2024/9/23

1.

设 $f(\boldsymbol{x})$ 在区域 $D \subset \mathbb{R}^n$ 上各偏导连续, 有界.

(1) 如果 D 是凸的, 证明 $f(\boldsymbol{x})$ 在区域 D 上一致连续.
Answer:

由于 D 是凸域, 知对 $\forall (x_1, \dots, x_n), (x_1, \dots, x_k, y_{k+1}, \dots, y_n) \in D$, 有
$\forall (x_1, \dots, \theta_{k+1}, \dots, x_n), \dots, (x_1, \dots, x_k, y_{k+1}, \dots, \theta_n) \in D$ 从而 $\exists M > 0, s.t.$

$$\begin{aligned}
&f(x_1, \dots, x_n) - f(x_1, \dots, x_k, y_{k+1}, \dots, y_n) \\
=&\sum_{i=k+1}^{n} f(x_1, \dots, x_k, y_{k+1}, \dots, y_{i-1}, x_i, \dots) - f(x_1, \dots, x_k, y_{k+1}, \dots, y_i, x_{i+1}, \dots) \\
=&\sum_{i=k+1}^{n} f'_i(x_1, \dots, \theta_i, \dots)(x_i - y_i) \\
\leq& M \sum_{i=k+1}^{n} (x_i - y_i)
\end{aligned}$$

故对 $\forall \epsilon > 0, \boldsymbol{x}, \boldsymbol{y} \in D, \exists \delta = \frac{\epsilon}{Mn} > 0 \quad s.t. |\boldsymbol{x} - \boldsymbol{y}| < \delta$, 有:

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq |M \sum_{i \in [n]} (x_i - y_i)| \leq Mn|\boldsymbol{x} - \boldsymbol{y}| < \epsilon$$

故一致连续.

(2) 如果 D 不是凸的, 举例说明 $f(\boldsymbol{x})$ 在区域 D 上有可能不一致连续.
Answer:
考虑定义在 $N(0,1) \setminus \{(x_1, \dots, x_{n-1}, 0) \mid 0 \leq x_1, \dots, x_{n-1} < 1\}$ 上的函数:

$$f(x_1, x_2, \dots, x_n) = \begin{cases} 0, & x_1, \dots, x_n > 0, \\ x_n^2, & \text{o.w.} \end{cases}$$

$f(x)$ 在 D 上存在 n 个连续的偏导数并且各个偏导数都有界, 但 $f(x)$ 在 D 上不一致连续, 证毕.

2.

设定义在凸区域 $D \subseteq \mathbb{R}^n$ 上的可微映射 $\boldsymbol{f}$ 满足 $\boldsymbol{f}'(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in D$,
证明 $\boldsymbol{f}(\boldsymbol{x}) = (c, \ldots, c)^T$ 为常值映射.
Answer:
取 $\boldsymbol{a}, \boldsymbol{b} \in D$, 考虑 $g(\boldsymbol{x}) = \langle \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}(\boldsymbol{x}) \rangle$, 显然 g 在 D 上可微.
由微分中值定理知 , 在 $\boldsymbol{a}, \boldsymbol{b}$ 的连线上 $\exists \boldsymbol{\theta}\ s.t.$

$$g(\boldsymbol{a}) - g(\boldsymbol{b}) = g'(\boldsymbol{\theta})(\boldsymbol{a} - \boldsymbol{b}) = (\boldsymbol{a} - \boldsymbol{b}) \langle \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}'(\boldsymbol{\theta}) \rangle$$

从而有

$$\begin{aligned}
\|\boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b})\|^2 &= \langle \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}) \rangle = g(\boldsymbol{a}) - g(\boldsymbol{b}) \\
&= (\boldsymbol{a} - \boldsymbol{b}) \langle \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}'(\boldsymbol{\theta}) \rangle \\
&\leq (\boldsymbol{a} - \boldsymbol{b}) \| \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}) \rangle \| \| \boldsymbol{f}'(\boldsymbol{\theta}) \| \\
&= 0
\end{aligned}$$

因此 $\boldsymbol{f}$ 是常值函数.

3.

设 $u(x, y), v(x, y) \in C^1(\mathbb{R}^2)$,
且存在常数 $C > 0$ 使得:

$$\begin{gathered}
(u_1 - u_2)^2 + (v_1 - v_2)^2 \geq C\left((x_1 - x_2)^2 + (y_1 - y_2)^2\right), \forall (x_i, y_i) \in \mathbb{R}^2, \\
u_i = u(x_i, y_i), v_i = v(x_i, y_i), i = 1, 2.
\end{gathered}$$

证明:

$$\left| \frac{\partial(u, v)}{\partial(x, y)} \right| \neq 0, \forall (x, y) \in \mathbb{R}^2.$$

Answer:
设$\boldsymbol{f}(\boldsymbol{x})=(u(x,y),v(x,y))$
反证法,若 $\exists\boldsymbol{x}\in\mathbb{R}^2\ s.t.\ \det J\boldsymbol{f}(\boldsymbol{x})=0$, 则由 $\boldsymbol{f}$ 连续可微知, $\exists\boldsymbol{h}\neq 0,\forall t\to 0\ s.t.\ J\boldsymbol{f}(\boldsymbol{x})(t\boldsymbol{h})=0$, 此时

$$\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})=J\boldsymbol{f}(\boldsymbol{x})(t\boldsymbol{h})+o(t\boldsymbol{h})=o(t\boldsymbol{h})$$

从而有,

$$\lim_{t\to 0}\frac{\|\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{h})-f(\boldsymbol{x})\|}{\|\boldsymbol{h}\||t|}=0$$

由题设条件知 $\frac{\|\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{h})-f(\boldsymbol{x})\|}{\|\boldsymbol{h}\||t|}\geq C>0$, 矛盾

4.

设 f 具有二阶连续导数, 求函数 $z=f(x^2+y^2,xy)$ 的所有二阶偏导数.
Answer:

$$\frac{\partial z}{\partial x}=2xf_1'+yf_2',\frac{\partial z}{\partial y}=2yf_1'+xf_2'\Rightarrow$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=2f_1'+2x(2xf_11''+yf_{21}'')+y(2yf_{12}''+xf_{22}'')\\&=4x^2f_{11}''+4xyf_{12}''+y^2f_{22}''+2f_1'\\\frac{\partial^2 z}{\partial y^2}&=2f_1'+2y(2yf_11''+xf_{21}'')+x(2xf_{12}''+yf_{22}'')\\&=4y^2f_{11}''+4xyf_{12}''+x^2f_{22}''+2f_1'\\\frac{\partial^2 z}{\partial x\partial y}&=2x(2yf_{11}''+xf_{21}'')+f_2'+y(2yf_{12}''+xf_{22}'')\\&=4xyf_{11}''+(2x^2+2y^2)f_{12}''+xyf_{22}''+f_2'\end{aligned}$$

5.

设 $f(x_1, x_2, \ldots, x_n) = \ln\left(\sum_{i=1}^n a_i x_i\right)$, 其中 $a_i, i = 1, 2, \ldots, n$ 为常数. 求函数的高阶偏导数:

$$\frac{\partial^{m_1+m_2+\cdots+m_n} f(\boldsymbol{x})}{\partial x_1^{m_1} \partial x_2^{m_2} \ldots \partial x_n^{m_n}}.$$

Answer:

$$(-1)^{\sum_{i\in[n]} m_i - 1} \left(\sum_{i\in[n]} m_i - 1\right)! \left(\prod_{i\in[n]} a_i^{m_i}\right) \left(\sum_{i\in[n]} a_i x_i\right)^{-\sum_{i\in[n]} m_i}$$

2024/10/09

1.

设 $f(t)$ 是 $\mathbb{R}$ 上的二次可导函数. 如果对于任何调和函数 $u(x,y)$（即满足拉普拉斯方程 $\Delta u = 0$ 的函数）.都有 $F(x,y) = f(u(x,y))$ 仍是调和函数.则 $f(t) = at + b$.其中 $a, b \in \mathbb{R}$ 是常数.

Answer:
注意到

$$\begin{aligned}
0 = \Delta F &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial^2 F}{\partial u^2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)
\end{aligned}$$

因此当 u 不是常值函数时, 有 $\frac{\partial^2 F}{\partial u^2} = 0$, 从而

$$f(t) = at + b, \quad a, b \in \mathbb{R}$$

2.

设 $z = z(x,y) \in C^2(\mathbb{R}^2)$ 满足方程

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 3\frac{\partial^2 z}{\partial y^2} = 0,$$

求在变换

$$\begin{cases} u = 3x + y, \\ v = x + y \end{cases}$$

后 $z = z(u,v)$ 所满足的方程.并由此给出方程的解 $z = z(x,y)$ 的表达式.

Answer:

注意到 $\frac{\partial(u,v)}{\partial(x,y)}$ 是常值矩阵,因此有

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 \\ &= 9\frac{\partial^2 z}{\partial u^2} + 6\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \end{aligned}$$

同理,

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= 3\frac{\partial^2 z}{\partial u^2} + 4\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial^2 z}{\partial y^2} &= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \end{aligned}$$

代入原方程并化简得到

$$\frac{\partial^2 z}{\partial u \partial v} = 0$$

这意味着,

$$z=f(u)+g(v),\quad (f,g\in C^1(\mathbb{R}))$$

从而有

$$z=f(3x+y)+g(x+y)$$

3.

求函数

$$f(x,y)=\frac{1+x+y+2xy}{1+x^2+y^2}$$

在原点处的直到四次项的 Peano 余项型 Taylor 公式.

Answer:

$$\begin{aligned}\frac{1+x+y+2xy}{1+x^2+y^2}=&(1+x+y+2xy)\left(1-(x^2+y^2)+(x^2+y^2)^2+o((x^2+y^2)^2)\right)\\=&1+x+y-x^2+2xy-y^2-x^3-x^2y-xy^2-y^3+x^4-2x^3y\\&+2x^2y^2-2xy^3+y^4+o((x^2+y^2)^2)(\sqrt{x^2+y^2}\to 0).\end{aligned}$$

4.

设 $f(x,y)=e^{xy}$.对任意 $k\in\mathbb{N}$.求 $f(x,y)$ 在 $(0,0)$ 处的所有 k 阶偏导数.

Answer:

注意到

$$e^{xy}=\sum_{i=0}^{+\infty}\frac{x^ky^k}{k!}$$

由泰勒展开式的唯一性,知

$$\frac{1}{(2k)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{2k}f(0,0)=\frac{x^ky^k}{k!}$$

系数一一对应,因此

$$\frac{\partial^m f(0,0)}{\partial x^k \partial y^{m-k}} = \begin{cases} k!, & m = 2k \\ 0, & \text{o.w.} \end{cases}$$