2024/10/16

1.

设函数 $z = F(x, y)$ 在区域 D 内具有连续偏导数且处处成立 $F_x(x, y) \neq 0, F_y(x, y) \neq 0$. 证明: 对于 $(x_0, y_0) \in D$, 方程 $F(x, y) = F(x_0, y_0)$ 在 (x_0, y_0) 的邻域内确定的隐函数 $y = f(x)$ 及 $x = g(y)$ 互为反函数.

Answer:
考虑 $G(x, y) = F(x, y) - F(x_0, y_0)$, 由隐函数存在定理以及 G 在 (x_0, y_0) 邻域上连续且偏导数连续且偏导数在该点值不为0可知, $G(x, y) = 0$ 唯一确定两个隐函数 $y = f(x)$ 和 $x = g(y)$.
注意到 $y = f(x)$ 在 x_0 邻域上不变号, 故存在反函数, 由唯一性知 g 是 f 的反函数.

2.

证明三元方程 $x^2 - 2xy + z + xe^z = 0$ 在 $(1, 1, 0)$ 点的邻域内唯一确定一个隐函数 $z = f(x, y)$, 并求 $f(x, y)$ 在 $(1, 1)$ 处的 Taylor 公式（直到二阶）.
Answer:
设 $F(x, y, z) = x^2 - 2xy + z + xe^z$, 注意到

$$\begin{cases} F(1,1,0) = 0 \\ F'_z(x,y,z) = e^z + 1 \text{在邻域上连续} \\ F'_z(1,1,0) = 2 \neq 0 \end{cases}$$

由隐函数存在定理知 $z = f(x, y)$ 被唯一确定, 设 $x' = x - 1, y' = x - 1$, 设 f 在 $(1, 1)$ 处Taylor公式为

$$f = ax' + by' + cx'^2 + dx'y' + ey'^2 + o(x'^2 + y'^2)$$

则有

$$\begin{aligned} F(x,y,f(x,y)) = (2a+1)x' + (2b-2)y' + \left(\frac{a^2}{2} + a + 2c + 1\right)x'^2 \\ +(ab + b + 2d - 2)x'y' + \left(\frac{b^2}{2} + 2e\right)y'^2 + o(x'^2 + y'^2) = 0 \end{aligned}$$

解得 $a=-\frac{1}{2}, b=1, c=-\frac{5}{16}, d=\frac{3}{4}, e=-\frac{1}{4}$
故$f(x,y)$ 在 $(1,1)$ 处的 Taylor 公式为

$$
\begin{aligned}
& f(x,y)=-\frac{1}{2}(x-1)+(y-1)-\frac{5}{16}(x-1)^2+\frac{3}{4}(x-1)(y-1) \\
& -\frac{1}{4}(y-1)^2+o((x-1)^2+(y-1)^2)(\sqrt{(x-1)^2+(y-1)^2}\to 0)
\end{aligned}
$$

3.

设 $\Omega\subset\mathbb{R}^3$, $F\in C^2(\Omega), u=F(xy,y+z,xz)$ 满足 $\frac{\partial u}{\partial z}=0$, 对于 $(x,y,z)\in\Omega$. 对 $F(xy,y+z,xz)=0$ 所确定的隐函数 $z=z(x,y)$ 计算全部的一阶、二阶偏导数.

Answer:

$$
\begin{cases} yF_1'+\frac{\partial z}{\partial x}F_2'+(z+x\frac{\partial z}{\partial x})F_3'=0 \\ xF_1'+(1+\frac{\partial z}{\partial y})F_2'+(x\frac{\partial z}{\partial y})F_3'=0 \end{cases} \Rightarrow \begin{cases} \frac{\partial z}{\partial x}&=-\frac{yF_1'+zF_3'}{F_2'+xF_3'} \\ \frac{\partial z}{\partial y}&=-\frac{xF_1'+F_2'}{F_2'+xF_3'} \end{cases}
$$

注意到

$$
\begin{cases}
\frac{\partial F_1'}{\partial x}=yF_{11}''+\frac{\partial z}{\partial x}F_{12}''+(z+x\frac{\partial z}{\partial x})F_{13}'' \\
\frac{\partial F_1'}{\partial y}=xF_{11}''+(1+\frac{\partial z}{\partial y})F_{12}''+(x\frac{\partial z}{\partial y})F_{13}'' \\
\frac{\partial F_2'}{\partial x}=yF_{12}''+\frac{\partial z}{\partial x}F_{22}''+(z+x\frac{\partial z}{\partial x})F_{23}'' \\
\frac{\partial F_2'}{\partial y}=xF_{12}''+(1+\frac{\partial z}{\partial y})F_{22}''+(x\frac{\partial z}{\partial y})F_{23}'' \\
\frac{\partial F_3'}{\partial x}=yF_{13}''+\frac{\partial z}{\partial x}F_{23}''+(z+x\frac{\partial z}{\partial x})F_{33}'' \\
\frac{\partial F_3'}{\partial y}=xF_{13}''+(1+\frac{\partial z}{\partial y})F_{23}''+(x\frac{\partial z}{\partial y})F_{33}''
\end{cases}
$$

设 $f=yF_1'+zF_3', g=F_2'+xF_3', h=xF_1'+F_2'$, 则

$$
\begin{cases}
\frac{\partial f}{\partial x}=y\frac{\partial F_1'}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial F_3'}{\partial x} \\
\frac{\partial g}{\partial x}=\frac{\partial F_2'}{\partial x}+F_3'+x\frac{\partial F_3'}{\partial x} \\
\frac{\partial g}{\partial y}=\frac{\partial F_2'}{\partial y}+x\frac{\partial F_3'}{\partial y} \\
\frac{\partial h}{\partial y}=\frac{\partial F_2'}{\partial y}+x\frac{\partial F_1'}{\partial y}
\end{cases}
$$

下求二阶导, 有

$$\begin{cases} \frac{\partial^2 z}{\partial x^2} = -\frac{\partial f/g}{\partial x} = \frac{\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g}{g^2} \\ \frac{\partial^2 z}{\partial x \partial y} = -\frac{\partial f/g}{\partial y} = \frac{\frac{\partial g}{\partial y} f - \frac{\partial f}{\partial y} g}{g^2} \\ \frac{\partial^2 z}{\partial y^2} = -\frac{\partial h/g}{\partial y} = \frac{\frac{\partial g}{\partial y} h - \frac{\partial h}{\partial y} g}{g^2} \end{cases}$$

全部代入化简即可.

4.

设 $f(u)$ 可导, $u = u(x, y)$ 可微, $z = z(x, y)$ 由$\begin{cases} (z - f(u))^2 = x^2(y^2 - u^2) \\ (z - f(u))\, f'(u) = ux^2 \end{cases}$ 给定.

验证: $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = xy$.

Answer:

在上式中对 x, y 求导得到,

$$(z - f(u))(\frac{\partial z}{\partial x} - f'(u)\frac{\partial u}{\partial x}) = x(y^2 - u^2) - x^2 u\frac{\partial u}{\partial x}$$

联立下式解得

$$\frac{\partial z}{\partial x} = \frac{(y^2 - u^2) f'(u)}{ux}, \frac{\partial z}{\partial y} = \frac{y f'(u)}{u}$$

因此

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \frac{(y^2 - u^2) f'(u)^2 y}{u^2 x} = \frac{u^2 x^4 y}{u^2 x^3} = xy$$

5.

设 $n > m \geq 1$, 证明: 不存在 $\mathbb{R}^n$ 到 $\mathbb{R}^m$ 的 C^1 同胚.

Answer:

假设存在这样的同胚 f, 则恒同映射 $(f^{-1}f)'(x) = f'(x)(f^{-1})'(f(x))$ 是一个 $n \times m$ 矩阵和 $m \times n$ 矩阵的乘积, 因此 $n = \mathrm{rank}((f^{-1}f)'(x)) \leq m$, 矛盾,故不存在$\mathbb{R}^n$ 到 $\mathbb{R}^m$ 的 C^1 同胚