Homework 1

Due: March 7th, 2025 (in class)

Problem 1

Let P be a matrix such that $P^2 = P$. Show that the eigenvalues of P are either 0 or 1.

Solution

Let λ be an eigenvalue of P with corresponding eigenvector $v \neq 0$. This means:

$$Pv = \lambda v$$

Since $P^2 = P$, we can multiply both sides of the eigenvalue equation by P:

$$\begin{aligned}
P(Pv) &= P(\lambda v) \\
P^2 v &= \lambda P v \\
Pv &= \lambda P v \quad \text{(using the fact that } P^2 = P\text{)} \\
\lambda v &= \lambda P v \quad \text{(substituting } Pv = \lambda v\text{)} \\
\lambda v &= \lambda^2 v
\end{aligned}$$

Since $v \neq 0$ (as it is an eigenvector), we can cancel v on both sides to get:

$$\lambda = \lambda^2$$

This quadratic equation $\lambda^2 - \lambda = 0$ or $\lambda(\lambda - 1) = 0$ has only two solutions: $\lambda = 0$ or $\lambda = 1$. Therefore, any eigenvalue of a matrix P that satisfies $P^2 = P$ must be either 0 or 1.

Problem 2

Let A be symmetric. Show that A is positive definite if and only if all eigenvalues of A are positive.

Solution

We need to prove both directions of the if and only if statement.

(⇒) If A is positive definite, then all eigenvalues of A are positive.

Let A be positive definite, which means that for any non-zero vector x, we have $x^T A x > 0$. Let λ be an eigenvalue of A with corresponding eigenvector $v \neq 0$. Then:

$$Av = \lambda v$$

Consider the quadratic form $v^T Av$:

$$\begin{aligned} v^T Av &= v^T(\lambda v) \\ &= \lambda v^T v \\ &= \lambda \|v\|^2 \end{aligned}$$

Since v is an eigenvector, it is non-zero, so $\|v\|^2 > 0$. Since A is positive definite, we know that $v^T Av > 0$. Therefore:

$$\begin{aligned} \lambda \|v\|^2 &> 0 \\ \lambda &> 0 \end{aligned}$$

This shows that all eigenvalues of A must be positive.

($\Leftarrow$) If all eigenvalues of A are positive, then A is positive definite.

Let's assume all eigenvalues of A are positive. Since A is symmetric, we can diagonalize it using an orthogonal matrix Q:

$$A = QDQ^T$$

where D is a diagonal matrix containing the eigenvalues of A, and Q is an orthogonal matrix ($Q^T Q = I$) whose columns are the eigenvectors of A.

For any non-zero vector x, consider the quadratic form $x^T Ax$:

$$\begin{aligned} x^T Ax &= x^T(QDQ^T)x \\ &= (Q^T x)^T D(Q^T x) \end{aligned}$$

Let $y = Q^T x$. Since Q is orthogonal, if $x \neq 0$, then $y \neq 0$. Now:

$$\begin{aligned} x^T Ax &= y^T Dy \\ &= \sum_{i=1}^{n} \lambda_i y_i^2 \end{aligned}$$

where λ_i are the eigenvalues of A and y_i are the components of y.

Since all $\lambda_i > 0$ (by our assumption) and at least one $y_i \neq 0$ (since $y \neq 0$), we have:

$$\sum_{i=1}^{n} \lambda_i y_i^2 > 0$$

Thus, $x^T Ax > 0$ for all non-zero vectors x, which means A is positive definite.

Therefore, A is positive definite if and only if all eigenvalues of A are positive.

Problem 3

Let A be symmetric and positive semidefinite. Show that there exists a symmetric and positive semidefinite matrix B such that $A = B^2$.

Solution

Since A is symmetric, it can be diagonalized by an orthogonal matrix. That is, there exists an orthogonal matrix P such that:

$$A = PDP^T$$

where D is a diagonal matrix containing the eigenvalues of A.

Since A is positive semidefinite, all its eigenvalues are non-negative. Let's denote the diagonal elements of D as $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_i \geq 0$ for all i.

We can define a diagonal matrix $D^{1/2}$ whose diagonal elements are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}$. Note that since $\lambda_i \geq 0$, these square roots are all well-defined real numbers.

Now, let's define matrix B as:

$$B = PD^{1/2}P^T$$

Let's verify that B satisfies our requirements:

1. First, we show that B is symmetric:

$$\begin{aligned} B^T &= (PD^{1/2}P^T)^T \\ &= (P^T)^T(D^{1/2})^T P^T \\ &= PD^{1/2}P^T \\ &= B \end{aligned}$$

where we used the fact that $D^{1/2}$ is diagonal, so $(D^{1/2})^T = D^{1/2}$.

2. Next, we show that $B^2 = A$:

$$\begin{aligned} B^2 &= B \cdot B \\ &= (PD^{1/2}P^T)(PD^{1/2}P^T) \\ &= PD^{1/2}(P^T P)D^{1/2}P^T \\ &= PD^{1/2}ID^{1/2}P^T \quad \text{(since } P \text{ is orthogonal, } P^T P = I) \\ &= PD^{1/2}D^{1/2}P^T \\ &= PDP^T \\ &= A \end{aligned}$$

3. Finally, we show that B is positive semidefinite. For any non-zero vector x:

$$\begin{aligned} x^T Bx &= x^T(PD^{1/2}P^T)x \\ &= (P^T x)^T D^{1/2}(P^T x) \end{aligned}$$

Let $y = P^T x$. Since P is orthogonal, if $x \neq 0$, then $y \neq 0$. Now:

$$\begin{aligned} x^T Bx &= y^T D^{1/2} y \\ &= \sum_{i=1}^{n} \sqrt{\lambda_i} \cdot y_i^2 \end{aligned}$$

Since all $\lambda_i \geq 0$, we have $\sqrt{\lambda_i} \geq 0$ for all i. Since $y \neq 0$, at least one $y_i \neq 0$. Therefore:

$$\sum_{i=1}^{n} \sqrt{\lambda_i} \cdot y_i^2 \geq 0$$

This shows that $x^T Bx \geq 0$ for all non-zero vectors x, which means B is positive semidefinite.

Therefore, we have shown that there exists a symmetric and positive semidefinite matrix $B = PD^{1/2}P^T$ such that $A = B^2$.

Problem 4

Let (X, d) be a metric space, and $\mathcal{A}$ be a collection of subsets of X. Prove the following properties for open and closed sets:

1. If $A_\alpha \in \mathcal{A}$ is open $\forall \alpha \in I$, then $\displaystyle\bigcup_{\alpha \in I} A_\alpha$ is open. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ are open, then $\displaystyle\bigcap_{i=1}^{n} A_i$ is open.
2. If $A_\alpha \in \mathcal{A}$ is closed $\forall \alpha \in I$, then $\displaystyle\bigcap_{\alpha \in I} A_\alpha$ is closed. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ are closed, then $\displaystyle\bigcup_{i=1}^{n} A_i$ is closed.

Solution

Part 1: Properties of Open Sets

(a) Arbitrary Union of Open Sets is Open: Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of open sets in the metric space (X, d). We need to show that $\bigcup_{\alpha \in I} A_\alpha$ is open.

Let $x \in \bigcup_{\alpha \in I} A_\alpha$. Then $x \in A_\beta$ for some $\beta \in I$. Since A_β is open, there exists $\epsilon > 0$ such that the open ball $B(x, \epsilon) \subset A_\beta$. But $A_\beta \subset \bigcup_{\alpha \in I} A_\alpha$, so $B(x, \epsilon) \subset \bigcup_{\alpha \in I} A_\alpha$.

Thus, for any point x in the union, there exists an $\epsilon > 0$ such that $B(x, \epsilon)$ is contained in the union. Therefore, $\bigcup_{\alpha \in I} A_\alpha$ is open.

(b) Finite Intersection of Open Sets is Open: Let $A_1, A_2, \ldots, A_n$ be open sets in (X, d). We need to show that $\bigcap_{i=1}^{n} A_i$ is open.

Let $x \in \bigcap_{i=1}^{n} A_i$. Then $x \in A_i$ for all $i \in \{1, 2, \ldots, n\}$. Since each A_i is open, for each i there exists $\epsilon_i > 0$ such that $B(x, \epsilon_i) \subset A_i$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Since we're taking the minimum of a finite number of positive values, $\epsilon > 0$.

Now, for any $i \in \{1, 2, \ldots, n\}$, we have $B(x, \epsilon) \subset B(x, \epsilon_i) \subset A_i$. Therefore, $B(x, \epsilon) \subset \bigcap_{i=1}^{n} A_i$.

Thus, for any point x in the intersection, there exists an $\epsilon > 0$ such that $B(x, \epsilon)$ is contained in the intersection. Therefore, $\bigcap_{i=1}^{n} A_i$ is open.

Part 2: Properties of Closed Sets

(a) Arbitrary Intersection of Closed Sets is Closed: Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of closed sets in the metric space (X, d). We need to show that $\bigcap_{\alpha \in I} A_\alpha$ is closed.

A set is closed if and only if its complement is open. Let's consider:

$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$

Since each A_α is closed, each A_α^c is open. By part 1(a), the arbitrary union of open sets is open. Therefore, $\bigcup_{\alpha \in I} A_\alpha^c$ is open, which means $\bigcap_{\alpha \in I} A_\alpha$ is closed.

(b) Finite Union of Closed Sets is Closed: Let $A_1, A_2, \ldots, A_n$ be closed sets in (X, d). We need to show that $\bigcup_{i=1}^{n} A_i$ is closed.

Again, a set is closed if and only if its complement is open. Let's consider:

$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$

Since each A_i is closed, each A_i^c is open. By part 1(b), the finite intersection of open sets is open. Therefore, $\bigcap_{i=1}^{n} A_i^c$ is open, which means $\bigcup_{i=1}^{n} A_i$ is closed.