2025 年数理经济学笔记

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第 1 章 Comparative Statics and Envelope Theorem

Keywords

- Generalized Comparative Statics 广义比较静态
- Cramer's Rule 克拉默法则
- Envelope Theorem 包络定理

1.1 Comparative Statics

笔记 经济学中的比较静态分析是指在给定一个经济模型的情况下, 研究当 **exogenous variables** 发生变化时, **endogenous variables** 的值如何变化. 比如当给定消费者收入去讨论市场供需模型中的均衡价格, 给定税率去讨论对 Monopoly 的影响, 前者作为外生变量 (经济模型的输入), 后者作为内生变量 (经济模型的输出).

定义 1.1 (Generalized Comparative Statics)

We have an economic model, the equilibrium solution of which is given by the form:

$$F(x^*, \alpha) = 0$$

where x^* is the equilibrium solution of endogenous variables x, and α is a vector of exogenous variables. The key objective is to find the derivative $\frac{\partial x_i^*}{\partial \alpha_j}$ and identify its sign. ♣

笔记 这里向量函数 f 的个数应当与内生变量的个数相同, 设为 n.

定理 1.1 (Cramer's Rule)

Let $F(x^*(\alpha), \alpha) = 0$ be a system of n equations in n unknowns. The Jacobian matrix of the system is given by:

$$\det J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

We have:

$$J\frac{\partial x^*}{\partial \alpha_j} + \frac{\partial F}{\partial \alpha_j} = 0, \forall j$$

The Cramer's Rule states that the derivative of the equilibrium solution with respect to the exogenous variable α_j is given by:

$$\frac{\partial x_i^*}{\partial \alpha_j} = -\frac{\det J_{ij}}{\det J}$$

where J_{ij} is the matrix obtained by replacing the i-th column of J with the vector $\frac{\partial F}{\partial \alpha_j}$. ♡

1.1.1 Comparative Statics for Unconstrained Optimization

这时我们考虑一个最优化问题, 其形式为:

$$\max_x f(x; a)$$

其中 $f : \mathbb{R}^n \times R^m \to \mathbb{R}$ 是一个可微函数[1]. m 以下视为 1 (不考虑外生变量之间的影响).

[1] 请注意 $\nabla f = F$: 这里 $\nabla f = 0$ 是 1.1.1 小节中函数的 FOC, F 是广义比较静态的模型函数, 我们不是第一次用类似的记号

FOC $\nabla f = 0$ 可以这样写:

$$\frac{\partial f}{\partial x_i}(x_1^*, \ldots, x_n^*; a) = 0, \forall i$$

$f(\cdot)$ 的 FOC 的 Jacobian 矩阵也就是 $f(\cdot)$ 的 Hessian 矩阵:

$$\det J(x^*; a) = \frac{\partial^2 f}{\partial x^2}(x^*; a)$$

命题 1.1 (Implict Function Theorem)

If $\det J(x^*; a) \neq 0$, then the system implicitly defines differentiable functions:

$$x_i^* : a \to x_i^*(a), \forall i$$

And the derivatives of these functions are given by[a]:

$$\frac{\partial x_i^*}{\partial a} = -\frac{\det J_i(x^*; a)}{\det J(x^*; a)}$$

where J_i is the matrix obtained by replacing the i-th column of J with the vector $\frac{\partial f}{\partial a}$.

[a]这个式子和上面的 Cramer's Rule 的结论是一样的, 区别是在优化问题中, 我们将向量函数指定为被优化函数的梯度, 那么 Jacobian 矩阵成为了一个特例, 也就是 Hessian 矩阵.

笔记 隐函数定理揭示了二阶条件与比较静态的存在性的关联

1.1.2 Comparative Statics for Equality Constrained Optimization

在有约束优化中, 我们先做拉格朗日再用同样的隐函数定理方法来进行比较分析, 实际上还是一样的, 因为要对 Lagrangian 函数求一阶条件.

$$\begin{aligned}
&L = f + \lambda h \\
&\frac{\partial L}{\partial \lambda}(x^*, \lambda^*; a) = h(x^*; a) = 0 \\
&\frac{\partial L}{\partial x_i}(x^*, \lambda^*; a) = \frac{\partial f}{\partial x_i}(x^*; a) + \lambda^* \frac{\partial h}{\partial x_i}(x^*; a) = 0 \quad \text{for } i = 1, 2, ..., n
\end{aligned}$$

Jacobian 是 Bordered Hessian 矩阵:

$$\det J_L(x^*; a) = \begin{bmatrix} 0 & \frac{\partial h}{\partial x}(x^*; a) \\ \frac{\partial h}{\partial x}(x^*; a) & \frac{\partial^2 f}{\partial x^2}(x^*; a) \end{bmatrix}$$

利用隐函数定理:

$$\frac{\partial x_i^*}{\partial a} = -\frac{\det J_i(x^*; a)}{\det J(x^*; a)}$$

其中 J_i 是通过将 J_L 的第 i 列替换为 $\frac{\partial L}{\partial a}$ 得到的矩阵.

1.2 Envelope Theorem

还是之前的最优化问题:

$$V(a) = \max_x f(x; a)$$

- $F : \mathbb{R}^n \times R^m \to \mathbb{R}$ is differentiable.
- a is an exogenous variable.
- $x^*(a)$ is a local solution with differentiable components $x_i^*(a) : \mathbb{R} \to \mathbb{R}$.

定理 1.2 (Envelope Theorem)

The derivative of the value function with respect to the exogenous variable a is given by:

$$
\begin{aligned}
\frac{\partial V(a)}{\partial a} &= \frac{\partial f}{\partial a}(x^*(a);a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x^*(a);a)\frac{\partial x_i^*(a)}{\partial a} \\
&= \underbrace{\frac{\partial f}{\partial a}}_{\text{直接影响}} + \underbrace{\nabla_x f \cdot \frac{\partial x^*(a)}{\partial a}}_{\text{间接影响}}
\end{aligned}
$$

where $\nabla_x f$ is the gradient of the objective function with respect to the endogenous variables, and $\frac{\partial x^*(a)}{\partial a}$ is the derivative of the equilibrium solution with respect to the exogenous variable a.

笔记 包络定理揭示了: 当外生参数变化时, 只需考虑该参数的直接影响, 而无需额外计算内生变量调整[2]带来的间接影响. 因为由一阶条件, $\nabla_x f$ 项在最优解处趋于零.

[2]如果是有约束优化, λ 的选取也包含在内