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1.

设 $f(\boldsymbol{x})$ 在区域 $D \subset \mathbb{R}^n$ 上各偏导连续, 有界.

(1) 如果 D 是凸的, 证明 $f(\boldsymbol{x})$ 在区域 D 上一致连续.
Answer:

由于 D 是凸域, 知对 $\forall (x_1, \dots, x_n), (x_1, \dots, x_k, y_{k+1}, \dots, y_n) \in D$, 有 $\forall (x_1, \dots, \theta_{k+1}, \dots, x_n), \dots, (x_1, \dots, x_k, y_{k+1}, \dots, \theta_n) \in D$ 从而 $\exists M > 0, s.t.$

$$\begin{aligned}
& f(x_1, \dots, x_n) - f(x_1, \dots, x_k, y_{k+1}, \dots, y_n) \\
= & \sum_{i=k+1}^{n} f(x_1, \dots, x_k, y_{k+1}, \dots, y_{i-1}, x_i, \dots) - f(x_1, \dots, x_k, y_{k+1}, \dots, y_i, x_{i+1}, \dots) \\
= & \sum_{i=k+1}^{n} f_i'(x_1, \dots, \theta_i, \dots)(x_i - y_i) \\
\leq & M \sum_{i=k+1}^{n} (x_i - y_i)
\end{aligned}$$

故对 $\forall \epsilon > 0, \boldsymbol{x}, \boldsymbol{y} \in D, \exists \delta = \frac{\epsilon}{Mn} > 0 \quad s.t. |\boldsymbol{x} - \boldsymbol{y}| < \delta$, 有:

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq |M \sum_{i \in [n]} (x_i - y_i)| \leq Mn|\boldsymbol{x} - \boldsymbol{y}| < \epsilon$$

故一致连续.

(2) 如果 D 不是凸的, 举例说明 $f(\boldsymbol{x})$ 在区域 D 上有可能不一致连续.
Answer:
考虑定义在 $N(0,1) \setminus \{(x_1, \dots, x_{n-1}, 0) \mid 0 \leq x_1, \dots, x_{n-1} < 1\}$ 上的函数:

$$f(x_1, x_2, \dots, x_n) = \begin{cases} 0, & x_1, \dots, x_n > 0, \\ x_n^2, & \text{o.w.} \end{cases}$$

$f(x)$ 在 D 上存在 n 个连续的偏导数并且各个偏导数都有界, 但 $f(x)$ 在 D 上不一致连续, 证毕.

2.

设定义在凸区域 $D \subseteq \mathbb{R}^n$ 上的可微映射 $\boldsymbol{f}$ 满足 $\boldsymbol{f}'(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in D$,
证明 $\boldsymbol{f}(\boldsymbol{x}) = (c, \ldots, c)^T$ 为常值映射.
Answer:
取 $\boldsymbol{a}, \boldsymbol{b} \in D$, 考虑 $g(\boldsymbol{x}) = \langle \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}(\boldsymbol{x}) \rangle$, 显然 g 在 D 上可微.
由微分中值定理知 , 在 $\boldsymbol{a}, \boldsymbol{b}$ 的连线上 $\exists \boldsymbol{\theta}\ s.t.$

$$g(\boldsymbol{a}) - g(\boldsymbol{b}) = g'(\boldsymbol{\theta})(\boldsymbol{a} - \boldsymbol{b}) = (\boldsymbol{a} - \boldsymbol{b})\langle \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}'(\boldsymbol{\theta}) \rangle$$

从而有

$$\begin{aligned} \|\boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b})\|^2 &= \langle \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}) \rangle = g(\boldsymbol{a}) - g(\boldsymbol{b}) \\ &= (\boldsymbol{a} - \boldsymbol{b})\langle \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}'(\boldsymbol{\theta}) \rangle \\ &\le (\boldsymbol{a} - \boldsymbol{b})\|\boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{f}(\boldsymbol{b}) \rangle\| \|\boldsymbol{f}'(\boldsymbol{\theta})\| \\ &= 0 \end{aligned}$$

因此 $\boldsymbol{f}$ 是常值函数.

3.

设 $u(x, y), v(x, y) \in C^1(\mathbb{R}^2)$,
且存在常数 $C > 0$ 使得:

$$\begin{gathered} (u_1 - u_2)^2 + (v_1 - v_2)^2 \ge C\left((x_1 - x_2)^2 + (y_1 - y_2)^2\right), \forall (x_i, y_i) \in \mathbb{R}^2, \\ u_i = u(x_i, y_i), v_i = v(x_i, y_i), i = 1, 2. \end{gathered}$$

证明:

$$\left|\frac{\partial(u, v)}{\partial(x, y)}\right| \ne 0, \forall (x, y) \in \mathbb{R}^2.$$

Answer:

设$\boldsymbol{f}(\boldsymbol{x}) = (u(x,y), v(x,y))$

反证法,若 $\exists \boldsymbol{x} \in \mathbb{R}^2 \ s.t. \ \det J\boldsymbol{f}(\boldsymbol{x}) = 0$, 则由 $\boldsymbol{f}$ 连续可微知, $\exists \boldsymbol{h} \neq 0, \forall t \to 0 \ s.t. \ J\boldsymbol{f}(\boldsymbol{x})(t\boldsymbol{h}) = 0$, 此时

$$\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) = J\boldsymbol{f}(\boldsymbol{x})(t\boldsymbol{h}) + o(t\boldsymbol{h}) = o(t\boldsymbol{h})$$

从而有,

$$\lim_{t\to 0} \frac{\|\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{h}) - f(\boldsymbol{x})\|}{\|\boldsymbol{h}\||t|} = 0$$

由题设条件知 $\frac{\|\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{h}) - f(\boldsymbol{x})\|}{\|\boldsymbol{h}\||t|} \geq C > 0$, 矛盾

4.

设 f 具有二阶连续导数, 求函数 $z = f(x^2+y^2, xy)$ 的所有二阶偏导数.

Answer:

$$\frac{\partial z}{\partial x} = 2xf_1' + yf_2', \frac{\partial z}{\partial y} = 2yf_1' + xf_2' \Rightarrow$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= 2f_1' + 2x(2xf_11'' + yf_{21}'') + y(2yf_{12}'' + xf_{22}'') \\
&= 4x^2 f_{11}'' + 4xyf_{12}'' + y^2 f_{22}'' + 2f_1' \\
\frac{\partial^2 z}{\partial y^2} &= 2f_1' + 2y(2yf_11'' + xf_{21}'') + x(2xf_{12}'' + yf_{22}'') \\
&= 4y^2 f_{11}'' + 4xyf_{12}'' + x^2 f_{22}'' + 2f_1' \\
\frac{\partial^2 z}{\partial x \partial y} &= 2x(2yf_{11}'' + xf_{21}'') + f_2' + y(2yf_{12}'' + xf_{22}'') \\
&= 4xyf_{11}'' + (2x^2 + 2y^2)f_{12}'' + xyf_{22}'' + f_2'
\end{aligned}$$

5.

设 $f(x_1, x_2, \ldots, x_n) = \ln\left(\sum_{i=1}^{n} a_i x_i\right)$, 其中 a_i, $i = 1, 2, \ldots, n$ 为常数. 求函数的高阶偏导数:

$$\frac{\partial^{m_1+m_2+\cdots+m_n} f(\boldsymbol{x})}{\partial x_1^{m_1} \partial x_2^{m_2} \ldots \partial x_n^{m_n}}.$$

Answer:

$$(-1)^{\sum_{i\in[n]} m_i - 1} \left(\sum_{i\in[n]} m_i - 1\right)! \left(\prod_{i\in[n]} a_i^{m_i}\right) \left(\sum_{i\in[n]} a_i x_i\right)^{-\sum_{i\in[n]} m_i}$$