Măgurele Summer School for Computing in a Rapidly Evolving Society: Parallel Algorithms and Optimizations

One and two-qubit applications on IBM's quantum computers

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Extreme Light Infrastructure - Nuclear Physics (ELI-NP)

July 10, 2025

The (small) quantum part of this summer school:

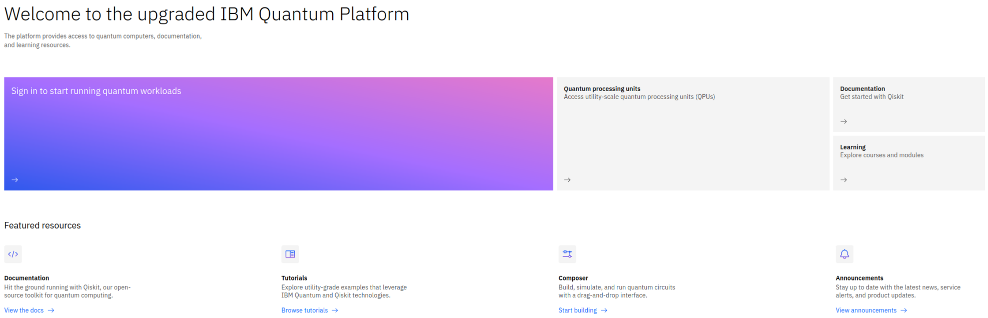
10/07/2025:

- 9h00 – 10h30 The future is quantum by Radu Ionicioiu. ✓
- 11h00 – 12h30 Quantum information: foundations and entanglement by Radu Ionicioiu. ✓
- 14h00 – 15h30 One and two-qubit quantum gates, quantum circuits & measurement by S. A. ✓
- 16h00 – 17h30 One and two-qubit applications on IBM's quantum computers by S. A.

11/07/2025:

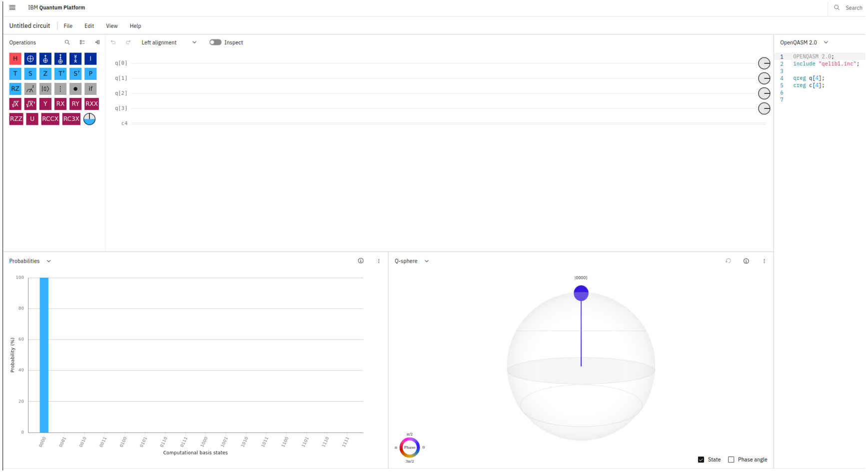
- 9h00–10h30 Quantum information: protocols and applications by Radu Ionicioiu.
- 11h00–12h30 Quantum algorithms on IBM's quantum computers (Deutsch, Bernstein-Vazirani, Grover) by S. A.

The IBM Quantum Platform (as of 2025)



Click on **Composer**.

The IBM Quantum Platform (as of 2025)



If you see this, then all good, we can start.

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Dirac bra-ket notation – the qubit

In the "normal" z-basis,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we have the generic qubit,

$$\boxed{|\psi\rangle = c_0|0\rangle + c_1|1\rangle}$$

with $c_0, c_1 \in \mathbb{C}$ and for **normalization**:

$$|c_0|^2 + |c_1|^2 = 1$$

More qubit examples

Let us choose $c_1 = c_0 = \frac{1}{\sqrt{2}}$. We have the qubit

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

and we will call this **“the plus state”**.

Set $c_0 = \frac{1}{\sqrt{2}}$ and $c_1 = -\frac{1}{\sqrt{2}}$, we get

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

and we will call this **“the minus state”**.

The X gate

The first quantum gate we consider is Pauli's σ_x matrix. We call it simply the X gate:

$$\mathrm{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So we had the X gate:

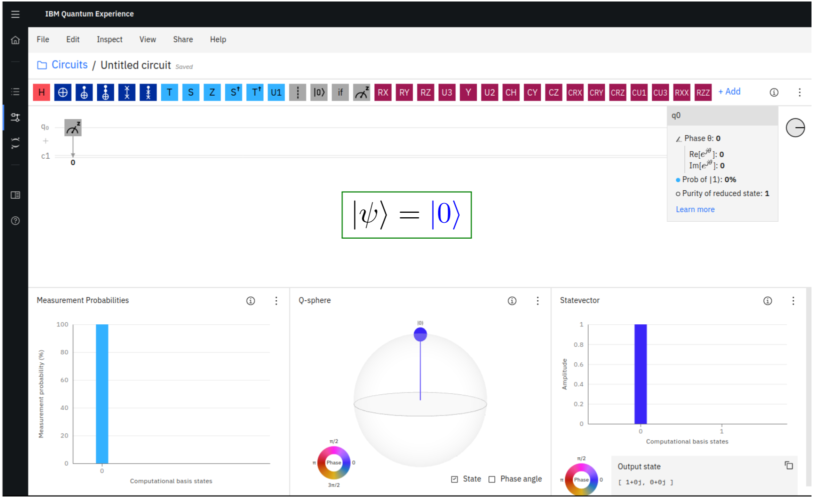
$$\begin{cases} \mathrm{X}|0\rangle = |1\rangle \\ \mathrm{X}|1\rangle = |0\rangle \end{cases}$$

As a quantum circuit this is:

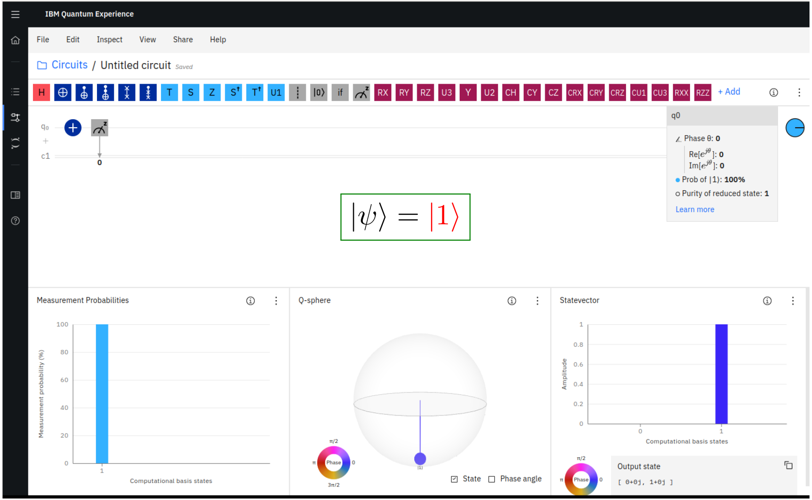
$$|0\rangle \; — \boxed{X} — \; |1\rangle$$

$$|1\rangle \; — \boxed{X} — \; |0\rangle$$

$|0\rangle$ on IBM's Quantum Platform



$|1\rangle$ on IBM's Quantum Platform



$$\boxed{|\psi\rangle = |1\rangle}$$

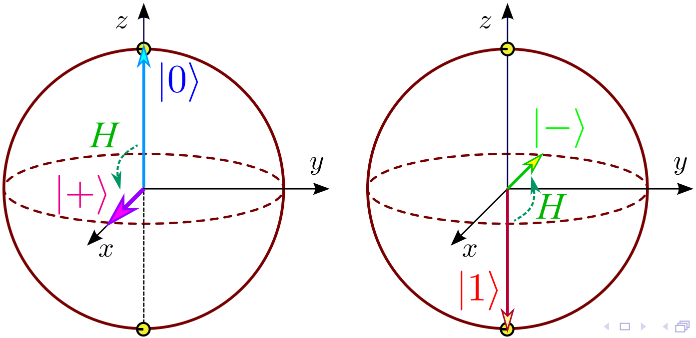
The Hadamard gate

The Hadamard (H) gate:

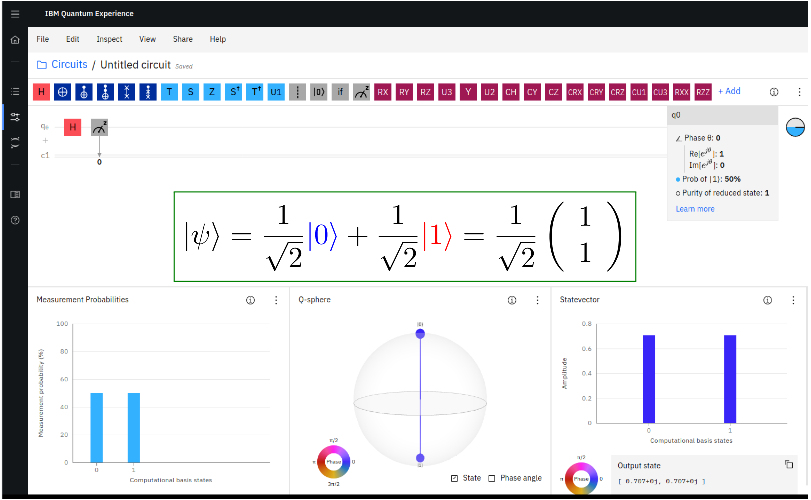
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|0\rangle \; \text{—}\boxed{H}\text{—} \; \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle \; \text{—}\boxed{H}\text{—} \; \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$



$|0\rangle + |1\rangle$ in IBM's Quantum



$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The Y and Z gates

Pauli's σ_y / Y gate si defined by

$$\mathsf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

And finally, the last Pauli gate is

$$\mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

I also recall Pauli's σ_x matrix *i.e.* the X gate,

$$\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The phase gate

Meet the “phase gate” called P in IBM’s Quantum Experience,

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

It is a very handy gate when you want to create a state

$$|\psi'\rangle = c_0|0\rangle + c_1 e^{i\phi}|1\rangle$$

Indeed: start with $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$. Apply the phase gate:

$$P(\phi)|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 e^{i\phi} \end{pmatrix} = c_0|0\rangle + c_1 e^{i\phi}|1\rangle$$

The phase gate

So we have

$$P\left(\phi\right)|\psi\rangle = c_0|0\rangle + c_1 e^{i\phi}|1\rangle$$

Another way to look at the phase gate P,

$$P\left(\phi\right) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

is that it has the effect:

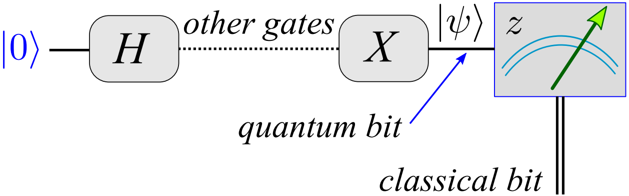
$$\begin{cases} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow e^{i\phi}|1\rangle \end{cases}$$

This is quite easy to remember.

Measurement of a qubit

Recall Max Born (1926)

Measurement is probabilistic.



Obvious fact:

The measurement of a qubit yields a (single) classical bit.

Measurement of a qubit

Recall: Max Born (1926)

(Much to Einstein's dislike.) Measurement is probabilistic.

Specifically, for our qubit:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

The probability to find $|0\rangle$ *i. e.* "0" after a measurement is $p_0 = |c_0|^2$.
The probability to find $|1\rangle$ *i. e.* "1" after a measurement is $p_1 = |c_1|^2$.

Remark:

The total probability is the probability to find $|0\rangle$ plus the probability to find $|1\rangle$. This is $p_0 + p_1 = |c_0|^2 + |c_1|^2 = 1$.

Measurement of a qubit - formalized

Specifically, for our qubit:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

The probability to find $|0\rangle$ after a measurement is

$$p_0 = |\langle 0|\psi\rangle|^2 = |c_0|^2$$

The probability to find $|1\rangle$ after a measurement is

$$p_1 = |\langle 1|\psi\rangle|^2 = |c_1|^2$$

Measurement of a qubit – exercises

Consider the state

$$|\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

What is the probability to find $|0\rangle$ after a measurement?
What is the probability to find $|1\rangle$ after a measurement?

Consider now the state

$$|\psi\rangle = |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Again, what is the probability to find $|0\rangle$ after a measurement?
What is the probability to find $|1\rangle$ after a measurement?

Are these results different? How do you explain?

Measurement of a qubit – solutions

Consider the state

$$|\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

The probability to find $|0\rangle$ after a measurement is $p_0 = |\langle 0|\psi\rangle|^2 = |c_0|^2 = \frac{1}{2}$.
The probability to find $|1\rangle$ after a measurement is $p_1 = |\langle 1|\psi\rangle|^2 = |c_1|^2 = \frac{1}{2}$.

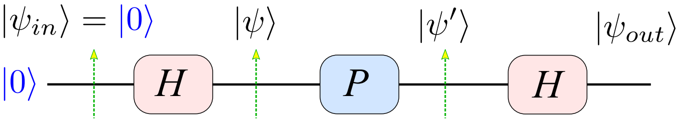
Consider now the state

$$|\psi\rangle = |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

The probability to find $|0\rangle$ after a measurement is $p_0 = |\langle 0|\psi\rangle|^2 = |c_0|^2 = \frac{1}{2}$.
The probability to find $|1\rangle$ after a measurement is $p_1 = |\langle 1|\psi\rangle|^2 = |c_1|^2 = \frac{1}{2}$.

Phase measurement - calculations

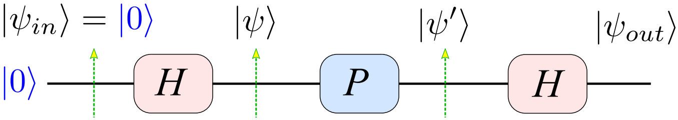
So we actually have this



and need to compute $|\psi_{out}\rangle$.
The first Hadamard gate does the transformation

$$|\psi\rangle = H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

Phase measurement - calculations



Next, the phase gate introduces the phase shift:

$$\begin{cases} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow e^{i\phi}|1\rangle \end{cases}$$

so we have the state

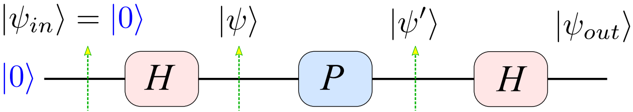
$$|\psi'\rangle = P(\phi)\,|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\phi}|1\rangle\Big)$$

Now apply a new Hadamard gate on $|\psi'\rangle$ to get the final state

$$|\psi_{out}\rangle = H\,|\psi'\rangle = \frac{1}{\sqrt{2}}\Big(H|0\rangle + e^{i\phi}H|1\rangle\Big)$$

We're not done yet, we need $H|0\rangle$ and $H|1\rangle$.

Phase measurement - calculations



But we recall that

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

therefore

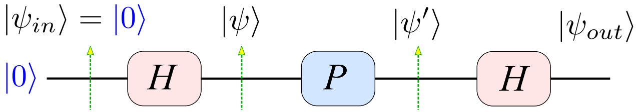
$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(H|0\rangle + e^{i\phi}H|1\rangle\right) = \frac{1}{2}\left(\left(|0\rangle + |1\rangle\right) + e^{i\phi}\left(|0\rangle - |1\rangle\right)\right)$$

so in the end

$$|\psi_{out}\rangle = \frac{1+e^{i\phi}}{2}|0\rangle + \frac{1-e^{i\phi}}{2}|1\rangle$$

It can still be simplified. How about to force a global phase $e^{i\frac{\phi}{2}}$? (It will make sense soon.)

Phase measurement - calculations



So we *force* a global phase $e^{i\frac{\phi}{2}}$, thus

$$|\psi_{out}\rangle = \frac{1+e^{i\phi}}{2}|0\rangle + \frac{1-e^{i\phi}}{2}|1\rangle$$

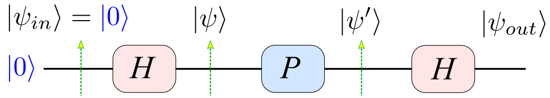
becomes

$$|\psi_{out}\rangle = e^{i\frac{\phi}{2}}\left(\frac{e^{-i\frac{\phi}{2}}+e^{i\frac{\phi}{2}}}{2}|0\rangle + \frac{e^{-i\frac{\phi}{2}}-e^{i\frac{\phi}{2}}}{2}|1\rangle\right) = e^{i\frac{\phi}{2}}\left(\cos\left(\frac{\phi}{2}\right)|0\rangle - i\sin\left(\frac{\phi}{2}\right)|1\rangle\right)$$

and ignoring the global phase we have

$$|\psi_{out}\rangle = \cos\left(\frac{\phi}{2}\right)|0\rangle - i\sin\left(\frac{\phi}{2}\right)|1\rangle$$

Phase measurement - calculations



$$|\psi_{out}\rangle = \cos\left(\frac{\phi}{2}\right)|0\rangle - i\sin\left(\frac{\phi}{2}\right)|1\rangle$$

So we actually estimated the phase because the probability to find $|0\rangle$ after a measurement is

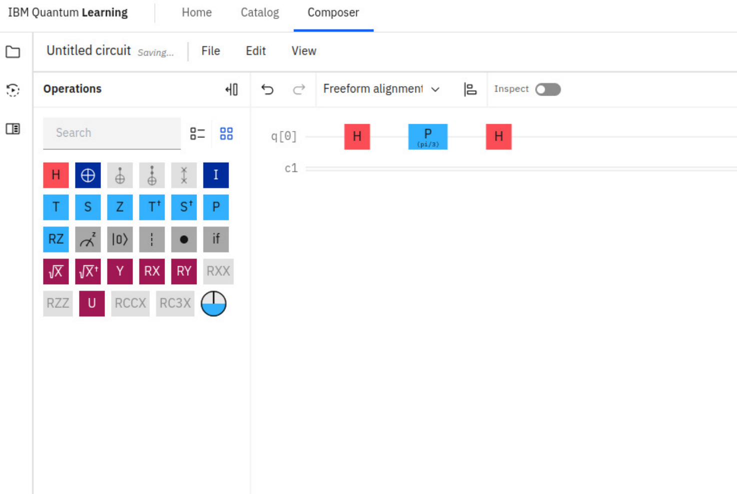
$$\boxed{p_0 = \cos^2\left(\frac{\phi}{2}\right)}$$

and similarly, the probability to find $|1\rangle$ after a measurement is

$$\boxed{p_1 = \sin^2\left(\frac{\phi}{2}\right)}$$

Phase measurement in IBM QE

Open IBM Quantum and implement the following quantum circuit:



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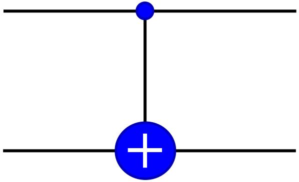
Two qubits. Two qubit gates

The CNOT gate

This is probably the most important gate we will use. It is called the controlled-NOT or simply CNOT gate. It has a **control qubit** (upper one) and a **target qubit** (the lower one):

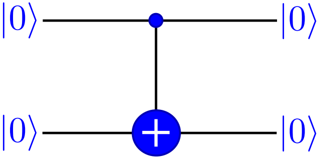


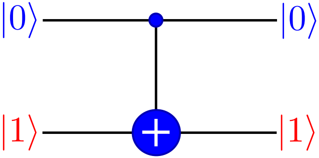
Remark:

The control qubit as it name says, *controls*. It is not affected by the CNOT gate.

The CNOT gate

Set the control qubit to $|0\rangle$. Not much happens:

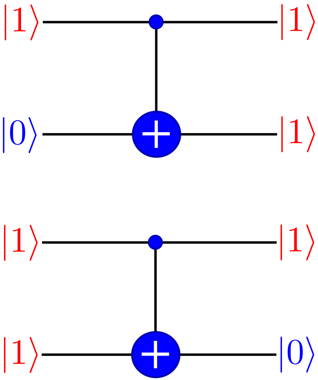




Boring.

The CNOT gate

Set the control qubit to $|1\rangle$. Now interesting things happen:



Question: what gate has the same behaviour on the target bit?

The CNOT gate - matrix representation

Being a two-qubit gate it must have a 4×4 matrix representation. Here it comes:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathbb{I}_2 & \mathbf{0}_2 \\ \hline \mathbf{0}_2 & X \end{array}\right)$$

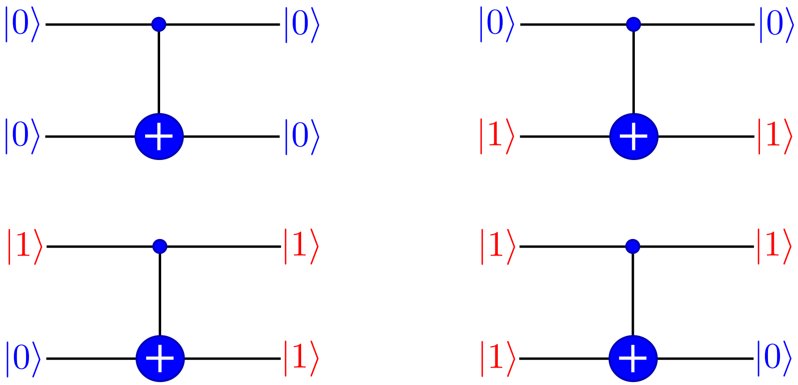
I don't suggest to use it, except in the beginning. I would go for the truth table instead:

$$\begin{cases} |00\rangle \to |00\rangle \\ |01\rangle \to |01\rangle \\ |10\rangle \to |11\rangle \\ |11\rangle \to |10\rangle \end{cases}$$

Remark:

You can use the matrix representation of the qubits to check it.

The CNOT gate in a single slide



The control-Z gate

We saw the CNOT gate:

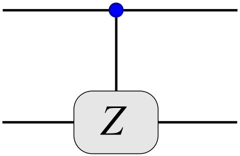
$$\text{CNOT} = \left(\begin{array}{c|c} \mathbb{I}_2 & \mathbf{0}_2 \\ \hline \mathbf{0}_2 & X \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

We can maybe imagine a gate

$$\text{CZ} = \left(\begin{array}{c|c} \mathbb{I}_2 & \mathbf{0}_2 \\ \hline \mathbf{0}_2 & Z \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

The control-Z gate

Meet the CZ gate:



Truth table:

$$\begin{cases} |00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |01\rangle \\ |10\rangle \rightarrow |10\rangle \\ |11\rangle \rightarrow -|11\rangle \end{cases}$$

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The maximally entangled state - examples

The EPR (also called Bell) state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|{\color{red}\blacksquare}_{Alice}{\color{red}\blacksquare}_{Bob}\rangle + |{\color{blue}\blacksquare}_{Alice}{\color{blue}\blacksquare}_{Bob}\rangle\right)$$

can be made of **two photons** having horizontal/vertical (H/V) polarization

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|{\color{red}H}_{Alice}{\color{red}H}_{Bob}\rangle + |{\color{blue}V}_{Alice}{\color{blue}V}_{Bob}\rangle\right) = \frac{1}{\sqrt{2}}\left(|{\color{red}HH}\rangle + |{\color{blue}VV}\rangle\right)$$

or, in quantum computing, two maximally entangles qubits

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|{\color{red}0}_{Alice}{\color{red}0}_{Bob}\rangle + |{\color{blue}1}_{Alice}{\color{blue}1}_{Bob}\rangle\right) = \frac{1}{\sqrt{2}}\left(|{\color{red}00}\rangle + |{\color{blue}11}\rangle\right)$$

The Bell states

There are actually 4 maximally entangled states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

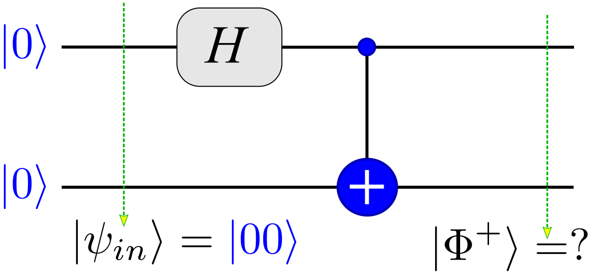
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

They are called **the Bell states**. They are all orthogonal among themselves.

Back to the previous question

You have the quantum circuit below.

What is the output state?

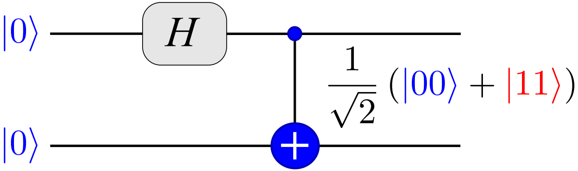


Any guesses?

The Bell states - $|\Phi^+\rangle$

I claim that this scheme

generates the $|\Phi^+\rangle$ state. Do you agree?



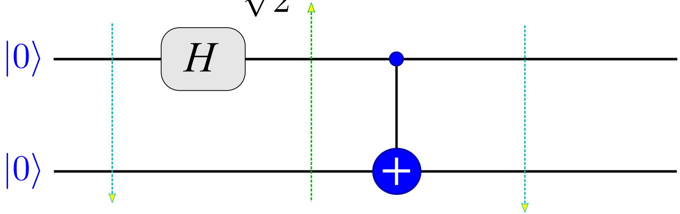
We know the Hadamard gate by now. Recall the CNOT gate:

$$\begin{cases} |00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |01\rangle \\ |10\rangle \rightarrow |11\rangle \\ |11\rangle \rightarrow |10\rangle \end{cases}$$

The Bell state generation

Let's start describing the circuit, from left to right:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle$$



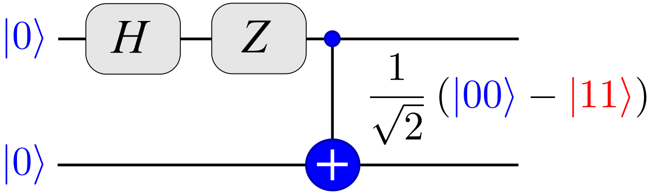
$$|\psi_{in}\rangle = |00\rangle \qquad |\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Everyone agrees? Did you get the same result?

Let's implement it in IBM Quantum Learning!

The Bell states

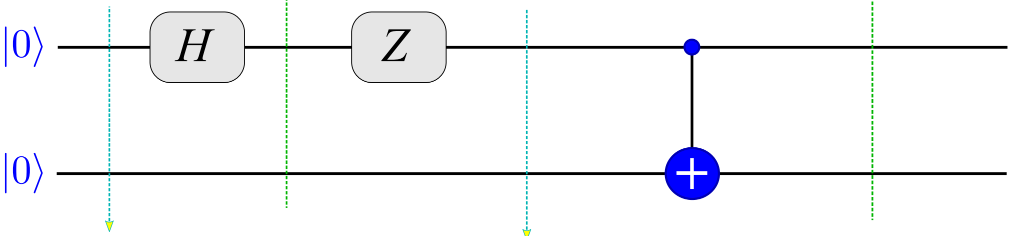
I claim that this scheme generates the $|\Phi^{-}\rangle$ state.



The Bell state generation - $|\Phi^-\rangle$

Let's see:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$



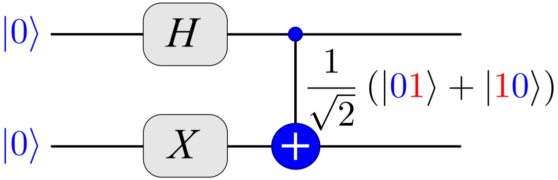
$$|\psi_{in}\rangle = |00\rangle \qquad |\psi''\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|0\rangle$$

Let's implement it in IBM QE!

The Bell states - $|\Psi^+\rangle$

I claim that this scheme

generates the $|\Psi^+\rangle$ state. Do you agree?

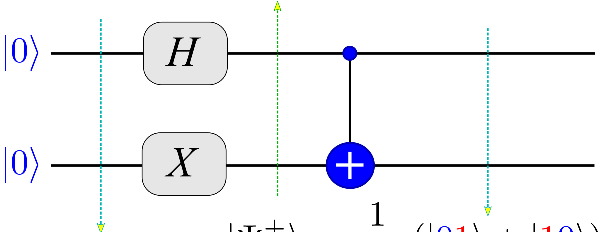


Can you prove this result? Give it a try!

The Bell state generation - $|\Psi^+\rangle$

Let's see:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|1\rangle$$



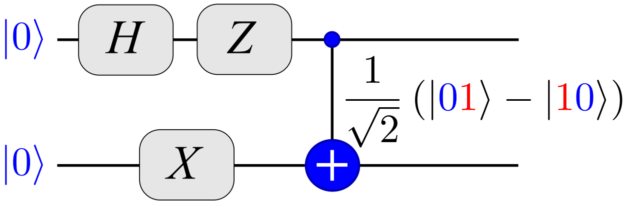
$$|\psi_{in}\rangle = |00\rangle \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

Implement it in IBM QE! **Go!**

The Bell states - $|\Psi^-\rangle$

I claim that this scheme

generates the $|\Psi^-\rangle$ state. Do you agree?



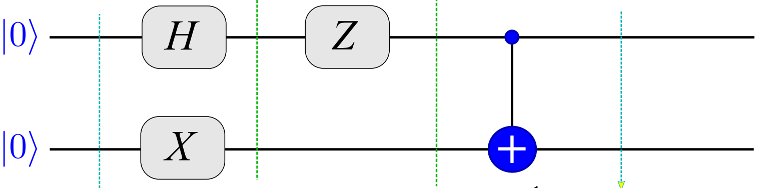
Can you prove this result? Give it a try!

The Bell state generation - $|\Psi^-\rangle$

Let's see:

$$|\psi''\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|1\rangle$$

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|1\rangle$$



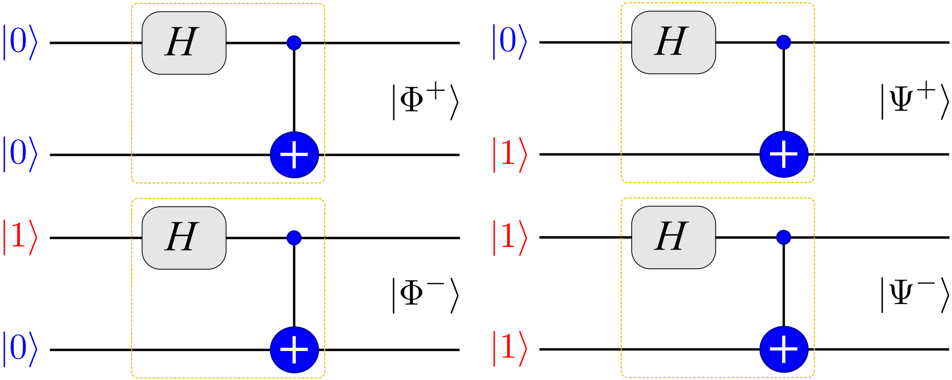
$$|\psi_{in}\rangle = |00\rangle$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Implement it in IBM QE! **Go!**

The Bell state generation - homework

Show that:



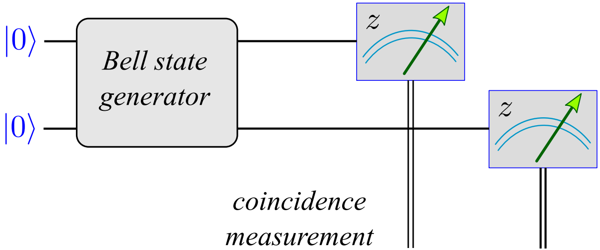
The Bell state measurement

Very subtle point

All Bell states are entangled i. e. not separable. **Single measurements on one qubit will yield totally random values**.

Measure **both** qubits!

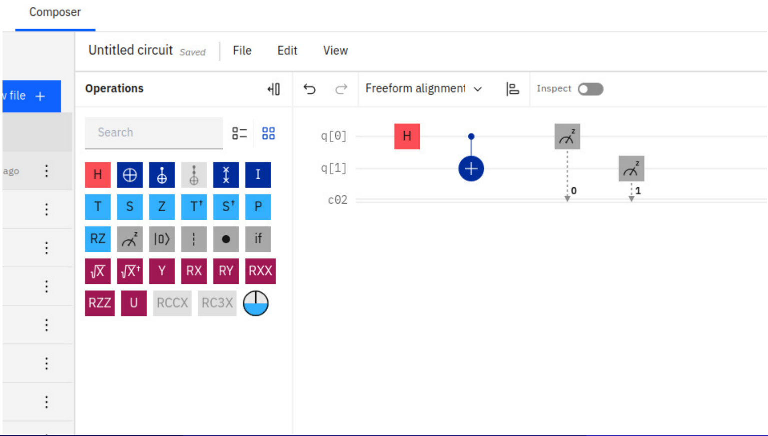
We need **coincidence measurements**.



The Bell state measurement in IBM's Quantum simulator

Implementing this measurement scheme in IBM QE.

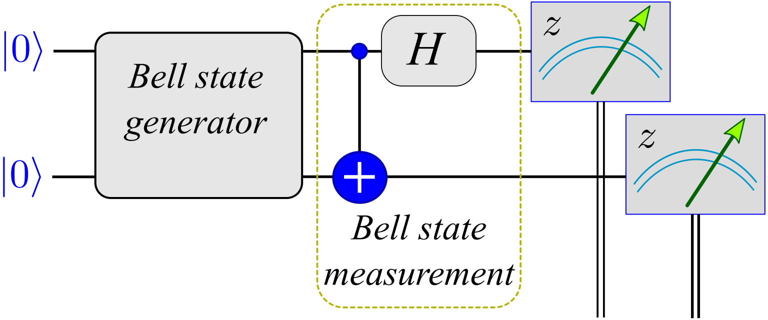
Let's test all four Bell states.



The Bell state measurement

Claim:

The circuit below distinguishes among the 4 Bell states.

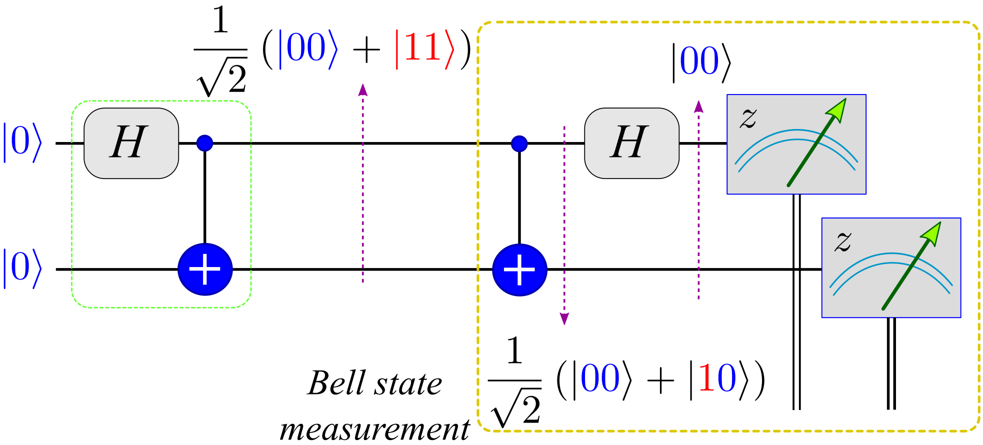


Can you prove it?

Start with the $|\Phi^+\rangle$ state!

The Bell state measurement

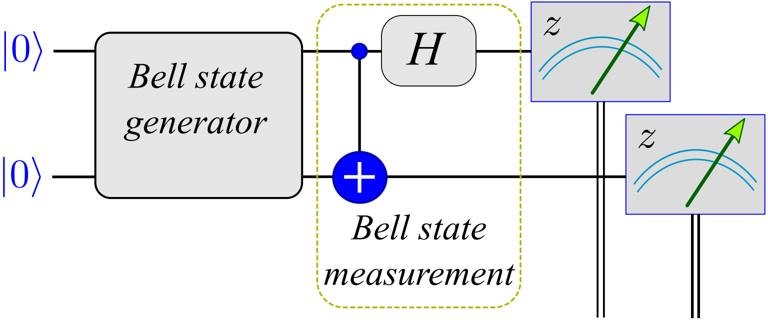
The $|\Phi^+\rangle$ state. Here we go:



Bell state measurement in IBM QE

Use IBM QE:

Implement all for Bell states and use the circuit below to measure them in IBM QE.

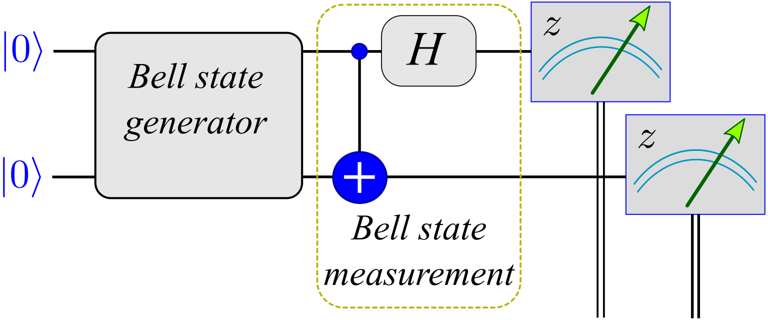


Can you distinguish now among the 4 Bell states?

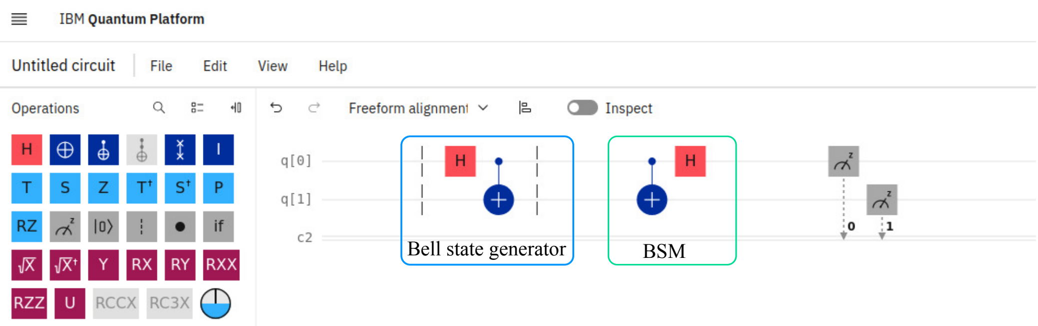
Are your results probabilistic or deterministic? Comment on your findings!

IBM Quantum implementation

Show that the circuit below distinguishes among the 4 Bell states.



IBM Quantum implementation – solution



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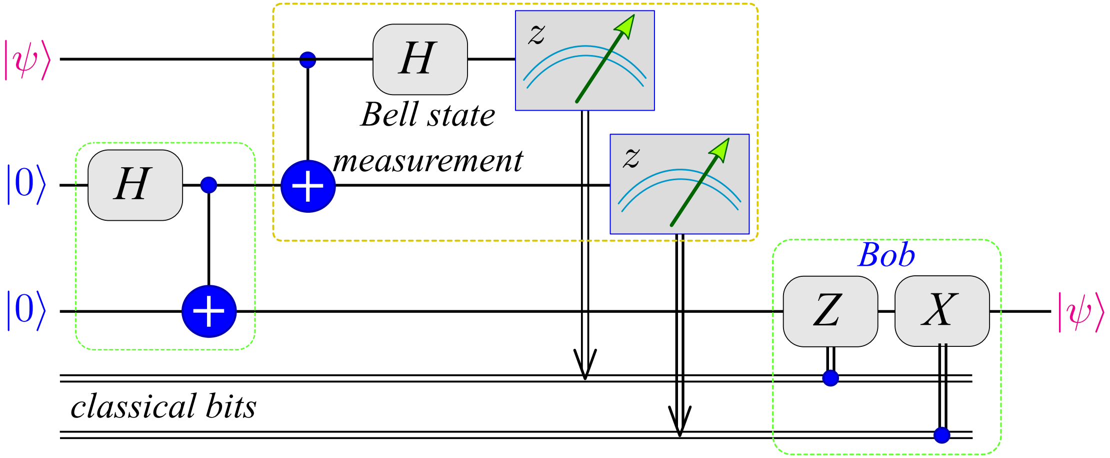
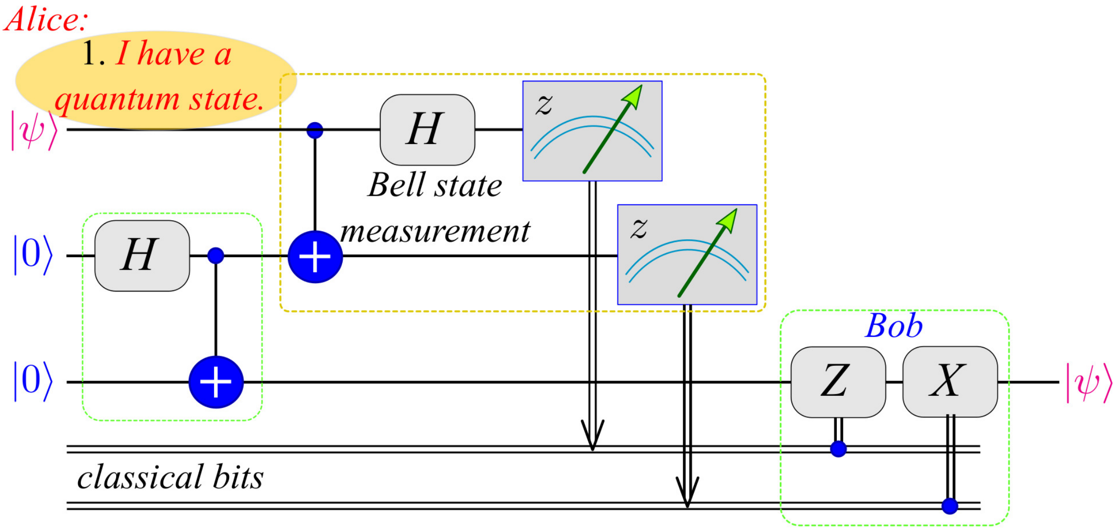
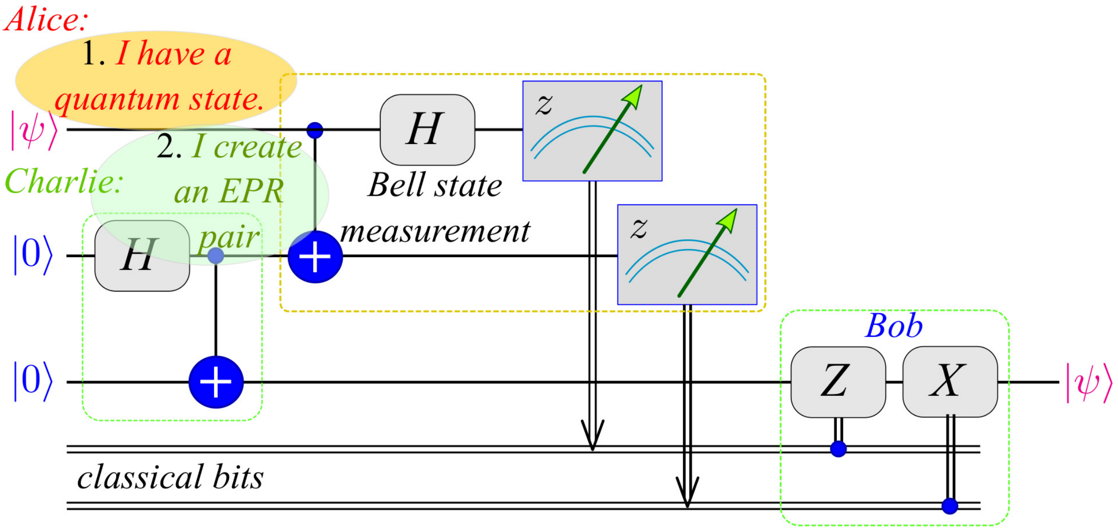
The Quantum teleportation



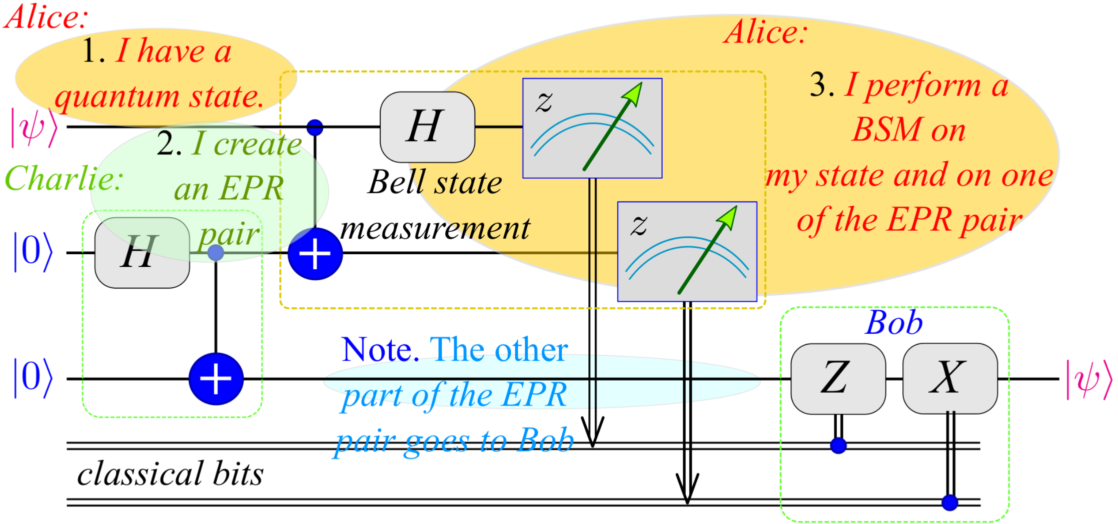
The Quantum teleportation – 1. Alice has a $|\psi\rangle$.



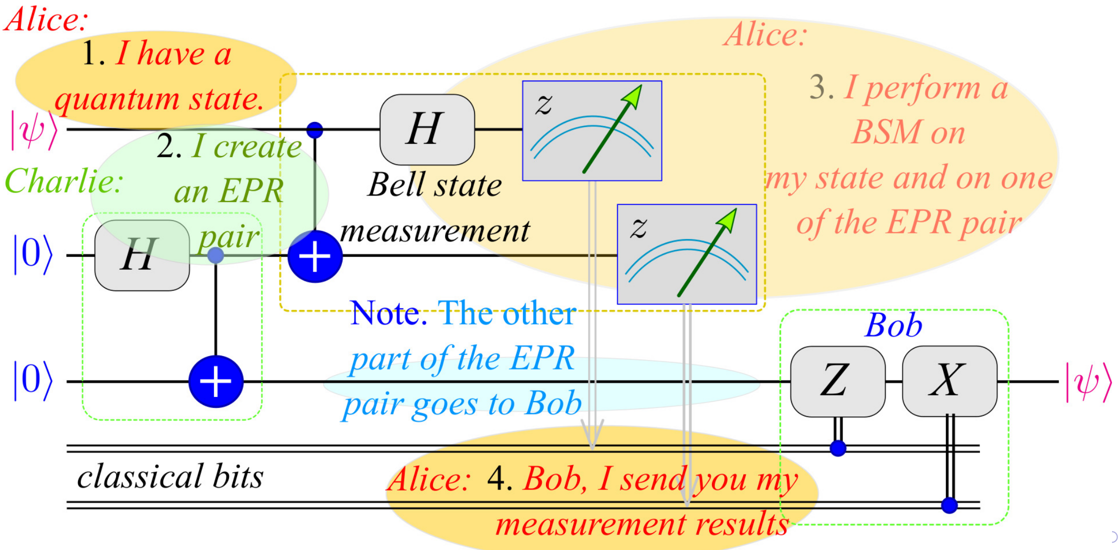
The Quantum teleportation – 2. Charlie creates $|\Phi^+\rangle$.



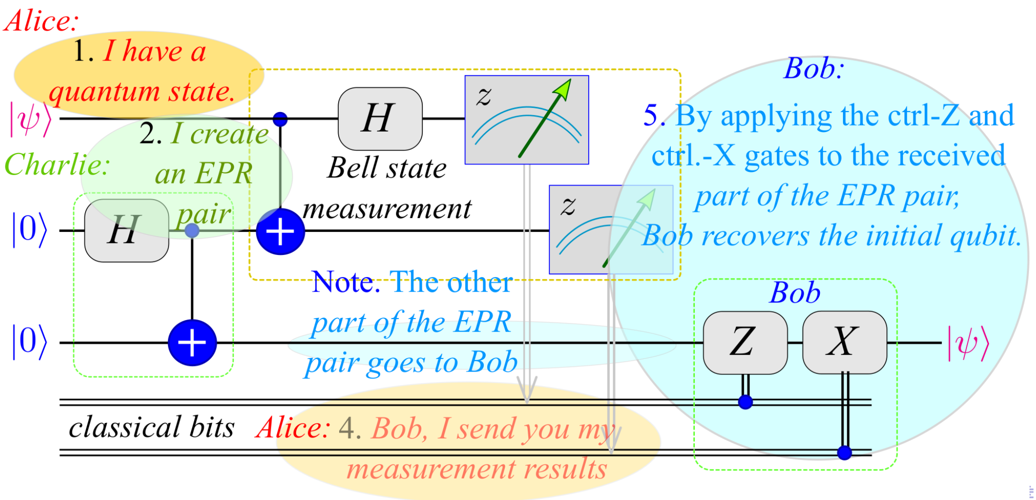
The Quantum teleportation – 3. Alice performs a BSM.



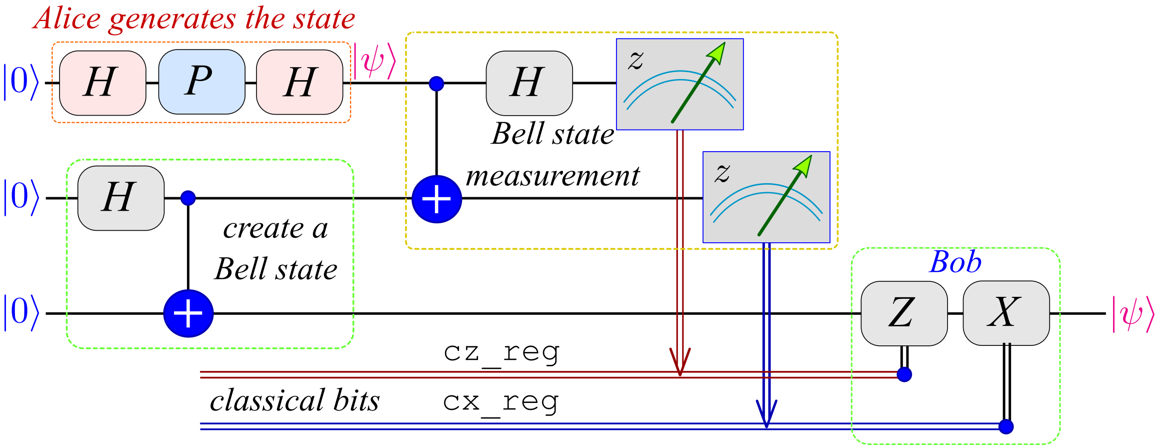
The Quantum teleportation – 4. Alice sends her measurements to Bob.



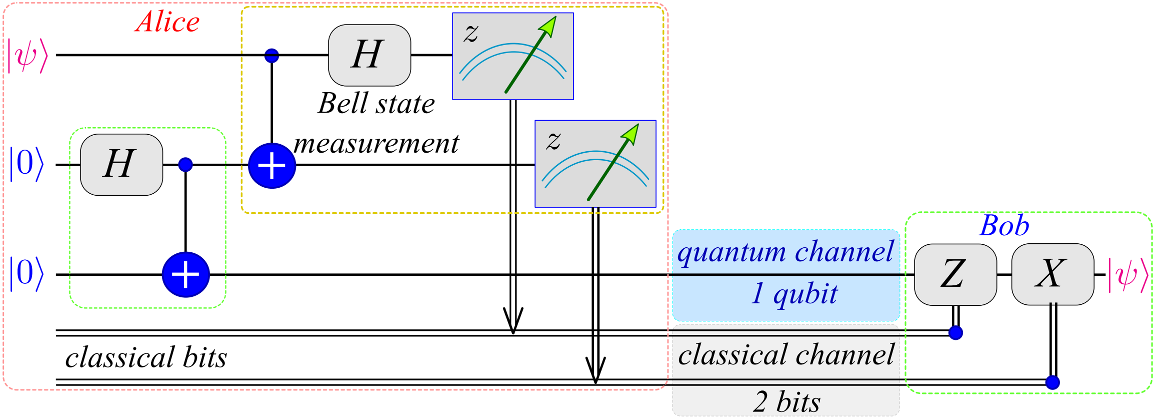
The Quantum teleportation – 5. Bob uses the quantum plus classical channels.



The Quantum teleportation – more practical implementation for us

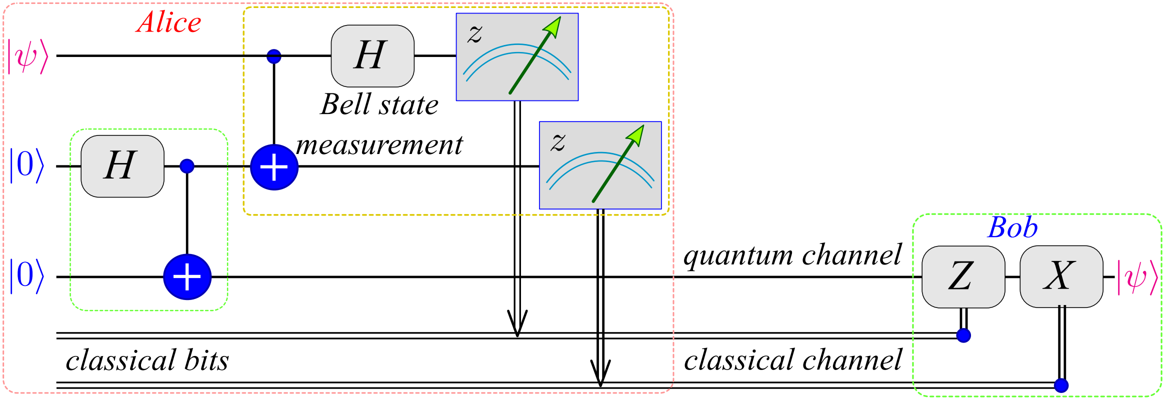


Quantum teleportation - the channels

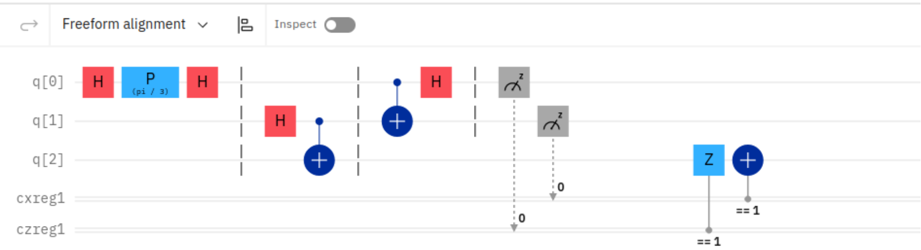


The "quantum channel" can be *e. g.* a fiber.
The "classical channel" can be your usual phone line.

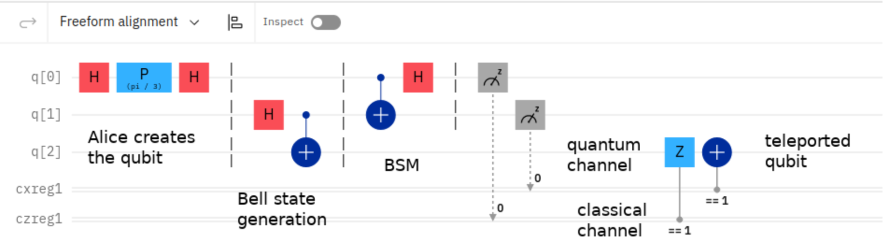
Quantum teleportation - the channels



Quantum teleportation in IBM Quantum



Quantum teleportation in IBM QE

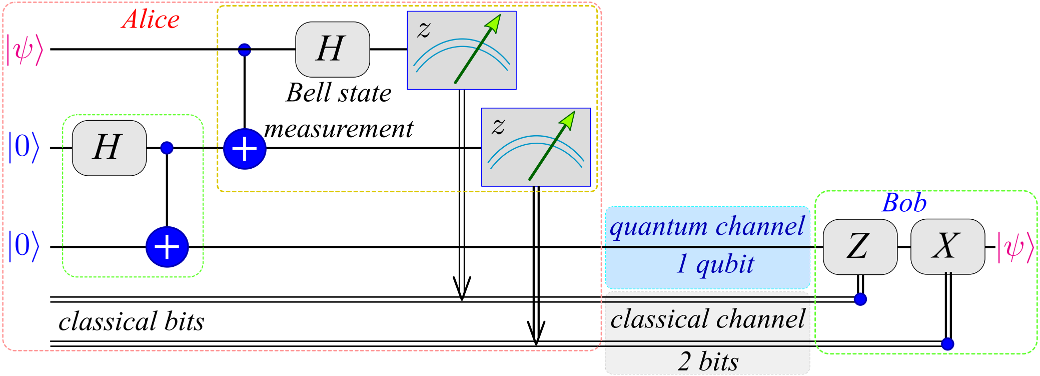


Now implement yourself the scheme

Good luck!

Hint: for control-Z and control-X (classical control) put first the quantum gate and the drag then the `if` from the left panel on top. Then configure the control `Value` to 1.

Quantum teleportation - the big picture



OK, but why does it work?

We implemented it on the IBM Quantum platform. If you did it correctly, it should work. Well done!

That's it for today.

Thank you for your attention!

Questions are welcome.