2025 年数理经济学笔记

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Keywords

- Generalized Comparative Statics 广义比较静态
- Cramer's Rule 克拉默法则
- Envelope Theorem 包络定理

1.1 Comparative Statics

笔记 经济学中的比较静态分析是指在给定一个经济模型的情况下, 研究当 **exogenous variables** 发生变化时, **endogenous variables** 的值如何变化. 比如当给定消费者收入去讨论市场供需模型中的均衡价格, 给定税率去讨论对 Monopoly 的影响, 前者作为外生变量 (经济模型的输入), 后者作为内生变量 (经济模型的输出).

定义 1.1 (Generalized Comparative Statics)

We have an economic model, the equilibrium solution of which is given by the form:

$$F(x^*, \alpha) = 0$$

where x^* is the equilibrium solution of endogenous variables x, and α is a vector of exogenous variables. The key objective is to find the derivative $\frac{\partial x_i^*}{\partial \alpha_j}$ and identify its sign. ♣

笔记 这里向量函数 f 的个数应当与内生变量的个数相同, 设为 n.

定理 1.1 (Cramer's Rule)

Let $F(x^*(\alpha), \alpha) = 0$ be a system of n equations in n unknowns. The Jacobian matrix of the system is given by:

$$\det J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

We have:

$$J\frac{\partial x^*}{\partial \alpha_j} + \frac{\partial F}{\partial \alpha_j} = 0, \forall j$$

The Cramer's Rule states that the derivative of the equilibrium solution with respect to the exogenous variable α_j is given by:

$$\frac{\partial x_i^*}{\partial \alpha_j} = -\frac{\det J_{ij}}{\det J}$$

where J_{ij} is the matrix obtained by replacing the i-th column of J with the vector $\frac{\partial F}{\partial \alpha_j}$. ♡

1.1.1 Comparative Statics for Unconstrained Optimization

这时我们考虑一个最优化问题, 其形式为:

$$\max_x f(x; a)$$

其中 $f : \mathbb{R}^n \times R^m \to \mathbb{R}$ 是一个可微函数[1]. m 以下视为 1 (不考虑外生变量之间的影响).

[1] 请注意 $\nabla f = F$: 这里 $\nabla f = 0$ 是 1.1.1 小节中函数的 FOC, F 是广义比较静态的模型函数, 我们不是第一次用类似的记号

FOC $\nabla f=0$ 可以这样写:

$$\frac{\partial f}{\partial x_i}(x_1^*,\ldots,x_n^*;a)=0,\forall i$$

$f(\cdot)$ 的 FOC 的 Jacobian 矩阵也就是 $f(\cdot)$ 的 Hessian 矩阵:

$$\det J(x^*;a)=\frac{\partial^2 f}{\partial x^2}(x^*;a)$$

命题 1.1 (Implict Function Theorem)

If $\det J(x^*;a)\neq 0$, then the system implicitly defines differentiable functions:

$$x_i^*:a\to x_i^*(a),\forall i$$

And the derivatives of these functions are given by[a]:

$$\frac{\partial x_i^*}{\partial a}=-\frac{\det J_i(x^*;a)}{\det J(x^*;a)}$$

where J_i is the matrix obtained by replacing the i-th column of J with the vector $\frac{\partial f}{\partial a}$.

[a]这个式子和上面的 Cramer's Rule 的结论是一样的, 区别是在优化问题中, 我们将向量函数指定为被优化函数的梯度, 那么 Jacobian 矩阵成为了一个特例, 也就是 Hessian 矩阵.

笔记 隐函数定理揭示了二阶条件与比较静态的存在性的关联

1.1.2 Comparative Statics for Equality Constrained Optimization

在等式约束的优化问题中, 我们先做拉格朗日再用同样的隐函数定理方法来进行比较分析, 实际上还是一样的, 因为要对 Lagrangian 函数求一阶条件.

1.2 Envelope Theorem

还是之前的最优化问题:

$$V(a)=\max_x f(x;a)$$

- $F:\mathbb{R}^n\times R^m\to\mathbb{R}$ is differentiable.
- a is an exogenous variable.
- $x^*(a)$ is a local solution with differentiable components $x_i^*(a):\mathbb{R}\to\mathbb{R}$.

定理 1.2 (Envelope Theorem)

The derivative of the value function with respect to the exogenous variable a is given by:

$$\begin{aligned}\frac{\partial V(a)}{\partial a}&=\frac{\partial f}{\partial a}(x^*(a);a)+\sum_{i=1}^n\frac{\partial f}{\partial x_i}(x^*(a);a)\frac{\partial x_i^*(a)}{\partial a}\\&=\frac{\partial f}{\partial a}+\nabla_x f\cdot\frac{\partial x^*(a)}{\partial a}\end{aligned}$$

where $\nabla_x f$ is the gradient of the objective function with respect to the endogenous variables, and $\frac{\partial x^*(a)}{\partial a}$ is the derivative of the equilibrium solution with respect to the exogenous variable a.