2025 年数理经济学笔记

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第 1 章 Linear Algebra[1]

内容提要

- Leading Principal Minor : 顺序主子式
- Orthogonal matrix : 正交矩阵
- Symmetric matrix : 对称矩阵
- Positive definite matrix : 正定矩阵
- Positive semi-definite matrix : 半正定矩阵
- Determinant : 行列式

定义 1.1

For $N \times N$ matrix $A = (a_{ij})$, using any row or column:

$$\det A = \sum_{i=1}^{N} (-1)^{i+j} a_{ij} \det A_{ij}$$

where A_{ij} is the $(N-1) \times (N-1)$ matrix obtained by deleting the i-th row and j-th column of A. ♣

定理 1.1

$$A^{-1} = \frac{1}{\det A} \tilde{A}$$

where $\tilde{a_{mn}} = (-1)^{m+n} \det A_{nm}$. ♡

定义 1.2

Orthogonal matrix : $P^T P = I$.

Symmetric matrix : $A^T = A$.

Positive definite matrix : $x^T A x > 0$ for all $x \neq 0$.

Positive semi-definite matrix : $x^T A x \geq 0$ for all x. ♣

定义 1.3

Leading Principal Minor : determinant of the first $k \times k$ submatrix of A. For real symmetric matrix A, A is positive definite if and only if all its leading principal minors are positive. ♣

定义 1.4

$$Av = \lambda v$$

where v is **eigenvector**, λ is **eigenvalue**. λ is a root of the **characteristic polynomial** $\det(A - \lambda I) = 0$. ♣

定义 1.5

Complex inner product :

$$\langle x, y \rangle = x^* y = \sum_{i=1}^{n} \bar{x}_i y_i$$

where $\bar{x}$ is the **complex conjugate** and x^* is the **conjugate transpose** (adjoint). ♣

[1] 只记一些矩阵分解吧, 以防忘了

定义 1.6

Hermitian matrix : $A^* = A$.

For real matrices, Hermitian matrix is symmetric.

For Hermitian matrix, all eigenvalues are real.

♣

定理 1.2 (Diagonalization of Symmetric Matrices)

$$P^T AP = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

where P is orthogonal matrix, λ_i are eigenvalues of A.

♡

第 2 章 Topology of $\mathbb{R}^N$[1]

Keywords

- Topology 拓扑
- Metric Space 度量空间
- Convergence 收敛
- interior 内部
- closure 闭包
- boundary 边界
- compact set 紧集
- cluster point 聚点
- Lipschitz continuity 利普希茨连续
- semicontinuity 半连续
- Bolzano-Weierstrass Theorem 博尔扎诺-魏尔斯特拉斯定理
- Heine-Borel Theorem 海涅-波雷尔定理
- Contraction Mapping Theorem 压缩映射定理
- Intermediate Value Theorem 中值定理

2.1 Metric Spaces

2.1.1 Definition of Metric Spaces

定义 2.1

Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** (or **distance**) on X if :

1. (positivity) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
2. (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

♣

A set X together with a metric d is called a **metric space**, denoted by (X, d).

2.1.2 Examples of metrics in $\mathbb{R}^N$

- **Euclidean metric**: $d(x, y) = \sqrt{\sum_{i=1}^N (x_i - y_i)^2}$.
- L^p **metric** (for $p \geq 1$): $d(x, y) = (\sum_{i=1}^N |x_i - y_i|^p)^{1/p}$.
- **Sup norm** (when $p = \infty$): $d(x, y) = \max_{i=1}^N |x_i - y_i|$.

2.2 Convergence of sequences

2.2.1 Definition of Convergence

定义 2.2

Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to **converge** to a point $x \in X$ if for every $\epsilon > 0$, there exists an integer N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$. A sequence that converges is called **convergent**, otherwise it is called **divergent**.

♣

[1] 点集拓扑对应数分高代这一级别的数学基础课, 只有 sms 和图班的同学学过, 所以我简单记一下, 无需关注证明, 数理经济学只用到结论

定义 2.3

When metric space is $\mathbb{R}^N$, we say that $\{x_n\}$ is **bounded** if there exists a real number M such that $\|x_k\| \leq M$ for all n.

♣

2.2.2 Cauchy Sequences and Complete Metric Spaces

- **Cauchy sequence**: A sequence $\{x_n\}$ in a metric space (X, d) is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists an integer N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.
- **Complete metric space**: A metric space (X, d) is called **complete** if every Cauchy sequence in X converges to a point in X.

定理 2.1

Any convergent sequence in a metric space is a Cauchy sequence.

♡

2.2.3 Example: Cauchy Sequence Not Convergent in $\mathbb{Q}$

Consider the metric space $(\mathbb{Q}, d)$, where $d(x, y) = |x - y|$.

Fibonacci sequence : Let $\{F_k\}$ be the Fibonacci sequence, defined by

$$F_1 = F_2 = 1, F_{k+1} = F_k + F_{k-1}, k \geq 2$$

A Special Sequence: Define $a_k = \frac{F_{k+1}}{F_k}$. Then $\{a_k\}$ is a Cauchy sequence in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$.

2.2.4 Properties of Convergent Sequences in $\mathbb{R}^N$

Consider $\mathbb{R}^N$ with the Euclidean metric. Let $\{x_n\}$ and $\{y_n\}$ be two sequences.

- **Preservation of Addition/Subtraction**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty}(x_n \pm y_n) = x \pm y$.
- **Preservation of Multiplication**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$.
- **Preservation of Division**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.
- **Preservation of Inequality**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $x_n \leq y_n$ for all n implies $x \leq y$.

2.2.5 Properties of Sequences in $\mathbb{R}^N$

性质 A convergent sequence in $\mathbb{R}^N$ is bounded.

A sequence $\{x_{n_k}\}$ is called a **subsequence** of $\{x_n\}$ if $n_1 < n_2 < n_3 < \cdots$.

性质 subsequences of a convergent sequence in $\mathbb{R}^N$ also converge to the same limit.

2.2.6 Limit Superior and Limit Inferior

定义 2.4

Let $\{x_n\}$ be a sequence in $\mathbb{R}^N$. The **limit superior** of $\{x_n\}$ is defined by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

The **limit inferior** of $\{x_n\}$ is defined by

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

♣

2.3 Topological properties

2.3.1 Open and Closed Sets

定义 2.5

In a metric space (X, d), a set $U \subset X$ is called **open** if for every $x \in U$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A set $F \subset X$ is called **closed** if its complement $F^c \stackrel{\text{def}}{=} X \backslash F$ is open. ♣

性质 For open sets:

1. The union of any collection of open sets is open.
2. The intersection of finitely many open sets is open.

For closed sets:

1. The intersection of any collection of closed sets is closed.
2. The union of finitely many closed sets is closed.

2.3.2 Interior, Closure, and Boundary of Sets

定义 2.6

The **interior** of a set $A \subset X$ is defined as:

$$\text{int}(A) = \bigcup \{U \subset A : U \text{ is open}\}$$

The **closure** of a set $A \subset X$ is defined as:

$$\overline{A} = \bigcap \{F \supset A : F \text{ is closed}\}$$

The **boundary** of a set $A \subset X$ is defined as:

$$\partial A = \overline{A} \backslash \text{int}(A)$$

♣

命题 2.1

- $A \subset X$ is open if and only if $\partial A \subset A$.
- $A \subset X$ is closed if and only if $\partial A \subset A$.

♠

2.3.3 Bounded Sets and Compact Sets in $\mathbb{R}^N$

定义 2.7

A set $A \subset \mathbb{R}^N$ is called **bounded** if there exists a real number M such that $\|x\| \leq M$ for all $x \in A$.
A set $A \subset \mathbb{R}^N$ is called **compact** if for any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ that converges to a point in A. ♣

定理 2.2 (Heine-Borel Theorem)

In $\mathbb{R}^N$, a set A is compact if and only if it is closed and bounded. ♡

2.4 Continuous functions

2.4.1 Cluster Points in Metric Spaces

定义 2.8

Let (X, d) be a metric space and $A \subset X$. A point $x \in X$ is called a **cluster point** of A if for every $\epsilon > 0$, there exists a point $y \in A$ such that $d(x, y) < \epsilon$ and $x \neq y$.
Equivalently, x is a cluster point of A if there exists a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = x$ and $x_n \neq x$ for all n. ♣

2.4.2 Limits of Functions at Cluster Points

定义 2.9

Let (X, d) and (Y, ρ) be metric spaces, $A \subset X$, $f : A \to Y$, and x be a cluster point of A. We say that f has a **limit** $y \in Y$ at x if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\rho(f(x_0), y) < \epsilon$ for all $x_0 \in A$ such that $0 < d(x_0, x) < \delta$.
Equivalently, using neighborhoods: f has a limit y at x if for every neighborhood V of y, there exists a neighborhood U of x such that $f(U \cap A) \subset V$. ♣

性质

1. $\lim_{x\to\bar{x}} f(x) = f(\bar{x})$ if and only if for every sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = \bar{x}$, we have $\lim_{n\to\infty} f(x_n) = f(\bar{x})$.
2. If f has a limit at x, then the limit is unique.

2.4.3 Continuity of Functions

定义 2.10

Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$.

- f is **continuous at** $\bar{x} \in X$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in X, d(x, \bar{x}) < \delta \Rightarrow \rho(f(x), f(\bar{x})) < \varepsilon$$

  Equivalently:

$$\forall \varepsilon > 0, \exists \delta > 0 : f(B_\delta(\bar{x})) \subseteq B_\varepsilon(f(\bar{x}))$$

- f is **continuous on** X (or simply **continuous**) if:

$$\forall \bar{x} \in X, f \text{ is continuous at } \bar{x}$$

♣

命题 2.2

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $x \in X$. The following are equivalent:

1. f is continuous at x.
2. For every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.
3. For every open set $V \subset Y$, $f^{-1}(V)$ is open in X.

♠

2.4.4 Bolzano-Weierstrass Theorem

定理 2.3 (Bolzano-Weierstrass Theorem)

If $K \subset \mathbb{R}^N$ is compact and nonempty, and $f : K \to \mathbb{R}^M$ is continuous, then :

1. $f(K)$ is compact.
2. f attains its maximum and minimum on K.

♡

2.4.5 Semicontinuity

定义 2.11

For $f : \mathbb{R}^N \to \mathbb{R}^M$:

- f is **upper semicontinuous** at x if :

$$f(x) \leq \limsup_{y \to x} f(y) \text{ for all } x \in \mathbb{R}^N$$

- f is **lower semicontinuous** at x if :

$$f(x) \geq \liminf_{y \to x} f(y) \text{ for all } x \in \mathbb{R}^N$$

♣

注 f is upper semicontinuous $\Leftrightarrow -f$ is lower semicontinuous.

定理 2.4 (Extrema of semicontinuous Functions)

Let $K \subset \mathbb{R}^N$ be compact and $f : K \to \mathbb{R}$ be upper semicontinuous. Then f attains its maximum on K. If f is lower semicontinuous, then f attains its minimum on K.

♡

2.4.6 Lipschitz Continuity

定义 2.12

A function $f : \mathbb{R}^N \to \mathbb{R}^M$ is called **Lipschitz continuous** if there exists a constant $K > 0$ such that:

$$\|f(x) - f(y)\| \leq K\|x - y\| \text{ for all } x, y \in \mathbb{R}^N$$

where K is called the **Lipschitz constant** of f. If $K < 1$, then f is called a **contraction mapping**.

♣

注 Lipschitz continuity implies uniform continuity, but the converse is not true. For example, $f(x) = x^2$.

定理 2.5 (Contraction Mapping Theorem)

Let (X, d) be a complete metric space and $f : X \to X$ be a contraction mapping. Then f has a unique fixed point $x^* \in X$, i.e., $f(x^*) = x^*$.

♡

定理 2.6 (Intermediate Value Theorem)

Let $f : D \to \mathbb{R}$ be a continuous function and $D \subset \mathbb{R}$. If :

- $[a, b] \subset D$ (closed interval)
- y is between $f(a)$ and $f(b)$

then there exists a point $c \in [a, b]$ such that $f(c) = y$.

♡

第 3 章 Multi-Variable Calculus[1]

Keywords

❑ gradient 梯度

3.1 introduction

3.1.1 Motivation and Insight

- Many practical problems involve optimization with multple variables.
- Real-world applications often require optimizing several variables simultaneoutly.
- Linear functions are easy to understand and manipulate.
  - Not all interesting functions are linear, but many can be approximated by linear functions.
  - The gradient is a generalization of the derivative to functions of multiple variables.

[1] 多元微分还能有不会的吗, 这真不用记了吧

第 4 章 Multi-Variable Unconstrained Optimization

Keywords

- ❑ First Order Condition 一阶条件
- ❑ Newton's Method 牛顿法
- ❑ Bisection Method 二分法

4.1 First Order Condition

An Unconstrained Optimization Problem is :

$$\min_{x \in \mathbb{R}^n} f(x)$$

定义 4.1

First Order Condition (FOC): $\nabla f(x^*) = 0$.

- x^* is a **stationary point** (驻点) of f.
- It is necessary but not sufficient.

global minimum: $f(x^*) \leq f(x)$ for all $x \in \mathbb{R}^n$.

local minimum: $f(x^*) \leq f(x)$ for all $x \in B(x^*, \epsilon)$ for some $\epsilon > 0$.

♣

命题 4.1

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and $\nabla f(x^*) = 0$. If $\nabla^2 f(x^*)$ is:

- positive definite, then x^* is a local minimum.
- negative definite, then x^* is a local maximum.
- indefinite, then x^* is a **saddle point**. (鞍点)

♠

4.2 Convex Optimization

定义 4.2 (Convex Function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

♣

定理 4.1

A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its Hessian matrix $\nabla^2 f(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$.

♡

命题 4.2

Let f be differentiable. Then f is (strictly) convex if and only if:

$$f(y) - f(x)(>) \geq \nabla f(x) \cdot (y - x)$$

for all $x, y \in \mathbb{R}^n$.

♠

定理 4.2 (Minimum/maximum Characterization)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex (concave) function. Then x^* is a local minimum (maximum) if and only if:

$$\nabla f(x^*) = 0$$

- If f is strictly **convex**, then x^* is a global minimum.
- If f is strictly **concave**, then x^* is a global maximum.

♡

4.2.1 Bisection Method

定义 4.3 (Bisection Method)

A simple, robust method for finding roots of continuous functions on bounded intervals

- Start with an interval $[a, b]$ such that $f(a)f(b) < 0$.
- Compute the midpoint $c = \frac{a+b}{2}$, and evaluate $f(c)$.
- Replace a or b with c based on the sign of $f(c)$.
- Iterate until desired precision.

♣

定义 4.4 (Convergence Rate and Order 收敛速度和阶)

For iteration x_n approaching the root r, the convergence rate C and order ρ are defined as:

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^\rho} = C$$

- Linear convergence: $\rho = 1$, $C < 1$.
- Quadratic convergence: $\rho = 2$, $C < 1$.
- Superlinear convergence: $\rho > 1$, $C < 1$.

♣

Method	**Definition**	**Rate**	**Order**
Bisection	Iteratively bisects an interval and selects a subinterval	Linear ($C = 0.5$)	1
Secant	Root approximation via secant line through two points	Superlinear ($C \approx 1.618$)	1.618
False Position	Bisection variant with linear interpolation updates	Linear	1
Newton-Raphson	Derivative-based iterative root-finding	Quadratic ($C \propto f''$)	2
Gradient method	Function minimization via negative gradient direction	Linear ($C \propto \kappa$)	1

表 4.1: Compact Comparison of Numerical Methods

4.3 Numerical Optimization