2025 年数理经济学笔记

授课: 杨佳楠老师

作者：徐靖
组织：PKU
时间：Febuary 27, 2025
声明：请勿用于个人学习外其他用途!

个人笔记, 如有谬误, 欢迎指正! 联系方式: 2200012917@stu.pku.edu.cn

目录

第 1 章 Topology of $\mathbb{R}^N$[1]

Keywords

- ❏ Topology 拓扑
- ❏ Metric Space 度量空间
- ❏ Convergence 收敛
- ❏ interior 内部
- ❏ closure 闭包
- ❏ boundary 边界
- ❏ compact set 紧集
- ❏ cluster point 聚点
- ❏ Lipschitz continuity 利普希茨连续
- ❏ semicontinuity 半连续
- ❏ Bolzano-Weierstrass Theorem 博尔扎诺-魏尔斯特拉斯定理
- ❏ Heine-Borel Theorem 海涅-波雷尔定理
- ❏ Contraction Mapping Theorem 压缩映射定理
- ❏ Intermediate Value Theorem 中值定理

1.1 Metric Spaces

1.1.1 Definition of Metric Spaces

定义 1.1

Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** (or **distance**) on X if :

1. (positivity) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
2. (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

♣

A set X together with a metric d is called a **metric space**, denoted by (X, d).

1.1.2 Examples of metrics in $\mathbb{R}^N$

- **Euclidean metric**: $d(x, y) = \sqrt{\sum_{i=1}^N (x_i - y_i)^2}$.
- L^p **metric** (for $p \geq 1$): $d(x, y) = (\sum_{i=1}^N |x_i - y_i|^p)^{1/p}$.
- **Sup norm** (when $p = \infty$): $d(x, y) = \max_{i=1}^N |x_i - y_i|$.

1.2 Convergence of sequences

1.2.1 Definition of Convergence

定义 1.2

Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to **converge** to a point $x \in X$ if for every $\epsilon > 0$, there exists an integer N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$. A sequence that converges is called **convergent**, otherwise it is called **divergent**.

♣

[1] 点集拓扑对应数分高代这一级别的数学基础课, 只有 sms 和图班的同学学过, 所以我简单记一下, 无需关注证明, 数理经济学只用到结论

定义 1.3

When metric space is $\mathbb{R}^N$, we say that $\{x_n\}$ is **bounded** if there exists a real number M such that $\|x_k\| \le M$ for all n. ♣

1.2.2 Cauchy Sequences and Complete Metric Spaces

- **Cauchy sequence**: A sequence $\{x_n\}$ in a metric space (X, d) is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists an integer N such that $d(x_n, x_m) < \epsilon$ for all $n, m \ge N$.
- **Complete metric space**: A metric space (X, d) is called **complete** if every Cauchy sequence in X converges to a point in X.

定理 1.1

Any convergent sequence in a metric space is a Cauchy sequence. ♡

1.2.3 Example: Cauchy Sequence Not Convergent in $\mathbb{Q}$

Consider the metric space $(\mathbb{Q}, d)$, where $d(x, y) = |x - y|$.

Fibonacci sequence : Let $\{F_k\}$ be the Fibonacci sequence, defined by

$$F_1 = F_2 = 1, F_{k+1} = F_k + F_{k-1}, k \ge 2$$

A Special Sequence: Define $a_k = \frac{F_{k+1}}{F_k}$. Then $\{a_k\}$ is a Cauchy sequence in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$.

1.2.4 Properties of Convergent Sequences in $\mathbb{R}^N$

Consider $\mathbb{R}^N$ with the Euclidean metric. Let $\{x_n\}$ and $\{y_n\}$ be two sequences.

- **Preservation of Addition/Subtraction**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty}(x_n \pm y_n) = x \pm y$.
- **Preservation of Multiplication**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$.
- **Preservation of Division**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.
- **Preservation of Inequality**: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $x_n \le y_n$ for all n implies $x \le y$.

1.2.5 Properties of Sequences in $\mathbb{R}^N$

性质 A convergent sequence in $\mathbb{R}^N$ is bounded.

A sequence $\{x_{n_k}\}$ is called a **subsequence** of $\{x_n\}$ if $n_1 < n_2 < n_3 < \cdots$.

性质 subsequences of a convergent sequence in $\mathbb{R}^N$ also converge to the same limit.

1.2.6 Limit Superior and Limit Inferior

定义 1.4

Let $\{x_n\}$ be a sequence in $\mathbb{R}^N$. The **limit superior** of $\{x_n\}$ is defined by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\ge n} x_k\right)$$

The **limit inferior** of $\{x_n\}$ is defined by

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\ge n} x_k\right)$$

♣

1.3 Topological properties

1.3.1 Open and Closed Sets

定义 1.5

In a metric space (X, d), a set $U \subset X$ is called **open** if for every $x \in U$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A set $F \subset X$ is called **closed** if its complement $F^c \stackrel{\text{def}}{=} X \backslash F$ is open. ♣

性质 For open sets:

1. The union of any collection of open sets is open.
2. The intersection of finitely many open sets is open.

For closed sets:

1. The intersection of any collection of closed sets is closed.
2. The union of finitely many closed sets is closed.

1.3.2 Interior, Closure, and Boundary of Sets

定义 1.6

The **interior** of a set $A \subset X$ is defined as:

$$\text{int}(A) = \bigcup\{U \subset A : U \text{ is open}\}$$

The **closure** of a set $A \subset X$ is defined as:

$$\overline{A} = \bigcap\{F \supset A : F \text{ is closed}\}$$

The **boundary** of a set $A \subset X$ is defined as:

$$\partial A = \overline{A} \backslash \text{int}(A)$$

♣

命题 1.1

- $A \subset X$ is open if and only if $\partial A \subset A$.
- $A \subset X$ is closed if and only if $\partial A \subset A$.

♠

1.3.3 Bounded Sets and Compact Sets in $\mathbb{R}^N$

定义 1.7

A set $A \subset \mathbb{R}^N$ is called **bounded** if there exists a real number M such that $\|x\| \leq M$ for all $x \in A$.
A set $A \subset \mathbb{R}^N$ is called **compact** if for any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ that converges to a point in A. ♣

定理 1.2 (Heine-Borel Theorem)

In $\mathbb{R}^N$, a set A is compact if and only if it is closed and bounded. ♡

1.4 Continuous functions

1.4.1 Cluster Points in Metric Spaces

定义 1.8

Let (X, d) be a metric space and $A \subset X$. A point $x \in X$ is called a **cluster point** of A if for every $\epsilon > 0$, there exists a point $y \in A$ such that $d(x, y) < \epsilon$ and $x \neq y$.
Equivalently, x is a cluster point of A if there exists a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = x$ and $x_n \neq x$ for all n. ♣

1.4.2 Limits of Functions at Cluster Points

定义 1.9

Let (X, d) and (Y, ρ) be metric spaces, $A \subset X$, $f : A \to Y$, and x be a cluster point of A. We say that f has a **limit** $y \in Y$ at x if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\rho(f(x_0), y) < \epsilon$ for all $x_0 \in A$ such that $0 < d(x_0, x) < \delta$.
Equivalently, using neighborhoods: f has a limit y at x if for every neighborhood V of y, there exists a neighborhood U of x such that $f(U \cap A) \subset V$. ♣

性质

1. $\lim_{x\to\bar{x}} f(x) = f(\bar{x})$ if and only if for every sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = \bar{x}$, we have $\lim_{n\to\infty} f(x_n) = f(\bar{x})$.
2. If f has a limit at x, then the limit is unique.

1.4.3 Continuity of Functions

定义 1.10

Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$.

- f is **continuous at** $\bar{x} \in X$ if:
$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in X, d(x, \bar{x}) < \delta \Rightarrow \rho(f(x), f(\bar{x})) < \varepsilon$$
Equivalently:
$$\forall \varepsilon > 0, \exists \delta > 0 : f(B_\delta(\bar{x})) \subseteq B_\varepsilon(f(\bar{x}))$$
- f is **continuous on** X (or simply **continuous**) if:
$$\forall \bar{x} \in X, f \text{ is continuous at } \bar{x}$$
♣

命题 1.2

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $x \in X$. The following are equivalent:

1. f is continuous at x.
2. For every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.
3. For every open set $V \subset Y$, $f^{-1}(V)$ is open in X. ♠

1.4.4 Bolzano-Weierstrass Theorem

定理 1.3 (Bolzano-Weierstrass Theorem)

If $K \subset \mathbb{R}^N$ is compact and nonempty, and $f : K \to \mathbb{R}^M$ is continuous, then :

1. $f(K)$ is compact.
2. f attains its maximum and minimum on K.

♡

1.4.5 Semicontinuity

定义 1.11

For $f : \mathbb{R}^N \to \mathbb{R}^M$:

- f is **upper semicontinuous** at x if :

$$f(x) \leq \limsup_{y \to x} f(y) \text{ for all } x \in \mathbb{R}^N$$

- f is **lower semicontinuous** at x if :

$$f(x) \geq \liminf_{y \to x} f(y) \text{ for all } x \in \mathbb{R}^N$$

♣

注 f is upper semicontinuous $\Leftrightarrow -f$ is lower semicontinuous.

定理 1.4 (Extrema of semicontinuous Functions)

Let $K \subset \mathbb{R}^N$ be compact and $f : K \to \mathbb{R}$ be upper semicontinuous. Then f attains its maximum on K. If f is lower semicontinuous, then f attains its minimum on K.

♡

1.4.6 Lipschitz Continuity

定义 1.12

A function $f : \mathbb{R}^N \to \mathbb{R}^M$ is called **Lipschitz continuous** if there exists a constant $K > 0$ such that:

$$\|f(x) - f(y)\| \leq K\|x - y\| \text{ for all } x, y \in \mathbb{R}^N$$

where K is called the **Lipschitz constant** of f. If $K < 1$, then f is called a **contraction mapping**.

♣

注 Lipschitz continuity implies uniform continuity, but the converse is not true. For example, $f(x) = x^2$.

定理 1.5 (Contraction Mapping Theorem)

Let (X, d) be a complete metric space and $f : X \to X$ be a contraction mapping. Then f has a unique fixed point $x^* \in X$, i.e., $f(x^*) = x^*$.

♡

定理 1.6 (Intermediate Value Theorem)

Let $f : D \to \mathbb{R}$ be a continuous function and $D \subset \mathbb{R}$. If :

- $[a, b] \subset D$ (closed interval)
- y is between $f(a)$ and $f(b)$

then there exists a point $c \in [a, b]$ such that $f(c) = y$.

♡