2024/11/4

1.

设函数 $f(x)$ 在区间 $[a, b]$ 上连续, 且对任意的 $x \in [a, b]$, $f(x) \geq a > 0$. 记 $D = [a, b] \times [a, b]$, 证明: $\iint_D f(x)/f(y)\mathrm{d}\sigma \geq (b-a)^2$

Answer:
由柯西-施瓦茨不等式, 我们有:

$$
\begin{aligned}
\iint_D \frac{f(x)}{f(y)}\mathrm{d}\sigma &= \int_a^b f(x)\mathrm{d}x \int_a^b f^{-1}(y)\mathrm{d}y \\
&= \int_a^b f(x)\mathrm{d}x \int_a^b f^{-1}(x)\mathrm{d}x \\
&\geq \left(\int_a^b \mathrm{d}x\right)^2 \\
&= (a-b)^2
\end{aligned}
$$

2.

计算积分: $\int_0^{\sqrt{3}} \mathrm{d}x \int_0^1 \frac{8x}{(x^2+y^2+1)^2} \mathrm{d}y$

Answer:

$$
\begin{aligned}
\int_0^{\sqrt{3}} \mathrm{d}x \int_0^1 \frac{8x}{(x^2+y^2+1)^2}\mathrm{d}y &= \int_0^1 \mathrm{d}y \int_0^{\sqrt{3}} \frac{8x}{(x^2+y^2+1)^2}\mathrm{d}x \\
&= \int_0^1 \left(\frac{4}{y^2+1} - \frac{4}{y^2+4}\right)\mathrm{d}y \\
&= \pi - 2\arctan\frac{1}{2}
\end{aligned}
$$

3.

计算积分: $\iint_D \sin(y^3)\mathrm{d}\sigma$, 其中 D 由 $y=\sqrt{x}, y=2$ 与 $x=0$ 所围

Answer:

$$\begin{aligned}\iint_D \sin(y^3)\mathrm{d}\sigma &= \int_0^2 \mathrm{d}y \int_0^{y^2} \sin(y^3)\mathrm{d}x \\ &= \int_0^2 y^2 \sin(y^3)\mathrm{d}y \\ &= -\frac{\cos y^3}{3}\Big|_0^2 \\ &= \frac{1-\cos 8}{3}\end{aligned}$$

4.

设 $f(x)\in C^1[a,b]$, 且 $f(a)=0$, 证明: $\int_a^b f^2(x)\,\mathrm{d}x \le \frac{1}{2}\int_a^b f'(x)^2\left((b-a)^2-(x-a)^2\right)\,\mathrm{d}x$

Answer:

$$\begin{aligned}\text{RHS} =& \frac{1}{2}\int_a^b \left((b-a)^2-(x-a)^2\right)\mathrm{d}\left(\int_a^x f'(t)^2\mathrm{d}t\right) \\ =& \frac{1}{2}\left((b-a)^2-(x-a)^2\right)\left(\int_a^x f'(t)^2\mathrm{d}t\right)\Bigg|_a^b \\ +& \int_a^b (x-a)\int_a^x f'(t)^2\mathrm{d}t\mathrm{d}x \\ =& \int_a^b \left(\int_a^x \mathrm{d}t \int_a^x f'(t)^2\right)\mathrm{d}t\mathrm{d}x \\ \ge& \int_a^b f^2(x)\mathrm{d}x\end{aligned}$$

5.

写出积分: $\int_0^1 \mathrm{d}y \int_{-y}^{y} \mathrm{d}z \int_{-\sqrt{y^2-z^2}}^{\sqrt{y^2-z^2}} f(x,y,z)\mathrm{d}x$ 的其他各种累次积分

Answer:

$$\int_{-1}^{1} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}z \int_{\sqrt{x^2+z^2}}^{1} f(x,y,z)\mathrm{d}y, \int_{-1}^{1} \mathrm{d}z \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \mathrm{d}x \int_{\sqrt{x^2+z^2}}^{1} f(x,y,z)\mathrm{d}y$$

$$\int_{-1}^{1} \mathrm{d}x \int_{x}^{1} \mathrm{d}y \int_{-\sqrt{y^2-x^2}}^{\sqrt{y^2-x^2}} f(x,y,z)\mathrm{d}z, \int_{-1}^{1} \mathrm{d}z \int_{z}^{1} \mathrm{d}y \int_{-\sqrt{y^2-z^2}}^{\sqrt{y^2-z^2}} f(x,y,z)\mathrm{d}x$$

$$\int_{0}^{1} \mathrm{d}y \int_{-y}^{y} \mathrm{d}x \int_{-\sqrt{y^2-x^2}}^{\sqrt{y^2-x^2}} f(x,y,z)\mathrm{d}z$$

6.

计算重积分: $\iiint_\Omega \cos x \cos y \cos z \mathrm{d}v$, 其中 Ω 为闭区域 $|x|+|y|+|z| \leq 1$

$$\begin{aligned}
\iiint_\Omega \cos x \cos y \cos z \mathrm{d}v &= \int_{-1}^{1} \cos x \mathrm{d}x \int_{-1+|x|}^{1-|x|} \cos y \mathrm{d}y \int_{-1+|x|+|y|}^{1-|x|-|y|} \cos z \mathrm{d}z \\
&= 8 \int_0^1 \cos x \mathrm{d}x \int_0^{1-x} \cos y \mathrm{d}y \int_0^{1-x-y} \cos z \mathrm{d}z \\
&= 8 \int_0^1 \cos x \mathrm{d}x \int_0^{1-x} \cos y \sin(1-x-y) \mathrm{d}y \\
&= 8 \int_0^1 \frac{(1-x)\sin(1-x)\cos x}{2} \mathrm{d}x \\
&= 2\sin 1 - \cos 1
\end{aligned}$$

2024/11/6

1.

计算积分:$\iint_D y\mathrm{d}\sigma$, 其中 D 是心脏线 $r=2(1+\cos\theta)$ 落在 $r=2$ 外部的区域.

Answer:

$$\iint_D y\mathrm{d}x\mathrm{d}y=\iint_D r^2\sin\theta\mathrm{d}r\mathrm{d}\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\mathrm{d}\theta\int_2^{2(1+\cos\theta)}r^2\mathrm{d}r=0$$

2.

设 $D=\{(x,y)\mid 0\le x+y\le 1, 0\le x-y\le 1\}$, 计算:$I=\iint_D(x+y)^2e^{x^2-y^2}d\sigma_{xy}$

做变量替换 $\begin{cases}u=x+y\\v=x-y\end{cases}$, 由 $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{2}$, 有

$$\begin{aligned}\iint_D(x+y)^2e^{x^2-y^2}d\sigma_{xy}&=\iint_D\frac{1}{2}m^2e^{mn}\mathrm{d}m\mathrm{d}n\\&=\int_0^1\frac{1}{2}m(e^m-1)\mathrm{d}m\\&=\frac{1}{4}\end{aligned}$$

3.

求曲面 $z=x^2+y^2$, $x^2+y^2=x$ 及 $x^2+y^2=2x$, $z=0$ 所围立体的体积.

Answer:

作变量替换: $\begin{cases}x=r\cos\theta,\\y=r\sin\theta,\\z=z\end{cases}$, 由 $\left|\frac{\partial(x,y,z)}{\partial(u,v,z)}\right|=r$, 有

$$V=\iiint_D\mathrm{d}D=\iiint_D r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_{\cos\theta}^{2\cos\theta}\mathrm{d}r\int_0^{r^3}r\mathrm{d}z=\frac{45\pi}{32}$$

4.

计算积分:$\iiint_{\Omega}(x^2+y^2)\,\mathrm{d}v$, 其中 Ω 为由曲面 $z=12-2x^2-2y^2$ 与 $z=x^2+y^2$ 所围区域.

Answer:

作变量替换: $\begin{cases} x=r\cos\theta, \\ y=r\sin\theta, \\ z=z \end{cases}$, 由 $\left|\frac{\partial(x,y,z)}{\partial(u,v,z)}\right|=r$, 有

$$V=\iiint_D(x^2+y^2)dD=\iiint_D r^3\mathrm{d}r\mathrm{d}\theta\mathrm{d}z=\int_0^{2\pi}\mathrm{d}\theta\int_0^2\mathrm{d}r\int_{r^2}^{12-2r^2}r^3\mathrm{d}z=32\pi$$

5.

求由曲面 $\left(\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{3}}\right)^2+\frac{z^2}{2}=1$ 与三个坐标面所围成的立体在第一象限部分的体积.

作变量替换: $\begin{cases} x=\sqrt{2}rs\cos\theta, \\ y=\sqrt{3}r(1-s)\sin\theta, \\ z=\sqrt{2}r\cos\theta \end{cases}$, 由 $\left|\frac{\partial(x,y,z)}{\partial(r,s,\theta)}\right|=2\sqrt{3}r^2\sin\theta$, 有

$$V=\iiint_D\mathrm{d}D=\iiint_D 2\sqrt{3}r^2\sin\theta\mathrm{d}r\mathrm{d}s\mathrm{d}\theta=\int_0^1\mathrm{d}r\int_0^1\mathrm{d}s\int_0^{\frac{\pi}{2}}2\sqrt{3}r^2\sin\theta\mathrm{d}\theta=\frac{2\sqrt{3}}{3}$$