Homework 6 Solutions

April 12, 2025

Problem 1

Given the consumer utility functions as:

(a) $u = x_1 x_2$

(b) $u = \sqrt{x_1} + x_2$

With budget constraint $p_1 x_1 + p_2 x_2 = m$, derive the corresponding Slutsky decomposition for $\frac{\partial x_1^*}{\partial p_1}$ and $\frac{\partial x_1^*}{\partial p_2}$. Discuss the sign of income and substitution effect.

Solution

For the utility function $u = x_1 x_2$:

Applying the Lagrangean method gives the first-order conditions

$$
\begin{aligned}
x_2 - \lambda^* p_1 &= 0 \\
x_1 - \lambda^* p_2 &= 0 \\
m - p_1 x_1^* - p_2 x_2^* &= 0
\end{aligned}
$$

Applying the standard method, we obtain the effects of changes in prices and income on the demand of good 1:

$$
\begin{aligned}
\frac{\partial x_1^*}{\partial p_1} &= \begin{vmatrix} 0 & 0 & -p_1 \\ 0 & 0 & -p_2 \\ x_1^* & -p_2 & 0 \end{vmatrix} / |D| \\
&= -\frac{p_2^2 x_1^*}{|D|} < 0
\end{aligned}
$$

$$
\frac{\partial x_1^*}{\partial m} = \begin{vmatrix} 0 & 0 & -p_1 \\ 0 & 0 & -p_2 \\ 1 & -p_2 & 0 \end{vmatrix} / |D| = \frac{p_2 p_1}{|D|} > 0
$$

where

$$
|D| = \begin{vmatrix} 0 & x_2^* & -p_1 \\ x_2^* & 0 & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} = 2 p_1 p_2 x_2^*
$$

The income effect is positive, which is expected for normal goods. Similarly we obtain for good 2

$$
\begin{aligned}
\frac{\partial x_2^*}{\partial p_2} &= \begin{vmatrix} 0 & 0 & -p_1 \\ 0 & 0 & -p_2 \\ -p_1 & x_2^* & 0 \end{vmatrix} / |D| \\
&= -\frac{p_1^2 x_2^*}{|D|} < 0
\end{aligned}
$$

The income effect for good 2 is also positive, and we have $\partial x_2^*/\partial p_2 < 0$ as expected.

For the cross-price effects:

$$\begin{aligned}\frac{\partial x_1^*}{\partial p_2} &= \begin{vmatrix} 0 & 0 & -p_1 \\ 0 & 0 & -p_2 \\ -p_1 & x_1^* & 0 \end{vmatrix} / |D| \\ &= -\frac{p_1^2 x_1^*}{|D|} < 0\end{aligned}$$

This indicates that goods 1 and 2 are complements, which is characteristic of Cobb-Douglas utility functions.

Using the Slutsky equation, we can decompose the price effect into substitution and income effects:

$$\begin{aligned}\frac{\partial x_1^*}{\partial p_1} &= \frac{\partial h_1}{\partial p_1} - \frac{\partial x_1}{\partial m} \cdot x_1 \\ &= -\frac{p_2^2 x_1^*}{2p_1p_2x_2^*} - \frac{p_2p_1}{2p_1p_2x_2^*} \cdot \frac{m}{2p_1} \\ &= -\frac{x_1^*}{p_1}\end{aligned}$$

Which gives us the Slutsky equation for the Cobb-Douglas utility function.

[(b)] For the utility function $u = \sqrt{x_1} + x_2$:

Applying the Lagrangean method gives the first-order conditions

$$\begin{aligned}0.5(x_1^*)^{-0.5} - \lambda^* p_1 &= 0 \\ 1 - \lambda^* p_2 &= 0 \\ m - p_1 x_1^* - p_2 x_2^* &= 0\end{aligned}$$

Applying the standard method, we obtain the effects of changes in prices and income on the demand of good 1:

$$\begin{aligned}\frac{\partial x_1^*}{\partial p_1} &= \begin{vmatrix} \lambda^* & 0 & -p_1 \\ 0 & 0 & -p_2 \\ x_1^* & -p_2 & 0 \end{vmatrix} / |D| \\ &= -\frac{\lambda p_2^2}{|D|} < 0\end{aligned}$$

$$\frac{\partial x_1^*}{\partial m} = \begin{vmatrix} 0 & 0 & -p_1 \\ 0 & 0 & -p_2 \\ 1 & -p_2 & 0 \end{vmatrix} / |D| = \frac{0}{|D|} = 0$$

where

$$|D| = \begin{vmatrix} -0.25(x_1^*)^{-1.5} & 0 & -p_1 \\ 0 & 0 & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} = 0.25 p_2^2 x_1^{-1.5}$$

Since the income effect is zero, we must have $\partial x_1^*/\partial p_1 0$. Similarly we obtain for good 2

$$\begin{aligned}\frac{\partial x_2^*}{\partial p_2} &= \begin{vmatrix} -0.25(x_1^*)^{-1.5} & 0 & -p_1 \\ 0 & \lambda^* & -p_2 \\ -p_1 & x_2^* & 0 \end{vmatrix} / |D| \\ &= -\frac{\lambda^* p_1^2}{|D|} - x_2^* \frac{0.25 x_1^{-1.5} p_2}{|D|}\end{aligned}$$

The income effect $\left(-x_2^* \frac{0.25 x_1^{-1.5} p_2}{|D|}\right)$ is negative, which implies that $\partial x_2^*/\partial p_2 < 0$.

Problem 2

A competitive firm seeks to maximize its profit given by:

$$\pi = py - wL - rK$$

where y is output, L is labor input, K is capital input, and p, w, and r are output price, wage rate, and capital rental rate, respectively. The production function is given by $y = f(L, K)$.

(a) Set up the Lagrangian for this constrained optimization problem.

(b) Derive the first-order conditions.

(c) Use the envelope theorem to find expressions for $\frac{\partial V}{\partial p}$, $\frac{\partial V}{\partial w}$, and $\frac{\partial V}{\partial r}$, where $V(p, w, r)$ is the value function (maximum profit).

(d) Provide economic interpretations for these derivatives.

Solution

(a) The Lagrangian for this profit maximization problem is:

$$\mathcal{L} = py - wL - rK + \lambda[f(L, K) - y]$$

(b) The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial y} = p - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial L} = -w + \lambda f_L(L, K) = 0$$
$$\frac{\partial \mathcal{L}}{\partial K} = -r + \lambda f_K(L, K) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = f(L, K) - y = 0$$

From these conditions, we can derive the input demand functions $L(p, w, r)$ and $K(p, w, r)$, as well as the output supply function $y(p, w, r)$.

(c) The value function (maximum profit) is:

$$V(p, w, r) = py(p, w, r) - wL(p, w, r) - rK(p, w, r)$$

Using the envelope theorem, we can find the partial derivatives:

$$\frac{\partial V}{\partial p} = \frac{\partial \mathcal{L}}{\partial p} = y(p, w, r)$$
$$\frac{\partial V}{\partial w} = \frac{\partial \mathcal{L}}{\partial w} = -L(p, w, r)$$
$$\frac{\partial V}{\partial r} = \frac{\partial \mathcal{L}}{\partial r} = -K(p, w, r)$$

(d) Economic interpretations:

- $\frac{\partial V}{\partial p} = y$: The change in maximum profit with respect to a marginal increase in output price equals the quantity of output. This is the firm's supply function.
- $\frac{\partial V}{\partial w} = -L$: The change in maximum profit with respect to a marginal increase in wage rate equals the negative of labor input. This is the firm's labor demand function (with a negative sign).

- $\frac{\partial V}{\partial r} = -K$: The change in maximum profit with respect to a marginal increase in capital rental rate equals the negative of capital input. This is the firm's capital demand function (with a negative sign).

These relationships are collectively known as Hotelling's lemma, which states that the derivatives of the profit function with respect to prices yield the supply and input demand functions.