Homework 2

Due: March 14th, 2025 (in class)

Problem 1

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x, y) = x^2 + y^2$$

Given a nonempty and compact set $K \subset \mathbb{R}^2$, defined as:

$$K = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

1. Prove that the set K is compact.
2. Prove that the function f attains its maximum and minimum values over K and determine the points where f attains its maximum and minimum values over K.
3. Consider the function $g(x, y) = (x - 1)^2 + (y - 1)^2$ on the same set K. Repeat the above steps to find the maximum and minimum values of g over K and the corresponding points.

Problem 2

Consider the function $f : [0, 1] \to [0, 1]$ defined by:

$$f(x) = \frac{1}{2}x(1 - x)$$

1. Show that f is a contraction mapping.
2. Determine the fixed point(s) of the function f and verify its uniqueness.
3. Prove that for any $K \in (0, 1)$, the function $g(x) = Kx(1 - x)$ also has a unique fixed point in the interval $[0, 1]$, and find this fixed point.

Problem 3

Let A, B be symmetric positive definite matrices and $f(x) = \langle x, y\rangle - \frac{1}{2}\langle x, Ax\rangle$, where y is a (fixed) vector.

1. Compute $\nabla f(x)$.
2. Compute the maximum of $f(x)$ over $x \in \mathbb{R}^N$.

Problem 4

Consider the Cobb-Douglas production function $y = Ax_1^\alpha x_2^\beta x_3^\gamma$. $A > 0$ and $\alpha, \beta, \gamma < 1$.

1. Compute the first-order partial derivatives and determine the signs of these derivatives. Explain the economic implications.
2. Compute the second-order partial derivatives and determine the signs of these derivatives. Explain the economic implications.