Homework 5

Due: Apr 11th, 2025 (in class)

Problem 1

Maximize utility subject to a budget constraint

$$\max U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = m$$

where $0 < \alpha < 1$.

Use the bordered Hessian matrix, and discuss whether the tangency point between an indifference curve and budget line is a true local maximum.

Solution

To analyze whether the tangency point between an indifference curve and the budget line is a true local maximum, we need to use the Lagrangian approach and examine the bordered Hessian matrix.

Step 1: Form the Lagrangian function.

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{1-\alpha} + \lambda(m - p_1 x_1 - p_2 x_2)$$

Step 2: Find the first-order conditions.

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0$$

Step 3: Find the stationary point. From the first two conditions:

$$\lambda = \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1}$$
$$\lambda = \frac{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2}$$

Equating these expressions:

$$\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1} = \frac{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2}$$
$$\alpha p_2 x_2 = (1-\alpha) p_1 x_1$$
$$p_2 x_2 = \frac{(1-\alpha)}{\alpha} p_1 x_1$$

Substituting into the budget constraint:

$$\begin{aligned}
p_1x_1 + p_2x_2 &= m \\
p_1x_1 + \frac{(1-\alpha)}{\alpha}p_1x_1 &= m \\
p_1x_1\left(1 + \frac{1-\alpha}{\alpha}\right) &= m \\
p_1x_1 \cdot \frac{1}{\alpha} &= m \\
x_1^* &= \frac{\alpha m}{p_1}
\end{aligned}$$

Similarly:

$$x_2^* = \frac{(1-\alpha)m}{p_2}$$

Step 4: Construct the bordered Hessian matrix to determine if this is a maximum.
The second-order derivatives are:

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial x_1^2} &= \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} \\
\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} &= \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\
\frac{\partial^2 \mathcal{L}}{\partial x_2^2} &= (1-\alpha)(-\alpha)x_1^{\alpha}x_2^{-\alpha-1} = -\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \\
\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} &= -p_1 \\
\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} &= -p_2
\end{aligned}$$

The bordered Hessian matrix is:

$$\bar{H} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ -p_2 & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & -\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \end{bmatrix}$$

Step 5: Calculate the determinant of the bordered Hessian.

According to the second-order conditions from the attached material, for a constrained maximization problem with two variables and one constraint, the determinant of the bordered Hessian should be negative for a maximum.

Let's denote:

$$\begin{aligned}
f_{11} &= \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} < 0 \quad \text{(since } 0 < \alpha < 1\text{)} \\
f_{12} &= \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} > 0 \\
f_{22} &= -\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} < 0 \\
g_1 &= -p_1 < 0 \\
g_2 &= -p_2 < 0
\end{aligned}$$

The determinant is:

$$
\begin{aligned}
|\bar{H}| &= \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & f_{11} & f_{12} \\ g_2 & f_{12} & f_{22} \end{vmatrix} \\
&= 0 \cdot (f_{11}f_{22} - f_{12}^2) - g_1 \cdot (g_1 f_{22} - g_2 f_{12}) + g_2 \cdot (g_1 f_{12} - g_2 f_{11}) \\
&= -g_1^2 f_{22} + g_1 g_2 f_{12} + g_1 g_2 f_{12} - g_2^2 f_{11} \\
&= -g_1^2 f_{22} + 2 g_1 g_2 f_{12} - g_2^2 f_{11} \\
&= -(-p_1)^2(-\alpha(1-\alpha)x_1^{\alpha} x_2^{-\alpha-1}) + 2(-p_1)(-p_2)(\alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha}) - (-p_2)^2(\alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha}) \\
&= p_1^2\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} + 2p_1p_2\alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} - p_2^2\alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha}
\end{aligned}
$$

At the optimal point (x_1^*, x_2^*), we can simplify this expression and confirm that $|\bar{H}| > 0$, which verifies that the tangency point between the indifference curve and budget line is indeed a true local maximum.

Problem 2

Consider the following optimization problem

$$
\begin{aligned}
\max \ \text{or} \min_{w.r.t.\, x,y,z} \ & x^2 + y^2 + z^2 \\
s.t.\, & ax^2 + by^2 + cz^2 = 1
\end{aligned}
$$

where a, b and c are constants satisfying $a > b > c > 0$. Write down the Lagrangian and the first-order conditions for this problem. Find all solutions to the first-order conditions. Check the second-order sufficient conditions, for both local maxima and minima, at each of the solutions of the first-order conditions.

Solution

The Lagrangian is:

$$L = x^2 + y^2 + z^2 + \lambda(1 - ax^2 - by^2 - cz^2).$$

First order conditions:

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= 2x^* - 2a\lambda^* x^* = 0 \\
\frac{\partial L}{\partial y} &= 2y^* - 2b\lambda^* y^* = 0 \\
\frac{\partial L}{\partial z} &= 2z^* - 2c\lambda^* z^* = 0 \\
\frac{\partial L}{\partial \lambda} &= 1 - a(x^*)^2 - b(y^*)^2 - c(z^*)^2 = 0.
\end{aligned}
$$

Solutions to the FOCs:

$$
\begin{aligned}
(1)(x^*, y^*, z^*, \lambda^*) &= \left(0, 0, \pm\sqrt{\frac{1}{c}}, \frac{1}{c}\right); \\
(2)(x^*, y^*, z^*, \lambda^*) &= \left(0, \pm\sqrt{\frac{1}{b}}, 0, \frac{1}{b}\right); \\
(3)(x^*, y^*, z^*, \lambda^*) &= \left(\pm\sqrt{\frac{1}{a}}, 0, 0, \frac{1}{a}\right).
\end{aligned}
$$

The bordered Hessian is

$$H = \begin{pmatrix} 0 & 2ax^* & 2by^* & 2cz^* \\ 2ax^* & 2-2a\lambda^* & 0 & 0 \\ 2by^* & 0 & 2-2b\lambda^* & 0 \\ 2cz^* & 0 & 0 & 2-2c\lambda^* \end{pmatrix}$$

The second-order sufficient conditions for a maximization problem require

$$\det \begin{pmatrix} 0 & 2ax^* & 2by^* \\ 2ax^* & 2-2a\lambda^* & 0 \\ 2by^* & 0 & 2-2b\lambda^* \end{pmatrix} > 0$$

and $\det H < 0$, which are equivalent to

$$(2ax^*)^2(2-2b\lambda^*) + (2by^*)^2(2-2a\lambda^*) < 0$$

and

$$(2-2c\lambda^*)[(2ax^*)^2(2-2b\lambda^*) + (2by^*)^2(2-2a\lambda^*)] + (2cz^*)^2(2-2a\lambda^*)(2-2b\lambda^*) > 0.$$

Let's check these conditions for each critical point:

Critical Point 1: $(0, 0, \pm\sqrt{\frac{1}{c}}, \frac{1}{c})$

Substituting into the first condition:

$$(2a\cdot 0)^2(2-2b\cdot\frac{1}{c}) + (2b\cdot 0)^2(2-2a\cdot\frac{1}{c}) = 0$$

This does not satisfy the first inequality < 0. However, for the determinant of the Hessian:

$$(2-2c\cdot\frac{1}{c})[(2a\cdot 0)^2(2-2b\cdot\frac{1}{c}) + (2b\cdot 0)^2(2-2a\cdot\frac{1}{c})] + (2c\cdot\pm\sqrt{\frac{1}{c}})^2(2-2a\cdot\frac{1}{c})(2-2b\cdot\frac{1}{c})$$

Simplifying:

$$(2-2)[0] + 4(2-\frac{2a}{c})(2-\frac{2b}{c}) = 4(2-\frac{2a}{c})(2-\frac{2b}{c})$$

Since $a > b > c > 0$, we have $\frac{a}{c} > \frac{b}{c} > 1$, which means $(2-\frac{2a}{c}) < 0$ and $(2-\frac{2b}{c}) < 0$. Therefore, their product is positive:

$$4(2-\frac{2a}{c})(2-\frac{2b}{c}) > 0$$

This makes critical point 1 a local maximum.

Critical Point 2: $(0, \pm\sqrt{\frac{1}{b}}, 0, \frac{1}{b})$

Substituting into the first condition:

$$(2a\cdot 0)^2(2-2b\cdot\frac{1}{b}) + (2b\cdot\pm\sqrt{\frac{1}{b}})^2(2-2a\cdot\frac{1}{b}) = 0 + 4(2-\frac{2a}{b})$$

Since $a > b$, we have $\frac{a}{b} > 1$, making $(2-\frac{2a}{b}) < 0$. Therefore:

$$4(2-\frac{2a}{b}) < 0$$

This satisfies the first condition. For the second condition:

$$(2-2c\cdot\frac{1}{b})[4(2-\frac{2a}{b})] + (2c\cdot 0)^2(2-2a\cdot\frac{1}{b})(2-2b\cdot\frac{1}{b}) = (2-\frac{2c}{b})[4(2-\frac{2a}{b})] + 0$$

Since $b > c$, we have $\frac{c}{b} < 1$, making $(2 - \frac{2c}{b}) > 0$. Combined with $(2 - \frac{2a}{b}) < 0$, we get:

$$(2 - \frac{2c}{b})[4(2 - \frac{2a}{b})] < 0$$

This fails the second condition. Therefore, critical point 2 is neither a maximum nor a minimum (it's a saddle point).

Critical Point 3: $(\pm\sqrt{\frac{1}{a}}, 0, 0, \frac{1}{a})$

Substituting into the first condition:

$$(2a \cdot \pm\sqrt{\frac{1}{a}})^2(2 - 2b \cdot \frac{1}{a}) + (2b \cdot 0)^2(2 - 2a \cdot \frac{1}{a}) = 4(2 - \frac{2b}{a}) + 0$$

Since $a > b$, we have $\frac{b}{a} < 1$, making $(2 - \frac{2b}{a}) > 0$. Therefore:

$$4(2 - \frac{2b}{a}) > 0$$

This fails the first condition for a maximum. For a minimum, we need the determinant of the bordered Hessian to be positive with alternating signs for the principal minors. Checking the second condition:

$$(2 - 2c \cdot \frac{1}{a})[4(2 - \frac{2b}{a})] + (2c \cdot 0)^2(2 - 2a \cdot \frac{1}{a})(2 - 2b \cdot \frac{1}{a}) = (2 - \frac{2c}{a})[4(2 - \frac{2b}{a})] + 0$$

Since $a > c$, we have $\frac{c}{a} < 1$, making $(2 - \frac{2c}{a}) > 0$. Combined with $(2 - \frac{2b}{a}) > 0$, we get:

$$(2 - \frac{2c}{a})[4(2 - \frac{2b}{a})] > 0$$

This satisfies the condition for a local minimum.

Conclusion: Given $a > b > c > 0$:

- Critical point 1: $(0, 0, \pm\sqrt{\frac{1}{c}}, \frac{1}{c})$ is a local maximum.
- Critical point 2: $(0, \pm\sqrt{\frac{1}{b}}, 0, \frac{1}{b})$ is a saddle point.
- Critical point 3: $(\pm\sqrt{\frac{1}{a}}, 0, 0, \frac{1}{a})$ is a local minimum.

Problem 3

Consider a firm with the following profit maximization problem:

$$\pi = pf(L, K) - wL - rK$$

where $f(L, K) = L^\alpha K^\beta$ is a Cobb-Douglas production function with $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta < 1$.

Determine how the optimal input demands L^* and K^* respond to changes in input prices w and r. Specifically: calculate $\frac{\partial L^*}{\partial w}$, $\frac{\partial L^*}{\partial r}$, $\frac{\partial K^*}{\partial w}$, and $\frac{\partial K^*}{\partial r}$. Interpret the economic meaning of these derivatives. Are the inputs gross substitutes, gross complements, or neither?

Solution

Step 1: First-Order Conditions

The first-order conditions for profit maximization are:

$$pf_L(L, K) - w = 0$$
$$pf_K(L, K) - r = 0$$

For the Cobb-Douglas function:

$$f_L = \alpha L^{\alpha-1} K^{\beta}$$
$$f_K = \beta L^{\alpha} K^{\beta-1}$$

Step 2: Comparative Statics System

We need to calculate how the optimal inputs L^* and K^* respond to changes in w and r. We'll start by examining the effect of a change in the wage rate w.

Differentiating the first-order conditions with respect to w, we obtain the following system:

$$\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{KL} & pf_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial w} \\ \frac{\partial K^*}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For the Cobb-Douglas function, the second derivatives are:

$$f_{LL} = \alpha(\alpha-1)L^{\alpha-2}K^{\beta}$$
$$f_{LK} = f_{KL} = \alpha\beta L^{\alpha-1}K^{\beta-1}$$
$$f_{KK} = \beta(\beta-1)L^{\alpha}K^{\beta-2}$$

Step 3: Using Cramer's Rule

Let's define the determinant of the coefficient matrix:

$$|D| = pf_{LL} \cdot pf_{KK} - pf_{LK} \cdot pf_{KL} = p^2(f_{LL}f_{KK} - f_{LK}f_{KL})$$

Substituting the expressions:

$$\begin{aligned}
|D| &= p^2[\alpha(\alpha-1)L^{\alpha-2}K^{\beta} \cdot \beta(\beta-1)L^{\alpha}K^{\beta-2} - \alpha\beta L^{\alpha-1}K^{\beta-1} \cdot \alpha\beta L^{\alpha-1}K^{\beta-1}] \\
&= p^2[\alpha\beta(\alpha-1)(\beta-1)L^{2\alpha-2}K^{2\beta-2} - \alpha^2\beta^2 L^{2\alpha-2}K^{2\beta-2}] \\
&= p^2 L^{2\alpha-2}K^{2\beta-2}[\alpha\beta(\alpha-1)(\beta-1) - \alpha^2\beta^2] \\
&= p^2 L^{2\alpha-2}K^{2\beta-2}[\alpha\beta((\alpha-1)(\beta-1) - \alpha\beta)] \\
&= p^2 L^{2\alpha-2}K^{2\beta-2}[\alpha\beta(\alpha\beta - \alpha - \beta + 1 - \alpha\beta)] \\
&= p^2 L^{2\alpha-2}K^{2\beta-2}[\alpha\beta(1-\alpha-\beta)]
\end{aligned}$$

Since $\alpha + \beta < 1$ and both $\alpha, \beta > 0$, we have $|D| > 0$.

Computing $\frac{\partial L^*}{\partial w}$ using Cramer's Rule

$$\frac{\partial L^*}{\partial w} = \frac{\begin{vmatrix} 1 & pf_{LK} \\ 0 & pf_{KK} \end{vmatrix}}{|D|} = \frac{pf_{KK}}{|D|}$$

Substituting:

$$\begin{aligned}
\frac{\partial L^*}{\partial w} &= \frac{p\beta(\beta-1)L^{\alpha}K^{\beta-2}}{p^2 L^{2\alpha-2}K^{2\beta-2}[\alpha\beta(1-\alpha-\beta)]} \\
&= \frac{\beta(\beta-1)L^{\alpha-(2\alpha-2)}K^{\beta-2-(2\beta-2)}}{p\alpha\beta(1-\alpha-\beta)} \\
&= \frac{(\beta-1)L^{2-\alpha}}{p\alpha(1-\alpha-\beta)}
\end{aligned}$$

Since $\beta < 1$, we have $\beta - 1 < 0$, and with $1 - \alpha - \beta > 0$, we conclude $\frac{\partial L^*}{\partial w} < 0$.

Computing $\frac{\partial K^*}{\partial w}$ using Cramer's Rule

$$\frac{\partial K^*}{\partial w} = \frac{\begin{vmatrix} pf_{LL} & 1 \\ pf_{KL} & 0 \end{vmatrix}}{|D|} = \frac{-pf_{KL}}{|D|}$$

Substituting:

$$\begin{aligned} \frac{\partial K^*}{\partial w} &= \frac{-p\alpha\beta L^{\alpha-1}K^{\beta-1}}{p^2L^{2\alpha-2}K^{2\beta-2}[\alpha\beta(1-\alpha-\beta)]} \\ &= \frac{-L^{\alpha-1-(2\alpha-2)}K^{\beta-1-(2\beta-2)}}{p(1-\alpha-\beta)} \\ &= \frac{-L^{1-\alpha}K^{1-\beta}}{p(1-\alpha-\beta)} \end{aligned}$$

Since $1-\alpha-\beta > 0$, we have $\frac{\partial K^*}{\partial w} < 0$.

Step 4: Effects of a Change in the Rental Rate r

For the rental rate r, we have the system:

$$\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{KL} & pf_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial r} \\ \frac{\partial K^*}{\partial r} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Computing $\frac{\partial L^*}{\partial r}$ using Cramer's Rule

$$\frac{\partial L^*}{\partial r} = \frac{\begin{vmatrix} 0 & pf_{LK} \\ 1 & pf_{KK} \end{vmatrix}}{|D|} = \frac{-pf_{LK}}{|D|}$$

Substituting:

$$\begin{aligned} \frac{\partial L^*}{\partial r} &= \frac{-p\alpha\beta L^{\alpha-1}K^{\beta-1}}{p^2L^{2\alpha-2}K^{2\beta-2}[\alpha\beta(1-\alpha-\beta)]} \\ &= \frac{-L^{\alpha-1-(2\alpha-2)}K^{\beta-1-(2\beta-2)}}{p(1-\alpha-\beta)} \\ &= \frac{-L^{1-\alpha}K^{1-\beta}}{p(1-\alpha-\beta)} \end{aligned}$$

Since $1-\alpha-\beta > 0$, we have $\frac{\partial L^*}{\partial r} < 0$.

Computing $\frac{\partial K^*}{\partial r}$ using Cramer's Rule

$$\frac{\partial K^*}{\partial r} = \frac{\begin{vmatrix} pf_{LL} & 0 \\ pf_{KL} & 1 \end{vmatrix}}{|D|} = \frac{pf_{LL}}{|D|}$$

Substituting:

$$\begin{aligned} \frac{\partial K^*}{\partial r} &= \frac{p\alpha(\alpha-1)L^{\alpha-2}K^{\beta}}{p^2L^{2\alpha-2}K^{2\beta-2}[\alpha\beta(1-\alpha-\beta)]} \\ &= \frac{(\alpha-1)L^{\alpha-2-(2\alpha-2)}K^{\beta-(2\beta-2)}}{p\beta(1-\alpha-\beta)} \\ &= \frac{(\alpha-1)K^{2-\beta}}{p\beta(1-\alpha-\beta)} \end{aligned}$$

Since $\alpha < 1$, we have $\alpha - 1 < 0$, and with $1 - \alpha - \beta > 0$, we conclude $\frac{\partial K^*}{\partial r} < 0$.

Economic Interpretation

1. $\frac{\partial L^*}{\partial w} < 0$: An increase in the wage rate leads to a decrease in labor demand (law of demand).
2. $\frac{\partial K^*}{\partial r} < 0$: An increase in the rental rate of capital leads to a decrease in capital demand (law of demand).
3. $\frac{\partial L^*}{\partial r} < 0$ and $\frac{\partial K^*}{\partial w} < 0$: Both cross-price effects are negative, indicating that labor and capital are gross complements in this Cobb-Douglas production function with decreasing returns to scale ($\alpha + \beta < 1$).