{5}$

$\therefore A + C = 0$

$\Rightarrow A = -\frac{1}{5}$

$V - (iv \times 4)$

$$4B + 9D = 0$$
$$4B + 4D = 0$$

$\Rightarrow 5D = 0$

$D = 0$

$\therefore$

$D + B = 0$

$B = 0.$

Our Equation:

$$\Rightarrow \frac{s}{s^2+9} + \frac{c}{s^2+9} - \frac{s}{5(s^2+9)} + \frac{s}{5(s^2+4)}$$

$$\Rightarrow \frac{4}{5}\left(\frac{s}{s^2+9}\right) + \frac{c}{s^2+9} + \frac{s}{5(s^2+4)}$$

$$\Rightarrow \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2$$