1. General Properties

P8.1: In free space, a wave propagating radially away from an antenna at the origin has

$$\mathbf{H}_s = \frac{-I_s}{r}\cos^2\theta\ \mathbf{a}_\theta,$$

where the driving current phasor $I_s = I_o e^{j\alpha}$. Determine (a) $\mathbf{E}_s$, (b) $\mathbf{P}(r,\theta,\phi)$ and (c) R_{rad}.

$$\mathbf{E}_s = -\eta\mathbf{a}_P \times \mathbf{H}_s = -\eta_o\mathbf{a}_r \times \left(\frac{-I_s}{r}\cos^2\theta\mathbf{a}_\theta\right),$$

$$(a)\mathbf{E}_s = \frac{\eta_o I_s}{r}\cos^2\theta\mathbf{a}_\phi$$

$$\mathbf{P} = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}_s \times \mathbf{H}_s^*\right] = \frac{1}{2}\mathrm{Re}\left[\frac{\eta_o I_o e^{j\alpha}}{r}\cos^2\theta\mathbf{a}_\phi \times \frac{-I_o e^{-j\alpha}}{r}\cos^2\theta\mathbf{a}_\theta\right]$$

$$\text{(b) } \mathbf{P}(r,\theta,\phi) = \frac{1}{2}\eta_o\left(\frac{I_o}{r}\right)^2\cos^4\theta\mathbf{a}_r$$

Now to find R_{rad}:

$$P_{rad} = \int \mathbf{P}(r,\theta,\phi)\cdot d\mathbf{S} = \frac{1}{2}I_o^2 R_{rad},$$

$$P_{rad} = \frac{1}{2}\eta_o I_o^2\int\frac{\cos^4\theta}{r^2}\mathbf{a}_r\cdot r^2\sin\theta d\theta d\phi\mathbf{a}_r = \frac{1}{2}\eta_o I_o^2\int_0^\pi\cos^4\theta\sin\theta d\theta\int_0^{2\pi}d\phi$$

$$P_{rad} = \frac{\pi\eta_o I_o^2}{5}\left(-\cos^5\theta\right)\Big|_0^\pi = \frac{2}{5}\pi\eta_o I_o^2 = \frac{1}{2}I_o^2 R_{rad}$$

Solving:

$$R_{rad} = \frac{\frac{2}{5}\pi(120\pi)I_o^2}{\frac{1}{2}I_o^2} = 96\pi^2\Omega$$

$$(c)R_{rad} = 950\Omega$$

P8.5: You are given the following normalized radiation intensity:

$$P_n(\theta,\phi) = \sin^2\theta\sin^3\phi \text{ for } 0 \le \phi \le \pi,$$

$$0 \text{ otherwise.}$$

Find the beamwidth, pattern solid angle, and directivity.

The beam is pointing in the $\mathbf{a}_y$ direction, and we have $BW = \frac{1}{2}\left(BW_\theta + BW_\phi\right)$.

To find BW_θ, we fix $\phi = \pi/2$ and set $\sin^2\theta$ equal to ½. Then,

$$\theta = \sin^{-1}\left(\sqrt{\frac{1}{2}}\right) = 45^\circ, \quad \text{so } BW_\theta = \left(180^\circ - 45^\circ\right) - 45^\circ = 90^\circ.$$

To find BW_ϕ, we fix $\theta = \pi/2$, and set $\sin^3\phi$ = ½, giving us

$$\theta = \sin^{-1}\left((1/2)^{1/3}\right) = 52.5^\circ, \quad \text{so } BW_\phi = \left(180^\circ - 52.5^\circ\right) - 52.5^\circ = 75^\circ.$$

Finally, $BW = \frac{1}{2}\left(90^\circ + 75^\circ\right) = 82.5^\circ.$

The pattern solid angle is

$$\Omega_P = \iint P_n d\Omega = \iint \left(\sin^2\theta \sin^3\phi\right)\sin\theta d\theta d\phi,$$

$$\Omega_P = \int_0^\pi \sin^3\theta d\theta \int_0^\pi \sin^3\phi d\phi, \quad \text{(note limits on } \phi\text{)}$$

Each integral is solved as follows:

$$y = \int_0^\pi \sin^3 x dx = \int_0^\pi \left(1-\cos^2 x\right)\sin x dx = \int_0^\pi \sin x dx - \int_0^\pi \cos^2 x \sin x dx.$$

$$\int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = 2$$

$$\int_0^\pi \cos^2 x \sin x dx = -\int u^2 du = -\frac{1}{3}u^3, \quad \text{where } u = \cos x,\ du = -\sin x dx.$$

$$\text{so } \int_0^\pi \cos^2 x \sin x dx = -\frac{1}{3}\cos^3 x\Big|_0^\pi = -\frac{1}{3}(-1-1) = \frac{2}{3}.$$

So we have

$$y = \int_0^\pi \sin^3 x dx = 2 - \frac{2}{3} = \frac{4}{3},$$

$$\text{and} \quad \Omega_P = \int_0^\pi \sin^3\theta d\theta \int_0^\pi \sin^3\phi d\phi = \left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = \frac{16}{9} = 1.78 sr.$$

$$D_{\max} = \frac{4\pi}{\Omega_P} = \frac{4\pi}{1.78} = 7.1$$

P8.6: You are given the following normalized radiation intensity:

$$P_n\left(\theta,\phi\right) = \sin^2\theta \sin\frac{\phi}{2}.$$

Determine the beamwidth, direction of maximum radiation, pattern solid angle and directivity.

$$BW = \frac{1}{2}\left(BW_\theta + BW_\phi\right),$$

BW_θ: Fix $\phi = \pi$, $\sin^2\theta = 1/2$, $\theta = 45°$, $BW_\theta = (180° – 45°) – 45° =90°$.
BW_ϕ: Fix $\theta = \pi/2$, $\sin(\phi/2)=1/2$, $BW_\phi = (360° – 60°) – 60° = 240°$
BW = (90°+240°)/2=165°

By inspection, the direction of maximum radiation is at $\phi = \pi$ and $\theta = \pi/2$. (i.e. the $-\mathbf{a}_x$ direction).

$$\Omega_P = \iint \sin^2\theta \sin\frac{\phi}{2}\sin\theta d\theta d\phi = \int_0^{2\pi}\sin\frac{\phi}{2}d\phi\int_0^{\pi}\sin^3\theta d\theta$$

Do each integral separately:

$$\int_0^{2\pi}\sin\frac{\phi}{2}d\phi = -2\cos\frac{\phi}{2}\Big|_0^{2\pi} = -2(-1-1) = 4$$

$$\int_0^{\pi}\sin^3\theta d\theta = \int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta dx = \int_0^{\pi}\sin\theta d\theta - \int_0^{\pi}\cos^2\theta\sin\theta d\theta$$

$$= -\cos\theta\Big|_0^{\pi} + \frac{1}{3}\cos^3\theta\Big|_0^{\pi} = -(-1-1) + \frac{1}{3}(-1-1) = 2 - \frac{2}{3} = \frac{4}{3}$$

So $\Omega_P = (4)\left(\frac{4}{3}\right) = \frac{16}{3}$, and $D_{\max} = \frac{4\pi}{\Omega_P} = \frac{3\pi}{4} = 2.4$

P8.9: Suppose a Hertzian dipole antenna is 1.0 cm long and is excited by a 10 mA amplitude current source at 100. MHz. What is the maximum power density radiated by this antenna at a 1.0 km distance? What is the antenna's radiation resistance?

$$c = \lambda f, \quad \lambda = \frac{c}{f} = \frac{3x10^8\, m/s}{100x10^6\, 1/s} = 3m.$$

$$P_{\max} = \frac{\eta_o\beta^2 I_o^2 \ell^2}{32\pi^2 r^2} = \frac{120\pi}{32}\frac{(2\pi)^2}{3^2}\frac{(0.010)^2}{\pi^2}\frac{(0.010)^2}{1000^2} = 0.052\frac{pW}{m^2}$$

$$R_{rad} = 80\pi^2\left(\frac{\ell}{\lambda}\right)^2 = 80\pi^2\left(\frac{0.01}{3}\right)^2 = 8.8m\Omega$$

P8.10: A 1.0 cm long, 1.0 mm diameter copper wire is used as a Hertzian dipole radiator at 1.0 GHz. (a) Find R_{rad}. (b) Estimate R_{diss} by considering the skin effect resistance of the wire. (c) Find efficiency, e. (d) Find the maximum power gain G_{max}.

$$\lambda = \frac{c}{f} = \frac{3x10^8\, m/s}{1x10^9\, 1/s} = 0.3m$$

$$R_{rad} = 80\pi^2\left(\frac{\ell}{\lambda}\right)^2 = 80\pi^2\left(\frac{0.01}{.3}\right)^2 = 0.877\Omega$$

From Example 8.2 we have $\delta_{Cu}\big|_{1GHz} = 2.09x10^{-6}m$

$$S = \pi d\delta_{Cu} = \pi(0.001m)\left(2.09x10^{-6}m\right) = 6.57x10^{-9}m^2$$

$$R_{diss} = \frac{1}{\sigma}\frac{\ell}{S} = \frac{1}{\left(5.8x10^7\right)}\frac{0.01}{6.57x10^{-9}} = 0.026\Omega$$

$$e = \frac{R_{rad}}{R_{rad}+R_{diss}} = \frac{0.877}{0.877+0.026} = 0.97$$

$$G_{\max} = eD_{\max} = 0.97(1.5) = 1.46$$

P8.13: Suppose in the far-field for an antenna at the origin,

$$\mathbf{H}_{os} = \frac{\beta I_s}{4\pi}\frac{e^{-j\beta r}}{r}\sin\theta\cos\phi\,\mathbf{a}_\phi$$

where $I_s = I_o e^{j\alpha}$. What is the radiation resistance of this antenna at 100 MHz?

$$\mathbf{E}_{OS} = -\eta_o \mathbf{a}_r \times \mathbf{H}_{OS} = \frac{\eta_o \beta I_s}{4\pi}\frac{e^{-j\beta r}}{r}\sin\theta\cos\phi\,\mathbf{a}_\theta$$

$$\mathbf{P}(r,\theta,\phi) = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}_{OS}\times\mathbf{H}_{OS}^*\right] = \frac{1}{2}\eta_o\left(\frac{\beta I_o}{4\pi r}\right)^2 \sin^2\theta\cos^2\phi\,\mathbf{a}_r$$

Note also that $\mathbf{P}(r,\theta,\phi) = P_{\max} P_n(\theta,\phi)\mathbf{a}_r$, where here

$$P_{\max} = \frac{1}{2}\eta_o\left(\frac{\beta I_o}{4\pi r}\right)^2, \quad \text{and } P_n(\theta,\phi) = \sin^2\theta\cos^2\phi.$$

Then, $\Omega_P = \iint P_n(\theta,\phi)d\Omega.$

Referring to P8.5, $\int_0^\pi \sin^3\theta d\theta = 2 - \frac{2}{3} = \frac{4}{3}$, and

$$\int_0^{2\pi}\cos^2\phi d\phi = \frac{1}{2}\left[\int d\phi + \int \cos 2\phi d\phi\right] = \pi.$$

So, $\Omega_P = \iint P_n(\theta,\phi)d\Omega = \frac{4\pi}{3}sr$

$$P_{rad} = r^2 P_{\max}\Omega_P = r^2\left(\frac{1}{2}\eta_o\left(\frac{\beta I_o}{4\pi r}\right)^2\right)\left(\frac{4\pi}{3}\right).$$

Using $\eta_o = 120\pi$ and $\beta = 2\pi/\lambda$, we find

$$P_{rad} = \frac{20\pi^2 I_o^2}{\lambda^2}.$$

Finally, $P_{rad} = \frac{1}{2}I_o^2 R_{rad}$, so $R_{rad} = \frac{2P_{rad}}{I_o^2} = \frac{40\pi^2}{\lambda^2}$,

and since for this problem, $\lambda = c/f = 3m$, $R_{rad} = \left(40\pi^2/9\right) = 44\Omega$

P8.14: Suppose in the far-field for a particular antenna at the origin, the electric field is

$$\mathbf{E}_{os} = \eta_o I_o \frac{e^{-j\beta r}}{\pi r}\sin\theta\,\mathbf{a}_\theta.$$

What is the radiation resistance of this antenna?

We'll use: $P_{rad} = \frac{1}{2}I_o^2 R_{rad} = r^2 P_{\max}\Omega_P$, so we must find P_{max} and Ω_p.

$$\mathbf{H}_S = \frac{1}{\eta_o}\mathbf{a}_r \times \eta_o I_o \frac{e^{-j\beta r}}{\pi r}\sin\theta\,\mathbf{a}_\theta = I_o\frac{e^{-j\beta r}}{\pi r}\sin\theta\,\mathbf{a}_\phi$$

$$\mathbf{P} = \frac{1}{2}\text{Re}\left[\mathbf{E}_{OS} \times \mathbf{H}_{OS}^*\right] = \frac{1}{2}\eta_o I_o^2 \frac{1}{(\pi r)^2}\sin^2\theta \mathbf{a}_r,$$

so

$$P_{\max} = \frac{1}{2}\eta_o I_o^2 \frac{1}{(\pi r)^2}, \quad \text{and} \quad P_n = \sin^2\theta.$$

$$\Omega_P = \iint P_n d\Omega = \int_0^{\pi}\sin^3\theta d\theta \int_0^{2\pi} d\phi = 2\pi\int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta d\theta$$

$$\Omega_P = 2\pi\left[\int_0^{\pi}\sin\theta d\theta - \int_0^{\pi}\cos^2\theta\sin\theta d\theta\right] = 2\pi\left[-\cos\theta\Big|_0^{\pi} + \frac{1}{3}\cos^2\theta\Big|_0^{\pi}\right] = \frac{8\pi}{3}sr$$

$$R_{rad} = \frac{2}{I_o^2}r^2 P_{\max}\Omega_P = \frac{2}{I_o^2}r^2\frac{1}{2}\eta_o I_o^2\frac{1}{(\pi r)^2}\frac{8\pi}{3} = \eta_o\frac{8}{3\pi} = \frac{(120\pi)8}{3\pi} = 320\Omega$$

P8.15: Derive the expressions for radiated power and radiation resistance for a small loop antenna.

We'll use: $P_{rad} = \frac{1}{2}I_o^2 R_{rad} = r^2 P_{\max}\Omega_P$

We have $P_{\max} = \dfrac{\omega^2\mu_o^2 I_o^2 S^2\beta^2}{32\eta_o\pi^2 r^2}$

and

$$\Omega_P = \iint \sin^2\theta d\Omega = \int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}d\phi = \frac{8\pi}{3}sr \text{ (see integral solution of P8.14)}$$

Now,

$$P_{rad} = r^2 P_{\max}\Omega_P = r^2\left(\frac{\omega^2\mu_o^2 I_o^2 S^2\beta^2}{32\eta_o\pi^2 r^2}\right)\left(\frac{8\pi}{3}\right)$$

Using the conversions: $\beta = \omega\sqrt{\mu_o\varepsilon_o}$, $\beta = 2\pi/\lambda$, and $\eta_o = \sqrt{\mu_o/\varepsilon_o}$

we arrive at:

$$P_{rad} = \frac{4}{3}\eta_o\pi^3 I_o^2\left(\frac{S}{\lambda^2}\right)^2 = \frac{1}{2}I_o^2 R_{rad}$$

Solving for R_{rad},

$$R_{rad} = 320\pi^4\left(\frac{S}{\lambda^2}\right)^2 \Omega$$

P8.17: How long is a 1.5λ long dipole antenna at 1.0 GHz? Suppose this antenna is constructed using AWG#20 (0.406 mm radius) copper wire. Determine R_{diss}, e, and G_{max}.

$$\lambda = \frac{c}{f} = \frac{3x10^8}{1x10^9} = 0.3m, \quad L = 1.5\lambda = 0.45m$$

$R_{diss} = \frac{1}{\sigma}\frac{\ell}{S}$

The skin depth for this wire at 1 GHz is 2.09x10^{-6}m. Then, the cross-sectional surface over which we consider the current to be conducted is: $S = 2\pi r\delta_{Cu} = 5.33x10^{-9}m^2$

Then:

$$R_{diss} = \frac{1}{5.8x10^7\ 1/\Omega m}\frac{0.45m}{5.33x10^{-9}m^2} = 1.456\Omega$$

Now we need radiation resistance, $R_{rad} = \frac{30}{\pi}F(\theta)_{\max}\Omega_P$, and we use Matlab 0804 to find Ω_P = 8.08 (and D_{max} = 1.55), and F_{max} = 1.366. Therefore, R_{rad} = 105Ω.

The efficiency is

$$e = \frac{R_{rad}}{R_{rad} + R_{diss}} = 0.986$$

Finally, G_{max} = e D_{max} = 1.53.

P8.31: Determine the pattern solid angle, directivity and radiation resistance for a half-wave monopole antenna.

We have the following for a 1λ dipole:

Ω_P = 5.21 sr

D_{max} = 2.41

R_{rad} = 200 Ω

Now, for a λ/2 monopole,

$$\Omega_p\big|_{monopole} = \frac{1}{2}\Omega_p\big|_{dipole},\ D_{\max}\big|_{monopole} = 2D_{\max}\big|_{dipole},\ R_{rad}\big|_{monopole} = \frac{1}{2}R_{rad}\big|_{dipole},$$

So,

Ω_P = 2.6 sr

D_{max} = 4.8

R_{rad} = 100 Ω