Homework 4 Solutions

Due: March 28th, 2025 (in class)

Problem 1

Let $f(x_1, x_2) = x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_2$ and consider the optimization problem

$$\textit{minimize} \quad f(x_1, x_2)$$

Try start the gradient algorithm from $x_0 = (0, 0)$, and perform three iterations to find x_3

Solution

Step 1: Iteration 0

- Compute $g_0 = \nabla f(x_0) = \nabla f(0, 0) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$
- Define $\phi(t) = f(x_0 - tg_0) = f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - t\begin{pmatrix} 0 \\ -2 \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 \\ 2t \end{pmatrix}\right)$
- $\phi(t) = f(0, 2t) = 0^2 + 2(2t)^2 - 2(0)(2t) - 2(2t) = 8t^2 - 4t$
- Minimize $\phi(t)$ for $t > 0$: $\phi'(t) = 16t - 4 = 0 \implies t = \frac{1}{4}$
- The optimal step size is $t_0 = \frac{1}{4}$
- Update: $x_1 = x_0 - t_0 g_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$

Step 2: Iteration 1

- Compute $g_1 = \nabla f(x_1) = \nabla f(0, 0.5) = \begin{pmatrix} 2(0) - 2(0.5) \\ 4(0.5) - 2(0) - 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
- Define $\phi(t) = f(x_1 - tg_1) = f\left(\begin{pmatrix} 0 \\ 0.5 \end{pmatrix} - t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} t \\ 0.5 \end{pmatrix}\right)$
- $\phi(t) = f(t, 0.5) = t^2 + 2(0.5)^2 - 2t(0.5) - 2(0.5) = t^2 - t + 0.5 - 1 = t^2 - t - 0.5$
- Minimize $\phi(t)$ for $t > 0$: $\phi'(t) = 2t - 1 = 0 \implies t = \frac{1}{2}$
- The optimal step size is $t_1 = \frac{1}{2}$
- Update: $x_2 = x_1 - t_1 g_1 = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

Step 3: Iteration 2

- Compute $g_2 = \nabla f(x_2) = \nabla f(0.5, 0.5) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

- Define $\phi(t) = f(x_2 - tg_2) = f\left(\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} - t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 0.5 \\ 0.5+t \end{pmatrix}\right)$
- $\phi(t) = f(0.5, 0.5+t) = (0.5)^2 + 2(0.5+t)^2 - 2(0.5)(0.5+t) - 2(0.5+t) = 0.25 + 2(0.25+t+t^2) - (0.5+t) - 2(0.5+t) = 0.25 + 0.5 + 2t + 2t^2 - 0.5 - t - 1 - 2t = 0.25 - 1 - t + 2t^2 = 2t^2 - t - 0.75$
- Minimize $\phi(t)$ for $t > 0$: $\phi'(t) = 4t - 1 = 0 \implies t = \frac{1}{4}$
- The optimal step size is $t_2 = \frac{1}{4}$
- Update: $x_3 = x_2 - t_2 g_2 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.75 \end{pmatrix}$

We would continue in this manner, and after several more iterations, the algorithm would converge to the minimum $(1, 1)$.

Final Solution The minimum of $f(x_1, x_2) = x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_2$ occurs at $(1, 1)$ with value $f(1, 1) = 1 + 2 - 2 - 2 = -1$.

Problem 2

Consider a simple Cournot duopoly model in the previous homework, in which the inverse demand for a good is

$$P(q) = q^{-1/\eta}$$

and the two firms producing the good face cost functions

$$C_i(q_i) = \frac{1}{2}c_i q_i^2, \quad \text{for } i = 1, 2$$

1. Base on your answer to Problem 2 in Homework 3, assuming $\eta = 1.6$, $c_1 = 0.6$, and $c_2 = 0.8$, with an initial guess of $q_1 = q_2 = 0.2$, write down the procedure of finding the numerical answer with Newton Method with tolerance level 1.510^8.

Solution

Given $\eta = 1.6$, $c_1 = 0.6$, $c_2 = 0.8$, and initial guess $q_1 = q_2 = 0.2$, we will apply the Newton method to find the equilibrium values.

Let's define a system of equations $F(q) = 0$ where:

$$F_1(q_1, q_2) = (q_1 + q_2)^{-1/1.6}\left(1 - \frac{q_1}{1.6(q_1 + q_2)}\right) - 0.6q_1$$
$$F_2(q_1, q_2) = (q_1 + q_2)^{-1/1.6}\left(1 - \frac{q_2}{1.6(q_1 + q_2)}\right) - 0.8q_2$$

The Newton method requires the Jacobian matrix:

$$J(q) = \begin{pmatrix} \frac{\partial F_1}{\partial q_1} & \frac{\partial F_1}{\partial q_2} \\ \frac{\partial F_2}{\partial q_1} & \frac{\partial F_2}{\partial q_2} \end{pmatrix}$$

The Newton iteration is given by:

$$q^{(k+1)} = q^{(k)} - J(q^{(k)})^{-1} F(q^{(k)})$$

Starting with $q^{(0)} = (0.2, 0.2)$ and iterating until $\|F(q^{(k)})\| < 1.5 \times 10^{-8}$.

The Newton method procedure:

1. Initialize $q^{(0)} = (0.2, 0.2)$
2. For $k = 0, 1, 2, \ldots$ until convergence:
   (a) Compute $F(q^{(k)})$
   (b) Compute the Jacobian $J(q^{(k)})$
   (c) Compute $\Delta q^{(k)} = -J(q^{(k)})^{-1} F(q^{(k)})$
   (d) Update $q^{(k+1)} = q^{(k)} + \Delta q^{(k)}$
   (e) Check if $\|F(q^{(k+1)})\| < 1.5 \times 10^{-8}$. If yes, stop; otherwise, continue.

Using numerical calculations, the equilibrium quantities converge to:

$$q_1^* \approx 0.8396$$
$$q_2^* \approx 0.6888$$

Problem 3

Solve the following constrained maximization and minimization problems:

1. $\max y = x_1^{0.25} x_2^{0.75}$ subject to $2x_1^2 + 5x_2^2 = 10$
2. $\min y = 2x_1 + 4x_2$ subject to $x_1^{0.25} x_2^{0.75} = 10$
3. $\max y = x_1 + x_2$ subject to $x_1^2 + 2x_2^2 + x_3^2 = 1, \quad x_1 + x_2 + x_3 = 1$

Solution

1. $\max y = x_1^{0.25} x_2^{0.75}$ subject to $2x_1^2 + 5x_2^2 = 10$

The Lagrangian function is

$$\mathcal{L} = x_1^{0.25} x_2^{0.75} + \lambda(10 - 2x_1^2 - 5x_2^2)$$

The first-order conditions are:

$$0.25 x_1^{-0.75} x_2^{0.75} - 4\lambda x_1 = 0$$

$$0.75 x_1^{0.25} x_2^{-0.25} - 10\lambda x_2 = 0$$

$$10 - 2x_1^2 - 5x_2^2 = 0$$

Solving the first two of these equations to eliminate λ yields

$$x_2^2 = 1.2 x_1^2$$

Inserting into the third equation gives the solution

$$x_1^* = \frac{\sqrt{5}}{2} \quad x_2^* = \frac{\sqrt{6}}{2}$$

2. $\min y = 2x_1 + 4x_2$ subject to $x_1^{0.25} x_2^{0.75} = 10$

The Lagrangian function is

$$\mathcal{L} = 2x_1 + 4x_2 + \lambda(10 - x_1^{0.25} x_2^{0.75})$$

The first-order conditions are

$$
\begin{aligned}
2 - 0.25\lambda x_1^{-0.75} x_2^{0.75} &= 0 \\
4 - 0.75\lambda x_1^{0.25} x_2^{-0.25} &= 0 \\
10 - x_1^{0.25} x_2^{0.75} &= 0
\end{aligned}
$$

Solving the first two of these equations to eliminate λ yields

$$x_2 = 1.5x_1$$

Inserting into the third equation gives

$$x_1^* = 7.38 \quad x_2^* = 11.07$$

3. $\max y = x_1 + x_2$ subject to $x_1^2 + 2x_2^2 + x_3^2 = 1, \quad x_1 + x_2 + x_3 = 1$

The Lagrangrean function with two multipliers λ_1 and λ_2 is:

$$\mathcal{L}(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1 + x_2 + \lambda_1(1 - x_1^2 - 2x_2^2 - x_3^2) + \lambda_2(1 - x_1 - x_2 - x_3)$$

First-order conditions:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= 1 - 2\lambda_1 x_1 - \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= 1 - 4\lambda_1 x_2 - \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_3} &= -2\lambda_1 x_3 - \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= 1 - x_1^2 - 2x_2^2 - x_3^2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_2} &= 1 - x_1 - x_2 - x_3 = 0
\end{aligned}
$$

From the first three equations:

$$
\begin{aligned}
1 - 2\lambda_1 x_1 - \lambda_2 = 0 \quad &\Rightarrow \quad 2\lambda_1 x_1 = 1 - \lambda_2 \\
1 - 4\lambda_1 x_2 - \lambda_2 = 0 \quad &\Rightarrow \quad 4\lambda_1 x_2 = 1 - \lambda_2 \\
-2\lambda_1 x_3 - \lambda_2 = 0 \quad &\Rightarrow \quad x_3 = -\frac{\lambda_2}{2\lambda_1}
\end{aligned}
$$

From the first two of these equations:

$$\frac{x_1}{x_2} = \frac{1 - \lambda_2}{2\lambda_1} \cdot \frac{4\lambda_1}{1 - \lambda_2} = 2 \quad \Rightarrow \quad x_1 = 2x_2$$

Using the constraint $x_1 + x_2 + x_3 = 1$:

$$
\begin{aligned}
2x_2 + x_2 + x_3 &= 1 \\
3x_2 + x_3 &= 1
\end{aligned}
$$

Substituting for x_1 in the quadratic constraint:

$$
\begin{aligned}
(2x_2)^2 + 2x_2^2 + x_3^2 &= 1 \\
4x_2^2 + 2x_2^2 + x_3^2 &= 1 \\
6x_2^2 + x_3^2 &= 1
\end{aligned}
$$

From $3x_2 + x_3 = 1$, we get $x_3 = 1 - 3x_2$. Substituting:

$$\begin{aligned}
6x_2^2 + (1 - 3x_2)^2 &= 1 \\
6x_2^2 + 1 - 6x_2 + 9x_2^2 &= 1 \\
15x_2^2 - 6x_2 &= 0 \\
3x_2(5x_2 - 2) &= 0
\end{aligned}$$

This gives $x_2 = 0$ or $x_2 = \frac{2}{5}$. If $x_2 = 0$, then $x_3 = 1$ and $x_1 = 0$, making $y = 0$.
If $x_2 = \frac{2}{5}$, then $x_1 = 2x_2 = \frac{4}{5}$ and $x_3 = 1 - 3x_2 = 1 - \frac{6}{5} = -\frac{1}{5}$.
We need to verify if this critical point satisfies the constraints:

$$\begin{aligned}
x_1^2 + 2x_2^2 + x_3^2 &= \left(\frac{4}{5}\right)^2 + 2\left(\frac{2}{5}\right)^2 + \left(-\frac{1}{5}\right)^2 \\
&= \frac{16}{25} + \frac{8}{25} + \frac{1}{25} = 1\checkmark \\
x_1 + x_2 + x_3 &= \frac{4}{5} + \frac{2}{5} - \frac{1}{5} = 1\checkmark
\end{aligned}$$

The maximum value is:

$$y = x_1 + x_2 = \frac{4}{5} + \frac{2}{5} = \frac{6}{5} = 1.2$$

Therefore, the maximum value of the objective function is $\boxed{y = \frac{6}{5}}$ at the point $\left(\frac{4}{5}, \frac{2}{5}, -\frac{1}{5}\right)$.