1、

(a) 错误。反例：$x_n=\begin{cases}0, & n\text{奇}\\ 1, & n\text{偶}\end{cases}$

$y_n=\begin{cases}1, & n\text{奇}\\ 0, & n\text{偶}\end{cases}$

(b) 错误。反例：$f: [0,1] \to [-1,1]$

$x \to \sqrt{x}$

$g: [-1,1] \to [0,1]$

$x \to x^2$

(c) 正确。 $2nx_{2n} \leq 4038(x_n + x_{n+1} + \cdots + x_{2n-1})$

$\to 0$

$(2n-1)x_{2n-1} \leq 4038(x_n + x_{n+1} + \cdots + x_{2n-1})$

$\to 0$

$\therefore \lim\limits_{n\to\infty} n x_n = 0$

2、

(a) $\left|X_{2n}-\frac{1}{3}\right| = \frac{1}{2}\left|X_{2n-1}-\frac{2}{3}\right|$

$$= \frac{1}{2}\left|\frac{1+X_{2n-2}}{2}-\frac{2}{3}\right| = \frac{1}{4}\left|X_{2n-2}-\frac{1}{3}\right|$$

$\Rightarrow X_{2n}\to\frac{1}{3}$ ， $X_{2n+1}\to\frac{2}{3}$

$\therefore \{X_n\}$ 不收敛 $\square$

(b) $\because \lim\limits_{n\to\infty} a_n = a$ $\therefore$ 对 $\forall \varepsilon>0$，$\exists N_1$，当 $n>N_1$ 时

$|a_n - a| < \frac{\varepsilon}{2}$

$\because \frac{a}{2} = \frac{n(n+1)a}{2n^2} - \frac{a}{2n}$

$\therefore \left|\frac{a_1+2a_2+\cdots+na_n}{n^2} - \frac{a}{2}\right|$

$$\le \left|\frac{(a_1-a)+2(a_2-a)+\cdots+n(a_n-a)}{n^2}\right| + \frac{|a|}{2n}$$

$$\le \left|\frac{(a_1-a)+2(a_2-a)+\cdots+N_1(a_{N_1}-a)}{n^2}\right|$$

$$+ \left|\frac{(N_1+1)(a_{N_1+1}-a)+\cdots+n(a_n-a)}{n^2}\right| + \frac{|a|}{2n} \quad \cdots (*)$$

$\therefore \exists N_2$，当 $n > N_2$ 时，第一项与第三项 $< \frac{\varepsilon}{4}$

对第二项，有：

$$\left|\frac{(N_1+1)(a_{N_1+1}-a)+\cdots+n(a_n-a)}{n^2}\right|$$

$$\leq \frac{1}{n^2}\sum_{i=N_1+1}^{n} i|a_i-a| < \frac{1}{n^2}\cdot\frac{\varepsilon}{2}\cdot\frac{n(n+1)}{2}$$

$$< \frac{1}{n^2}\cdot\frac{\varepsilon}{2}\cdot n^2 < \frac{\varepsilon}{2}$$

$\therefore$ 当 $n > \max\{N_1, N_2\}$ 时

$(*)$式 $< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$

$\square$

(c)

令 $E_1 = \{(1+2^{2n})^{\frac{1}{2n}}\}$

$E_2 = \{(1+2^{-(2n-1)})^{\frac{1}{2n-1}}\}$

则 $E = E_1 \cup E_2$

$\because x_n = (1+2^{2n})^{\frac{1}{2n}} = 2^{\frac{\log_2(1+2^{2n})}{2n}}$

$= 2^{\frac{1+\log_2(1+\frac{1}{2^{2n}})}{2n}} \downarrow$

$\therefore \sup\{x_n\} = \sqrt{5}$, $\inf\{x_n\} = 2$

$\because y_n = (1+2^{-(2n-1)})^{\frac{1}{2n-1}} = 2^{\frac{\log_2(1+\frac{1}{2^{2n-1}})}{2n-1}} \downarrow$

$\therefore \sup\{y_n\} = \frac{3}{2}$, $\inf\{y_n\} = 1$

$\therefore \inf E = 1$, $\sup E = \sqrt{5}$

$\square$

(d)

$$a_n - 1 = \sin(a_{n-1} - 1)$$

令 $b_n = a_n - 1 \Rightarrow \begin{cases} b_n = \sin b_{n-1} \\ b_0 = -1 \end{cases}$

令 $f(x) = \sin x - x$

$f'(x) = \cos x - 1$

当 $x \in [-1, 0]$ 时，$f'(x) \le 0$，$f(x) \downarrow$ (或者利用几何性质)

$\therefore f(x) \geq f(0) = 0 \quad (x \in [-1, 0])$

$\therefore b_n = \sin b_{n-1} \geq b_{n-1}$，$\therefore \{b_n\} \uparrow$

又 $\{b_n\}$ 有界 1，$\therefore \lim\limits_{n\to\infty} b_n$ $\exists$，记为 b

在 $b_n = \sin b_{n-1}$ 两端取极限，由 $y = \sin x$ 的连续性

$\Rightarrow b = \sin b \Rightarrow b = 0$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n + 1 = 1$

$\square$

3.(a)

叙述：设 $\lim\limits_{n\to\infty} a_n = a$，$\lim\limits_{n\to\infty} b_n = b$，$b\neq 0, b_n\neq 0$

则有：$\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \frac{a}{b}$

解法一：先证乘法定理，再证 $\lim\limits_{n\to\infty}\frac{1}{b_n} = \frac{1}{b}$

解法二：当 $a=0$ 时，$\lim\limits_{n\to\infty}\frac{a_n}{b} = 0$

$\forall \varepsilon>0$ $\exists N$，$n>N$ 时，$|a_n| < \frac{b}{2}\varepsilon$，$|b_n| > \frac{b}{2}$.

$\Rightarrow \left|\frac{a_n}{b_n}\right| < \varepsilon \Rightarrow \lim\limits_{n\to\infty}\frac{a_n}{b_n} = 0$

当 $a\neq 0$ 时，$\forall \varepsilon>0$，$\exists N_1$，当 $n>N_1$ 时 $|a_n - a| < \frac{|b|}{4}\varepsilon$

$\exists N_2$，当 $n>N_2$ 时，$|b_n - b| < \frac{|b|^2\varepsilon}{4|a|}$

$\exists N_3$，当 $n>N_3$ 时，$|b_n| > \frac{|b|}{2}$

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{ba_n - ab_n}{bb_n}\right| = \left|\frac{ba_n - ab + ab - ab_n}{bb_n}\right|$$

$$\leq \left|\frac{a_n - a}{b_n}\right| + \frac{|a||b_n - b|}{|bb_n|} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \lim\limits_{n\to\infty}\frac{a_n}{b_n} = \frac{a}{b}$

$\square$

3、(b)

设 $\lim\limits_{n\to\infty}\dfrac{a_n-a_{n-1}}{b_n-b_{n-1}}=l$ （l 为有限数或 ∞）

当 l 为有限数时，$\forall\varepsilon>0$，$\exists N$，当 $n>N$ 时

$$\left|\frac{a_n-a_{n-1}}{b_n-b_{n-1}}-l\right|<\varepsilon$$

又 $b_n<b_{n-1}$

$\therefore (l-\varepsilon)(b_{n-1}-b_n)<a_{n-1}-a_n<(l+\varepsilon)(b_{n-1}-b_n)$

$\forall$ 取 $m>n$，将 $m-n-1$ 个式子叠加

$\Rightarrow (l-\varepsilon)(b_n-b_m)<a_n-a_m<(l+\varepsilon)(b_n-b_m)$

$\Rightarrow \left|\dfrac{a_n-a_m}{b_n-b_m}-l\right|<\varepsilon$，令 $m\to\infty$

$\Rightarrow \left|\dfrac{a_n}{b_n}-l\right|\leq\varepsilon \Rightarrow \lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=l$

对 l 为 $+\infty$ 时同理 $\square$