



# 2025 年数理经济学笔记

授课: 杨佳楠老师

作者: 徐靖

组织: PKU

时间: February 28, 2025

声明: 请勿用于个人学习外其他用途!



# 目录

|       |   |   |
|-------|---|---|
| 第 1 章 | Linear Algebra  | 1 |
| 第 2 章 | Topology of $\mathbb{R}^N$                              | 3 |
| 2.1   | Metric Spaces   | 3 |
| 2.1.1 | Definition of Metric Spaces                             | 3 |
| 2.1.2 | Examples of metrics in $\mathbb{R}^N$                   | 3 |
| 2.2   | Convergence of sequences                                | 3 |
| 2.2.1 | Definition of Convergence                               | 3 |
| 2.2.2 | Cauchy Sequences and Complete Metric Spaces             | 4 |
| 2.2.3 | Example: Cauchy Sequence Not Convergent in $\mathbb{Q}$ | 4 |
| 2.2.4 | Properties of Convergent Sequences in $\mathbb{R}^N$    | 4 |
| 2.2.5 | Properties of Sequences in $\mathbb{R}^N$               | 4 |
| 2.2.6 | Limit Superior and Limit Inferior                       | 4 |
| 2.3   | Topological properties                                  | 5 |
| 2.3.1 | Open and Closed Sets                                    | 5 |
| 2.3.2 | Interior, Closure, and Boundary of Sets                 | 5 |
| 2.3.3 | Bounded Sets and Compact Sets in $\mathbb{R}^N$         | 5 |
| 2.4   | Continuous functions                                    | 6 |
| 2.4.1 | Cluster Points in Metric Spaces                         | 6 |
| 2.4.2 | Limits of Functions at Cluster Points                   | 6 |
| 2.4.3 | Continuity of Functions                                 | 6 |
| 2.4.4 | Bolzano-Weierstrass Theorem                             | 7 |
| 2.4.5 | Semicontinuity  | 7 |
| 2.4.6 | Lipschitz Continuity                                    | 7 |
| 第 3 章 | Multi-Variable Calculus                                 | 8 |
| 3.1   | introduction  | 8 |
| 3.1.1 | Motivation and Insight                                  | 8 |

# 第 1 章 Linear Algebra<sup>1</sup>

## 内容提要

- Leading Principal Minor : 顺序主子式
- Orthogonal matrix : 正交矩阵
- Symmetric matrix : 对称矩阵
- Positive definite matrix : 正定矩阵
- Positive semi-definite matrix : 半正定矩阵
- Determinant : 行列式

### 定义 1.1

For  $N \times N$  matrix  $A = (a_{ij})$ , using any row or column:

$$\det A = \sum_{i=1}^N (-1)^{i+j} a_{ij} \det A_{ij}$$

where  $A_{ij}$  is the  $(N-1) \times (N-1)$  matrix obtained by deleting the  $i$ -th row and  $j$ -th column of  $A$ .



### 定理 1.1

$$A^{-1} = \frac{1}{\det A} \tilde{A}$$

where  $a_{mn}^{\tilde{}} = (-1)^{m+n} \det A_{nm}$ .



### 定义 1.2

**Orthogonal** matrix :  $P^T P = I$ .

**Symmetric** matrix :  $A^T = A$ .

**Positive definite** matrix :  $x^T A x > 0$  for all  $x \neq 0$ .

**Positive semi-definite** matrix :  $x^T A x \geq 0$  for all  $x$ .



### 定义 1.3

**Leading Principal Minor** : determinant of the first  $k \times k$  submatrix of  $A$ . For real symmetric matrix  $A$ ,  $A$  is positive definite if and only if all its leading principal minors are positive.



### 定义 1.4

$$Av = \lambda v$$

where  $v$  is **eigenvector**,  $\lambda$  is **eigenvalue**.  $\lambda$  is a root of the **characteristic polynomial**  $\det(A - \lambda I) = 0$ .



### 定义 1.5

**Complex inner product** :

$$\langle x, y \rangle = x^* y = \sum_{i=1}^n \bar{x}_i y_i$$

where  $\bar{x}$  is the **complex conjugate** and  $x^*$  is the **conjugate transpose** (adjoint).



<sup>1</sup> 只记一些矩阵分解吧, 以防忘了

### 定义 1.6

**Hermitian matrix** :  $A^* = A$ .

For real matrices, Hermitian matrix is symmetric.

For Hermitian matrix, all eigenvalues are real.



### 定理 1.2 (Diagonalization of Symmetric Matrices)

$$P^T A P = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

where  $P$  is orthogonal matrix,  $\lambda_i$  are eigenvalues of  $A$ .



## 第 2 章 Topology of $\mathbb{R}^N$ <sup>1</sup>

### Keywords

- Topology 拓扑
- Metric Space 度量空间
- Convergence 收敛
- interior 内部
- closure 闭包
- boundary 边界
- compact set 紧集
- cluster point 聚点
- Lipschitz continuity 利普希茨连续
- semicontinuity 半连续
- Bolzano-Weierstrass Theorem 博尔扎诺-魏尔斯特拉斯定理
- Heine-Borel Theorem 海涅-波雷尔定理
- Contraction Mapping Theorem 压缩映射定理
- Intermediate Value Theorem 中值定理

## 2.1 Metric Spaces

### 2.1.1 Definition of Metric Spaces

#### 定义 2.1

Let  $X$  be a set. A function  $d : X \times X \rightarrow \mathbb{R}$  is called a **metric** (or **distance**) on  $X$  if :

1. (positivity)  $d(x, y) \geq 0$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ .
2. (symmetry)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .
3. (triangle inequality)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .



A set  $X$  together with a metric  $d$  is called a **metric space**, denoted by  $(X, d)$ .

### 2.1.2 Examples of metrics in $\mathbb{R}^N$

- **Euclidean metric**:  $d(x, y) = \sqrt{\sum_{i=1}^N (x_i - y_i)^2}$ .
- $L^p$  **metric** (for  $p \geq 1$ ):  $d(x, y) = (\sum_{i=1}^N |x_i - y_i|^p)^{1/p}$ .
- **Sup norm** (when  $p = \infty$ ):  $d(x, y) = \max_{i=1}^N |x_i - y_i|$ .

## 2.2 Convergence of sequences

### 2.2.1 Definition of Convergence

#### 定义 2.2

Let  $(X, d)$  be a metric space. A sequence  $\{x_n\}$  in  $X$  is said to **converge** to a point  $x \in X$  if for every  $\epsilon > 0$ , there exists an integer  $N$  such that  $d(x_n, x) < \epsilon$  for all  $n \geq N$ . In this case, we write  $\lim_{n \rightarrow \infty} x_n = x$ . A sequence that converges is called **convergent**, otherwise it is called **divergent**.



<sup>1</sup>点集拓扑对应数分高代这一级别的数学基础课, 只有 sms 和图班的同学学过, 所以我简单记一下, 无需关注证明, 数理经济学只用到结论

**定义 2.3**

When metric space is  $\mathbb{R}^N$ , we say that  $\{x_n\}$  is **bounded** if there exists a real number  $M$  such that  $\|x_k\| \leq M$  for all  $n$ .

**2.2.2 Cauchy Sequences and Complete Metric Spaces**

- **Cauchy sequence:** A sequence  $\{x_n\}$  in a metric space  $(X, d)$  is called a **Cauchy sequence** if for every  $\epsilon > 0$ , there exists an integer  $N$  such that  $d(x_n, x_m) < \epsilon$  for all  $n, m \geq N$ .
- **Complete metric space:** A metric space  $(X, d)$  is called **complete** if every Cauchy sequence in  $X$  converges to a point in  $X$ .

**定理 2.1**

Any convergent sequence in a metric space is a Cauchy sequence.

**2.2.3 Example: Cauchy Sequence Not Convergent in  $\mathbb{Q}$** 

Consider the metric space  $(\mathbb{Q}, d)$ , where  $d(x, y) = |x - y|$ .

**Fibonacci sequence :** Let  $\{F_k\}$  be the Fibonacci sequence, defined by

$$F_1 = F_2 = 1, F_{k+1} = F_k + F_{k-1}, k \geq 2$$

**A Special Sequence:** Define  $a_k = \frac{F_{k+1}}{F_k}$ . Then  $\{a_k\}$  is a Cauchy sequence in  $\mathbb{Q}$  but does not converge in  $\mathbb{Q}$ .

**2.2.4 Properties of Convergent Sequences in  $\mathbb{R}^N$** 

Consider  $\mathbb{R}^N$  with the Euclidean metric. Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences.

- **Preservation of Addition/Subtraction:** If  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} y_n = y$ , then  $\lim_{n \rightarrow \infty} (x_n \pm y_n) = x \pm y$ .
- **Preservation of Multiplication:** If  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} y_n = y$ , then  $\lim_{n \rightarrow \infty} (x_n \cdot y_n) = x \cdot y$ .
- **Preservation of Division:** If  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} y_n = y \neq 0$ , then  $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$ .
- **Preservation of Inequality:** If  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} y_n = y$ , then  $x_n \leq y_n$  for all  $n$  implies  $x \leq y$ .

**2.2.5 Properties of Sequences in  $\mathbb{R}^N$** 

**性质** A convergent sequence in  $\mathbb{R}^N$  is bounded.

A sequence  $\{x_{n_k}\}$  is called a **subsequence** of  $\{x_n\}$  if  $n_1 < n_2 < n_3 < \dots$ .

**性质** subsequences of a convergent sequence in  $\mathbb{R}^N$  also converge to the same limit.

**2.2.6 Limit Superior and Limit Inferior****定义 2.4**

Let  $\{x_n\}$  be a sequence in  $\mathbb{R}^N$ . The **limit superior** of  $\{x_n\}$  is defined by

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right)$$

The **limit inferior** of  $\{x_n\}$  is defined by

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right)$$



## 2.3 Topological properties

### 2.3.1 Open and Closed Sets

#### 定义 2.5

In a metric space  $(X, d)$ , a set  $U \subset X$  is called **open** if for every  $x \in U$ , there exists an  $\epsilon > 0$  such that  $B(x, \epsilon) \subset U$ .  
A set  $F \subset X$  is called **closed** if its complement  $F^c \stackrel{\text{def}}{=} X \setminus F$  is open.



性质 For open sets:

1. The union of any collection of open sets is open.
2. The intersection of finitely many open sets is open.

For closed sets:

1. The intersection of any collection of closed sets is closed.
2. The union of finitely many closed sets is closed.

### 2.3.2 Interior, Closure, and Boundary of Sets

#### 定义 2.6

The **interior** of a set  $A \subset X$  is defined as:

$$\text{int}(A) = \bigcup \{U \subset A : U \text{ is open}\}$$

The **closure** of a set  $A \subset X$  is defined as:

$$\bar{A} = \bigcap \{F \supset A : F \text{ is closed}\}$$

The **boundary** of a set  $A \subset X$  is defined as:

$$\partial A = \bar{A} \setminus \text{int}(A)$$



#### 命题 2.1

- $A \subset X$  is open if and only if  $\partial A \subset A$ .
- $A \subset X$  is closed if and only if  $\partial A \subset A$ .



### 2.3.3 Bounded Sets and Compact Sets in $\mathbb{R}^N$

#### 定义 2.7

A set  $A \subset \mathbb{R}^N$  is called **bounded** if there exists a real number  $M$  such that  $\|x\| \leq M$  for all  $x \in A$ .

A set  $A \subset \mathbb{R}^N$  is called **compact** if for any sequence  $\{x_n\}$  in  $A$ , there exists a subsequence  $\{x_{n_k}\}$  that converges to a point in  $A$ .



#### 定理 2.2 (Heine-Borel Theorem)

In  $\mathbb{R}^N$ , a set  $A$  is compact if and only if it is closed and bounded.





## 2.4 Continuous functions

### 2.4.1 Cluster Points in Metric Spaces

#### 定义 2.8

Let  $(X, d)$  be a metric space and  $A \subset X$ . A point  $x \in X$  is called a **cluster point** of  $A$  if for every  $\epsilon > 0$ , there exists a point  $y \in A$  such that  $d(x, y) < \epsilon$  and  $x \neq y$ .

Equivalently,  $x$  is a cluster point of  $A$  if there exists a sequence  $\{x_n\}$  in  $A$  such that  $\lim_{n \rightarrow \infty} x_n = x$  and  $x_n \neq x$  for all  $n$ .



### 2.4.2 Limits of Functions at Cluster Points

#### 定义 2.9

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $A \subset X$ ,  $f : A \rightarrow Y$ , and  $x$  be a cluster point of  $A$ . We say that  $f$  has a **limit**  $y \in Y$  at  $x$  if for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\rho(f(x_0), y) < \epsilon$  for all  $x_0 \in A$  such that  $0 < d(x_0, x) < \delta$ .

Equivalently, using neighborhoods:  $f$  has a limit  $y$  at  $x$  if for every neighborhood  $V$  of  $y$ , there exists a neighborhood  $U$  of  $x$  such that  $f(U \cap A) \subset V$ .



#### 性质

1.  $\lim_{x \rightarrow \bar{x}} f(x) = f(\bar{x})$  if and only if for every sequence  $\{x_n\}$  in  $A$  such that  $\lim_{n \rightarrow \infty} x_n = \bar{x}$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = f(\bar{x})$ .
2. If  $f$  has a limit at  $x$ , then the limit is unique.

### 2.4.3 Continuity of Functions

#### 定义 2.10

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces, and  $f : X \rightarrow Y$ .

- $f$  is **continuous at**  $\bar{x} \in X$  if:

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in X, d(x, \bar{x}) < \delta \Rightarrow \rho(f(x), f(\bar{x})) < \epsilon$$

Equivalently:

$$\forall \epsilon > 0, \exists \delta > 0 : f(B_\delta(\bar{x})) \subseteq B_\epsilon(f(\bar{x}))$$

- $f$  is **continuous on**  $X$  (or simply **continuous**) if:

$$\forall \bar{x} \in X, f \text{ is continuous at } \bar{x}$$



#### 命题 2.2

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $f : X \rightarrow Y$ , and  $x \in X$ . The following are equivalent:

1.  $f$  is continuous at  $x$ .
2. For every sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} x_n = x$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ .
3. For every open set  $V \subset Y$ ,  $f^{-1}(V)$  is open in  $X$ .





### 2.4.4 Bolzano-Weierstrass Theorem

#### 定理 2.3 (Bolzano-Weierstrass Theorem)

If  $K \subset \mathbb{R}^N$  is compact and nonempty, and  $f : K \rightarrow \mathbb{R}^M$  is continuous, then :

1.  $f(K)$  is compact.
2.  $f$  attains its maximum and minimum on  $K$ .



### 2.4.5 Semicontinuity

#### 定义 2.11

For  $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$  :

- $f$  is **upper semicontinuous** at  $x$  if :

$$f(x) \leq \limsup_{y \rightarrow x} f(y) \text{ for all } x \in \mathbb{R}^N$$

- $f$  is **lower semicontinuous** at  $x$  if :

$$f(x) \geq \liminf_{y \rightarrow x} f(y) \text{ for all } x \in \mathbb{R}^N$$



注  $f$  is upper semicontinuous  $\Leftrightarrow -f$  is lower semicontinuous.

#### 定理 2.4 (Extrema of semicontinuous Functions)

Let  $K \subset \mathbb{R}^N$  be compact and  $f : K \rightarrow \mathbb{R}$  be upper semicontinuous. Then  $f$  attains its maximum on  $K$ . If  $f$  is lower semicontinuous, then  $f$  attains its minimum on  $K$ .



### 2.4.6 Lipschitz Continuity

#### 定义 2.12

A function  $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$  is called **Lipschitz continuous** if there exists a constant  $K > 0$  such that:

$$\|f(x) - f(y)\| \leq K\|x - y\| \text{ for all } x, y \in \mathbb{R}^N$$

where  $K$  is called the **Lipschitz constant** of  $f$ . If  $K < 1$ , then  $f$  is called a **contraction mapping**.



注 Lipschitz continuity implies uniform continuity, but the converse is not true. For example,  $f(x) = x^2$ .

#### 定理 2.5 (Contraction Mapping Theorem)

Let  $(X, d)$  be a complete metric space and  $f : X \rightarrow X$  be a contraction mapping. Then  $f$  has a unique fixed point  $x^* \in X$ , i.e.,  $f(x^*) = x^*$ .



#### 定理 2.6 (Intermediate Value Theorem)

Let  $f : D \rightarrow \mathbb{R}$  be a continuous function and  $D \subset \mathbb{R}$ . If :

- $[a, b] \subset D$  (closed interval)
- $y$  is between  $f(a)$  and  $f(b)$

then there exists a point  $c \in [a, b]$  such that  $f(c) = y$ .



## 第 3 章 Multi-Variable Calculus<sup>1</sup>

### Keywords

□ gradient 梯度

### 3.1 introduction

#### 3.1.1 Motivation and Insight

- Many practical problems involve optimization with multiple variables.
- Real-world applications often require optimizing several variables simultaneously.
- Linear functions are easy to understand and manipulate.
  - Not all interesting functions are linear, but many can be approximated by linear functions.
  - The gradient is a generalization of the derivative to functions of multiple variables.

---

<sup>1</sup>多元微分还能有不会的吗, 这真不用记了吧