Math Review: Topology of $\mathbb{R}^N$

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Overview

1. Metric Spaces

2. Convergence of sequences

3. Topological properties

4. Continuous functions

Definition of a Metric

Definition

Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** (or **distance**) if:

1. (positivity) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$
2. (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
3. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

A set X together with a metric d is called a **metric space**, denoted by (X, d).

Common Metrics in $\mathbb{R}^N$

Examples of metrics in $\mathbb{R}^N$:

- **Euclidean distance:**

$$d(x, y) = \|x - y\| = \sqrt{\sum_{n=1}^{N} (x_n - y_n)^2}$$

- L^p **distance** (for $p \geq 1$):

$$d(x, y) = \left(\sum_{n=1}^{N} |x_n - y_n|^p \right)^{\frac{1}{p}}$$

- **Sup norm** (when $p = \infty$):

$$d(x, y) = \max_n |x_n - y_n|$$

Definition of Convergent Sequences

Definition

Let (X, d) be a metric space and $\{x_k\}_{k=1}^{\infty} \subset X$ be a sequence. We say that $\{x_k\}_{k=1}^{\infty}$ **converges** to $x \in X$, denoted by $x_k \to x$ or $\lim_{k \to \infty} x_k = x$, if:

A sequence that converges to some point is called a **convergent sequence**; otherwise, it is a **divergent sequence**.

Definition of Bounded Sequences

Definition

A sequence $\{x_k\}_{k=1}^{\infty}$ is **bounded** if:

When metric is defined by Euclidean distance in $\mathbb{R}^N$:

Cauchy Sequences and Complete Metric Spaces

Cauchy Sequence:
Let (X, d) be a metric space and $\{x_k\}_{k=1}^{\infty} \subset X$. The sequence $\{x_k\}_{k=1}^{\infty}$ is called a **Cauchy sequence** if:

Complete Metric Space:
A metric space (X, d) is complete if every Cauchy sequence in X converges to a point in X.

Theorem
Any convergent sequence in a metric space (X, d) is necessarily a Cauchy sequence.

Example: Cauchy Sequence Not Convergent in $\mathbb{Q}$

Consider the metric space (Q, α) of rational numbers with the metric $\alpha(x, y) = |x - y|$.
Fibonacci Sequence:
Let $\{F_k\}$ be the Fibonacci sequence defined by:

$$F_0 = 0, F_1 = 1, F_{k+1} = F_k + F_{k-1} \text{ for } k \geq 1$$

A Special Sequence:
Define $a_k = \frac{F_{k+1}}{F_k}$ for $k \geq 1$.
Key Properties:

- $\{a_k\}$ is a Cauchy sequence in (Q, α)
- However, $\{a_k\}$ does not converge in (Q, α)

Properties of Convergent Sequences in $\mathbb{R}$

Consider $\mathbb{R}$ with the Euclidean metric. Let $\{x_k\}$ and $\{y_k\}$ be two sequences:

1. **Preservation of Addition/Subtraction:**
2. **Preservation of Multiplication:**
3. **Preservation of Division:**
4. **Preservation of Inequality:**

Properties of Sequences in $\mathbb{R}^N$

Proposition *A convergent sequence is bounded.*

Proof.

$\square$

Properties of Sequences in $\mathbb{R}^N$

Subsequences:

- The sequence $\{x_{k_l}\}_{l=1}^{\infty}$ is a subsequence of $\{x_k\}_{k=1}^{\infty}$ if:

$$k_1 < k_2 < \cdots < k_l < \cdots$$

Proposition: *If $x_k \to x$ and $\{x_{k_l}\}_{l=1}^{\infty}$ is a subsequence of $\{x_k\}_{k=1}^{\infty}$, then $x_{k_l} \to x$.*

- In other words: subsequences of a convergent sequence converge to the same limit

Limit Superior and Limit Inferior

For a real sequence $\{x_k\} \subset \mathbb{R}$, define:

- $\alpha_l = \sup_{k \geq l} x_k$ (decreasing sequence)
- $\beta_l = \inf_{k \geq l} x_k$ (increasing sequence)

These sequences have limits $\alpha, \beta \in [-\infty, \infty]$:

$$\limsup_{k \to \infty} x_k := \alpha =$$

$$\liminf_{k \to \infty} x_k := \beta =$$

These are called the *limit superior* and *limit inferior* of $\{x_k\}$.

Open and Closed Balls in Metric Spaces

Definition

In a metric space (X, d), we denote by $B_r(x)$ the **open ball** with center $x \in X$ and radius $r > 0$, i.e.,

We denote by $C_r(x)$ the **closed ball** with center $x \in X$ and radius $r > 0$, i.e.,

Open and Closed Sets in Metric Spaces

Definition

Let (X, d) be a metric space.

- A set $A \subseteq X$ is **open** if
- A set $A \subseteq X$ is **closed** if its complement A^c is open, i.e.,

Properties of Open and Closed Sets

Let (X, d) be a metric space, and $\mathcal{A}$ be a collection of subsets of X.

1. For open sets:
   - Arbitrary union: If $A_\alpha \in \mathcal{A}$ is open $\forall \alpha \in I$, then $\bigcup\limits_{\alpha \in I} A_\alpha$ is open.
   - Finite intersection: If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ are open, then $\bigcap\limits_{i=1}^{n} A_i$ is open.
2. For closed sets:
   - Arbitrary intersection: If $A_\alpha \in \mathcal{A}$ is closed $\forall \alpha \in I$, then $\bigcap\limits_{\alpha \in I} A_\alpha$ is closed.
   - Finite union: If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ are closed, then $\bigcup\limits_{i=1}^{n} A_i$ is closed.

Interior, Closure, and Boundary of Sets

For any set A in a metric space, there exists a smallest closed set containing A and a largest open set contained in A.

Definitions:

- The **interior** of set A is defined as:

$$\mathrm{int}(A) =$$

- The **closure** of set A is defined as:

$$\overline{A} =$$

- The **boundary** of set A is defined as:

$$\partial A =$$

Characterization of Open and Closed Sets

Theorem

- *A set is closed if and only if* $A = \overline{A}$
- *A set is open if and only if* $A = int(A)$

Bounded Sets and Compact Sets in $\mathbb{R}^N$

Bounded Sets:

- $A \subset \mathbb{R}^N$ is *bounded* if

Compact Sets:

- $K \subset \mathbb{R}^N$ is *compact* if:
    -

Heine-Borel Theorem

Theorem (Heine-Borel)

A set $K \subset \mathbb{R}^N$ is compact if and only if it is closed and bounded.

This provides a simple characterization of compact sets in $\mathbb{R}^N$

Cluster Points in Metric Spaces

Definition

Let (X, d) be a metric space, $E \subset X$, and $\bar{x} \in X$.
$\bar{x}$ is called a **cluster point** of E if:

Equivalently:

Example

- If $X = \mathbb{R}^n$ and E is an open subset of $\mathbb{R}^n$, then every point in E is a cluster point of E.
- If $X = \mathbb{R}$ and $E = (0, 1)$, then every point in $[0, 1]$ is a cluster point of E.

Limits of Functions at Cluster Points

Definition

Let (X, d) and (Y, ρ) be metric spaces, $E \subseteq X$, $f : E \to Y$, and $\bar{x}$ be a cluster point of E.
We say $\lim\limits_{x \to \bar{x}} f(x) = y$ for some $y \in Y$ if:

Equivalently, using neighborhoods:

Then y is called the **limit** of function f at $\bar{x}$

Properties of Limits in Metric Spaces

Let (X, d) and (Y, ρ) be metric spaces, $f\colon X \to Y$, and $\bar{x}$ be a cluster point of X. Then:

1. $\lim_{x \to \bar{x}} f(x) = y$ if and only if for every sequence $\{x_k\}$ such that $x_k \to \bar{x}$ and $x_k \neq \bar{x}$ for all k, we have $f(x_k) \to y$
2. If the limit of f at $\bar{x}$ exists, then it is unique

Algebraic Properties of Limits

Let (X, d) be a metric space, $f\colon X \to \mathbb{R}$, $g : X \to \mathbb{R}$, and $\bar{x}$ be a cluster point of X. If $\lim_{x\to\bar{x}} f(x) = \alpha$ and $\lim_{x\to\bar{x}} g(x) = \beta$, then:

1.
2.
3.

Continuity in Metric Spaces

Definition

Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$.

- f is **continuous at** $\bar{x} \in X$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in X,\ d(x, \bar{x}) < \delta \Rightarrow \rho(f(x), f(\bar{x})) < \varepsilon$$

  Equivalently:

$$\forall \varepsilon > 0, \exists \delta > 0 : f(B_\delta(\bar{x})) \subseteq B_\varepsilon(f(\bar{x}))$$

- f is **continuous on** X (or simply **continuous**) if:

$$\forall \bar{x} \in X, f \text{ is continuous at } \bar{x}$$

Continuity of Vector-Valued Functions

Consider a function $f\colon E \subseteq \mathbb{R}^n \to \mathbb{R}^l$, composed of l component functions:
$f = (f^1, f^2, \ldots, f^l)$, where each component function $f^i : E \to \mathbb{R}$.
Then the function f is continuous at $x \in E$ if and only if each component function f^i is continuous at x.

Equivalent Characterizations of Continuity

Theorem

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $\bar{x} \in X$. The following statements are equivalent:

1. *f is continuous at $\bar{x}$;*
2. *For every open set $B \subseteq Y$ containing $f(\bar{x})$, there exists an open set $A \subseteq X$ containing $\bar{x}$ such that $A \subseteq f^{-1}(B)$;*
3. *For every sequence $\{x_k\}$ satisfying $x_k \to \bar{x}$, we have $f(x_k) \to f(\bar{x})$.*

Bolzano-Weierstrass Theorem

Theorem (Bolzano-Weierstrass)

If $K \subset \mathbb{R}^N$ is nonempty and compact and $f : K \to \mathbb{R}$ is continuous, then:

- *$f(K)$ is compact*
- *f attains its maximum and minimum over K*

Important for optimization: ensures existence of solutions on compact domains.

Semi-Continuous Functions

Definition: For $f : \mathbb{R}^N \to [-\infty, \infty]$:

- f is *lower semi-continuous* at x if:

$$f(x) \leq \liminf_{k \to \infty} f(x_k) \text{ for any } x_k \to x$$

- f is *upper semi-continuous* at x if:

$$f(x) \geq \limsup_{k \to \infty} f(x_k) \text{ for any } x_k \to x$$

Properties:

- f is upper semi-continuous $\iff$ $-f$ is lower semi-continuous

Semi-Continuous Functions

Theorem (Extrema of Semi-Continuous Functions)

Let K be compact. Then:

- *Lower semi-continuous $f\colon K \to [-\infty, \infty]$ attains its minimum*
- *Upper semi-continuous $f\colon K \to [-\infty, \infty]$ attains its maximum*

Lipschitz Continuity

Definition

Let (X, d) and (Y, ρ) be metric spaces, and $f\colon X \to Y$.
f is **Lipschitz continuous** if:

the constant K is called the **Lipschitz constant** of the function f.

- When a function is Lipschitz continuous with Lipschitz constant $K < 1$, it is called a **contraction mapping**.

Contraction Mapping Theorem

Theorem

Let (X, d) be a complete metric space and $f: X \to X$.

Hypothesis: f is a contraction mapping, i.e.,

Conclusion: f has a unique fixed point, i.e.,

Intermediate Value Theorem

Theorem (Intermediate Value Theorem)

Let $f : D \to \mathbb{R}$ be continuous and $D \subset \mathbb{R}$. If:

- $[a, b] \subset D$ *(closed interval)*
- y *is between* $f(a)$ *and* $f(b)$

Then $\exists c \in [a, b]$ *such that* $f(c) = y$.

Interpretation:

- If f is continuous on $[a, b]$
- Then f takes all intermediate values between $f(a)$ and $f(b)$
- That is, $y \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$ implies $y = f(c)$ for some $c \in [a, b]$

The End