Homework 2

Due: March 14th, 2025 (in class)

Problem 1

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x, y) = x^2 + y^2$$

Given a nonempty and compact set $K \subset \mathbb{R}^2$, defined as:

$$K = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

1. Prove that the set K is compact.
2. Prove that the function f attains its maximum and minimum values over K and determine the points where f attains its maximum and minimum values over K.
3. Consider the function $g(x, y) = (x - 1)^2 + (y - 1)^2$ on the same set K. Repeat the above steps to find the maximum and minimum values of g over K and the corresponding points.

Solution

Part 1: Prove that the set K is compact

To prove that K is compact, we must show that it is both closed and bounded.

- **Closed:** The set K contains all its limit points. If a sequence $\{(x_n, y_n)\}$ in K converges to (x, y), then by the continuity of the norm function, $\lim_{n\to\infty}(x_n^2 + y_n^2) = x^2 + y^2 \leq 1$. Hence, $(x, y) \in K$, proving that K is closed.
- **Bounded:** The set K is contained within the closed ball of radius 1 centered at the origin, hence it is bounded.

By the Heine-Borel Theorem, every closed and bounded subset of $\mathbb{R}^n$ is compact. Therefore, K is compact.

Part 2: Prove that the function f attains its maximum and minimum values over K

Since K is compact and f is continuous (as it is a polynomial function), by the Weierstrass Theorem, f attains its maximum and minimum values on K.

- **Minimum:** The function $f(x, y) = x^2 + y^2$ achieves its minimum value at the origin $(0, 0)$, where $f(0, 0) = 0$.
- **Maximum:** On the boundary of K, defined by $x^2 + y^2 = 1$, the function $f(x, y)$ reaches its maximum value. Since $x^2 + y^2 = 1$, the maximum value is $f(x, y) = 1$. This maximum is achieved at any point on the boundary.

Part 3: Consider the function $g(x, y) = (x-1)^2 + (y-1)^2$ on the same set K

We repeat the analysis for the function $g(x, y)$.

- **Minimum:** The function $g(x, y)$ achieves its minimum value at the point $(1, 1)$, where $g(1, 1) = 0$, since this point is the closest to the origin among all points in K shifted by $(1, 1)$.
- **Maximum:** On the boundary of K, the function $g(x, y)$ reaches its maximum value. The maximum value is $g(x, y) = 2$, achieved when (x, y) is diametrically opposite to $(1, 1)$ on the circle $x^2 + y^2 = 1$, i.e., at $(-1, -1)$.

Problem 2

Consider the function $f : [0, 1] \to [0, 1]$ defined by:

$$f(x) = \frac{1}{2}x(1-x)$$

1. Show that f is a contraction mapping.
2. Determine the fixed point(s) of the function f and verify its uniqueness.
3. Prove that for any $K \in (0, 1)$, the function $g(x) = Kx(1-x)$ also has a unique fixed point in the interval $[0, 1]$, and find this fixed point.

Solution

Part 1: Show that f is a contraction mapping

To show that f is a contraction mapping, we need to show that there exists a constant $K \in [0, 1)$ such that for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| \leq K|x - y|.$$

Let $x, y \in [0, 1]$. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{2}x(1-x) - \frac{1}{2}y(1-y) \right| \\ &= \frac{1}{2}|x - y||1 - (x + y)|. \end{aligned}$$

Since $x, y \in [0, 1]$, we have $0 \leq x + y \leq 2$, thus $-1 \leq 1 - (x + y) \leq 1$. Therefore,

$$|1 - (x + y)| \leq 1,$$

and hence,

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|.$$

Thus, f is a contraction mapping with $K = \frac{1}{2}$.

Part 2: Determine the fixed point(s) of the function f and verify its uniqueness

A fixed point x^* of f satisfies $f(x^*) = x^*$. Therefore,

$$\frac{1}{2}x^*(1 - x^*) = x^*.$$

Rearranging gives us the quadratic equation

$$x^*(1 - 2x^*) = 0.$$

Thus, $x^* = 0$ or $x^* = \frac{1}{2}$.

To verify uniqueness, we use the fact that f is a contraction mapping. By the contraction mapping theorem, a contraction mapping on a complete metric space has a unique fixed point. Since $[0, 1]$ is complete and f is a contraction, f has a unique fixed point in $[0, 1]$. Therefore, the fixed point is $x^* = \frac{1}{2}$.

Part 3: Prove that for any $K \in (0,1)$, the function $g(x) = Kx(1-x)$ also has a unique fixed point in the interval $[0,1]$, and find this fixed point

Consider the function $g(x) = Kx(1-x)$. A fixed point x^* of g satisfies $g(x^*) = x^*$. Therefore,

$$Kx^*(1-x^*) = x^*.$$

Rearranging gives us the quadratic equation

$$x^*(K(1-x^*) - 1) = 0.$$

Thus, $x^* = 0$ or $x^* = 1 - \frac{1}{K}$.

Since $K \in (0,1)$, $1 - \frac{1}{K}$ is outside the interval $[0,1]$. Therefore, the only fixed point in $[0,1]$ is $x^* = 0$.

To prove uniqueness, we note that $g(x)$ is also a contraction mapping for $K \in (0,1)$ with $K' = K$. By the contraction mapping theorem, g has a unique fixed point in $[0,1]$. Therefore, the fixed point is $x^* = 0$.

Problem 3

Let A, B be symmetric positive definite matrices and $f(x) = \langle x, y\rangle - \frac{1}{2}\langle x, Ax\rangle$, where y is a (fixed) vector.

1. Compute $\nabla f(x)$.
2. Let
$$f(x) = x_1 + x_2 - x_1^2 - x_1x_2 - x_2^2$$
Compute the gradient and the Hessian of f.

Solution

Part 1: Compute $\nabla f(x)$

Given $f(x) = \langle x, y\rangle - \frac{1}{2}\langle x, Ax\rangle$, the gradient $\nabla f(x)$ is:

$$\nabla f(x) = y - Ax$$

Part 2: Compute the gradient and the Hessian of f

For $f(x) = x_1 + x_2 - x_1^2 - x_1x_2 - x_2^2$, we first compute the gradient:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - 2x_1 - x_2 \\ 1 - x_1 - 2x_2 \end{bmatrix}$$

Next, we compute the Hessian matrix $H(f)$, which is the matrix of second partial derivatives:

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

Part 2's function is a specific instance of Part 1's general form with $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Problem 4

Consider the Cobb-Douglas production function $y = Ax_1^\alpha x_2^\beta x_3^\gamma$. $A > 0$ and $\alpha, \beta, \gamma < 1$.

1. Compute the first-order partial derivatives and determine the signs of these derivatives. Explain the economic implications.
2. Compute the second-order partial derivatives and determine the signs of these derivatives. Explain the economic implications.

Solution

Part 1: Compute the first-order partial derivatives

The first-order partial derivatives of the Cobb-Douglas production function are:

$$\frac{\partial y}{\partial x_1} = \alpha A x_1^{\alpha-1} x_2^{\beta} x_3^{\gamma}$$
$$\frac{\partial y}{\partial x_2} = \beta A x_1^{\alpha} x_2^{\beta-1} x_3^{\gamma}$$
$$\frac{\partial y}{\partial x_3} = \gamma A x_1^{\alpha} x_2^{\beta} x_3^{\gamma-1}$$

Since $\alpha, \beta, \gamma > 0$, the signs of these derivatives are positive. Economically, this implies that increasing any input will increase output.

Part 2: Compute the second-order partial derivatives

The second-order partial derivatives of the Cobb-Douglas production function are:

$$\frac{\partial^2 y}{\partial x_1^2} = \alpha(\alpha-1) A x_1^{\alpha-2} x_2^{\beta} x_3^{\gamma}$$
$$\frac{\partial^2 y}{\partial x_2^2} = \beta(\beta-1) A x_1^{\alpha} x_2^{\beta-2} x_3^{\gamma}$$
$$\frac{\partial^2 y}{\partial x_3^2} = \gamma(\gamma-1) A x_1^{\alpha} x_2^{\beta} x_3^{\gamma-2}$$

Since $\alpha, \beta, \gamma < 1$, the signs of these second-order partial derivatives are negative. Economically, this implies that the marginal product of inputs decreases as one input increases, reflecting diminishing returns to scale.