2024/12/16

习题 18

设函数 $f\left(x
ight)=\int_{1}^{+\infty}rac{\cos xy}{1+y^{2}}\mathrm{d}y$,求 $\lim_{x o0}f\left(x
ight)$, $\lim_{x o+\infty}f\left(x
ight)$..

Answer:

注意到 $\frac{\cos xy}{1+y^2}$ 在 $\mathbb{R} \times [1,+\infty)$ 上连续,且 $\left|\frac{\cos xy}{1+y^2}\right| < \frac{1}{y^2}$,后者的广义积分收敛,故 $\int_1^{+\infty} \left|\frac{\cos xy}{1+y^2}\right| \mathrm{d}y$ 收敛,因此 f(x) 在 \mathbb{R} 上连续. 从而

$$\lim_{x o0}f\left(x
ight)=f\left(0
ight)=\int_{1}^{+\infty}rac{1}{1+y^{2}}\mathrm{d}y=rac{\pi}{4}$$

由黎曼引理,且 $\frac{1}{1+v^2}$ 在 $\left[1,+\infty\right)$ 上黎曼可积,从而 $\lim_{x\to+\infty}f\left(x
ight)=0$.

习题 19

利用 $F(\alpha) = \int_0^{+\infty} \mathrm{e}^{-x^2 - rac{lpha^2}{x^2}} \mathrm{d}x$,求 $\int_0^{+\infty} \mathrm{e}^{-x^2 - x^{-2}} \mathrm{d}x$.

Answer:

注意到

$$F'(lpha)=\int_0^{+\infty}rac{-2lpha}{x^2}\mathrm{e}^{-x^2-rac{lpha^2}{x^2}}\mathrm{d}x=-2F(lpha)$$

解得 $F(\alpha) = C e^{-2\alpha}$, 带入 $F(0) = \frac{\sqrt{\pi}}{2}$, 得 $C = \frac{\sqrt{\pi}}{2}$, 从而 $F(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\alpha^2}$, 令 $\alpha = 1$, 得 $\int_0^{+\infty} e^{-x^2 - x^{-2}} \mathrm{d}x = \frac{\sqrt{\pi}}{2} e^{-2}$.

习题 20

求
$$\int_0^{+\infty} \frac{\sin^2 xy}{y^2} \mathrm{d}y$$

Answer:

不妨 x>0, 注意到 $F''(x)=2\int_0^{+\infty}\cos xy\mathrm{d}y=0$, 因此 F(x)=Cx+D, 令 x=0,1 可知,

$$F(x) = x \int_0^{+\infty} rac{\sin^2 y}{y^2} \mathrm{d}y = -rac{\sin^2 x}{x} igg|_0^{+\infty} + \int_0^{+\infty} rac{\sin 2x}{x} \mathrm{d}x$$
 $= \int_0^{+\infty} rac{\sin 2x}{2x} \mathrm{d}\left(2x
ight) = rac{\pi}{2}$

习题 21

求
$$\int_0^{+\infty} \frac{1-\cos x}{x^2} \mathrm{d}x$$

Answer:

$$I=\int_0^{+\infty}rac{1-\cos x}{x^2}\mathrm{d}x=\int_0^{+\infty}rac{\sin^2rac{x}{2}}{\left(rac{x}{2}
ight)^2}\mathrm{d}\left(rac{x}{2}
ight)=rac{\pi}{2}$$

2024/12/18

习题 30

计算下列广义积分 $(3)\int_0^1 \ln^n x dx$

Answer:

$$I = -\int_0^{+\infty} (-t)^n \mathrm{d} \mathrm{e}^{-t} = (-1)^n \Gamma(n+1) = (-1)^n \cdot n!$$

 $(4)\int_0^{rac{\pi}{2}} an^px\mathrm{d}x\,(|p|<1)$

Answer:

$$I = \int_0^{rac{\pi}{2}} \sin^{2 imes rac{p+1}{2}-1} x \cos^{2 imes rac{1-p}{2}-1} x \mathrm{d}x = rac{\mathrm{B}\left(rac{p+1}{2},rac{1-p}{2}
ight)}{2} = rac{\pi}{2\cosrac{\pi p}{2}}$$

$$(5)$$
 $\int_0^{+\infty} \frac{\mathrm{e}^{-2x}}{\sqrt{x}} \mathrm{d}x$

Answer:

$$I=rac{1}{\sqrt{2}}\int_0^{+\infty}(2x)^{-rac{1}{2}}\mathrm{e}^{-2x}\mathrm{d}2x=rac{1}{\sqrt{2}}\Gamma\left(rac{1}{2}
ight)=\sqrt{rac{\pi}{2}}$$

 $(6)\int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^5 \theta d\theta$

Answer:

$$I=rac{1}{2}\mathrm{B}\left(rac{5}{2},3
ight)=rac{\Gamma\left(rac{5}{2}
ight)\Gamma\left(3
ight)}{2\Gamma\left(rac{11}{2}
ight)}=rac{8}{315}$$

习题 33

求曲面 $x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}=a^{\frac{1}{2}}\,(a>0)$ 与坐标平面在第一象限所围立体的体积

Answer:

做变量替换
$$\begin{cases} x = r \sin^4 \varphi \cos^4 \theta \\ y = r \sin^4 \varphi \sin^4 \theta \text{ , in} \\ z = r \cos^4 \varphi \end{cases}$$

$$rac{\partial \left(x,y,z
ight) }{\partial \left(r,arphi, heta
ight) }=16r^{2}\cos ^{3}arphi \sin ^{7}arphi \cos ^{3} heta \sin ^{3} heta$$

可知,

$$\begin{split} V &= \iint_D \mathrm{d}x \mathrm{d}y \mathrm{d}z = \iint_\Omega 16 r^2 \cos^3 \varphi \sin^7 \varphi \cos^3 \theta \sin^3 \theta \mathrm{d}r \mathrm{d}\varphi \mathrm{d}\theta \\ &= \left(\int_0^\alpha 16 r^2 \mathrm{d}r \right) \left(\int_0^{\frac{\pi}{2}} \cos^3 \varphi \sin^7 \varphi \mathrm{d}\varphi \right) \left(\int_0^{\frac{\pi}{2}} \cos^3 \theta \sin^3 \theta \mathrm{d}\theta \right) \\ &= \frac{4a^3}{3} \mathrm{B}\left(2,4\right) \mathrm{B}\left(2,2\right) = \frac{4\Gamma^3 \left(2\right) \Gamma \left(4\right) a^3}{3\Gamma \left(6\right) \Gamma \left(4\right)} = \frac{4a^3}{3 \times 5!} = \frac{a^3}{90} \end{split}$$