Spring 2025 Mathematical Economics
Midterm Exam

Please write down the first-order condition(s) or the Lagrangian conditions clearly when solving any optimization problem.

中文说明：在求解最优化问题时，请清楚地写出一阶条件（First-order condition）或拉格朗日条件（Lagrangian conditions）。

Question 1 (Total 25 points)

Consider the function $f\colon [0,1] \to [0,1]$ defined by:

$$f(x) = \frac{1}{2}x(1-x)$$

考虑函数 $f\colon [0,1] \to [0,1]$，定义如下：

$$f(x) = \frac{1}{2}x(1-x)$$

(a) (5 points) Show that f is a contraction mapping.
中文翻译：证明函数 f 为压缩映射。

(b) (10 points) Determine the fixed point(s) of the function f and verify its uniqueness.
中文翻译：求函数 f 的不动点，并验证其唯一性。

(c) (10 points) Prove that for any $K \in (0,1)$, the function

$$g(x) = Kx(1-x)$$

also has a unique fixed point in the interval $[0,1]$, and find this fixed point.
中文翻译：证明对于任意 $K \in (0,1)$，函数

$$g(x) = Kx(1-x)$$

在区间 $[0,1]$ 内仅有唯一不动点，并求出该不动点。

Solution

Part 1: Show that f is a contraction mapping

To show that f is a contraction mapping, we need to show that there exists a constant $K \in [0,1)$ such that for all $x, y \in [0,1]$,

$$|f(x) - f(y)| \le K|x-y|.$$

Let $x, y \in [0,1]$. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{1}{2}x(1-x) - \frac{1}{2}y(1-y)\right| \\ &= \frac{1}{2}|x-y||1-(x+y)|. \end{aligned}$$

Since $x, y \in [0,1]$, we have $0 \le x+y \le 2$, thus $-1 \le 1-(x+y) \le 1$. Therefore,

$$|1-(x+y)| \le 1,$$

and hence,

$$|f(x)-f(y)| \le \frac{1}{2}|x-y|.$$

Thus, f is a contraction mapping with $K = \frac{1}{2}$.
评分标准：未按照准确的压缩映射定义酌情给 0-1 分，不动点计算错误扣 3-4 分。

Part 2: Determine the fixed point(s) of the function f and verify its uniqueness

A fixed point x^* of f satisfies $f(x^*) = x^*$. Therefore,

$$\frac{1}{2}x^*(1-x^*) = x^*.$$

Rearranging gives us the quadratic equation

$$x^*(1+x^*) = 0.$$

Thus, $x^* = 0$ or $x^* = -1$.

To verify uniqueness, we use the fact that f is a contraction mapping. By the contraction mapping theorem, a contraction mapping on a complete metric space has a unique fixed point. Since $[0,1]$ is complete and f is a contraction, f has a unique fixed point in $[0,1]$. Since -1 is outside the interval [0, 1]. Therefore, the fixed point is $x^* = 0$.
评分标准：此处不需要说明压缩映射定理，通过计算得出可行的不动点只有一个即为满分。

Part 3: Prove that for any $K \in (0,1)$, the function $g(x) = Kx(1-x)$ also has a unique fixed point in the interval $[0,1]$, and find this fixed point

Consider the function $g(x) = Kx(1-x)$. A fixed point x^* of g satisfies $g(x^*) = x^*$. Therefore,

$$Kx^*(1-x^*) = x^*.$$

Rearranging gives us the quadratic equation

$$x^*(K(1-x^*)-1) = 0.$$

Thus, $x^* = 0$ or $x^* = 1 - \frac{1}{K}$.

Since $K \in (0,1)$, $1-\frac{1}{K}$ is outside the interval $[0,1]$. Therefore, the only fixed point in $[0,1]$ is $x^* = 0$.

To prove uniqueness, we note that $g(x)$ is also a contraction mapping for $K \in (0,1)$ with $K' = K$. By the contraction mapping theorem, g has a unique fixed point in $[0,1]$. Therefore, the fixed point is $x^* = 0$.
评分标准：此处不需要说明压缩映射定理，通过不动点计算得出可行的不动点只有一个即为满分。

Question 2 (Total 20 points)

Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous with

$$\lim_{x \to \pm\infty} f(x) = +\infty.$$

设 $f\colon \mathbb{R} \to \mathbb{R}$ 为连续函数，且满足

$$\lim_{x \to \pm\infty} f(x) = +\infty$$

(a) (10 points) Use Weierstrass' theorem to prove that f has a minimum.
中文翻译：利用 Weierstrass 定理证明连续函数 f 存在最小值。

(b) (10 points) Conclude that every polynomial of even degree has either a minimum or a maximum, respectively, depending on the sign of the leading term.
中文翻译：推论每一偶次数多项式根据首项系数的正负分别具有最小值或最大值。

Solution

Step 1: Proving f has a minimum

Since f is continuous on $\mathbb{R}$ and $\lim_{x\to\pm\infty} f(x) = \infty$, this means that as x approaches either positive or negative infinity, the function values $f(x)$ also approach infinity.

1. Choose a real number $M > 0$: Since $\lim_{x\to\infty} f(x) = \infty$, there exists a real number $N > 0$ such that for all $x > N$, we have $f(x) > M$. Similarly, there exists N' such that for all $x < -N'$, we also have $f(x) > M$.

2. Define a closed interval $[a, b]$: Let $a = \max(-N', -N)$ and $b = \min(N, N')$. This interval $[a, b]$ is closed and bounded, therefore it is compact.

3. Apply Weierstrass' Theorem: Since f is continuous on the closed interval $[a, b]$, by Weierstrass' Theorem, f attains both a maximum and a minimum on $[a, b]$. In particular, f attains a minimum value m on $[a, b]$.

4. Conclusion: Since $f(x) > M$ for all $x \notin [a, b]$, and m is the minimum value of f on $[a, b]$, it follows that $min(m, M)$ is the minimum value of f on the entire real line $\mathbb{R}$.

评分标准：未准确使用 Weierstrass' Theorem 的酌情给 0-1 分，步骤 1-2 给 3-4 分，步骤 3 占 5 分，其中 Weierstrass' Theorem 的两个条件各占 2 分，步骤 4 给 1-2 分。

Step 2: Application to polynomials of even degree

For a polynomial $p(x)$ of even degree, we can write it in the general form:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is even and a_n is the leading coefficient.

1. Behavior as $x \to \pm\infty$: When x approaches either positive or negative infinity, the behavior of $p(x)$ is primarily determined by the highest-degree term $a_n x^n$. Since n is even, x^n is always non-negative.

2. Depending on the sign of a_n:

   - If $a_n > 0$, then $p(x) \to \infty$ as $x \to \pm\infty$. According to our proof above, $p(x)$ has a minimum.
   - If $a_n < 0$, then $p(x) \to -\infty$ as $x \to \pm\infty$. Since $p(x)$ is continuous and approaches negative infinity as $|x|$ becomes large, by Weierstrass' Theorem, $p(x)$ must attain a maximum value somewhere.

Therefore, every polynomial of even degree has either a minimum (when the leading coefficient is positive) or a maximum (when the leading coefficient is negative).

评分标准：分类讨论两种情况各占 5 分，最小值或者最大值的结论判断正确各占 1 分。对于 $a_n < 0$，若第一问证明错误此处直接写“同理可得”不给分，必须明确指出对称性，$q(x) = -p(x)$ 之类的逻辑说明，或者给出准确的证明。

Question 3 (Total 15 points)

Let

$$f(x_1, x_2) = x_1^2 - x_1x_2 - 6x_1 + x_2^3 - 3x_2.$$

(a) (5 points) Find the stationary point(s) of f.
中文翻译：求函数 $f(x_1, x_2)$ 的驻点。

(b) (10 points) Determine whether each stationary point is a local maximum, a local minimum, or a saddle point.
中文翻译：判定每个驻点是局部最大值、局部最小值，还是鞍点。

Solution

1. Find the stationary point(s) of f.

To find the stationary points, we compute the partial derivatives and set them equal to zero.

$$\frac{\partial f}{\partial x_1} = 2x_1 - x_2 - 6 = 0$$
$$\frac{\partial f}{\partial x_2} = -x_1 + 3x_2^2 - 3 = 0$$

From the first equation, we have:

$$2x_1 - x_2 - 6 = 0$$
$$\Rightarrow x_1 = \frac{x_2 + 6}{2}$$

Substituting this into the second equation:

$$-x_1 + 3x_2^2 - 3 = 0$$
$$-\frac{x_2 + 6}{2} + 3x_2^2 - 3 = 0$$
$$6x_2^2 - x_2 - 12 = 0$$

Solving this quadratic equation:

$$6x_2^2 - x_2 - 12 = 0$$
$$x_2 = \frac{1 \pm \sqrt{1 + 4 \cdot 6 \cdot 12}}{2 \cdot 6}$$
$$x_2 = \frac{1 \pm 17}{12}$$

So $x_2 = \frac{18}{12} = \frac{3}{2}$ or $x_2 = \frac{-16}{12} = -\frac{4}{3}$.

For $x_2 = \frac{3}{2}$, we have $x_1 = \frac{x_2+6}{2} = \frac{\frac{3}{2}+6}{2} = \frac{\frac{3}{2}+\frac{12}{2}}{2} = \frac{\frac{15}{2}}{2} = \frac{15}{4}$.

For $x_2 = -\frac{4}{3}$, we have $x_1 = \frac{x_2+6}{2} = \frac{-\frac{4}{3}+6}{2} = \frac{-\frac{4}{3}+\frac{18}{3}}{2} = \frac{\frac{14}{3}}{2} = \frac{7}{3}$.

Therefore, the stationary points are $\left(\frac{15}{4}, \frac{3}{2}\right)$ and $\left(\frac{7}{3}, -\frac{4}{3}\right)$.

评分标准：一阶导数正确得 2 分，驻点计算正确得 3 分。

Determine whether each stationary point is a local maximum, local minimum, or a saddle point.

To determine the nature of each stationary point, we need to analyze the Hessian matrix. The second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x_1^2} = 2$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -1$$
$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -1$$
$$\frac{\partial^2 f}{\partial x_2^2} = 6x_2$$

The Hessian matrix is:

$$H(x_1, x_2) = \begin{pmatrix} 2 & -1 \\ -1 & 6x_2 \end{pmatrix}$$

For a stationary point (x_1, x_2), the determinant of the Hessian is:

$$\det(H) = 2 \cdot 6x_2 - (-1)(-1) = 12x_2 - 1$$

For the stationary point $\left(\frac{15}{4}, \frac{3}{2}\right)$:

$$\begin{aligned} \det(H) &= 12 \cdot \frac{3}{2} - 1 \\ &= 18 - 1 \\ &= 17 > 0 \end{aligned}$$

Since $\frac{\partial^2 f}{\partial x_1^2} = 2 > 0$ and $\det(H) > 0$, this stationary point is a local minimum.

For the stationary point $\left(\frac{7}{3}, -\frac{4}{3}\right)$:

$$\begin{aligned} \det(H) &= 12 \cdot \left(-\frac{4}{3}\right) - 1 \\ &= -16 - 1 \\ &= -17 < 0 \end{aligned}$$

Since $\det(H) < 0$, this stationary point is a saddle point.

Therefore, the stationary point $\left(\frac{15}{4}, \frac{3}{2}\right)$ is a local minimum, and the stationary point $\left(\frac{7}{3}, -\frac{4}{3}\right)$ is a saddle point.

评分标准：Hessian 矩阵正确得 4 分，代入两个驻点计算正确行列式正确各得 2 分，判断各得 1 分。

Question 4 (Total 40 points)

Consider the expenditure minimization problem:

$$\begin{aligned} \text{minimize} \quad & p_1 x_1 + p_2 x_2 \\ \text{subject to} \quad & \alpha \log x_1 + (1-\alpha) \log x_2 = u, \end{aligned}$$

where $0 < \alpha < 1$ is a parameter, $p_1, p_2 > 0$ are prices, and $u \in \mathbb{R}$ is the desired utility level.

中文翻译：考虑以下支出最小化问题：

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad \alpha \log x_1 + (1-\alpha) \log x_2 = u$$

其中 $0 < \alpha < 1$ 为参数，$p_1, p_2 > 0$ 为价格，$u \in \mathbb{R}$ 为目标效用水平。

(a) (10 points) Solve the expenditure minimization problem.
中文翻译：求解此支出最小化问题。

(b) (10 points) Write down the bordered Hessian matrix of the Lagrangian. What is the second-order sufficient condition for this minimization problem?
中文翻译：写出拉格朗日函数的加边 Hessian 矩阵，并说明此最小化问题满足的二阶充分条件。

(c) (10 points) Let $e(p_1, p_2, u)$ be the value function. Write down the algebraic expression of the equation and compute the partial derivatives of e with respect to p_1 and p_2.
中文翻译：设 $e(p_1, p_2, u)$ 为值函数，写出其代数表达式，并求 e 对 p_1 和 p_2 的偏导数。

(d) (10 points) Verify *Shephard's lemma* $x_i = \frac{\partial e}{\partial p_i}$ for $i = 1, 2$, where x_i is the optimal demand for good i.
中文翻译：验证 Shephard 引理，即对于 $i = 1, 2$，有 $x_i = \frac{\partial e}{\partial p_i}$，其中 x_i 为商品 i 的最优需求量。

Solution

1. Solve the expenditure minimization problem.

To solve this problem, we form the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 - \lambda(\alpha \log x_1 + (1-\alpha)\log x_2 - u)$$

The first-order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= p_1 - \lambda \frac{\alpha}{x_1} = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= p_2 - \lambda \frac{1-\alpha}{x_2} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= -(\alpha \log x_1 + (1-\alpha)\log x_2 - u) = 0
\end{aligned}$$

From the first equation, we get $\lambda = \frac{p_1 x_1}{\alpha}$. From the second equation, $\lambda = \frac{p_2 x_2}{1-\alpha}$.
Setting these equal to each other:

$$\begin{aligned}
\frac{p_1 x_1}{\alpha} &= \frac{p_2 x_2}{1-\alpha} \\
\frac{x_2}{x_1} &= \frac{p_1(1-\alpha)}{p_2 \alpha}
\end{aligned}$$

From the third condition, we know:

$$\alpha \log x_1 + (1-\alpha)\log x_2 = u$$

Since at the optimum, the constraint will bind, we can solve for the optimal values of x_1 and x_2. Using the relationship between x_1 and x_2 derived above and substituting:

$$x_1 = e^u \left(\frac{p_2 \alpha}{p_1(1-\alpha)}\right)^{1-\alpha}$$

Similarly:

$$x_2 = e^u \left(\frac{p_1(1-\alpha)}{p_2 \alpha}\right)^{\alpha}$$

The expenditure function is thus:

$$\begin{aligned}
e(p_1, p_2, u) &= p_1 x_1 + p_2 x_2 \\
&= e^u \cdot p_1^{\alpha} \cdot p_2^{1-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)}
\end{aligned}$$

评分标准：拉格朗日条件正确得 2-3 分，解出最优值（显示解）得 7 分。

2. Write down the bordered Hessian matrix of the Lagrangian. What is the second-order sufficient condition for this minimization problem?

The bordered Hessian matrix is:

$$H = \begin{pmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} \end{pmatrix}$$

Where $g(x_1, x_2) = \alpha \log x_1 + (1-\alpha)\log x_2 - u$ is our constraint function.

Computing each element:

$$\frac{\partial g}{\partial x_1} = \frac{\alpha}{x_1}$$
$$\frac{\partial g}{\partial x_2} = \frac{1-\alpha}{x_2}$$
$$\frac{\partial^2 \mathcal{L}}{\partial x_1^2} = \lambda \frac{\alpha}{x_1^2}$$
$$\frac{\partial^2 \mathcal{L}}{\partial x_2^2} = \lambda \frac{1-\alpha}{x_2^2}$$
$$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 0$$

Therefore, the bordered Hessian is:

$$H = \begin{pmatrix} 0 & \frac{\alpha}{x_1} & \frac{1-\alpha}{x_2} \\ \frac{\alpha}{x_1} & \lambda\frac{\alpha}{x_1^2} & 0 \\ \frac{1-\alpha}{x_2} & 0 & \lambda\frac{1-\alpha}{x_2^2} \end{pmatrix}$$

For a constrained minimization problem with one constraint, the second-order sufficient condition is that the determinant of the bordered Hessian must be negative.

Let's calculate this determinant:

$$\det(H) = 0 \cdot \begin{vmatrix} \lambda\frac{\alpha}{x_1^2} & 0 \\ 0 & \lambda\frac{1-\alpha}{x_2^2} \end{vmatrix} - \frac{\alpha}{x_1} \cdot \begin{vmatrix} \frac{\alpha}{x_1} & \frac{1-\alpha}{x_2} \\ 0 & \lambda\frac{1-\alpha}{x_2^2} \end{vmatrix} + \frac{1-\alpha}{x_2} \cdot \begin{vmatrix} \frac{\alpha}{x_1} & \frac{1-\alpha}{x_2} \\ \lambda\frac{\alpha}{x_1^2} & 0 \end{vmatrix}$$

Simplifying:

$$\begin{aligned}
\det(H) &= -\frac{\alpha}{x_1} \cdot \left(\frac{\alpha}{x_1} \cdot \lambda\frac{1-\alpha}{x_2^2}\right) + \frac{1-\alpha}{x_2} \cdot \left(\frac{\alpha}{x_1} \cdot 0 - \frac{1-\alpha}{x_2} \cdot \lambda\frac{\alpha}{x_1^2}\right) \\
&= -\frac{\alpha^2\lambda(1-\alpha)}{x_1^2 x_2^2} - \frac{(1-\alpha)^2\lambda\alpha}{x_2^2 x_1^2} \\
&= -\frac{\lambda\alpha(1-\alpha)}{x_1^2 x_2^2}[\alpha + (1-\alpha)] \\
&= -\frac{\lambda\alpha(1-\alpha)}{x_1^2 x_2^2}
\end{aligned}$$

Since $\lambda > 0$ (from the first-order conditions), $\alpha \in (0,1)$, and $x_1, x_2 > 0$, we have $\det(H) < 0$, satisfying the second-order sufficient condition for a minimum.

评分标准：Broader Hessian 矩阵正确得 4 分，行列式正确接触显示表达式得 4 分，正确指出 λ 的符号并指出是否满足判断条件得 2 分。

3. Let $e(p_1, p_2, u)$ be the value function. Write down the algebraic expression of the equation and compute the partial derivatives of e with respect to p_1 and p_2.

We've already derived the algebraic expression for the expenditure function:

$$e(p_1, p_2, u) = e^u \cdot p_1^\alpha \cdot p_2^{1-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)}$$

Taking the partial derivative with respect to p_1:

$$\begin{aligned}
\frac{\partial e}{\partial p_1} &= \alpha \cdot e^u \cdot p_1^{\alpha-1} \cdot p_2^{1-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)} \\
&= \alpha \cdot \frac{e(p_1, p_2, u)}{p_1}
\end{aligned}$$

Taking the partial derivative with respect to p_2:

$$\begin{aligned}\frac{\partial e}{\partial p_2} &= (1-\alpha) \cdot e^u \cdot p_1^{\alpha} \cdot p_2^{(1-\alpha)-1} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)} \\ &= (1-\alpha) \cdot \frac{e(p_1, p_2, u)}{p_2}\end{aligned}$$

评分标准：得到指出函数的显示表达得 5 分，两个比较静态各占 2.5 分。未得出显示表达式的酌情给 2-3 分。

4. Verify Shephard's lemma $x_n = \frac{\partial e}{\partial p_i}$ for $i = 1, 2$, where x_i is the optimal demand of good n.

We need to verify that $x_n = \frac{\partial e}{\partial p_n}$ for $n = 1, 2$.

For $n = 1$:

$$\begin{aligned}\frac{\partial e}{\partial p_1} &= \alpha \cdot \frac{e(p_1, p_2, u)}{p_1} \\ &= \alpha \cdot \frac{e^u \cdot p_1^{\alpha} \cdot p_2^{1-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)}}{p_1} \\ &= \alpha \cdot e^u \cdot p_1^{\alpha-1} \cdot p_2^{1-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)} \\ &= \alpha e^u \cdot p_1^{-(1-\alpha)} \cdot p_2^{(1-\alpha)} \cdot \alpha^{-(1-\alpha)} \cdot (1-\alpha)^{-(1-\alpha)} \\ &= x_1\end{aligned}$$

For $n = 2$:

$$\begin{aligned}\frac{\partial e}{\partial p_2} &= (1-\alpha) \cdot \frac{e(p_1, p_2, u)}{p_2} \\ &= (1-\alpha) \cdot \frac{e^u \cdot p_1^{\alpha} \cdot p_2^{1-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)}}{p_2} \\ &= (1-\alpha) \cdot e^u \cdot p_1^{\alpha} \cdot p_2^{-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-(1-\alpha)} \\ &= (1-\alpha) e^u \cdot p_1^{\alpha} \cdot p_2^{-\alpha} \cdot \alpha^{-\alpha} \cdot (1-\alpha)^{-\alpha} \\ &= x_2\end{aligned}$$

Therefore, we have verified Shephard's lemma that $x_n = \frac{\partial e}{\partial p_n}$ for $n = 1, 2$.

评分标准：完成得到两个比较静态的表达式各占 4 分，表明 Shephard's lemma 正确各占 1 分。仅使用包络定理进行证明，且在（3）中未得出正确的显示表达式的酌情给 0-2 分。