Homework 3

Due: March 14th, 2025 (in class)

Problem 1

Use implicit differentiation to find the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ implied by the relationship:

$$F(x_1, x_2, y) = 3x_1x_2 + x_2y^2 + x_1^2x_2y - 10 = 0$$

Solution

First, we find

$$\begin{aligned}
F_{x_1} &= 3x_2 + 2x_1x_2y \\
F_{x_2} &= 3x_1 + y^2 + x_1^2y \\
F_y &= 2x_2y + x_1^2x_2
\end{aligned}$$

According to the implicit function theorem we can perceive this relationship as a function $y = f(x_1, x_2)$ within any neighborhood of a point, provided that $F_y \neq 0$ at that point. So, at any point where $F_y \neq 0$, we have

$$\frac{\partial y}{\partial x_1} = -\frac{F_{x_1}}{F_y} = -\frac{(3x_2 + 2x_1x_2y)}{(2x_2y + x_1^2x_2)}$$

and

$$\frac{\partial y}{\partial x_2} = -\frac{F_{x_2}}{F_y} = -\frac{(3x_1 + y^2 + x_1^2y)}{(2x_2y + x_1^2x_2)} \quad \blacksquare$$

Problem 2

Consider a simple Cournot duopoly model, in which the inverse demand for a good is

$$P(q) = q^{-1/\eta}$$

and the two firms producing the good face cost functions

$$C_i(q_i) = \frac{1}{2}c_iq_i^2, \quad \text{for } i = 1, 2$$

1. Write the profit for firm i
2. Illustrate the first order conditions of the equilibrium output level

Solution

1. Write the profit for firm i

The profit for firm i is given by:

$$\begin{aligned}\pi_i(q_i, q_j) &= q_i \cdot P(q) - C_i(q_i) \\ &= q_i \cdot (q_1 + q_2)^{-1/\eta} - \frac{1}{2} c_i q_i^2\end{aligned}$$

Where $q = q_1 + q_2$ represents the total output in the market.

2. Illustrate the first order conditions of the equilibrium output level

The first order condition for firm i is found by taking the derivative of the profit function with respect to q_i and setting it equal to zero:

$$\begin{aligned}\frac{\partial \pi_i}{\partial q_i} &= (q_1 + q_2)^{-1/\eta} + q_i \cdot \left(-\frac{1}{\eta}\right)(q_1 + q_2)^{-1/\eta - 1} - c_i q_i \\ &= (q_1 + q_2)^{-1/\eta} - \frac{q_i}{\eta}(q_1 + q_2)^{-1/\eta - 1} - c_i q_i \\ &= (q_1 + q_2)^{-1/\eta}\left(1 - \frac{q_i}{\eta(q_1 + q_2)}\right) - c_i q_i = 0\end{aligned}$$

For the Nash equilibrium, both firms' first order conditions must be satisfied simultaneously:

$$\begin{aligned}(q_1 + q_2)^{-1/\eta}\left(1 - \frac{q_1}{\eta(q_1 + q_2)}\right) - c_1 q_1 = 0 \\ (q_1 + q_2)^{-1/\eta}\left(1 - \frac{q_2}{\eta(q_1 + q_2)}\right) - c_2 q_2 = 0\end{aligned}$$

These equations constitute a system of nonlinear equations that define the equilibrium quantities q_1^* and q_2^*.

Problem 3

For each of the following functions, show whether it is convex, concave, or neither.

1. $f(x_1, x_2) = x_1^2 + 3x_1x_2 + 2x_2^2$.
2. $f(x_1, x_2) = e^{x_1} + e^{x_2}$.
3. $f(x_1, x_2) = \log(e^{x_1} + e^{x_2})$,

Solution

1. $f(x_1, x_2) = x_1^2 + 3x_1x_2 + 2x_2^2$

Computing the partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= 2x_1 + 3x_2 \\ \frac{\partial f}{\partial x_2} &= 3x_1 + 4x_2 \\ \frac{\partial^2 f}{\partial x_1^2} &= 2 \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} &= 3\end{aligned}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 3$$
$$\frac{\partial^2 f}{\partial x_2^2} = 4$$

The Hessian matrix is:

$$H = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

Determinant of $H = 2 \cdot 4 - 3^2 = 8 - 9 = -1 < 0$.

Since the determinant is negative and the first-order principal minor is positive, the Hessian is indefinite. Therefore, the function is **neither convex nor concave**.

2. $f(x_1, x_2) = e^{x_1} + e^{x_2}$

Computing the partial derivatives:

$$\frac{\partial f}{\partial x_1} = e^{x_1}$$
$$\frac{\partial f}{\partial x_2} = e^{x_2}$$
$$\frac{\partial^2 f}{\partial x_1^2} = e^{x_1} > 0$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$
$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$$
$$\frac{\partial^2 f}{\partial x_2^2} = e^{x_2} > 0$$

The Hessian matrix is: $H = \begin{pmatrix} e^{x_1} & 0 \\ 0 & e^{x_2} \end{pmatrix}$

The eigenvalues are e^{x_1} and e^{x_2}, both of which are positive for all $x_1, x_2 \in \mathbb{R}$. Therefore, the Hessian is positive definite, and the function is **convex**.

3. $f(x_1, x_2) = \log(e^{x_1} + e^{x_2})$

Computing the partial derivatives:

$$\frac{\partial f}{\partial x_1} = \frac{e^{x_1}}{e^{x_1} + e^{x_2}}$$
$$\frac{\partial f}{\partial x_2} = \frac{e^{x_2}}{e^{x_1} + e^{x_2}}$$

Let's define $p_1 = \frac{e^{x_1}}{e^{x_1}+e^{x_2}}$ and $p_2 = \frac{e^{x_2}}{e^{x_1}+e^{x_2}}$, noting that $p_1 + p_2 = 1$.

Second derivatives:

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{e^{x_1}(e^{x_1} + e^{x_2}) - e^{x_1} \cdot e^{x_1}}{(e^{x_1} + e^{x_2})^2} = \frac{e^{x_1} \cdot e^{x_2}}{(e^{x_1} + e^{x_2})^2} = p_1 p_2 > 0$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{e^{x_1} \cdot e^{x_2}}{(e^{x_1} + e^{x_2})^2} = -p_1 p_2 < 0$$
$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -\frac{e^{x_1} \cdot e^{x_2}}{(e^{x_1} + e^{x_2})^2} = -p_1 p_2 < 0$$
$$\frac{\partial^2 f}{\partial x_2^2} = \frac{e^{x_2} \cdot e^{x_1}}{(e^{x_1} + e^{x_2})^2} = p_1 p_2 > 0$$

The Hessian matrix is: $H = \begin{pmatrix} p_1 p_2 & -p_1 p_2 \\ -p_1 p_2 & p_1 p_2 \end{pmatrix}$

The determinant of H is $\det(H) = (p_1 p_2)^2 - (p_1 p_2)^2 = 0$, which means one eigenvalue is 0. The trace is $\mathrm{tr}(H) = 2p_1 p_2 > 0$, so the other eigenvalue is positive.

Since one eigenvalue is 0 and the other is positive, the Hessian is positive semidefinite. The function is **convex**.

Problem 4

Let $f(x_1, x_2) = x_1^2 - x_1x_2 + 2x_2^2 - x_1 - 3x_2$.

1. Compute the gradient and the Hessian of f.
2. Determine whether f is convex, concave, or neither.
3. Find the stationary point(s) of f.
4. Determine whether each stationary point is a maximum, minimum, or neither.

Solution

Let $f(x_1, x_2) = x_1^2 - x_1x_2 + 2x_2^2 - x_1 - 3x_2$.

1. Compute the gradient and the Hessian of f.

   First, let's compute the partial derivatives to find the gradient:

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^2 - x_1x_2 + 2x_2^2 - x_1 - 3x_2) \tag{1}$$
$$= 2x_1 - x_2 - 1 \tag{2}$$
$$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2}(x_1^2 - x_1x_2 + 2x_2^2 - x_1 - 3x_2) \tag{3}$$
$$= -x_1 + 4x_2 - 3 \tag{4}$$

   Therefore, the gradient of f is:

$$\nabla f(x_1, x_2) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 - 1 \\ -x_1 + 4x_2 - 3 \end{pmatrix} \tag{5}$$

   Now, let's compute the second-order partial derivatives to find the Hessian matrix:

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial}{\partial x_1}(2x_1 - x_2 - 1) = 2 \tag{6}$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial}{\partial x_2}(2x_1 - x_2 - 1) = -1 \tag{7}$$
$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right) = \frac{\partial}{\partial x_1}(-x_1 + 4x_2 - 3) = -1 \tag{8}$$
$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_2}\right) = \frac{\partial}{\partial x_2}(-x_1 + 4x_2 - 3) = 4 \tag{9}$$

   Therefore, the Hessian matrix of f is:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \tag{10}$$

2. Determine whether f is convex, concave, or neither.

   Let's check if the Hessian matrix is positive definite by computing the determinants of all leading principal minors:

   First leading principal minor:

$$\det\big((2)\big) = 2 > 0 \tag{11}$$

Second leading principal minor (which is the determinant of the entire Hessian):

$$\det(H) = \det\left(\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}\right) = 2 \cdot 4 - (-1) \cdot (-1) = 8 - 1 = 7 > 0 \tag{12}$$

Since all leading principal minors are positive, the Hessian matrix is positive definite at all points. Therefore, the function f is strictly convex.

3. Find the stationary point(s) of f.

   To find the stationary points, we set the gradient equal to zero:

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 - x_2 - 1 \\ -x_1 + 4x_2 - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

   This gives us the system of equations:

$$2x_1 - x_2 - 1 = 0 \tag{14}$$
$$-x_1 + 4x_2 - 3 = 0 \tag{15}$$

   Therefore, the only stationary point of f is $(1, 1)$.

4. Determine whether each stationary point is a maximum, minimum, or neither.

   We've already shown that the Hessian matrix $H = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$ is positive definite. Therefore, the stationary point $(1, 1)$ is indeed a global minimum of the function f.