Homework 1

Due: March 7th, 2025 (in class)

Problem 1

Let P be a matrix such that $P^2 = P$. Show that the eigenvalues of P are either 0 or 1.

Problem 2

Let A be symmetric. Show that A is positive definite if and only if all eigenvalues of A are positive.

Problem 3

Let A be symmetric and positive semidefinite. Show that there exists a symmetric and positive semidefinite matrix B such that $A = B^2$.

Problem 4

Let (X, d) be a metric space, and $\mathcal{A}$ be a collection of subsets of X. Prove the following properties for open and closed sets:

1. If $A_\alpha \in \mathcal{A}$ is open $\forall \alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha$ is open. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ are open, then $\bigcap_{i=1}^{n} A_i$ is open.
2. If $A_\alpha \in \mathcal{A}$ is closed $\forall \alpha \in I$, then $\bigcap_{\alpha \in I} A_\alpha$ is closed. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ are closed, then $\bigcup_{i=1}^{n} A_i$ is closed.