2025 年数理经济学笔记

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目录

第 1 章 Multi-Variable Unconstrained Optimization

Keywords

- ❑ First Order Condition 一阶条件
- ❑ Newton's Method 牛顿法
- ❑ Bisection Method 二分法

1.1 First Order Condition

An Unconstrained Optimization Problem is :

$$\min_{x \in \mathbb{R}^n} f(x)$$

定义 1.1

First Order Condition (FOC): $\nabla f(x^*) = 0$.

- x^* is a **stationary point** (驻点) of f.
- It is necessary but not sufficient.

global minimum: $f(x^*) \leq f(x)$ for all $x \in \mathbb{R}^n$.

local minimum: $f(x^*) \leq f(x)$ for all $x \in B(x^*, \epsilon)$ for some $\epsilon > 0$.

♣

命题 1.1

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and $\nabla f(x^*) = 0$. If $\nabla^2 f(x^*)$ is:

- positive definite, then x^* is a local minimum.
- negative definite, then x^* is a local maximum.
- indefinite, then x^* is a **saddle point**. (鞍点)

♠

1.2 Convex Optimization

定义 1.2 (Convex Function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

♣

定理 1.1

A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its Hessian matrix $\nabla^2 f(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$.

♡

命题 1.2

Let f be differentiable. Then f is (strictly) convex if and only if:

$$f(y) - f(x)(>) \geq \nabla f(x) \cdot (y - x)$$

for all $x, y \in \mathbb{R}^n$.

♠

定理 1.2 (Minimum/maximum Characterization)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex (concave) function. Then x^* is a local minimum (maximum) if and only if:

$$\nabla f(x^*) = 0$$

- If f is strictly **convex**, then x^* is a global minimum.
- If f is strictly **concave**, then x^* is a global maximum.

♡

1.2.1 Bisection Method

定义 1.3 (Bisection Method)

A simple, robust method for finding roots of continuous functions on bounded intervals

- Start with an interval $[a, b]$ such that $f(a)f(b) < 0$.
- Compute the midpoint $c = \frac{a+b}{2}$, and evaluate $f(c)$.
- Replace a or b with c based on the sign of $f(c)$.
- Iterate until desired precision.

♣

定义 1.4 (Convergence Rate and Order 收敛速度和阶)

For iteration x_n approaching the root r, the convergence rate C and order ρ are defined as:

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^\rho} = C$$

- Linear convergence: $\rho = 1, C < 1$.
- Quadratic convergence: $\rho = 2, C < 1$.
- Superlinear convergence: $\rho > 1, C < 1$.

♣

Method	**Definition**	**Rate**	**Order**
Bisection	Iteratively bisects an interval and selects a subinterval	Linear ($C = 0.5$)	1
Secant	Root approximation via secant line through two points	Superlinear ($C \approx 1.618$)	1.618
False Position	Bisection variant with linear interpolation updates	Linear	1
Newton-Raphson	Derivative-based iterative root-finding	Quadratic ($C \propto f''$)	2
Gradient method	Function minimization via negative gradient direction	Linear ($C \propto \kappa$)	1

表 **1.1:** Compact Comparison of Numerical Methods

1.3 Numerical Optimization