

2024/9/23

1.

设 $f(\mathbf{x})$ 在区域 $D \subset \mathbb{R}^n$ 上各偏导连续, 有界.

(1) 如果 D 是凸的, 证明 $f(\mathbf{x})$ 在区域 D 上一致连续.

Answer:

由于 D 是凸域, 知对 $\forall (x_1, \dots, x_n), (x_1, \dots, x_k, y_{k+1}, \dots, y_n) \in D$, 有
 $\forall (x_1, \dots, \theta_{k+1}, \dots, x_n), \dots, (x_1, \dots, x_k, y_{k+1}, \dots, \theta_n) \in D$ 从而 $\exists M > 0, s.t.$

$$\begin{aligned} & f(x_1, \dots, x_n) - f(x_1, \dots, x_k, y_{k+1}, \dots, y_n) \\ &= \sum_{i=k+1}^n f(x_1, \dots, x_k, y_{k+1}, \dots, y_{i-1}, x_i, \dots) - f(x_1, \dots, x_k, y_{k+1}, \dots, y_i, x_{i+1}, \dots) \\ &= \sum_{i=k+1}^n f'_i(x_1, \dots, \theta_i, \dots)(x_i - y_i) \\ &\leq M \sum_{i=k+1}^n (x_i - y_i) \end{aligned}$$

故对 $\forall \epsilon > 0, \mathbf{x}, \mathbf{y} \in D, \exists \delta = \frac{\epsilon}{Mn} > 0 \quad s.t. |\mathbf{x} - \mathbf{y}| < \delta$, 有:

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq |M \sum_{i \in [n]} (x_i - y_i)| \leq Mn |\mathbf{x} - \mathbf{y}| < \epsilon$$

故一致连续.

(2) 如果 D 不是凸的, 举例说明 $f(\mathbf{x})$ 在区域 D 上有可能不一致连续.

Answer:

考虑定义在 $N(0, 1) \setminus \{(x_1, \dots, x_{n-1}, 0) \mid 0 \leq x_1, \dots, x_{n-1} < 1\}$ 上的函数:

$$f(x_1, x_2, \dots, x_n) = \begin{cases} 0, & x_1, \dots, x_n > 0, \\ x_n^2, & \text{o.w.} \end{cases}$$

$f(x)$ 在 D 上存在 n 个连续的偏导数并且各个偏导数都有界, 但 $f(x)$ 在 D 上不一致连续, 证毕.

2.

设定义在凸区域 $D \subseteq \mathbb{R}^n$ 上的可微映射 f 满足 $f'(x) = 0, \forall x \in D$,
证明 $f(x) = (c, \dots, c)^T$ 为常值映射.

Answer:

取 $a, b \in D$, 考虑 $g(x) = \langle f(a) - f(b), f(x) \rangle$, 显然 g 在 D 上可微.

由微分中值定理知, 在 a, b 的连线上 $\exists \theta$ s.t.

$$g(a) - g(b) = g'(\theta)(a - b) = (a - b) \langle f(a) - f(b), f'(\theta) \rangle$$

从而有

$$\begin{aligned} \|f(a) - f(b)\|^2 &= \langle f(a) - f(b), f(a) - f(b) \rangle = g(a) - g(b) \\ &= (a - b) \langle f(a) - f(b), f'(\theta) \rangle \\ &\leq (a - b) \|f(a) - f(b)\| \|f'(\theta)\| \\ &= 0 \end{aligned}$$

因此 f 是常值函数.

3.

设 $u(x, y), v(x, y) \in C^1(\mathbb{R}^2)$,

且存在常数 $C > 0$ 使得:

$$\begin{aligned} (u_1 - u_2)^2 + (v_1 - v_2)^2 &\geq C ((x_1 - x_2)^2 + (y_1 - y_2)^2), \forall (x_i, y_i) \in \mathbb{R}^2, \\ u_i &= u(x_i, y_i), v_i = v(x_i, y_i), i = 1, 2. \end{aligned}$$

证明:

$$\left| \frac{\partial(u, v)}{\partial(x, y)} \right| \neq 0, \forall (x, y) \in \mathbb{R}^2.$$

Answer:

设 $\mathbf{f}(\mathbf{x}) = (u(x, y), v(x, y))$

反证法, 若 $\exists \mathbf{x} \in \mathbb{R}^2$ s.t. $\det J\mathbf{f}(\mathbf{x}) = 0$, 则由 \mathbf{f} 连续可微知, $\exists \mathbf{h} \neq 0, \forall t \rightarrow 0$ s.t. $J\mathbf{f}(\mathbf{x})(t\mathbf{h}) = 0$, 此时

$$\mathbf{f}(\mathbf{x} + t\mathbf{h}) - \mathbf{f}(\mathbf{x}) = J\mathbf{f}(\mathbf{x})(t\mathbf{h}) + o(t\mathbf{h}) = o(t\mathbf{h})$$

从而有,

$$\lim_{t \rightarrow 0} \frac{\|\mathbf{f}(\mathbf{x} + t\mathbf{h}) - \mathbf{f}(\mathbf{x})\|}{\|\mathbf{h}\||t|} = 0$$

由题设条件知 $\frac{\|\mathbf{f}(\mathbf{x} + t\mathbf{h}) - \mathbf{f}(\mathbf{x})\|}{\|\mathbf{h}\||t|} \geq C > 0$, 矛盾

4.

设 f 具有二阶连续导数, 求函数 $z = f(x^2 + y^2, xy)$ 的所有二阶偏导数.

Answer:

$$\frac{\partial z}{\partial x} = 2xf'_1 + yf'_2, \quad \frac{\partial z}{\partial y} = 2yf'_1 + xf'_2 \Rightarrow$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= 2f'_1 + 2x(2xf''_{11} + yf''_{21}) + y(2yf''_{12} + xf''_{22}) \\ &= 4x^2 f''_{11} + 4xy f''_{12} + y^2 f''_{22} + 2f'_1 \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= 2f'_1 + 2y(2yf''_{11} + xf''_{21}) + x(2xf''_{12} + yf''_{22}) \\ &= 4y^2 f''_{11} + 4xy f''_{12} + x^2 f''_{22} + 2f'_1 \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= 2x(2yf''_{11} + xf''_{21}) + f'_2 + y(2yf''_{12} + xf''_{22}) \\ &= 4xy f''_{11} + (2x^2 + 2y^2) f''_{12} + xy f''_{22} + f'_2 \end{aligned}$$

5.

设 $f(x_1, x_2, \dots, x_n) = \ln(\sum_{i=1}^n a_i x_i)$, 其中 $a_i, i = 1, 2, \dots, n$ 为常数. 求函数的高阶偏导数:

$$\frac{\partial^{m_1+m_2+\dots+m_n} f(\mathbf{x})}{\partial x_1^{m_1} \partial x_2^{m_2} \dots \partial x_n^{m_n}}.$$

Answer:

$$(-1)^{\sum_{i \in [n]} m_i - 1} \left(\sum_{i \in [n]} m_i - 1 \right)! \left(\prod_{i \in [n]} a_i^{m_i} \right) \left(\sum_{i \in [n]} a_i x_i \right)^{-\sum_{i \in [n]} m_i}$$

2024/10/09

1.

设 $f(t)$ 是 \mathbb{R} 上的二次可导函数. 如果对于任何调和函数 $u(x, y)$ (即满足拉普拉斯方程 $\Delta u = 0$ 的函数) 都有 $F(x, y) = f(u(x, y))$ 仍是调和函数. 则 $f(t) = at + b$. 其中 $a, b \in \mathbb{R}$ 是常数.

Answer:

注意到

$$\begin{aligned} 0 = \Delta F &= \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u} \frac{\partial u}{\partial y} \right) \\ &= \frac{\partial^2 F}{\partial u^2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{\partial F}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 F}{\partial u^2} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\partial F}{\partial u} \frac{\partial^2 u}{\partial y^2} \\ &= \frac{\partial^2 F}{\partial u^2} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) \end{aligned}$$

因此有 $\frac{\partial^2 F}{\partial u^2} = 0$, 从而

$$f(t) = at + b, \quad a, b \in \mathbb{R}$$

2.

设 $z = z(x, y) \in C^2(\mathbb{R}^2)$ 满足方程

$$\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 3 \frac{\partial^2 z}{\partial y^2} = 0,$$

求在变换

$$\begin{cases} u = 3x + y, \\ v = x + y \end{cases}$$

后 $z = z(u, v)$ 所满足的方程. 并由此给出方程的解 $z = z(x, y)$ 的表达式.

Answer:

注意到 $\frac{\partial(u, v)}{\partial(x, y)}$ 是常值矩阵, 因此有

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \right) \\ &= \frac{\partial^2 z}{\partial u^2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2} \left(\frac{\partial v}{\partial x} \right)^2 \\ &= 9 \frac{\partial^2 z}{\partial u^2} + 6 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \end{aligned}$$

同理,

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= 3 \frac{\partial^2 z}{\partial u^2} + 4 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial^2 z}{\partial y^2} &= \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \end{aligned}$$

代入原方程并化简得到

$$\frac{\partial^2 z}{\partial u \partial v} = 0$$

这意味着,

$$z = f(u) + g(v), \quad (f, g \in C^1(\mathbb{R}))$$

从而有

$$z = f(3x + y) + g(x + y)$$

3.

求函数

$$f(x, y) = \frac{1 + x + y + 2xy}{1 + x^2 + y^2}$$

在点处的直到四次项的 Peano 余项型 Taylor 公式.

Answer:

$$\begin{aligned} \frac{1 + x + y + 2xy}{1 + x^2 + y^2} &= (1 + x + y + 2xy) (1 - (x^2 + y^2) + (x^2 + y^2)^2 + o((x^2 + y^2)^2)) \\ &= 1 + x + y - x^2 + 2xy - y^2 - x^3 - x^2y - xy^2 - y^3 + x^4 - 2x^3y \\ &\quad + 2x^2y^2 - 2xy^3 + y^4 + o((x^2 + y^2)^2) (\sqrt{x^2 + y^2} \rightarrow 0). \end{aligned}$$

4.

设 $f(x, y) = e^{xy}$. 对任意 $k \in \mathbb{N}$. 求 $f(x, y)$ 在 $(0, 0)$ 处的所有 k 阶偏导数.

Answer:

注意到

$$e^{xy} = \sum_{k=0}^{+\infty} \frac{x^k y^k}{k!}$$

由泰勒展开式的唯一性, 知

$$\frac{1}{(2k)!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^{2k} f(0, 0) = \frac{x^k y^k}{k!}$$

系数一一对应,因此

$$\frac{\partial^k f(0,0)}{\partial x^i \partial y^{k-i}} = \begin{cases} k!, & k = 2i \\ 0, & \text{o.w.} \end{cases}$$