2025 年数理经济学笔记

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第 1 章 Linear Algebra[1]

内容提要

- Leading Principal Minor : 顺序主子式
- Orthogonal matrix : 正交矩阵
- Symmetric matrix : 对称矩阵
- Positive definite matrix : 正定矩阵
- Positive semi-definite matrix : 半正定矩阵
- Determinant : 行列式

定义 1.1

For $N \times N$ matrix $A = (a_{ij})$, using any row or column:

$$\det A = \sum_{i=1}^{N} (-1)^{i+j} a_{ij} \det A_{ij}$$

where A_{ij} is the $(N-1) \times (N-1)$ matrix obtained by deleting the i-th row and j-th column of A. ♣

定理 1.1

$$A^{-1} = \frac{1}{\det A} \tilde{A}$$

where $\tilde{a_{mn}} = (-1)^{m+n} \det A_{nm}$. ♡

定义 1.2

Orthogonal matrix : $P^T P = I$.

Symmetric matrix : $A^T = A$.

Positive definite matrix : $x^T A x > 0$ for all $x \neq 0$.

Positive semi-definite matrix : $x^T A x \geq 0$ for all x. ♣

定义 1.3

Leading Principal Minor : determinant of the first $k \times k$ submatrix of A. For real symmetric matrix A, A is positive definite if and only if all its leading principal minors are positive. ♣

定义 1.4

$$Av = \lambda v$$

where v is **eigenvector**, λ is **eigenvalue**. λ is a root of the **characteristic polynomial** $\det(A - \lambda I) = 0$. ♣

定义 1.5

Complex inner product :

$$\langle x, y \rangle = x^* y = \sum_{i=1}^{n} \bar{x}_i y_i$$

where $\bar{x}$ is the **complex conjugate** and x^* is the **conjugate transpose** (adjoint). ♣

[1] 只记一些矩阵分解吧, 以防忘了

定义 1.6

Hermitian matrix : $A^* = A$.

For real matrices, Hermitian matrix is symmetric.

For Hermitian matrix, all eigenvalues are real.

定理 1.2 (Diagonalization of Symmetric Matrices)

$$P^T AP = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$$

where P is orthogonal matrix, λ_i are eigenvalues of A.

第 2 章 Topology of $\mathbb{R}^N$[1]

Keywords

- Topology 拓扑
- Metric Space 度量空间
- Convergence 收敛
- interior 内部
- closure 闭包
- boundary 边界
- compact set 紧集
- cluster point 聚点

2.1 Metric Spaces

2.1.1 Definition of Metric Spaces

定义 2.1

Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** (or **distance**) on X if :

1. (positivity) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
2. (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

♣

A set X together with a metric d is called a **metric space**, denoted by (X, d).

2.1.2 Examples of metrics in $\mathbb{R}^N$

- **Euclidean metric**: $d(x, y) = \sqrt{\sum_{i=1}^N (x_i - y_i)^2}$.
- L^p **metric** (for $p \geq 1$): $d(x, y) = (\sum_{i=1}^N |x_i - y_i|^p)^{1/p}$.
- **Sup norm** (when $p = \infty$): $d(x, y) = \max_{i=1}^N |x_i - y_i|$.

2.2 Convergence of sequences

2.2.1 Definition of Convergence

定义 2.2

Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to **converge** to a point $x \in X$ if for every $\epsilon > 0$, there exists an integer N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$. A sequence that converges is called **convergent**, otherwise it is called **divergent**.

♣

定义 2.3

When metric space is $\mathbb{R}^N$, we say that $\{x_n\}$ is **bounded** if there exists a real number M such that $\|x_k\| \leq M$ for all n.

♣

[1] 点集拓扑对应数分高代这一级别的数学基础课, 只有 sms 和图班的同学学过, 所以我简单记一下, 无需关注证明, 数理经济学只用到结论

2.2.2 Cauchy Sequences and Complete Metric Spaces

- **Cauchy sequence**: A sequence $\{x_n\}$ in a metric space (X, d) is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists an integer N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.
- **Complete metric space**: A metric space (X, d) is called **complete** if every Cauchy sequence in X converges to a point in X.

定理 2.1

Any convergent sequence in a metric space is a Cauchy sequence. ♡

2.2.3 Example: Cauchy Sequence Not Convergent in $\mathbb{Q}$

Consider the metric space $(\mathbb{Q}, d)$, where $d(x, y) = |x - y|$.

Fibonacci sequence : Let $\{F_k\}$ be the Fibonacci sequence, defined by

$$F_1 = F_2 = 1, F_{k+1} = F_k + F_{k-1}, k \geq 2$$

A Special Sequence: Define $a_k = \frac{F_{k+1}}{F_k}$. Then $\{a_k\}$ is a Cauchy sequence in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$.

2.2.4 Properties of Convergent Sequences in $\mathbb{R}^N$

Consider $\mathbb{R}^N$ with the Euclidean metric. Let $\{x_n\}$ and $\{y_n\}$ be two sequences.

- **Preservation of Addition/Subtraction**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty}(x_n \pm y_n) = x \pm y$.
- **Preservation of Multiplication**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$.
- **Preservation of Division**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.
- **Preservation of Inequality**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $x_n \leq y_n$ for all n implies $x \leq y$.

2.2.5 Properties of Sequences in $\mathbb{R}^N$

性质 A convergent sequence in $\mathbb{R}^N$ is bounded.

A sequence $\{x_{n_k}\}$ is called a **subsequence** of $\{x_n\}$ if $n_1 < n_2 < n_3 < \cdots$.

性质 subsequences of a convergent sequence in $\mathbb{R}^N$ also converge to the same limit.

2.2.6 Limit Superior and Limit Inferior

定义 2.4

Let $\{x_n\}$ be a sequence in $\mathbb{R}^N$. The **limit superior** of $\{x_n\}$ is defined by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

The **limit inferior** of $\{x_n\}$ is defined by

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

♣

2.3 Topological properties

2.3.1 Open and Closed Sets

定义 2.5

In a metric space (X, d), a set $U \subset X$ is called **open** if for every $x \in U$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A set $F \subset X$ is called **closed** if its complement $F^c \stackrel{\text{def}}{=} X \backslash F$ is open.

♣

性质 For open sets:

1. The union of any collection of open sets is open.
2. The intersection of finitely many open sets is open.

For closed sets:

1. The intersection of any collection of closed sets is closed.
2. The union of finitely many closed sets is closed.

2.3.2 Interior, Closure, and Boundary of Sets

定义 2.6

The **interior** of a set $A \subset X$ is defined as:

$$\text{int}(A) = \bigcup\{U \subset A : U \text{ is open}\}$$

The **closure** of a set $A \subset X$ is defined as:

$$\overline{A} = \bigcap\{F \supset A : F \text{ is closed}\}$$

The **boundary** of a set $A \subset X$ is defined as:

$$\partial A = \overline{A} \backslash \text{int}(A)$$

♣

定理 2.2

- $A \subset X$ is open if and only if $\partial A \subset A$.
- $A \subset X$ is closed if and only if $\partial A \subset A$.

♡

2.3.3 Bounded Sets and Compact Sets in $\mathbb{R}^N$

定义 2.7

A set $A \subset \mathbb{R}^N$ is called **bounded** if there exists a real number M such that $\|x\| \leq M$ for all $x \in A$.
A set $A \subset \mathbb{R}^N$ is called **compact** if for any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ that converges to a point in A.

♣

定理 2.3 (Heine-Borel Theorem)

In $\mathbb{R}^N$, a set A is compact if and only if it is closed and bounded.

♡

2.4 Continuous functions

2.4.1 Cluster Points in Metric Spaces

定义 2.8

Let (X, d) be a metric space and $A \subset X$. A point $x \in X$ is called a **cluster point** of A if for every $\epsilon > 0$, there exists a point $y \in A$ such that $d(x, y) < \epsilon$ and $x \neq y$.
Equivalently, x is a cluster point of A if there exists a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = x$ and $x_n \neq x$ for all n.

♣

2.4.2 Limits of Functions at Cluster Points

定义 2.9

Let (X, d) and (Y, ρ) be metric spaces, $A \subset X$, $f : A \to Y$, and x be a cluster point of A. We say that f has a **limit** $y \in Y$ at x if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\rho(f(y), y) < \epsilon$ for all $y \in A$ with $0 < d(x, y) < \delta$.
Equivalently, using neighborhoods: f has a limit y at x if for every neighborhood V of y, there exists a neighborhood U of x such that $f(U \cap A) \subset V$.

♣