Măgurele Summer School for Computing in a Rapidly Evolving Society: Parallel Algorithms and Optimizations

One and two-qubit quantum gates, quantum circuits & measurement

by Stefan Ataman

Extreme Light Infrastructure - Nuclear Physics (ELI-NP)

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The (small) quantum part of this summer school:

10/07/2025:

- 9h00 – 10h30 The future is quantum by Radu Ionicioiu. ✓
- 11h00 – 12h30 Quantum information: foundations and entanglement by Radu Ionicioiu. ✓
- 14h00 – 15h30 **One and two-qubit quantum gates**, quantum circuits & measurement by S. A.
- 16h00 – 17h30 One and two-qubit applications on IBM's quantum computers by S. A.

11/07/2025:

- 9h00–10h30 Quantum information: protocols and applications by Radu Ionicioiu.
- 11h00–12h30 Quantum algorithms on IBM's quantum computers (Deutsch, Bernstein-Vazirani, Grover) by S. A.

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Recall the Dirac bra-ket notation

A *ket* can be imagined as a column vector, something like

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$$

A *bra* is then something like

$$\langle\phi| = \begin{pmatrix} \phi_0^* & \phi_1^* & \phi_2^* & \dots \end{pmatrix}$$

The bra-ket makes indeed sense from a mathematical point of view and we have

$$\langle\phi|\psi\rangle = \begin{pmatrix} \phi_0^* & \phi_1^* & \phi_2^* & \dots \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix} = \phi_0^*\psi_0 + \phi_1^*\psi_1 + \phi_2^*\psi_2 \dots$$

Dirac bra-ket notation – the qubit

Let us see the most basic ket/bra example, namely **the qubit**.

The ket is a vector.

You might have already seen:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The bra is a vector, too.

We have:

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \text{and} \quad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

You probably find all this pretty easy, since these objects are simply 1×2 or 2×1 **matrices**.

Dirac bra-ket notation – the qubit

From the basic ket examples,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we move to the generic qubit,

$$\boxed{|\psi\rangle = c_0|0\rangle + c_1|1\rangle}$$

with $c_0, c_1 \in \mathbb{C}$ and for **normalization**:

$$|c_0|^2 + |c_1|^2 = 1$$

Normalization: why is that? Let's recall.

Bra-ket of a qubit

Take the qubit,

$$\boxed{|\psi\rangle = c_0|0\rangle + c_1|1\rangle}$$

and make it a bra:

$$\boxed{\langle\psi| = c_0^*\langle 0| + c_1^*\langle 1|}$$

Take the bra-ket of the qubit i. e. the inner product:

$$\langle\psi|\psi\rangle = \langle\psi|\,|\psi\rangle = \left(c_0^*\langle 0| + c_1^*\langle 1|\right)\left(c_0|0\rangle + c_1|1\rangle\right)$$

We get:

$$\langle\psi|\psi\rangle = c_0c_0^*\langle 0|0\rangle + c_1c_1^*\langle 1|1\rangle + c_0c_1^*\langle 1|0\rangle + c_0^*c_1\langle 0|1\rangle$$

The postulates of QM impose the following constraint:

$$\langle\psi|\psi\rangle = 1.$$

Dirac bra-ket notation - the qubit

We defined the kets:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We defined the bras:

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \text{and} \quad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

and QM says that must have

$$\langle\psi|\psi\rangle = c_0 c_0^* \langle 0|0\rangle + c_1 c_1^* \langle 1|1\rangle + c_0 c_1^* \langle 1|0\rangle + c_0^* c_1 \langle 0|1\rangle = 1.$$

Okay. Let's compute the first term. We have

$$\langle 0|0\rangle = \underbrace{\langle 0|}_{\substack{\text{smash} \\ \text{the bra}}} \underbrace{|0\rangle}_{\substack{\text{into} \\ \text{the ket}}} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

Dirac bra-ket notation - the qubit

For the second term we have

$$\langle 1|1\rangle = \underbrace{\langle 1|}_{\substack{\text{smash}\\\text{the bra}}} \underbrace{|1\rangle}_{\substack{\text{into}\\\text{the ket}}} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

The last two terms are null. For example:

$$\langle 1|0\rangle = \underbrace{\langle 1|}_{\substack{\text{smash}\\\text{the bra}}} \underbrace{|0\rangle}_{\substack{\text{into}\\\text{the ket}}} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

So

$$c_0 c_0^* \underbrace{\langle 0|0\rangle}_{=1} + c_1 c_1^* \underbrace{\langle 1|1\rangle}_{=1} + c_0 c_1^* \underbrace{\langle 1|0\rangle}_{=0} + c_0^* c_1 \underbrace{\langle 0|1\rangle}_{=0} = c_0 c_0^* + c_1 c_1^* = \boxed{|c_0|^2 + |c_1|^2 = 1}$$

Qubit examples

We defined the states $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We have the general qubit,

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

together with the constraint

$$|c_0|^2 + |c_1|^2 = 1.$$

More qubit examples

Let us choose $c_0 = c_1 = \frac{1}{\sqrt{2}}$. We have the qubit

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and we will call this **"the plus state"**.

Set $c_0 = \frac{1}{\sqrt{2}}$ and $c_1 = -\frac{1}{\sqrt{2}}$, we get

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and we will call this **"the minus state"**.

Acting on qubits

Question: How to you *transform* a qubit (into another one)?

For example you have a qubit $|\psi\rangle$ and you want to *transform it* into another qubit, $|\psi'\rangle$.

How do you do it?

$$|\psi'\rangle = \underbrace{\blacksquare}_{=?} |\psi\rangle$$

For example: $|\psi\rangle = |0\rangle$ and I want to arrive at $|\psi'\rangle = |1\rangle$ that is

$$|1\rangle = \underbrace{\blacksquare}_{=?} |0\rangle.$$

Acting on qubits

Question: How do you act on one qubit?

Answer: **with operators** i. e. with 2×2 **matrices**!

Why is that?
Recall that a qubit is a 2×1 *matrix*. Matrix multiplications requires that:

$$\underbrace{\begin{pmatrix} \circ \\ \circ \end{pmatrix}}_{|\psi'\rangle} = \underbrace{\begin{pmatrix} \circ & \circ \\ \circ & \circ \end{pmatrix}}_{\text{operator or quantum gate}} \underbrace{\begin{pmatrix} \circ \\ \circ \end{pmatrix}}_{|\psi\rangle}$$

Here $\circ$ is a *generic* component of the above matrices.

Acting on qubits – the X gate

The first quantum gate we consider is Pauli's σ_x matrix. We call it simply the X gate:

$$\boxed{\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.}$$

Let's apply X on the qubit state $|1\rangle$! We have:

$$\mathsf{X}|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle.$$

Next, let's apply it on the state $|0\rangle$:

$$\mathsf{X}|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

The X gate as a quantum circuit

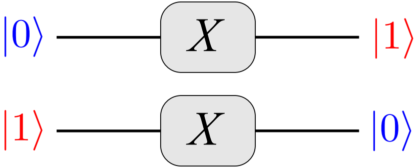
Similar to digital circuits, we can draw quantum circuits.

A single line implies a quantum bit. A double line means a classical bit (coming soon).

So we have the X gate “truth table”:

$$\begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{cases}$$

As a quantum circuit this is:

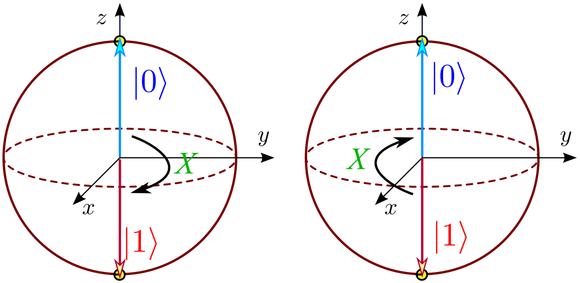


The X gate on the Bloch sphere

So we have the X gate "truth table":

$$\begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{cases}$$

On the Bloch sphere we get:



The X gate acting on a qubit

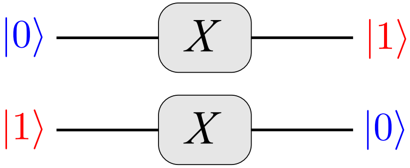
Let us assume that we have the usual qubit:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \quad |c_0|^2 + |c_1|^2 = 1$$

What do you get?

and that we would like to act with the X gate on it!
Recall:

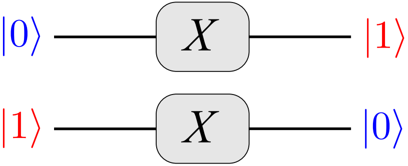
$$\begin{cases} \mathrm{X}|0\rangle = |1\rangle \\ \mathrm{X}|1\rangle = |0\rangle \end{cases}$$



The X gate acting on a qubit

We have:

$$|\psi'\rangle = \mathsf{X}\,|\psi\rangle = c_0\mathsf{X}|0\rangle + c_1\mathsf{X}|1\rangle$$



We get the output qubit:

$$|\psi'\rangle = c_0|1\rangle + c_1|0\rangle.$$

Remark: we acted in parallel on both $|0\rangle$ and $|1\rangle$.

The Hadamard gate

You probably know by now the Hadamard (H) gate:

$$\mathsf{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Applying H on the qubit $|0\rangle$ we get:

$$\mathsf{H}|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

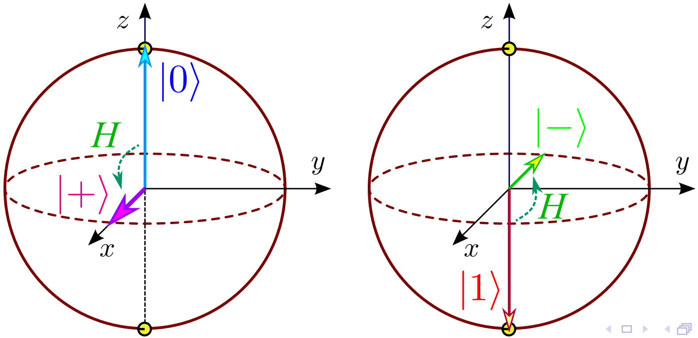
Applying H on the qubit $|1\rangle$ we have:

$$\mathsf{H}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$

The Hadamard gate

$$|0\rangle \;—\boxed{H}—\; \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle \;—\boxed{H}—\; \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$



Basic math stuff

Recall Euler's relation(s):

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

Some important values:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 \qquad e^{i2\theta} = 1$$

Ans some more values ($\theta = \pi/2$):

$$e^{i\pi/2} = i \qquad e^{-i\pi/2} = -i$$

And finally ($\theta = \pi/4$):

$$e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1+i}{\sqrt{2}} \qquad e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$$

The qubit on the Bloch sphere

We defined the qubit i. e. the *quantum bit*,

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

with $c_0, c_1 \in \mathbb{C}$ and for normalization i. e. $\langle\psi|\psi\rangle = 1$ we must have $|c_0|^2 + |c_1|^2 = 1$. But we can give geometrical meaning to these coefficients if we change the notation. Let:

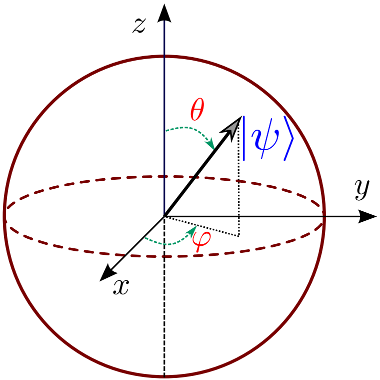
$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

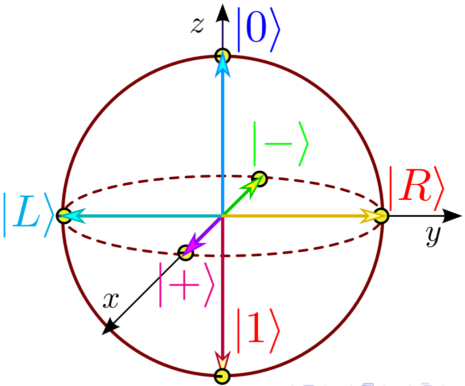
Remember the sphere and the coordinates r, θ, φ?

The qubit on the Bloch sphere

We have the qubit:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

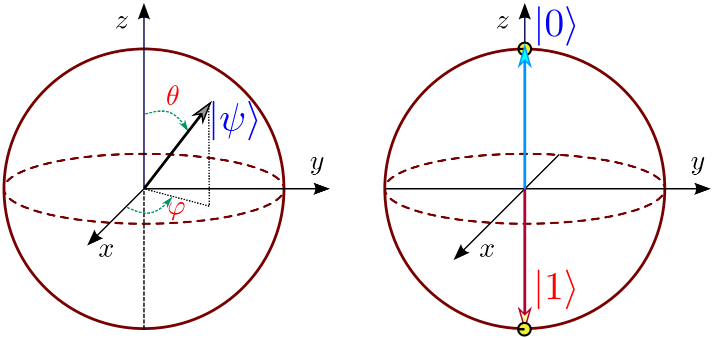




The qubit on the Bloch sphere - examples

For $\theta = 0$ and for any φ we have

$$|\psi\rangle = \cos 0|0\rangle + e^{i\varphi}\sin 0|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

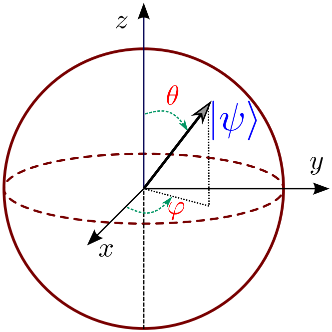


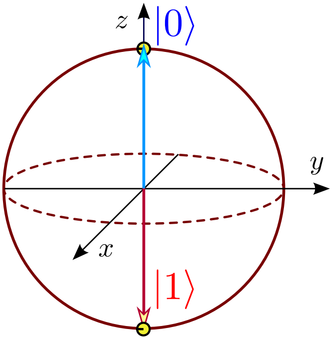
The qubit on the Bloch sphere - examples

For $\theta = \pi$ and for any φ we have

$$|\psi\rangle = \cos\frac{\pi}{2}|0\rangle + e^{i\varphi}\sin\frac{\pi}{2}|1\rangle = \begin{pmatrix} 0 \\ e^{i\varphi} \end{pmatrix} = e^{i\varphi}|1\rangle$$

Note: global phases are irrelevant in quantum mechanics.

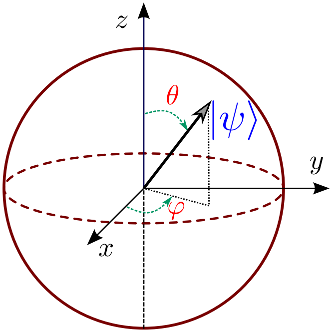


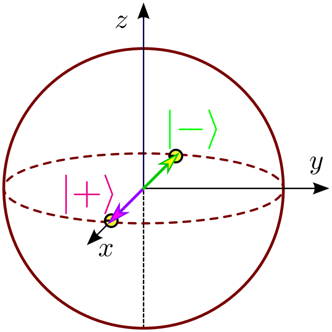


The qubit on the Bloch sphere - examples

For $\theta = \pi/2$ and for $\varphi = 0$ we have

$$|\psi\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i0}\sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

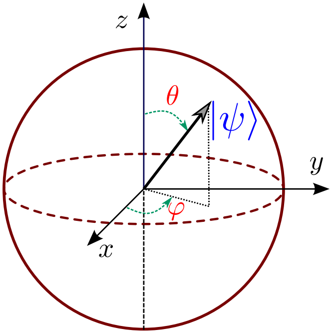


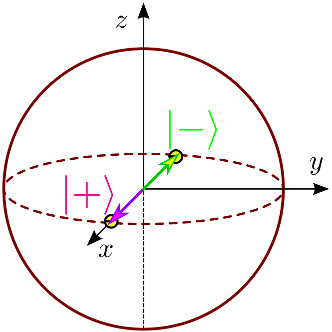


The qubit on the Bloch sphere - examples

For $\theta = \pi/2$ and for $\varphi = \pi$ we have

$$|\psi\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i\pi}\sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

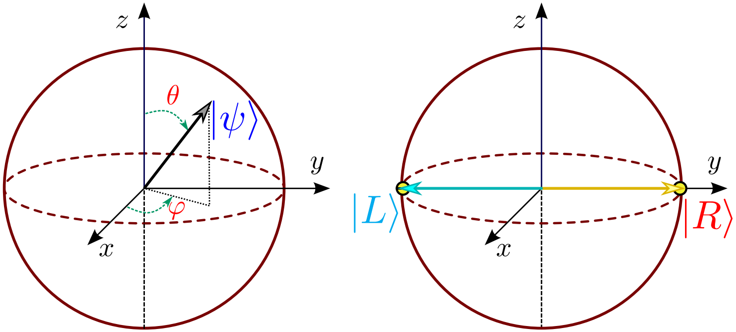




The qubit on the Bloch sphere - examples

For $\theta = \pi/2$ and for $\varphi = \pi/2$ we have

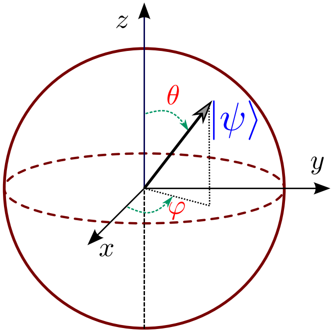
$$|\psi\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i\frac{\pi}{2}}\sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = |\mathsf{R}\rangle = |\mathsf{i}\rangle$$

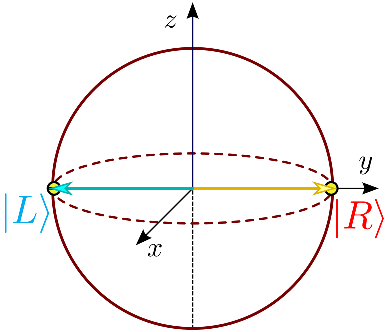


The qubit on the Bloch sphere - examples

For $\theta = \pi/2$ and for $\varphi = 3\pi/2$ we have

$$|\psi\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i\frac{3\pi}{2}}\sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = |\mathsf{L}\rangle = |\mathsf{-i}\rangle$$





The Y and Z gates

Pauli's σ_y / Y gate si defined by

$$\mathsf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

And finally, the last Pauli gate is

$$\mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

I also recall Pauli's σ_x matrix i .e the X gate,

$$\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Maybe as an exercise you computed: X^2, Y^2 and Z^2.

Do you recall the result?

The X, Y and Z gates - squared

You got

$$\mathsf{X}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}_2$$

and then

$$\mathsf{Y}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}_2$$

and finally

$$\mathsf{Z}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}_2$$

So we have the neat result

$$\boxed{X^2 = Y^2 = Z^2 = \mathbb{I}_2}$$

and as a consequence, for $\forall N \in \mathbb{N}$ we have

$$X^{2N+1} = X, \quad Y^{2N+1} = Y, \quad Z^{2N+1} = Z$$

The S gate

Could we find a quantum gate $\square$ so that

$$\square \cdot \square = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

or in other words

$$\square = \sqrt{Z}?$$

The answer is affirmative. Meet the "S gate":

$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

On the Bloch sphere, the S gate rotates your qubit with $\pi/2$ around the z-axis.

Obviously:

$$S^2 = Z$$

The phase gate

Meet the "phase gate" P,

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

It is a very handy gate when you want to create from $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ the state

$$|\psi'\rangle = c_0|0\rangle + c_1\boldsymbol{e^{i\phi}}|1\rangle$$

Indeed: start with $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$. Apply the phase gate:

$$P(\phi)|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 e^{i\phi} \end{pmatrix} = c_0|0\rangle + c_1 e^{i\phi}|1\rangle$$

The phase gate

So we have

$$P(\phi)\,|\psi\rangle = c_0|0\rangle + c_1 e^{i\phi}|1\rangle$$

Another way to look at the phase gate P,

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

is that it has the effect:

$$\begin{cases} |0\rangle \to |0\rangle \\ |1\rangle \to e^{i\phi}|1\rangle \end{cases}$$

This is quite easy to remember.

Some nice gate relations

Let us prove that:

$$\boxed{\mathsf{HXH} = \mathsf{Z}}$$

We compute the matrix multiplication HXH and find

$$\mathsf{HXH} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Other interesting relations:

$$\mathsf{HZH} = \mathsf{X} \qquad \mathsf{HYH} = -\mathsf{Y}$$

Some (more) nice gate relations

$$HXH = Z$$

$$HZH = X$$

$$HYH = -Y$$

$$XYX = -Y$$

$$YXY = -X$$

$$SXS^\dagger = Y$$

$$SYS^\dagger = -X$$

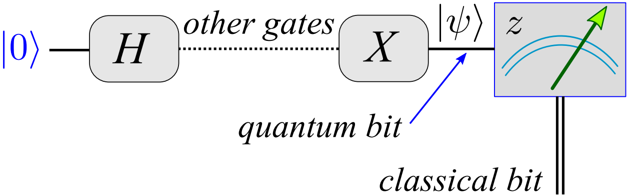
$$SZS^\dagger = Z$$

It would be a good exercise to prove them!

Measurement of a qubit

Recall Max Born (1926)

Measurement is probabilistic.



Obvious fact:

The measurement of a qubit yields a (single) classical bit.

Measurement of a qubit

Max Born (1926)

In QM, measurement results are **probabilistic**.

Specifically, for our qubit:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

The probability to find $|0\rangle$ *i. e.* "0" after a measurement is $p_0 = |c_0|^2$.
The probability to find $|1\rangle$ *i. e.* "1" after a measurement is $p_1 = |c_1|^2$.

Remark: The total probability is the probability to find $|0\rangle$ plus the probability to find $|1\rangle$.
This is $p_0 + p_1 = |c_0|^2 + |c_1|^2 = 1$.

Measurement of a qubit - formalized

Specifically, for our qubit:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

The probability to find $|0\rangle$ after a measurement is

$$p_0 = |\langle 0|\psi\rangle|^2 = |c_0|^2$$

The probability to find $|1\rangle$ after a measurement is

$$p_1 = |\langle 1|\psi\rangle|^2 = |c_1|^2$$

Quantum mechanics **is probabilistic** (most of the time).

Live with it!

Measurement of a qubit – exercises

Consider the state

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

What is the probability to find $|0\rangle$ after a measurement?
What is the probability to find $|1\rangle$ after a measurement?

Consider now the state

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Again, what is the probability to find $|0\rangle$ after a measurement?
What is the probability to find $|1\rangle$ after a measurement?

Are these results different? How do you explain?

Measurement of a qubit – solutions

Consider the state

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

The probability to find $|0\rangle$ after a measurement is $p_0 = |\langle 0|+\rangle|^2 = |c_0|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$.

The probability to find $|1\rangle$ after a measurement is $p_1 = |\langle 1|+\rangle|^2 = |c_1|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$.

Consider now the state

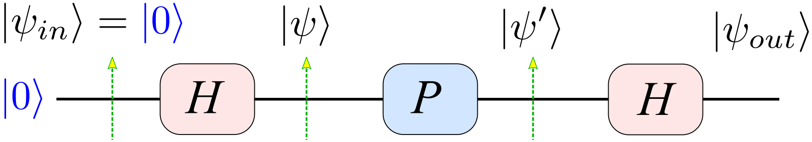
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

The probability to find $|0\rangle$ after a measurement is $p_0 = |\langle 0|-\rangle|^2 = |c_0|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$.

The probability to find $|1\rangle$ after a measurement is $p_1 = |\langle 1|-\rangle|^2 = |c_1|^2 = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$.

Phase measurement – with calculations

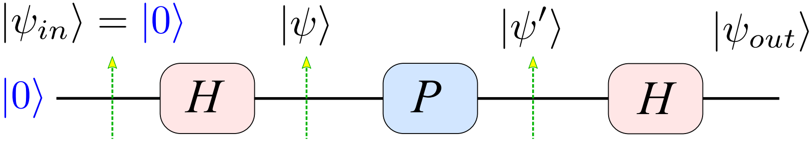
We have the following quantum circuit



and we need to compute the output state, $|\psi_{out}\rangle$.

We will perform this exercise in a step-by-step manner.

Phase measurement - calculations



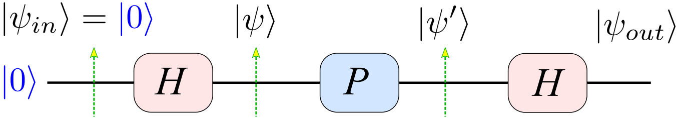
The first Hadamard gate performs the transformation

$$|\psi\rangle = \mathrm{H}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

We've already seen this.

So far, so good.

Phase measurement - calculations



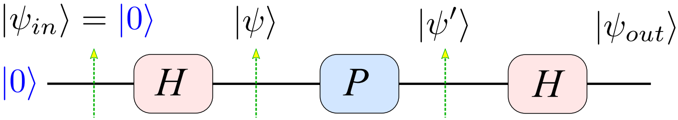
Next, the phase gate introduces the phase shift:

$$\begin{cases} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow e^{i\phi}|1\rangle \end{cases}$$

so we have the state

$$|\psi'\rangle = P(\phi)\,|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)$$

Phase measurement - calculations



So we had the state

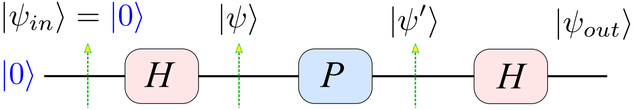
$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)$$

Good. Now apply a last Hadamard gate on $|\psi'\rangle$ to get the final state

$$|\psi_{out}\rangle = \mathsf{H}\,|\psi'\rangle = \frac{1}{\sqrt{2}}\left(\mathsf{H}|0\rangle + e^{i\phi}\mathsf{H}|1\rangle\right)$$

We're not done yet, we need $\mathsf{H}|0\rangle$ and $\mathsf{H}|1\rangle$.

Phase measurement - calculations



But we recall that $\begin{cases} \mathrm{H}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ \mathrm{H}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \end{cases}$ therefore we have

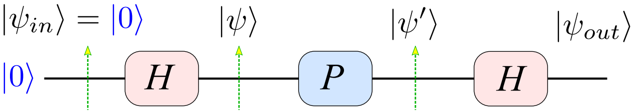
$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(H|0\rangle + e^{i\phi}H|1\rangle\right) = \frac{1}{2}\Big(\left(|0\rangle + |1\rangle\right) + e^{i\phi}\left(|0\rangle - |1\rangle\right)\Big)$$

so in the end

$$|\psi_{out}\rangle = \frac{1+e^{i\phi}}{2}|0\rangle + \frac{1-e^{i\phi}}{2}|1\rangle$$

It can still be simplified. How about to force a global phase $e^{i\frac{\phi}{2}}$? (It will make sense soon.)

Phase measurement - calculations



So we *force* a global phase $e^{i\frac{\phi}{2}}$, thus

$$|\psi_{out}\rangle = \frac{1+e^{i\phi}}{2}|0\rangle + \frac{1-e^{i\phi}}{2}|1\rangle$$

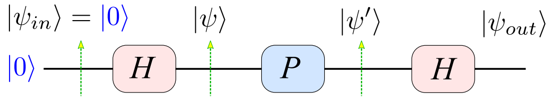
becomes

$$|\psi_{out}\rangle = e^{i\frac{\phi}{2}}\left(\frac{e^{-i\frac{\phi}{2}}+e^{i\frac{\phi}{2}}}{2}|0\rangle + \frac{e^{-i\frac{\phi}{2}}-e^{i\frac{\phi}{2}}}{2}|1\rangle\right) = e^{i\frac{\phi}{2}}\left(\cos\left(\frac{\phi}{2}\right)|0\rangle - i\sin\left(\frac{\phi}{2}\right)|1\rangle\right)$$

and ignoring the global phase we have

$$|\psi_{out}\rangle = \cos\left(\frac{\phi}{2}\right)|0\rangle - i\sin\left(\frac{\phi}{2}\right)|1\rangle$$

Phase measurement - calculations



$$|\psi_{out}\rangle = \cos\left(\frac{\phi}{2}\right)|0\rangle - i\sin\left(\frac{\phi}{2}\right)|1\rangle$$

So we actually estimated the phase because the probability to find $|0\rangle$ after a measurement is

$$\boxed{p_0 = \cos^2\left(\frac{\phi}{2}\right)}$$

and similarly, the probability to find $|1\rangle$ after a measurement is

$$\boxed{p_1 = \sin^2\left(\frac{\phi}{2}\right)}$$

Table of Contents

Welcome back, qubit!

Recall the qubit i. e. the *quantum bit*,

$$\boxed{|\psi\rangle = c_0|0\rangle + c_1|1\rangle}$$

with $c_0, c_1 \in \mathbb{C}$ and for normalization i. e. $\langle\psi|\psi\rangle = 1$ we must have $|c_0|^2 + |c_1|^2 = 1$.
Where:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

How about two qubits?

$$|\psi\rangle = ?$$

Two qubits!

We have the general two qubit state

$$\boxed{|\psi\rangle = c_0 |00\rangle + c_1 |01\rangle + c_2 |10\rangle + c_3 |11\rangle}$$

with $c_0, c_1, c_2, c_3 \in \mathbb{C}$ and for normalization i. e. $\langle\psi|\psi\rangle = 1$ we must have

$$|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$

If you want to be more posh, you can also write as

$$\boxed{|\psi\rangle = c_0 \underbrace{|0\rangle}_{\text{Alice}} \underbrace{|0\rangle}_{\text{Bob}} + c_1 |0\rangle |1\rangle + c_2 |1\rangle |0\rangle + c_3 |1\rangle |1\rangle}$$

Two qubits!

The general two qubit state

$$\boxed{|\psi\rangle = c_0\,|0\rangle\,|0\rangle + c_1\,|0\rangle\,|1\rangle + c_2\,|1\rangle\,|0\rangle + c_3\,|1\rangle\,|1\rangle}$$

If you insist to specify who has the first qubit (say Alice) and who has the second (say Bob), then specify this as

$$\boxed{|\psi\rangle = c_0\,|0\rangle_{\text{Alice}}\,|0\rangle_{\text{Bob}} + c_1\,|0\rangle_{\text{Alice}}\,|1\rangle_{\text{Bob}} + c_2\,|1\rangle_{\text{Alice}}\,|0\rangle_{\text{Bob}} + c_3\,|1\rangle_{\text{Alice}}\,|1\rangle_{\text{Bob}}}$$

It becomes kinda unreadable. How about

$$|\psi\rangle = c_0\,|00\rangle + c_1\,|01\rangle + c_2\,|10\rangle + c_3\,|11\rangle?$$

Back to the basics. Matrix representation of $|\psi\rangle$? Guesses?

Two qubits - matrix representation

We have the general two qubit state

$$\boxed{|\psi\rangle = c_0 |00\rangle + c_1 |01\rangle + c_2 |10\rangle + c_3 |11\rangle}$$

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Homework

Show that $\langle 00|00\rangle = 1$, $\langle 11|11\rangle = 1$ etc. Show also that any cross bra-kets are zero. For example $\langle 01|00\rangle = 0$, $\langle 11|00\rangle = 0$ etc.

Two qubits - mathematical sophistication

We have the general two qubit state – as posh as it cat get

$$\boxed{|\psi\rangle = c_0 |0\rangle \otimes |0\rangle + c_1 |0\rangle \otimes |1\rangle + c_2 |1\rangle \otimes |0\rangle + c_3 |1\rangle \otimes |1\rangle}$$

We define the **tensor product** ($\otimes$) of two qubits as

$$|\psi\phi\rangle = |\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \otimes \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \psi_0\phi_0 \\ \psi_0\phi_1 \\ \psi_1\phi_0 \\ \psi_1\phi_1 \end{pmatrix}$$

Two qubits - mathematical sophistication

We have the general two qubit state - as posh as it cat get

$$\boxed{|\psi\rangle = c_0\,|0\rangle \otimes |0\rangle + c_1\,|0\rangle \otimes |1\rangle + c_2\,|1\rangle \otimes |0\rangle + c_3\,|1\rangle \otimes |1\rangle}$$

because we have the **tensor products** ($\otimes$), namely

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Two qubits - mathematical sophistication

We have the general two qubit state - as posh as it cat get

$$\boxed{|\psi\rangle = c_0\,|0\rangle \otimes |0\rangle + c_1\,|0\rangle \otimes |1\rangle + c_2\,|1\rangle \otimes |0\rangle + c_3\,|1\rangle \otimes |1\rangle}$$

and the last two tensor products are

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We introduced $\otimes$ in order to avoid apoplexy attacks for purists and mathematicians.

Here's the thing:

You usually do not need the internal structure of a ket. Just use it's properties!

The sophistication in writing two qubits. Worth a meme

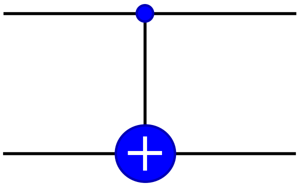
$$|\psi\rangle = |01\rangle$$

$$|\psi\rangle = |0\rangle|1\rangle$$

$$|\psi\rangle = |0\rangle \otimes |1\rangle$$

The CNOT gate

This is probably the most important gate we will use. It is called the controlled-NOT or simply CNOT gate. It has a **control qubit** (upper one) and a **target qubit** (the lower one):

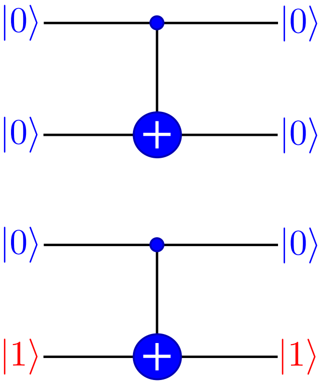


Remark:

The control qubit as it name says, *controls*. It is not affected by the CNOT gate.

The CNOT gate

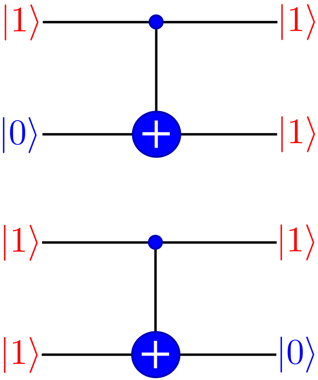
Set the control qubit to $|0\rangle$. Not much happens:



Boring.

The CNOT gate

Set the control qubit to $|1\rangle$. Now interesting things happen:



Question: what gate has the same behaviour on the target bit?

The CNOT gate - matrix representation

Being a two-qubit gate it must have a 4×4 matrix representation. Here it comes:

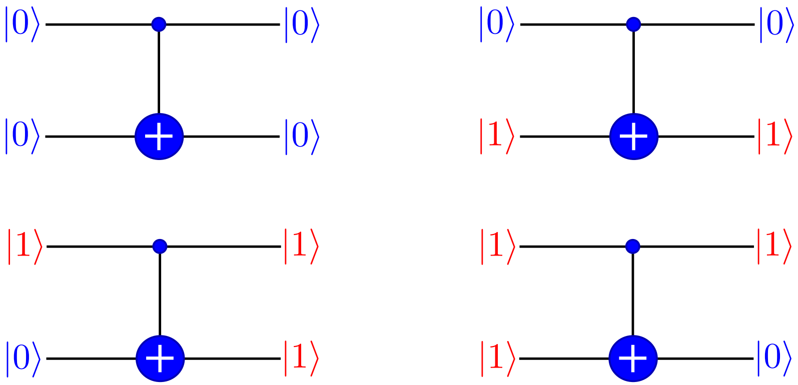
$$\mathsf{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathbb{I}_2 & \mathbf{0}_2 \\ \hline \mathbf{0}_2 & \mathsf{X} \end{array}\right)$$

I don't suggest to use it, except in the beginning. I would go for the truth table instead:

$$\begin{cases} |00\rangle \to |00\rangle \\ |01\rangle \to |01\rangle \\ |10\rangle \to |11\rangle \\ |11\rangle \to |10\rangle \end{cases}$$

Remark: You can use the matrix representation of the qubits to check it.

The CNOT gate in a single slide



The control-Z gate

We saw the CNOT gate:

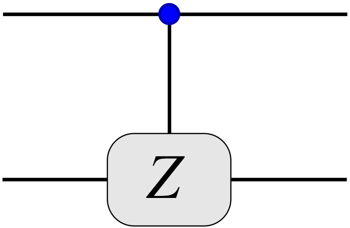
$$\text{CNOT} = \left(\begin{array}{c|c} \mathbb{I}_2 & \mathbf{0}_2 \\ \hline \mathbf{0}_2 & \text{X} \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

We can maybe imagine a gate

$$\text{CZ} = \left(\begin{array}{c|c} \mathbb{I}_2 & \mathbf{0}_2 \\ \hline \mathbf{0}_2 & Z \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

The control-Z gate

Meet the control-Z (CZ) gate:

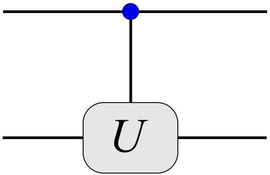


Truth table:

$$\begin{cases} |00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |01\rangle \\ |10\rangle \rightarrow |10\rangle \\ |11\rangle \rightarrow -|11\rangle \end{cases}$$

The control-U gate

Meet the control-U gate:

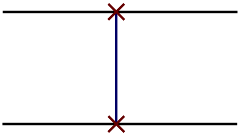


Truth table:

$$\begin{cases} |00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |01\rangle \\ |10\rangle \rightarrow U\,|10\rangle \\ |11\rangle \rightarrow U\,|11\rangle \end{cases}$$

The SWAP gate

Finally, meet the SWAP gate:



Matrix form and truth table:

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{cases} |00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |10\rangle \\ |10\rangle \rightarrow |01\rangle \\ |11\rangle \rightarrow |11\rangle \end{cases}$$

The maximally entangled state – if time allows, we discuss more in depth.

Suppose we have two photons. One goes to Alice, the other one to Bob.

We could generate them in the so-called EPR state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|{\color{blue}\blacksquare}_{Alice}{\color{blue}\blacksquare}_{Bob}\rangle + |{\color{red}\blacksquare}_{Alice}{\color{red}\blacksquare}_{Bob}\rangle\right)$$

What bothered Einstein: if Alice makes a measurement and obtains "${\color{red}\blacksquare}$", then Bob finds its state instantly changed to

$$|\psi_{Bob}\rangle = |{\color{red}\blacksquare}_{Bob}\rangle$$

Bob could even be in another Galaxy. So Alice measures, gets "${\color{red}\blacksquare}$" and – bang! – on the other side of the Universe, Bob will get "${\color{red}\blacksquare}$", too. With certainty!
Einstein called this "spooky action at a distance".

The maximally entangled state - examples

The EPR (also called Bell) state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\color{red}{\blacksquare}_{Alice}\color{red}{\blacksquare}_{Bob}\rangle + |\color{blue}{\blacksquare}_{Alice}\color{blue}{\blacksquare}_{Bob}\rangle\right)$$

can be made of **two photons** having horizontal/vertical (H/V) polarization

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|H_{Alice}H_{Bob}\rangle + |V_{Alice}V_{Bob}\rangle\right) = \frac{1}{\sqrt{2}}\left(|HH\rangle + |VV\rangle\right)$$

or, in quantum computing, two maximally entangles qubits

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0_{Alice}0_{Bob}\rangle + |1_{Alice}1_{Bob}\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

The Bell states

There are actually 4 maximally entangled states called "**the Bell states**". They are:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

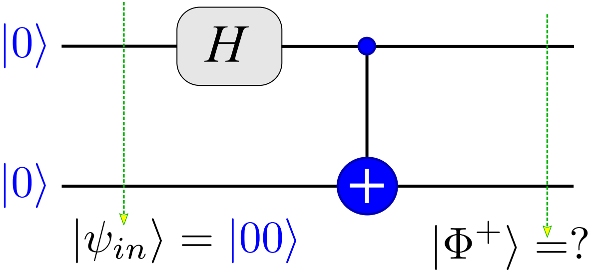
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

They are all orthogonal among themselves.

Back to the previous question

You have the quantum circuit below.

What output state does it generate?

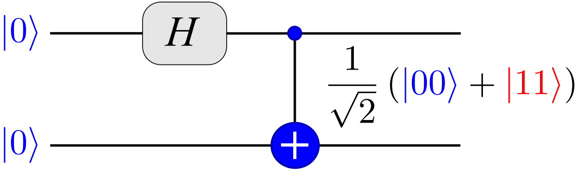


Any guesses?

The Bell states - $|\Phi^+\rangle$

I claim that this scheme generates the $|\Phi^+\rangle$ state.

Let's prove this!



We know the Hadamard gate by now.

$$\begin{cases} \mathrm{H}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ \mathrm{H}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \end{cases}$$

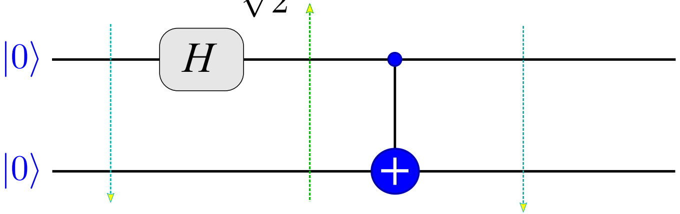
Let us recall the truth table of the CNOT gate:

$$\text{CNOT} \begin{cases} |00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |01\rangle \\ |10\rangle \rightarrow |11\rangle \\ |11\rangle \rightarrow |10\rangle \end{cases}$$

The Bell state generation

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle$$

Let's start describing the circuit, from left to right:



$$|\psi_{in}\rangle = |00\rangle$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

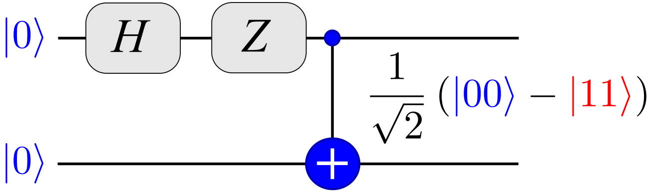
$$\mathrm{H}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\text{CNOT}\begin{cases} |00\rangle \rightarrow |00\rangle \\ |10\rangle \rightarrow |11\rangle \end{cases}$$

The Bell states

I claim that this scheme generates the $|\Phi^-\rangle$ state.

Let's prove this!

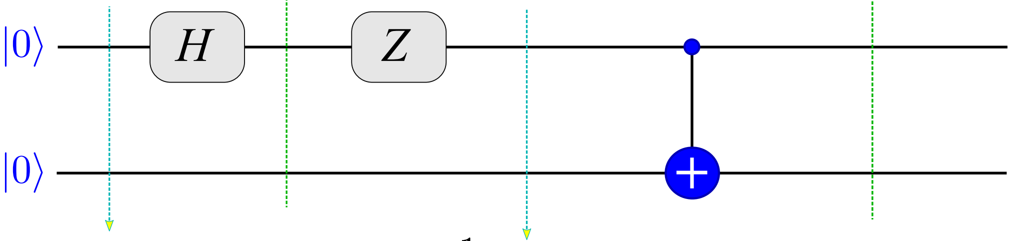


The Bell state generation - $|\Phi^-\rangle$

Let's see:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$



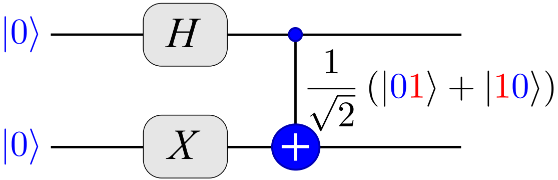
$$|\psi_{in}\rangle = |00\rangle$$

$$|\psi''\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|0\rangle$$

The Bell states - $|\Psi^+\rangle$

I claim that this scheme generates the $|\Psi^+\rangle$ state.

Let's prove this!

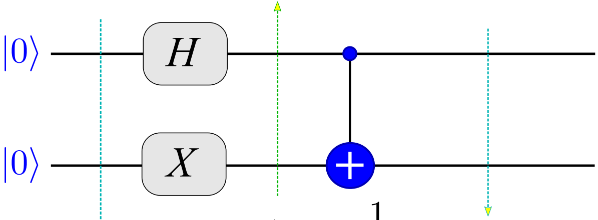


Can you prove this result? Give it a try!

The Bell state generation - $|\Psi^+\rangle$

Let's see:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|1\rangle$$

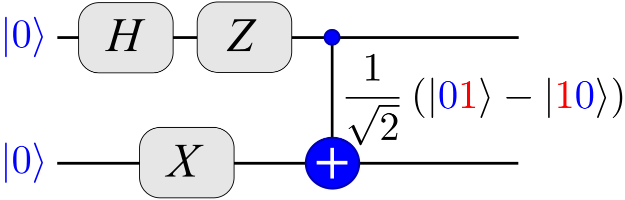


$$|\psi_{in}\rangle = |00\rangle \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

The Bell states - $|\Psi^-\rangle$

I claim that this scheme generates the $|\Psi^-\rangle$ state.

Let's prove this!



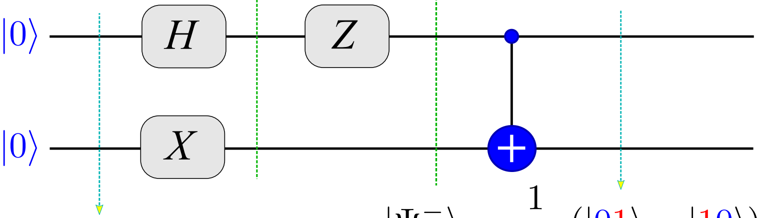
Can you prove this result? Give it a try!

The Bell state generation - $|\Psi^-\rangle$

Let's see:

$$|\psi''\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|1\rangle$$

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|1\rangle$$

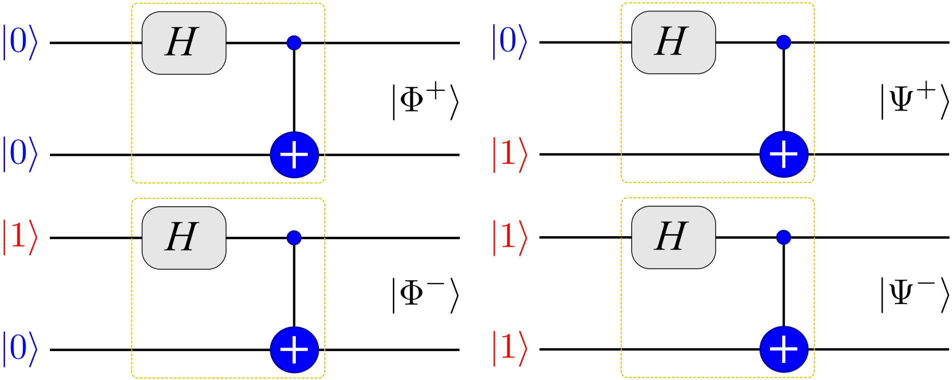


$$|\psi_{in}\rangle = |00\rangle$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

The Bell state generation - homework

Show that:

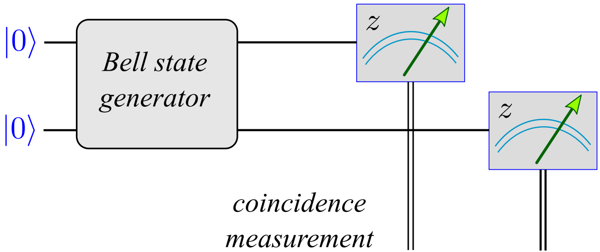


The Bell state measurement

All Bell states are entangled i. e. not separable. **Single measurements on one qubit will yield totally random values**.

Measure both qubits!

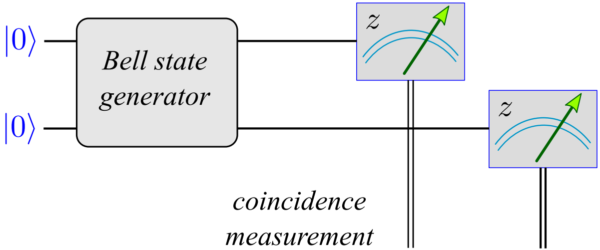
We need **coincidence measurements**.



The Bell state measurement – not so fast!

Even if you measure **both** qubits,

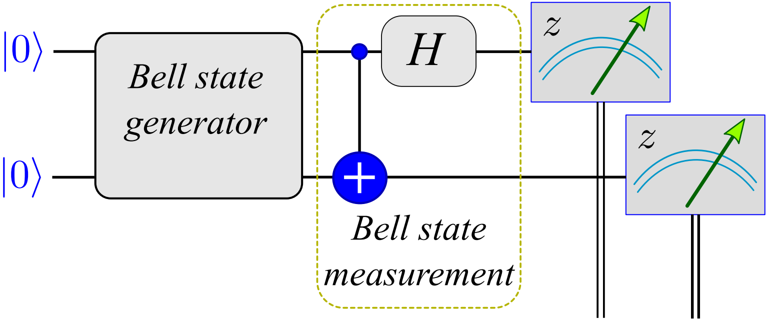
you will be able to distinguish only between $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$!



The Bell state measurement (BSM)

Claim:

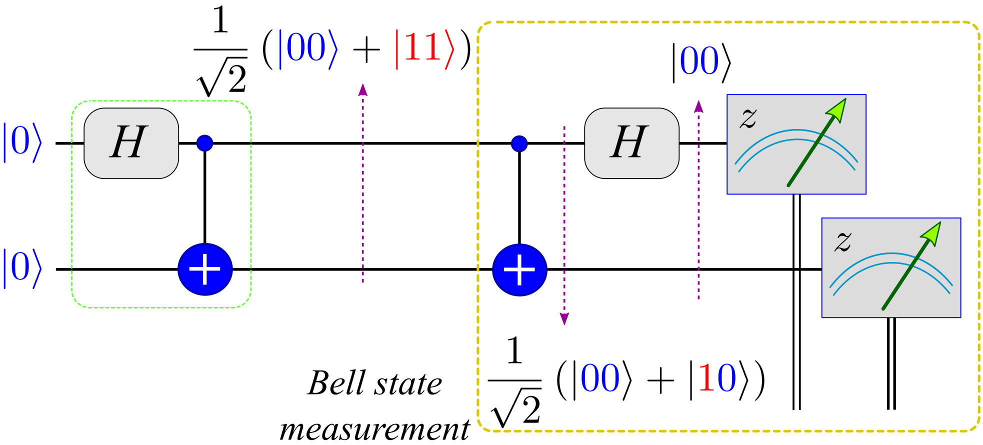
The circuit below distinguishes among **all** the 4 Bell states.



Can you prove it? Start with the $|\Phi^+\rangle$ state!

The Bell state measurement

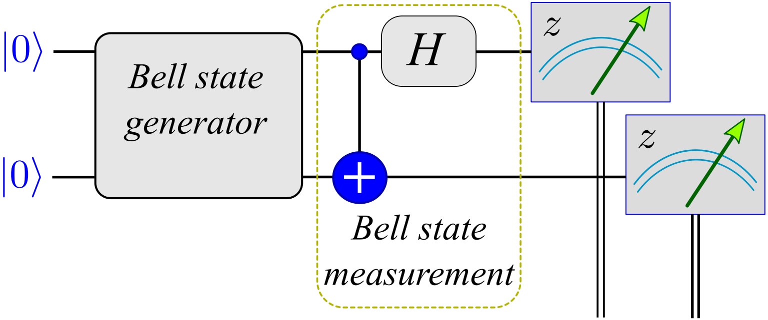
We start with a $|\Phi^+\rangle$ state and perform a BSM. Here we go:



Bell state measurement in IBM QE

Use IBM QE:

Implement all for Bell states and use the circuit below to measure them in IBM QE.

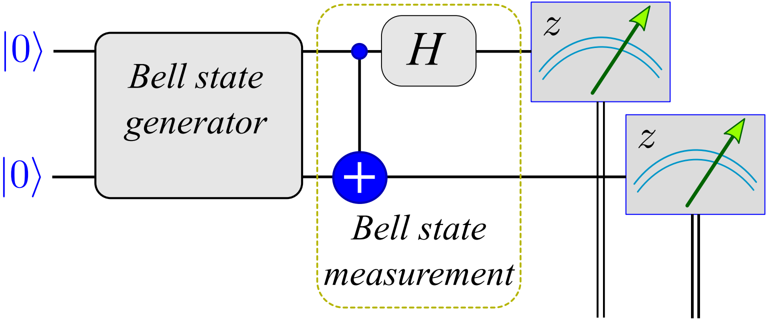


Can you distinguish now among the 4 Bell states?

Are your results probabilistic or deterministic? Comment on your findings!

Exercise – to be implemented soon in IBM Quantum

Show that the circuit below distinguishes among the 4 Bell states.



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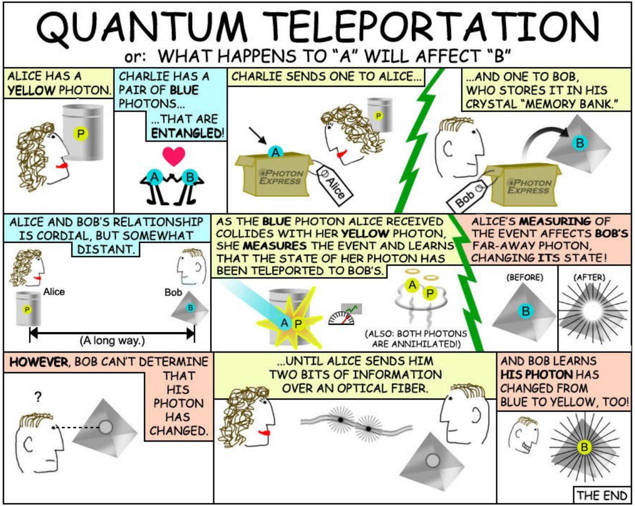
NASA explains quantum teleportation



Images source: NASA JPL, see the article Researchers Advance 'Quantum Teleportation'

Proposal and first experiments

The idea of quantum teleportation

Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters, *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*, Phys. Rev. Lett. **70**, 1895 (1993)
Note: the paper has over **12k citations** (as of July 2025).

Nice idea, but is this real?
Experiments:

Dik Bouwmeester, Jian-Wei Pan, Klaus Mattle, Manfred Eibl, Harald Weinfurter & Anton Zeilinger , *Experimental quantum teleportation*, Nature **390**, 575 (1997)

D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, *Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels*, Phys. Rev. Lett. 80, 1121 (1998)

More experiments

Teleportation – over longer and longer distances:

I. Marcikic, H. de Riedmatten, W. Tittel, H. Zbinden & N. Gisin, *Long-distance teleportation of qubits at telecommunication wavelengths*, Nature **421**, 509 (2003)

Rupert Ursin, Thomas Jennewein, Markus Aspelmeyer, Rainer Kaltenbaek, Michael Lindenthal, Philip Walther & Anton Zeilinger, *Quantum teleportation across the Danube*, Nature **430**, 849 (2004)

Ma, X., Herbst, T., Scheidl, T. et al. (A. Zeilinger group) *Quantum teleportation over 143 kilometres using active feed-forward*, Nature **489**, 269 (2012)

Ren, J., Xu, P., Yong, H. et al. (Jian-Wei Pan group) *Ground-to-satellite quantum teleportation*, Nature **549**, 70 (2017)

More experiments

More than the quantum state of a photon?
Yes, **teleport the quantum state of an atom**!

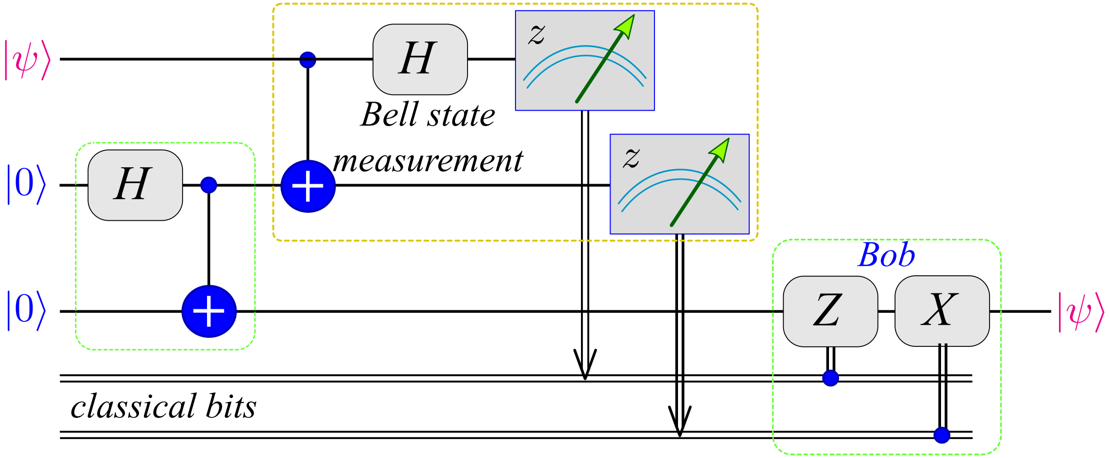
Teleporting atoms

M. Riebe, H. Häffner, C. F. Roos, W. Hänsel, J. Benhelm, G. P. T. Lancaster, T. W. Körber, C. Becher, F. Schmidt-Kaler, D. F. V. James & R. Blatt, *Deterministic quantum teleportation with atoms*, Nature **429**, 734 (2004)

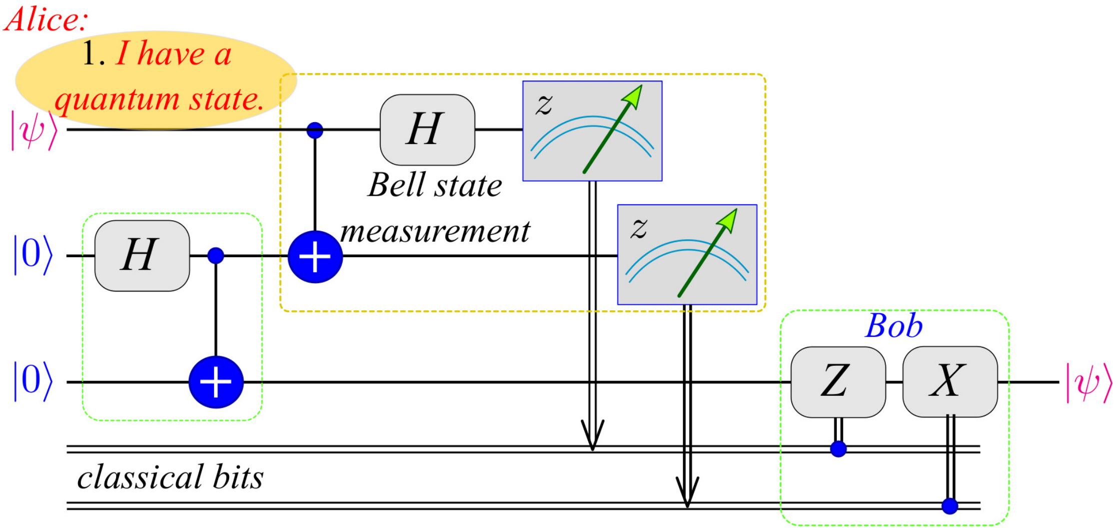
S. Olmschenk, D. N. Matsukevich, P. Maunz, D. Hayes, L.-M. Duan, C. Monroe, *Quantum Teleportation Between Distant Matter Qubits*, Science **323**, 486 (2009)

Just to be clear: we always teleport states, i. e. quantum states ($|\psi\rangle$) not the photons/atoms/molecules/cats/Klingons etc.

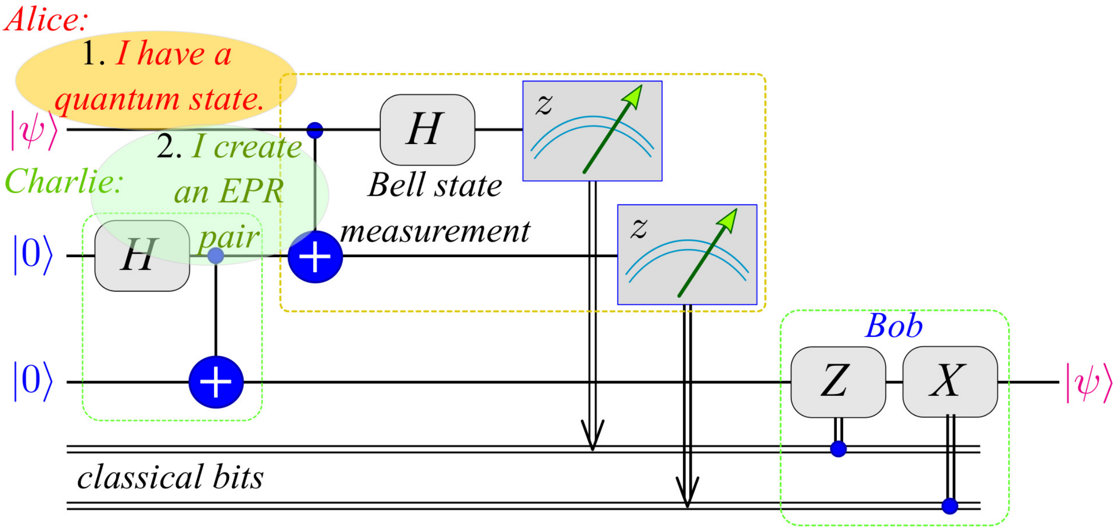
The Quantum teleportation



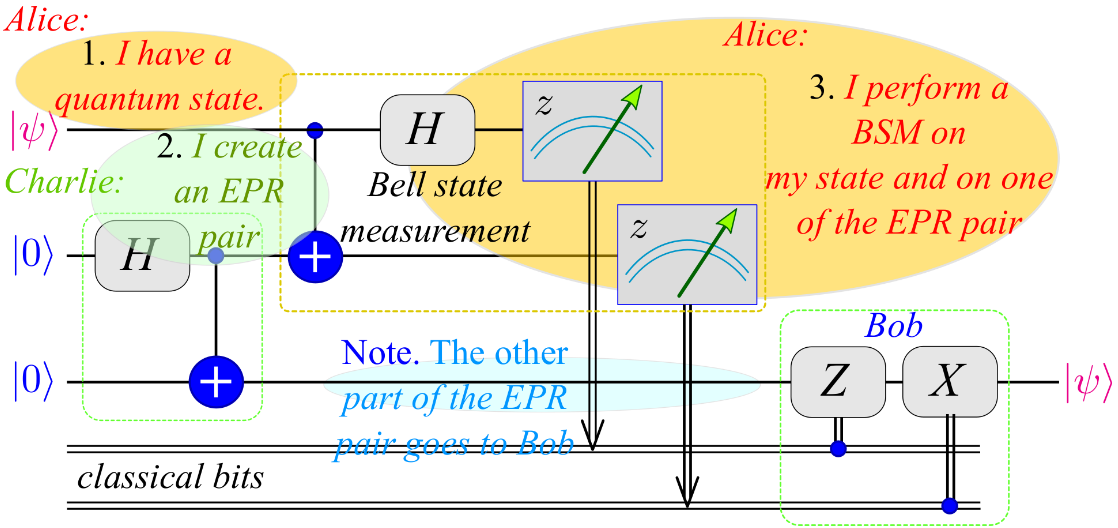
The Quantum teleportation – 1. Alice has a $|\psi\rangle$.



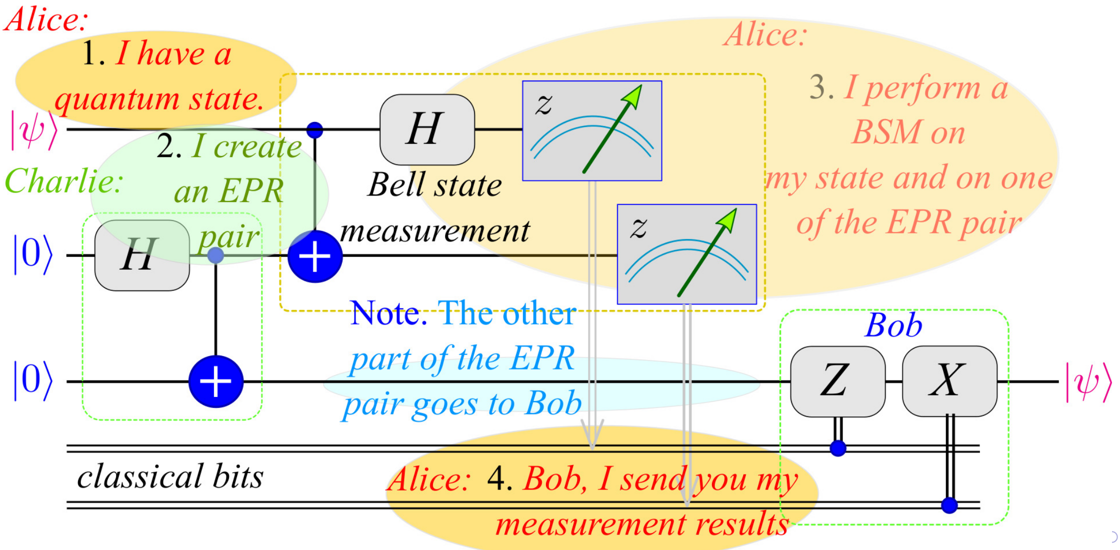
The Quantum teleportation – 2. Charlie creates $|\Phi^+\rangle$.



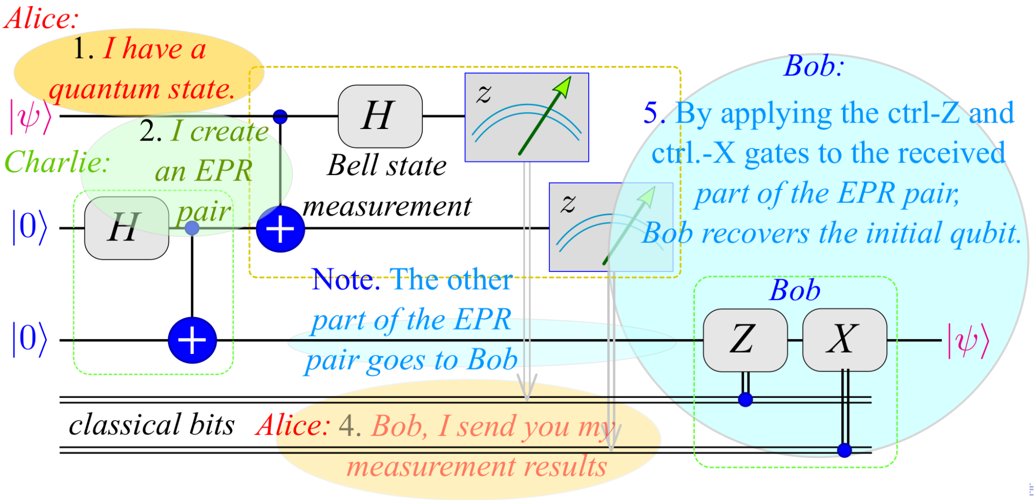
The Quantum teleportation – 3. Alice performs a BSM.



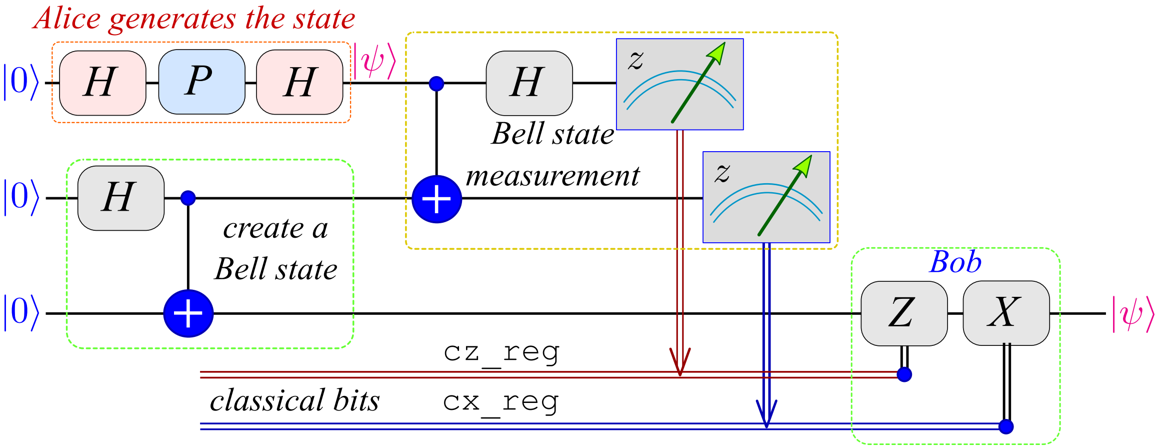
The Quantum teleportation – 4. Alice sends her measurements to Bob.



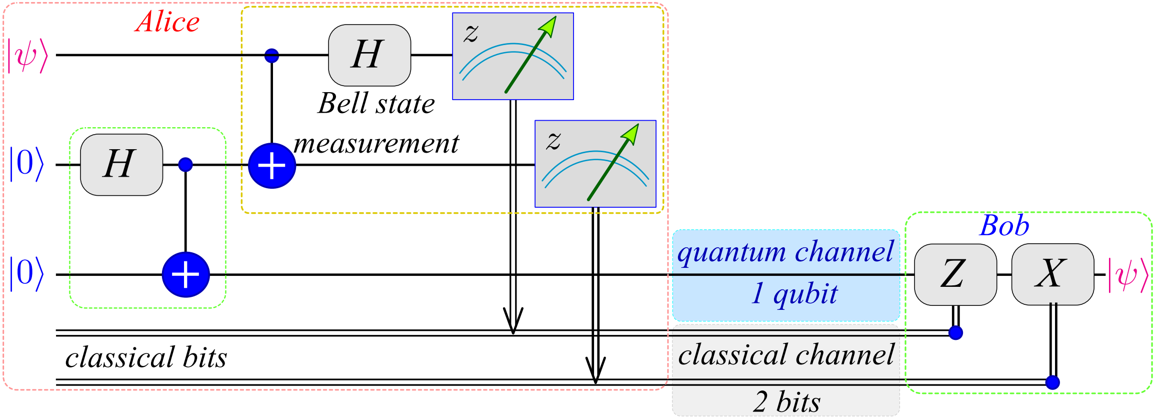
The Quantum teleportation – 5. Bob uses the quantum plus classical channels.



The Quantum teleportation – more practical implementation for us

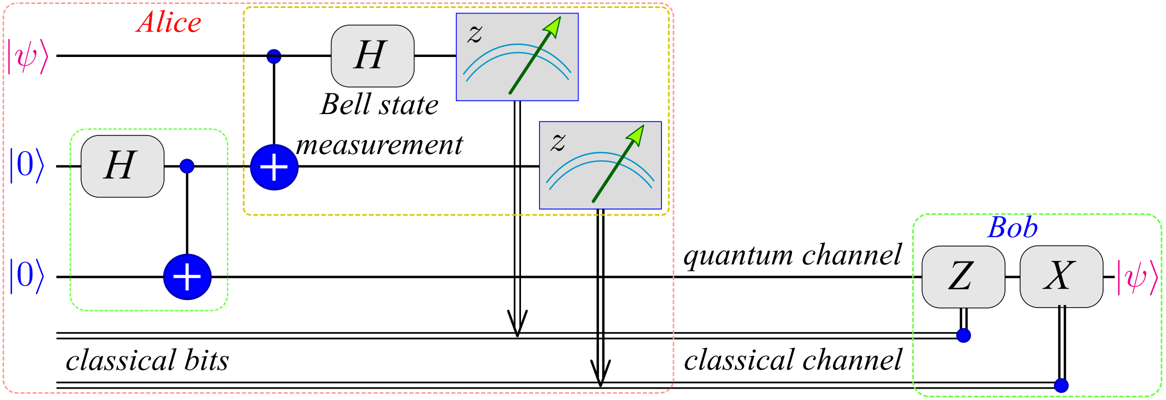


Quantum teleportation - the channels



The "quantum channel" can be *e. g.* a fiber.
The "classical channel" can be your usual phone line.

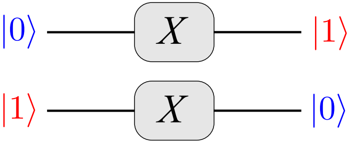
Quantum teleportation - the channels

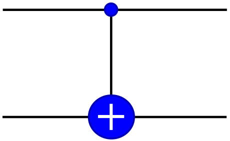


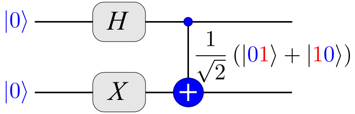
That's it for now.

Thank you for your attention!

Questions are welcome.







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1935: Einstein-Bohr debate – the later phase

Einstein to Bohr (1935): QM is incomplete!

Einstein, Podolsky and Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. **47**, 777 (1935)

Bohr to Einstein (1935):

No, it is not! It is nonlocal (whatever that means).

N. Bohr, *Can Quantum-Mechanical Description of Physical Reality be Considered Complete?*, Phys. Rev. **48**, 696 (1935)

What Einstein, Podolsky and Rosen did:

- → discover **entanglement**: QM allows such states
- → they imply "spooky action at a distance" (Einstein thought)
- → the authors didn't say it, but it is implicit: he predicts that hidden variables must exist

Einstein-Bohr debates – nobody cared!

Einstein versus Bohr: their debate had no immediate consequences.

It was mainly philosophical, so most physicists didn't get very involved.

There was a kind of "Quantum Orthodoxy":

Bohr's "Copenhagen interpretation" won the day and discussing the foundations of QM was considered a pure heresy.
Huge names like Schrödinger or de Broglie ended up marginalized.
Simply discussing the foundations of QM could jeopardize your career.

QM was very successful

There were good reasons not to reject QM. It was incredibly successful, in all domains. For example QED (Quantum Electro-Dynamics) emerged explaining the Lamb Shift, predicting the Casimir force, giving with extreme accuracy the anomalous momentum of the electron etc...
Not to mention the industrial successes with the transistor and the laser.

The maximally entangled state

Suppose we have two photons in this crazy state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\blacksquare_{Alice}\blacksquare_{Bob}\rangle + |\blacksquare_{Alice}\blacksquare_{Bob}\rangle\right)$$

One goes to Alice, the other one to Bob.

What bothered Einstein: if Alice makes a measurement and obtains "■", then Bob finds its state instantly changed to

$$|\psi_{Bob}\rangle = |\blacksquare_{Bob}\rangle$$

But: Bob can be many miles away from Alice!

Bob could even be in another Galaxy. So Alice measures, gets "■" and – bang! – on the other side of the Universe, Bob will get "■", too. With certainty!
Einstein called this "spooky action at a distance".

The maximally entangled state - examples

The EPR (also called Bell) state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|{\color{red}\blacksquare}_{Alice}{\color{red}\blacksquare}_{Bob}\rangle + |{\color{blue}\blacksquare}_{Alice}{\color{blue}\blacksquare}_{Bob}\rangle\right)$$

can be made of **two photons** having horizontal/vertical (H/V) polarization

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|{\color{red}H}_{Alice}{\color{red}H}_{Bob}\rangle + |{\color{blue}V}_{Alice}{\color{blue}V}_{Bob}\rangle\right) = \frac{1}{\sqrt{2}}\left(|{\color{red}HH}\rangle + |{\color{blue}VV}\rangle\right)$$

or, in quantum computing, two maximally entangles qubits

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|{\color{red}0}_{Alice}{\color{red}0}_{Bob}\rangle + |{\color{blue}1}_{Alice}{\color{blue}1}_{Bob}\rangle\right) = \frac{1}{\sqrt{2}}\left(|{\color{red}00}\rangle + |{\color{blue}11}\rangle\right)$$

The maximally entangled state

I claimed that the (EPR also called Bell) state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\color{red}{\blacksquare}_{Alice}\color{red}{\blacksquare}_{Bob}\rangle + |\color{blue}{\blacksquare}_{Alice}\color{blue}{\blacksquare}_{Bob}\rangle\right) = \frac{1}{\sqrt{2}}\left(|\color{red}{0}_{Alice}\color{red}{0}_{Bob}\rangle + |\color{blue}{1}_{Alice}\color{blue}{1}_{Bob}\rangle\right)$$

is bizarre and "spooky" from Einstein's point of view.

Indeed,
if Alice makes a measurement and obtains "$\color{red}{\blacksquare}$" *i. e.*, "$\color{red}{0}$",
then Bob finds its state instantly changed to

$$|\psi_{Bob}\rangle = |\color{red}{\blacksquare}_{Bob}\rangle = |\color{red}{1}_{Bob}\rangle$$

Are there "normal" states?

Yes! Almost **all** states are "normal". Entanglement is hard to create.

The non-entangled state

Suppose we have two photons. One goes to Alice, the other one to Bob. We could also generate them in the state:

$$|\Psi\rangle = \frac{1}{2}\left(|0_{Alice}0_{Bob}\rangle + |0_{Alice}1_{Bob}\rangle + |1_{Alice}0_{Bob}\rangle + |1_{Alice}1_{Bob}\rangle\right)$$

However, this state can be factored:

$$|\Psi\rangle = \frac{1}{2}\left(|0_{Alice}\rangle + |1_{Alice}\rangle\right) \otimes \left(|0_{Bob}\rangle + |1_{Bob}\rangle\right)$$

Compare it with the maximally entangled EPR state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0_{Alice}0_{Bob}\rangle + |1_{Alice}1_{Bob}\rangle\right)$$

This state **cannot be factorized**.

The non-entangled state

The totally non-entangled Alice-Bob state

$$
\begin{aligned}
|\Psi\rangle &= \frac{1}{2}\left(|0_{Alice}\rangle + |1_{Alice}\rangle\right) \otimes \left(|0_{Bob}\rangle + |1_{Bob}\rangle\right) \\
&= \underbrace{\frac{1}{\sqrt{2}}\left(|0_{Alice}\rangle + |1_{Alice}\rangle\right)}_{\text{Alice's wavevector}} \otimes \underbrace{\frac{1}{\sqrt{2}}\left(|0_{Bob}\rangle + |1_{Bob}\rangle\right)}_{\text{Bob's wavevector}} \\
&= |\psi_{Alice}\rangle \otimes |\psi_{Bob}\rangle
\end{aligned}
$$

Now: if Alice makes a measurement and obtains, say, "0", then Bob couldn't care less:

$$
|\psi_{Bob}\rangle = \frac{1}{\sqrt{2}}\left(|0_{Bob}\rangle + |1_{Bob}\rangle\right)
$$

There is no "spooky action at a distance".

The maximally entangled state - your classical explanations

We have the state: $|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|0_{Alice} 0_{Bob}\rangle + |1_{Alice} 1_{Bob}\rangle \right)$

Einstein's explanation

If Alice makes a measurement and obtains "0", then Bob finds its state **instantly** changed to

$$|\psi_{Bob}\rangle = |0_{Bob}\rangle$$

because the state was actually $|0_{Alice} 0_{Bob}\rangle$ *before* the measurement. There is no "spooky action at a distance" however Alice's and Bob's photons had a *secret agenda*.

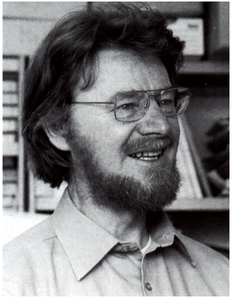
QM: absolutely not!

Measurement creates reality! There is no "color" in your $|\Psi\rangle$ before measurement!

Bell's inequalities - 1964

John Bell takes seriously Einstein's point of view and assumes that **local hidden variables** do exist. He goes on to assume the most general case and manages to obtain a result in contradiction with QM.

The Einstein-Bohr dispute can be tested in the laboratory!



The CHSH (Clauser-Horne-Shimony-Holt) inequality. If hidden variables exist, then:

$$S_{HVT} = \left| E\left(a, b\right) + E\left(a, b'\right) + E\left(a', b\right) - E\left(a', b'\right) \right| \leq 2$$

QM predicts a value well above 2, $S_{QM} = 2\sqrt{2} \approx 2.82$.

Hidden variables could save local realism

If there are hidden variables,

the fact that Bob obtains with certainty "■" on Alpha Centauri if Alice measures and obtains "■" on planet Earth would be not at all surprising.
In fact, when the entangled state was generated, secret (hidden) parameters were given to Alice and Bob. Before taking the measurement, they actually check those secret messages.

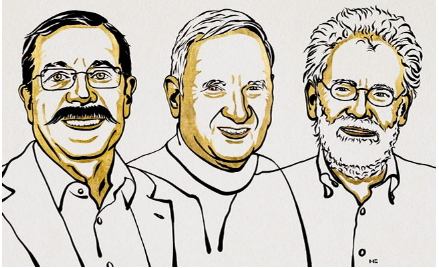
But if there are no hidden variables,

then we must concede that Bohr is right: **QM is non-local, there can be strong quantum correlations between Alice and Bob, correlations impossible to explain in a classical manner.**

The Nobel Prize in Physics 2022

The Nobel Prize in Physics 2022

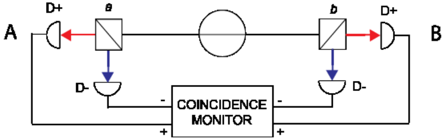
The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science".



Ill. Niklas Elmehed ©Nobel Prize Outreach

Bell inequality experiments

A typical CHSH apparatus sends the signals from the polarization analyzers to a coincidence monitor.



The coincidence monitor then counts four kinds of events, N++, N+-, N-+, and N--. Perfect correlation (and conservation of spin angular momentum) allows only +- and -+ events.

Freedman and Clauser (1972): Bell inequality violation. (Not very convincing, though.)

Aspect *et al.* (1980-1982): Bell inequality violation with time-varying experimental setup.

Other (more precise) experiments

Tittel and the Geneva group (1998), Rowe et al. (2001), Giustina et al. (2013), Larsson et al (2014). Hensen et al., Giustina et al., Shalm et al. (2015) – lopphole-free Bell test.

Bell inequality experiments

Aspect et al. $(1980 - 1982)$

Bell inequality violation with time-varying experimental setup.

Experimental Test of Bell's Inequalities Using Time-Varying Analyzers

Alain Aspect, Jean Dalibard,[a] and Gérard Roger
Institut d'Optique Théorique et Appliquée, F-91406 Orsay Cédex, France

(Received 27 September 1982)

Correlations of linear polarizations of pairs of photons have been measured with time-varying analyzers. The analyzer in each leg of the apparatus is an acousto-optical switch followed by two linear polarizers. The switches operate at incommensurate frequencies near 50 MHz. Each analyzer amounts to a polarizer which jumps between two orientations in a time short compared with the photon transit time. The results are in good agreement with quantum mechanical predictions but violate Bell's inequalities by 5 standard deviations.

PACS numbers: 03.65.Bz, 35.80.+s

Bell's inequalities apply to any correlated measurement on two correlated systems. For instance, in the optical version of the Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*,[1] a source emits pairs of photons (Fig. 1). Measurements of the correlations of linear polarizations are performed on two photons belonging to the same pair. For pairs emitted in suitable states, the correlations are strong. To account for these correlations, Bell[2] considered theories which invoke common properties of both members of the

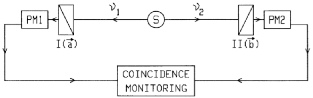


FIG. 1. Optical version of the Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*. The pair of photons ν_1 and ν_2 is analyzed by linear polarizers I and II (in orientations $\vec{a}$ and $\vec{b}$) and photomultipliers. The coincidence rate is monitored.

Bell inequality experiments

10 km Bell test – already in 1998!

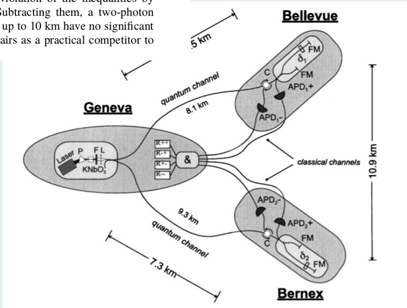
Violation of Bell Inequalities by Photons More Than 10 km Apart

W. Tittel,* J. Brendel, H. Zbinden, and N. Gisin

Group of Applied Physics, University of Geneva, 20, Rue de l'Ecole de Médecine, CH-1211 Geneva 4, Switzerland

(Received 10 June 1998)

A Franson-type test of Bell inequalities by photons 10.9 km apart is presented. Energy-time entangled photon pairs are measured using two-channel analyzers, leading to a violation of the inequalities by 16 standard deviations without subtracting accidental coincidences. Subtracting them, a two-photon interference visibility of 95.5% is observed, demonstrating that distances up to 10 km have no significant effect on entanglement. This sets quantum cryptography with photon pairs as a practical competitor to the schemes based on weak pulses. [S0031-9007(98)07478-X]



W. Tittel, J. Brendel, H. Zbinden, N. Gisin, *Violation of Bell Inequalities by Photons More Than 10 km Apart*, Phys. Rev. Lett. 81, 3563 (1998)

Bell inequality experiments – closing the loopholes

Other (more precise) experiments

Rowe et al., *Experimental violation of a Bell's inequality with efficient detection*, Nature **409** 791 (2001)

M. Giustina . . . and A. Zeilinger, *Bell violation using entangled photons without the fair-sampling assumption*, Nature **497**, 227 (2013)

Towards loophole-free experiments: 2015 all loopholes closed

J-A Larsson et al. *Bell-inequality violation with entangled photons, free of the coincidence-time loophole*, Phys. Rev. A **90**, 032107 (2014)

B. Hensen et al., *Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres*, Nature **526**, 682 (2015)

M. Giustina . . . and A. Zeilinger, *Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons*, Phys. Rev. Lett. **115**, 250401 (2015)

L. K. Shalm et al., *Strong Loophole-Free Test of Local Realism*, PRL **115**, 250402 (2015)

GHZ states and violation of local realism

Consider the state:

$$|\Psi\rangle = \frac{1}{2}\left(|\blacksquare_{Alice}\rangle|\blacksquare_{Bob}\rangle|\blacksquare_{Charlie}\rangle + |\blacksquare_{Alice}\rangle|\blacksquare_{Bob}\rangle|\blacksquare_{Charlie}\rangle\right)$$

This state is called a GHZ (Greenberger, Horne and Zeilinger) state.

Let's rewrite the state:

$$|\Psi\rangle = \frac{1}{2}\left(|H_1\rangle|H_2\rangle|H_3\rangle + |V_1\rangle|V_2\rangle|V_3\rangle\right)$$

So: Alice has photon 1 and it can be H (horizontal) or V (vertically) polarized. Same goes for Bob and Charlie.

But: If Alice measures her photon and finds H, then with certainty, Bob and Charlie will get H too.

GHZ states and the violation of local realism (LR)

The theoretical proposal:

D. Greenberger; M. Horne; A. Shimony; A. Zeilinger, *Bell's theorem without inequalities*, Am. J. Phys. **58**, 1131 (1990)

The point:

GHZ states violate local realism (LR) **without inequalities**. They violate the local hidden variables (LHV) model **every single time the experiment is performed**. This is way stronger than the Bell's inequalities!

The experimental result:

Jian-Wei Pan; D. Bouwmeester; M. Daniell; H. Weinfurter; A. Zeilinger, *Experimental test of quantum nonlocality in three-photon GHZ entanglement*, Nature **403**, 515 (2000)

GHZ states and violation of local realism

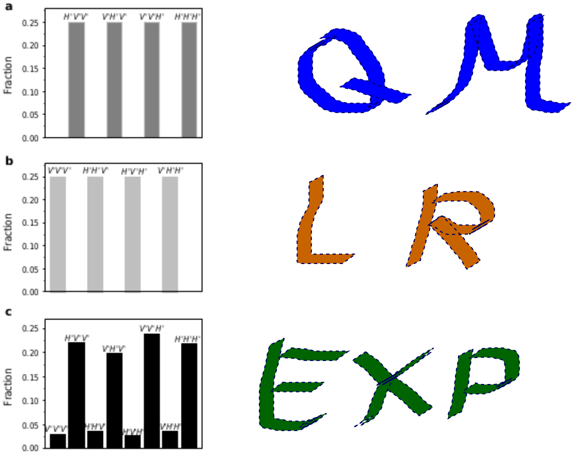


Figure from Pan et al, Nature **403**, 515 (2000)

Where do we stand?

Einstein had a reasonable point of view

Nature should be local and no "spooky" correlation should exist.
Reality should also exist in QM, even before measurements.
"Is the Moon there if nobody looks?" (Einstein to Pais)

But: as we stand today, Nature is not reasonable.

Nature is non-local in a funny way, much to Einstein's dislike.
We have to admit that Bohr won this dispute. In QM mechanics, **measurement creates reality**. There is no reality (*i. e.* result) before the measurement is performed.
"Unperformed experiments have no results" (A. Perez)

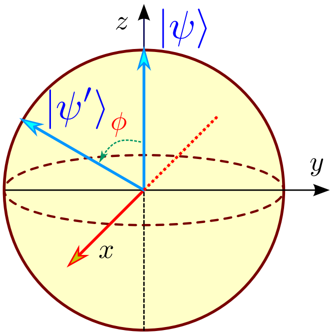
However:

Entanglement does not allow you to send faster-than-light signals.
Einstein's causality principle is not violated.

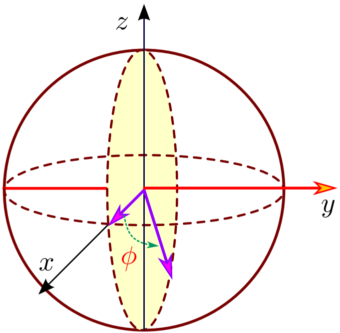
The rotation gates

Problem:

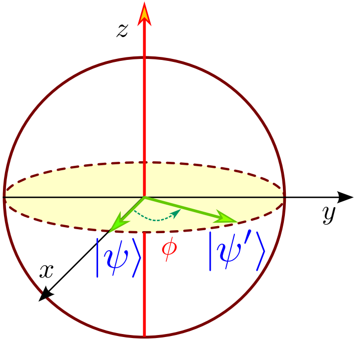
You have a given qubit $|\psi\rangle$. How do you rotate it on the Bloch sphere?



$$|\psi'\rangle = R_x(\phi)|\psi\rangle$$



$$|\psi'\rangle = R_y(\phi)|\psi\rangle$$



$$|\psi'\rangle = R_z(\phi)|\psi\rangle$$

The rotation gates

The operator (gate) the generates rotations around the x axis is:

$$R_x(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}X$$

The operator (gate) the generates rotations around the y axis is:

$$R_y(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Y$$

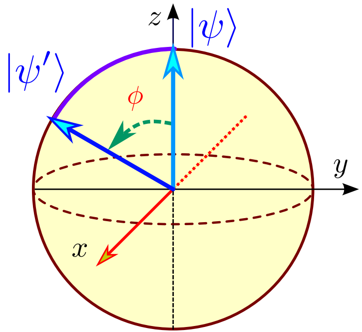
Finally, the operator (gate) the generates rotations around the z axis is:

$$R_z(\phi) = \begin{pmatrix} e^{-\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix} = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Z$$

The X-axis rotation

The operator (gate) the generates rotations around the x axis is:

$$R_x(\phi) = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}X$$

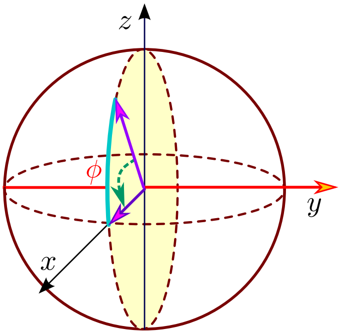


$$|\psi'\rangle = R_x(\phi)|\psi\rangle$$

The Y-axis rotation

The operator (gate) the generates rotations around the y axis is:

$$R_y(\phi) = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Y$$

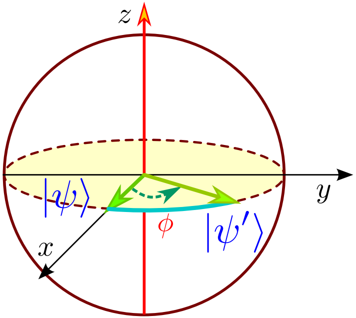


$$|\psi'\rangle = R_y(\phi)|\psi\rangle$$

The Z-axis rotation

The operator (gate) the generates rotations around the z axis is:

$$R_z(\phi) = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Z$$



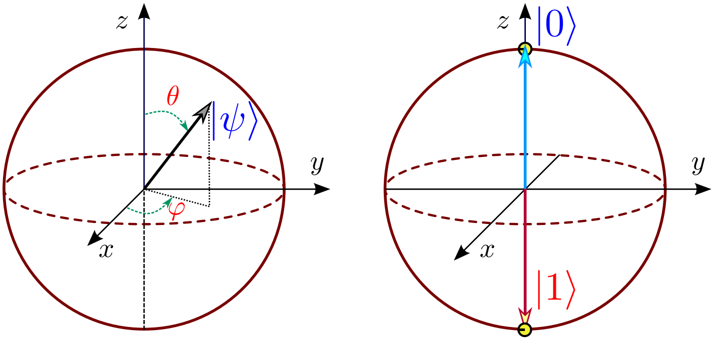
$$|\psi'\rangle = R_z(\phi)|\psi\rangle$$

The rotation gates - examples

Assume we have a qubit in the quantum state

$$\boxed{|\psi\rangle = |0\rangle}$$

Task: Rotate it around the z axis!



How to do it? Ideas?

The rotation gates - examples

So the quantum gate is actually the rotation operator

$$R_z\left(\phi\right) = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Z$$

We have to compute this new, rotated state,

$$|\psi'\rangle = R_z\left|\psi\right\rangle = R_z\left(\phi\right)|0\rangle = \cos\frac{\phi}{2}\mathbb{I}_2|0\rangle - i\sin\frac{\phi}{2}Z|0\rangle$$

So what is $\mathbb{I}_2|0\rangle$?
So what is $Z|0\rangle$?

The rotation gates - examples

So we have to compute (recall):

$$|\psi'\rangle = R_z(\phi)|0\rangle = \cos\frac{\phi}{2}\underbrace{\mathbb{I}_2}_{\text{matrix}}|0\rangle - i\sin\frac{\phi}{2}\underbrace{Z}_{\text{matrix}}|0\rangle$$

So, two (matrix) operations are on our task list:

$$\underbrace{\mathbb{I}_2}_{\text{matrix}}\underbrace{|0\rangle}_{\text{vector}} = |0\rangle$$

(recall the identity matrix) and

$$\underbrace{Z}_{\text{matrix}}\underbrace{|0\rangle}_{\text{vector}}$$

Okay, what yields $Z|0\rangle$?

Recall, we have the result: $\boxed{Z|0\rangle = |0\rangle}$.

The rotation gates - examples

So we had to compute:

$$|\psi'\rangle = R_z(\phi)\,|0\rangle = \cos\frac{\phi}{2}\mathbb{I}_2|0\rangle - i\sin\frac{\phi}{2}Z|0\rangle$$

and we found the result $\boxed{Z|0\rangle = |0\rangle}$. It is thus obvious that we find

$$|\psi'\rangle = \left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)|0\rangle$$

and this is (recall Euler's formula)

$$\boxed{|\psi'\rangle = e^{-i\frac{\phi}{2}}|0\rangle}$$

The rotation gates - examples

How about

$$R_z\left(\phi\right)|1\rangle?$$

So we have to compute:

$$|\psi'\rangle = R_z\left(\phi\right)|1\rangle = \cos\frac{\phi}{2}\mathbb{I}_2|1\rangle - i\sin\frac{\phi}{2}Z|1\rangle$$

Ok, so let's compute $Z|1\rangle$. We start from:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the Z gate:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The rotation gates - examples

We want to compute $Z|1\rangle$, we thus have the matrix multiplication:

$$Z\,|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\,|1\rangle$$

So we have that $\boxed{Z|1\rangle = -|1\rangle}$. Good, we had to compute:

$$|\psi'\rangle = R_z\,(\phi)\,|1\rangle = \cos\frac{\phi}{2}\mathbb{I}_2|1\rangle - i\sin\frac{\phi}{2}Z|1\rangle$$

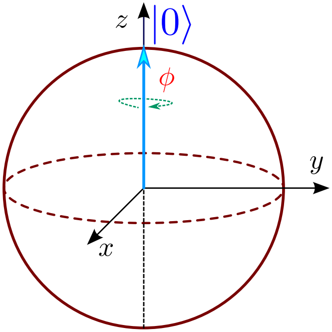
It is thus obvious that we find

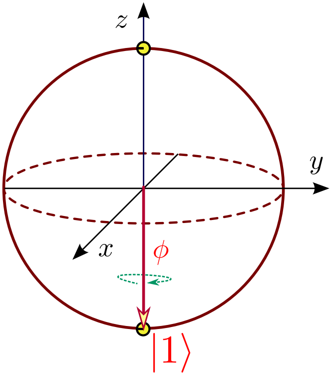
$$|\psi'\rangle = \left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)|1\rangle = e^{i\frac{\phi}{2}}|1\rangle$$

The rotation gates - examples

So we found $R_z(\phi)\,|0\rangle = e^{-\frac{i\phi}{2}}\,|0\rangle$

and similarly $R_z(\phi)\,|1\rangle = e^{\frac{i\phi}{2}}\,|1\rangle$





Conclusion: rotating $|0\rangle/|1\rangle$ around the z axis did nothing to our qubit.

The rotation gates - examples

But what if we rotate

$$|+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right)$$

around the z axis ? I. e. we want

$$R_z(\phi) \, |+\rangle .$$

Okay. Recall that we already computed:

$$\begin{cases} R_z(\phi) \, |0\rangle = e^{-\frac{i\phi}{2}} \, |0\rangle \\ R_z(\phi) \, |1\rangle = e^{\frac{i\phi}{2}} \, |1\rangle \end{cases}$$

So we need to add them and divide by $\sqrt{2}$.

The rotation gates - examples

So we have

$$R_z(\phi)|0\rangle = e^{-\frac{i\phi}{2}}|0\rangle$$

$$R_z(\phi)|1\rangle = e^{\frac{i\phi}{2}}|1\rangle$$

and adding them up yields

$$R_z(\phi)(|0\rangle + |1\rangle) = e^{-\frac{i\phi}{2}}|0\rangle + e^{\frac{i\phi}{2}}|1\rangle$$

now divide by $\sqrt{2}$,

$$R_z(\phi)\underbrace{\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)}_{=|+\rangle} = \frac{e^{-\frac{i\phi}{2}}|0\rangle + e^{\frac{i\phi}{2}}|1\rangle}{\sqrt{2}}$$

The rotation gates - rotation around the z axis

So we have

$$R_z(\phi)\left|+\right\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\phi}{2}}\left|0\right\rangle + e^{i\frac{\phi}{2}}\left|1\right\rangle\right) = \frac{1}{\sqrt{2}}e^{-i\frac{\phi}{2}}\left(\left|0\right\rangle + e^{i\phi}\left|1\right\rangle\right)$$

Very important remark: only global phases are irrelevant. But: relative phases are relevant!

$$R_z(\phi)\left|+\right\rangle = \frac{1}{\sqrt{2}}\underbrace{e^{-i\frac{\phi}{2}}}_{\text{this is irrelevant}}\left(\left|0\right\rangle + \underbrace{e^{i\phi}}_{\text{this is important}}\left|1\right\rangle\right)$$

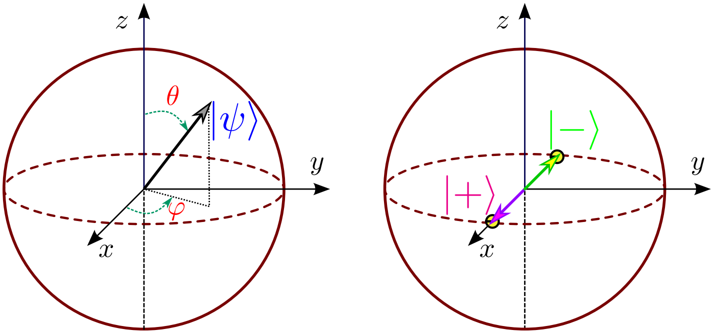
so, in its full glory:

$$\boxed{R_z(\phi)\left|+\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + e^{i\phi}\left|1\right\rangle\right)}$$

The rotation gates - rotation around the z axis

Set the rotation angle: $\phi = \pi$. We have:

$$R_z(\pi)\,|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$



The rotation gates - exercises

Exercise 1:

$R_z\left(\frac{\pi}{2}\right)|+\rangle = ?$ Recall: $R_z(\phi) = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Z$

Exercise 2:

$R_y\left(\frac{\pi}{2}\right)|0\rangle = ?$ Recall: $R_y(\phi) = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Y$

Exercise 3:

$R_y(\pi)|0\rangle = ?$

The rotation gates - Exercise 1

Exercise 1:

$R_z\left(\frac{\pi}{2}\right)|+\rangle =?$

Solution: So we have the Z-axis rotation operator

$$R_z(\phi) = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Z$$

thus

$$R_z\left(\frac{\pi}{2}\right)|+\rangle = \left(\cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Z\right)|+\rangle = \cos\frac{\phi}{2}|+\rangle - i\sin\frac{\phi}{2}Z|+\rangle$$

Now it is easy to see that $\boxed{Z|+\rangle = |-\rangle}$. If you don't believe me, let's compute:

$$\frac{1}{\sqrt{2}}Z\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\left(Z|0\rangle + Z|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

The rotation gates - Exercise 1

So we have

$$R_z\left(\frac{\pi}{2}\right)|+\rangle = \cos\frac{\phi}{2}|+\rangle - i\sin\frac{\phi}{2}|-\rangle$$

and expressing it in respect with the $\{|0\rangle, |1\rangle\}$ basis we have:

$$R_z\left(\frac{\pi}{2}\right)|+\rangle = \cos\frac{\phi}{2}\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) - i\sin\frac{\phi}{2}\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

so we can group some terms to have

$$R_z\left(\frac{\pi}{2}\right)|+\rangle = \frac{1}{\sqrt{2}}\underbrace{\left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)}_{\text{Euler's formula}=e^{-i\phi/2}}|0\rangle + \frac{1}{\sqrt{2}}\underbrace{\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)}_{\text{Euler's formula}=e^{i\phi/2}}|1 = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\phi}{2}}|0\rangle + e^{i\frac{\phi}{2}}|1\right)$$

The rotation gates - Exercise 1

So

$$R_z\left(\frac{\pi}{2}\right)|+\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\phi}{2}}|0\rangle + e^{i\frac{\phi}{2}}|1\rangle\right) = \frac{e^{-i\frac{\phi}{2}}}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)$$

and thus we remember this (Note: we gracefully disregard the global phase):

$$\boxed{R_z(\phi)|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)}$$

Now, in the exercise, the angle is $\phi = \pi/2$ so we have

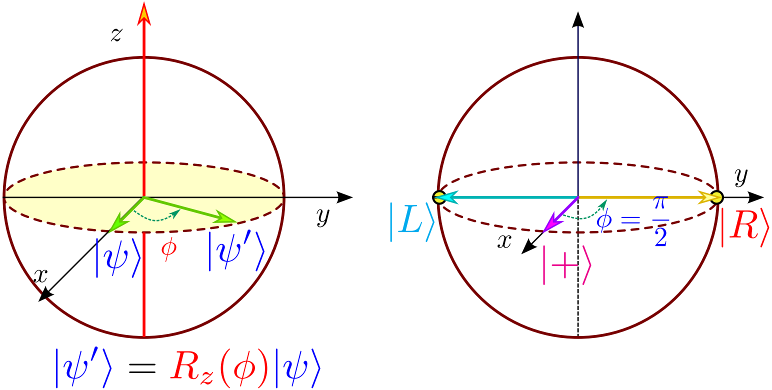
$$R_z\left(\frac{\pi}{2}\right)|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) = |R\rangle = |i\rangle$$

The rotation gates - Exercise 1

So we found out that

$$R_z(\pi/2)\,|+\rangle = |R\rangle$$

We can visualize this operation:



The rotation gates - exercises

Exercise 2:

$R_y\left(\frac{\pi}{2}\right)|0\rangle =?$

Ideas?

Suggestions how to calculate it?

Recall that the Y operator (gate) was defined by

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The rotation gates - Exercise 2

Exercise 2:

$R_y\left(\frac{\pi}{2}\right)|0\rangle = ?$

Solution: The operator (gate) the generates rotations around the y axis is:

$$R_y(\phi) = \cos\frac{\phi}{2}\mathbb{I}_2 - i\sin\frac{\phi}{2}Y$$

So we have

$$R_y(\phi)|0\rangle = \cos\frac{\phi}{2}\mathbb{I}_2|0\rangle - i\sin\frac{\phi}{2}Y|0\rangle = \cos\frac{\phi}{2}|0\rangle - i\sin\frac{\phi}{2}Y|0\rangle$$

What is $Y|0\rangle$? Let's compute it!

The rotation gates - Exercise 2

The Y operator (gate) is defined by

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{thus} \quad Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$$

So we have

$$\boxed{Y|0\rangle = i|1\rangle}$$

Don't get carried away, recall that we actually want to find out

$$R_y(\phi)|0\rangle = \cos\frac{\phi}{2}|0\rangle - i\sin\frac{\phi}{2}\underbrace{Y|0\rangle}_{=i|1\rangle}$$

and now it is obvious ($ii = i^2 = -1$, right?) that

$$\boxed{R_y(\phi)|0\rangle = \cos\frac{\phi}{2}|0\rangle + \sin\frac{\phi}{2}|1\rangle}$$

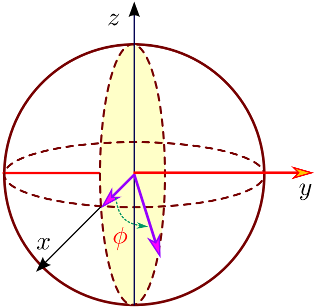
Now choose your ϕ!

The rotation gates - Exercise 2

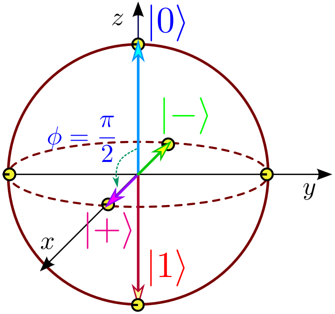
Setting $\phi = \pi/2$ yields

$$R_y\left(\pi/2\right)|0\rangle = \cos\frac{\pi}{4}|0\rangle + \sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$$

We can visualize this operation:



$$|\psi'\rangle = R_y(\phi)|\psi\rangle$$



The rotation gates - exercises

Exercise 3:

$R_y(\pi)\,|0\rangle = ?$

This should be...

...quite easy by now.

The rotation gates - Exercise 3

Exercise 3:

$R_y(\pi)\,|0\rangle = ?$

Solution: Recall that we obtained previously:

$$R_y(\phi)\,|0\rangle = \cos\frac{\phi}{2}|0\rangle + \sin\frac{\phi}{2}|1\rangle$$

Now put $\phi = \pi$. We have

$$R_y(\pi)\,|0\rangle = \cos\frac{\pi}{2}|0\rangle + \sin\frac{\pi}{2}|1\rangle = |1\rangle$$

Can you visualize this on the Bloch sphere?

We have the 4 the Bell states

We have the 4 maximally entangled i. e. Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Claim: any 2-qubit state can be written in the Bell basis

$$|\psi\rangle = c_{z+}|\Phi^+\rangle + c_{z-}|\Phi^-\rangle + c_{x+}|\Psi^+\rangle + c_{x-}|\Psi^-\rangle$$

Now stop for a moment

Claim: any 2-qubit state can be written in the Bell basis

$$|\psi\rangle = c_{z+}|\Phi^+\rangle + c_{z-}|\Phi^-\rangle + c_{x+}|\Psi^+\rangle + c_{x-}|\Psi^-\rangle$$

Your first thought is probably:

why on Earth would anyone want to do that?
(Legit thought!)

Short answer:

Because you just did a Bell state measurement on a state. When you do that, you actually *rewrite* the state in the Bell basis.
(Happy?)

Now stop for a moment

So, as I said, your first thought was probably:

why on Earth would anyone want to do that?

Recall that we have defined the general two qubit state as

$$\boxed{|\psi\rangle = c_0\,|00\rangle + c_1\,|01\rangle + c_2\,|10\rangle + c_3\,|11\rangle}$$

with $c_0, c_1, c_2, c_3 \in \mathbb{C}$ and for normalization i. e. $\langle\psi|\psi\rangle = 1$ we must have

$$|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$

$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is indeed a basis because any 2-qubit state can be expressed using these 4 elements.

The “normal” 2 qubit basis

How do you determine c_0, c_1, c_2 and c_3?
Well, assume that you have $|\psi\rangle$ and all you know is that it is a 2-qubit state.
Then, do the following operations:

$$\langle 00|\psi\rangle = c_0$$

$$\langle 01|\psi\rangle = c_1$$

$$\langle 10|\psi\rangle = c_2$$

$$\langle 11|\psi\rangle = c_3$$

So, finding the c_0, c_1, c_2 and c_3 coefficients in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is not so hard.

The Bell states as a basis

We can also turn around the relations to get:

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right)$$

$$|11\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle - |\Phi^-\rangle\right)$$

$$|01\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle + |\Psi^-\rangle\right)$$

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle - |\Psi^-\rangle\right)$$

Note: we did really basic manipulations to get these results.

Alice has a qubit. An EPR pair is also created.

Alice has a qubit (Congratulations Alice!). Her qubit reads

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

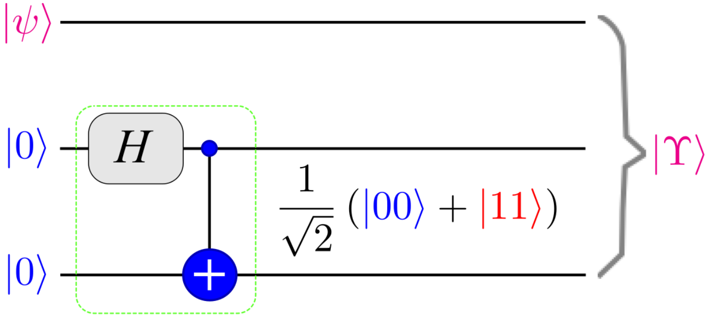
Alice (or somebody else, say Charlie) creates an entangled EPR pair (now you know how)

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

The total state (qubit plus EPR pair - 3 qubits, right?) is

$$|\Upsilon\rangle = |\psi\rangle \otimes |\Phi^+\rangle = |\psi\rangle|\Phi^+\rangle = \frac{1}{\sqrt{2}}\underbrace{\left(c_0|0\rangle + c_1|1\rangle\right)}_{\text{the qubit}}\underbrace{\left(|00\rangle + |11\rangle\right)}_{\text{the EPR state}}$$

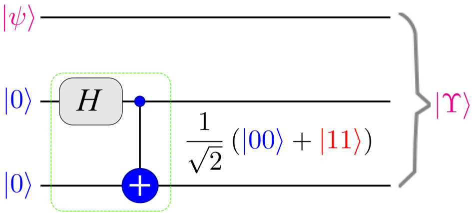
Alice's qubit + the EPR pair.



The total state

$$|\Upsilon\rangle = |\psi\rangle \otimes |\Phi^+\rangle = |\psi\rangle|\Phi^+\rangle = \frac{1}{\sqrt{2}} \underbrace{\left(c_0|0\rangle + c_1|1\rangle\right)}_{\text{the qubit}} \underbrace{\left(|00\rangle + |11\rangle\right)}_{\text{the EPR state}}$$

Alice has a qubit. And an EPR pair.



So the total state is $|\Upsilon\rangle = |\psi\rangle \otimes |\Phi^+\rangle = |\psi\rangle|\Phi^+\rangle = \frac{1}{\sqrt{2}} \underbrace{\left(c_0|0\rangle + c_1|1\rangle\right)}_{\text{the qubit}} \underbrace{\left(|00\rangle + |11\rangle\right)}_{\text{the EPR state}}$

and we simply expand it as

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}} \left(c_0|000\rangle + c_0\,|011\rangle + c_1\,|100\rangle + c_1\,|111\rangle\right)$$

So far, so good.

Alice rewrites her state

The total state (qubit plus EPR pair) is

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}}\left(c_0|000\rangle + c_0\,|011\rangle + c_1\,|100\rangle + c_1\,|111\rangle\right)$$

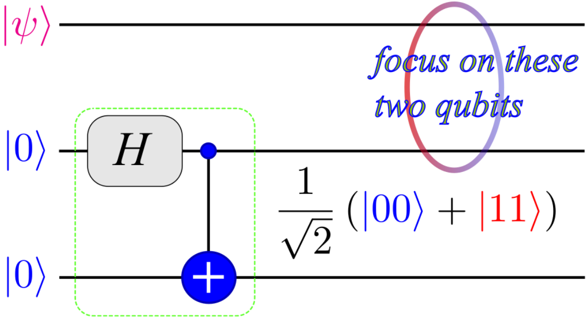
this can also be written

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}}\Bigg(c_0 \underbrace{|00\rangle}_{\text{her qubit plus one of the EPR pair}} \underbrace{|0\rangle}_{\text{qubit goes to Bob}} + c_0\,|01\rangle \underbrace{|1\rangle}_{\text{qubit goes to Bob}} + c_1\,|10\rangle \underbrace{|0\rangle}_{\text{qubit goes to Bob}} + c_1\,|11\rangle \underbrace{|1\rangle}_{\text{qubit goes to Bob}}\Bigg)$$

and putting the last term $(c_1\,|11\rangle\,|1\rangle)$ in the second position:

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}}\left(c_0|00\rangle\,|0\rangle + c_1\,|11\rangle\,|1\rangle + c_0\,|01\rangle\,|1\rangle + c_1\,|10\rangle\,|0\rangle\right)$$

Alice rewrites her state - and focuses on the upper 2 qubits



So the global state is:

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}}\left(c_0|00\rangle\,|0\rangle + c_1\,|11\rangle\,|1\rangle + c_0\,|01\rangle\,|1\rangle + c_1\,|10\rangle\,|0\rangle\right)$$

Alice rewrites even more her state

Alice is smart so she recalls that:

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right)$$

$$|11\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle - |\Phi^-\rangle\right)$$

$$|01\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle + |\Psi^-\rangle\right)$$

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle - |\Psi^-\rangle\right)$$

So she rewrites the (global) state

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}}\left(c_0|00\rangle\,|0\rangle + c_1\,|11\rangle\,|1\rangle + c_0\,|01\rangle\,|1\rangle + c_1\,|10\rangle\,|0\rangle\right)$$

with $|\Phi^+\rangle|/|\Phi^-\rangle/\Psi^+\rangle/|\Psi^-\rangle$. (Now this will hurt!)

Alice rewrites even more her state

So, Alice rewrites the (global) state

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}} \left(c_0 |00\rangle \, |0\rangle + c_1 \, |11\rangle \, |1\rangle + c_0 \, |01\rangle \, |1\rangle + c_1 \, |10\rangle \, |0\rangle \right)$$

and replaces her qubits with the Bell basis. She gets

$$|\Upsilon\rangle = \frac{1}{2} \Bigg(c_0 \Big(\underbrace{|\Phi^+\rangle + |\Phi^-\rangle}_{\substack{\text{this is } |00\rangle, \\ \text{right?}}} \Big) |0\rangle + c_1 \Big(\underbrace{|\Phi^+\rangle - |\Phi^-\rangle}_{\substack{\text{this is } |11\rangle, \\ \text{right?}}} \Big) |1\rangle$$

$$+ c_0 \Big(\underbrace{|\Psi^+\rangle + |\Psi^-\rangle}_{\substack{\text{this is } |01\rangle, \\ \text{right?}}} \Big) |1\rangle + c_1 \Big(\underbrace{|\Psi^-\rangle - |\Psi^-\rangle}_{\substack{\text{this is } |10\rangle, \\ \text{right?}}} \Big) |0\rangle \Bigg)$$

Horrible! Whatever, let's continue.

Alice rewrites even more her state

So we had

$$|\Upsilon\rangle = \frac{1}{2}\Big(c_0\big(\underbrace{|\Phi^+\rangle + |\Phi^-\rangle}_{=|00\rangle}\big)|0\rangle + c_1\big(\underbrace{|\Phi^+\rangle - |\Phi^-\rangle}_{=|11\rangle}\big)|1\rangle$$
$$+c_0\big(\underbrace{|\Psi^+\rangle + |\Psi^-\rangle}_{=|01\rangle}\big)|1\rangle + c_1\big(\underbrace{|\Psi^-\rangle - |\Psi^-\rangle}_{=|10\rangle}\big)|0\rangle\Big)$$

or, expanding all terms,

$$|\Upsilon\rangle = \frac{1}{2}\big(|\Phi^+\rangle c_0|0\rangle + |\Phi^-\rangle c_0|0\rangle + |\Phi^+\rangle c_1|1\rangle - |\Phi^-\rangle c_1|1\rangle$$
$$+|\Psi^+\rangle c_0|1\rangle + |\Psi^-\rangle c_0|1\rangle + |\Psi^-\rangle c_1|0\rangle - |\Psi^-\rangle c_1|0\rangle\big)$$

Question

Do you notice something?

Alice notices something

Changing the order of some terms takes us to

$$|\Upsilon\rangle = \frac{1}{2}\big(|\Phi^+\rangle c_0\,|0\rangle + |\Phi^+\rangle c_1\,|1\rangle + |\Phi^-\rangle c_0\,|0\rangle - |\Phi^-\rangle c_1\,|1\rangle \\ +|\Psi^+\rangle c_0\,|1\rangle + |\Psi^-\rangle c_1\,|0\rangle + |\Psi^-\rangle c_0\,|1\rangle - |\Psi^-\rangle c_1\,|0\rangle\big)$$

But **wait a minute**, this is something like

$$|\Upsilon\rangle = \frac{1}{2}\big(|\Phi^+\rangle\,(c_0\,|0\rangle + c_1\,|1\rangle) + |\Phi^-\rangle\,(c_0\,|0\rangle - c_1\,|1\rangle) \\ +|\Psi^+\rangle\,(c_0\,|1\rangle + c_1\,|0\rangle) + |\Psi^-\rangle\,(c_0\,|1\rangle - c_1\,|0\rangle)\big)$$

Question

Now do you notice something?

Alice notices something

Of course you do, this is actually

$$|\Upsilon\rangle = \frac{1}{2}\Bigg(|\Phi^+\rangle \underbrace{(c_0\,|0\rangle + c_1\,|1\rangle)}_{=|\psi\rangle} + |\Phi^-\rangle \underbrace{(c_0\,|0\rangle - c_1\,|1\rangle)}_{=Z|\psi\rangle}$$

$$+ |\Psi^+\rangle \underbrace{(c_0\,|1\rangle + c_1\,|0\rangle)}_{=X|\psi\rangle} + |\Psi^-\rangle \underbrace{(c_0\,|1\rangle - c_1\,|0\rangle)}_{=XZ|\psi\rangle}\Bigg)$$

Can you seen now what we've got?

Alice is ready to teleport

From the state

$$|\Upsilon\rangle = \frac{1}{2}\Big(|\Phi^+\rangle|\psi\rangle + |\Phi^-\rangle Z|\psi\rangle + |\Psi^+\rangle X|\psi\rangle + |\Psi^-\rangle XZ|\psi\rangle\Big)$$

Alice knows what to do.

Alice is ready to teleport

- Do a Bell state measurement.
- If you identify your state as $|\Phi^+\rangle$, then send to Bob on a classical channel (or quantum if you wish) "00" or "do nothing".
- If you measure $|\Phi^-\rangle$, then send to Bob "01" or "apply Z".
- etc.

The Quantum teleportation scheme