Homework #2

Due: 2024-10-21 23:59 | 7 Problems, 100 Pts

Name: XXX, ID: XXX

Problem 1 (20'). Consider a cube C with side length 1 in d-dimensional space.

(1) (4') Write down the radius of a ball B whose volume is equal to the cube's. You don't need to prove your result.

(2) (16') When the centers of the cube C and the ball B are both at the origin, calculate the volume of their intersection when $d \to +\infty$.

[Hint: Try to calculate $\mathbb{E}_{X\sim C}[\|X\|_2^2]$ and $Var_{X\sim C}[\|X\|_2^2]$. Then use Chebyshev's Inequality to analyze the concentration of $\|X\|_2^2$. You may use the Stirling's approximation

$$\lim_{n\to+\infty} \frac{\Gamma(n+1)}{\sqrt{2\pi n}(n/\mathrm{e})^n} = 1.$$

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Problem 2 (16'). You select a PE class this term, so you need to complete a total of 85km of extracurricular exercise. There are only n days before the deadline, but you still have m kilometers remaining. To simplify your work, you can run at most 10km every day, but there is no lower bound. You are mindless when running, so the distance you run every day is uniformly random. In other words, the number of kilometers you run every day is a real number uniformly selected from range $[0, 10]$. Prove that, the probability that you can complete m kilometers in n days, i.e., the probability that your total distance in n days is greater than or equal to m kilometers is

$$1 - \sum_{i=0}^{\min(\lfloor m/10 \rfloor, n)} \frac{(-1)^i}{i!(n-i)!} \left(\frac{m}{10} - i\right)^n.$$

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Problem 3 (10'). If A is square, show that $AA^\top$ and $A^\top A$ are similar.

[Hint: Use the SVD of A.] ◀

Problem 4 (12'). Calculate the SVD of matrix A, where

$$A = \begin{pmatrix} -4 & -6 \\ 3 & -8 \end{pmatrix}.$$

You need to calculate it in two different ways.

[Hint: Use the definition of singular vectors, or consider $A^\top A$.] ◀

Problem 5 (16'). Let $A = \sum_{i=1}^r \sigma_i u_i v_i^\top$ be the SVD of a rank-r matrix A, let $A_k := \sum_{i=1}^k \sigma_i u_i v_i^\top$ be the rank-k approximation of A for some $k < r$.

(1) (4') Express the following quantities in terms of the singular values $\{\sigma_i, 1 \leq i \leq r\}$. You don't need to prove your result.

(a) (2') $\|A_k\|_F^2$.

(b) (2') $\|A_k\|_2^2$.

(2) (3') Prove that, $\sigma_k \leq \frac{\|A\|_F}{\sqrt{k}}$ for $k = 1, 2, \cdots, r$.

(3) (9') Suppose $A \in \mathbb{R}^{m\times m}$. Let $B = \sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^\top$.

(a) (6') Show that $BAx = x$ for all x in the span of the right singular vectors of A. For this reason B is sometimes called the pseudo inverse of A and can play the role of A^{-1} in many applications.

(b) (3') Does $BAx = x$ always hold for all $x \in \mathbb{R}^m$? Prove or give a counterexample.

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Problem 6 (16'). In class, we introduced the best-fit subspace for a set of points. We extend this definition to probability densities instead of a set of points. If P is a probability density in $\mathbb{R}^d$, then we define the best-fit k-dimensional subspace ($k \leq d$)

$$V_k = \arg \max_{V, \dim(V)=k} \mathbb{E}_{X\sim P}(|\text{proj}(X, V)|^2),$$

where $\text{proj}(X, V)$ is the orthogonal projection of X onto V.

(1) (6') Find out the best-fit 1-dimensional subspace (i.e., a best-fit line through the origin) for probability density $\mathcal{N}\left(0, 0, 1, 2, \frac{1}{2}\right)$.

[Hint: Suppose random variable $(X_1, X_2) \sim \mathcal{N}\left(0, 0, 1, 2, \frac{1}{2}\right)$*, then we have*

$$X_1 \sim \mathcal{N}(0, 1),\ X_2 \sim \mathcal{N}(0, 2),\ \rho := \frac{\mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)}{\sqrt{Var(X_1)Var(X_2)}} = \frac{1}{2}.$$

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(2) (10') Prove that, for d-dimensional Gaussian distribution $\mathcal{N}(\mu, I_d)$, a k-dimensional subspace is a best-fit subspace if and only if it contains μ.

[Hint: Suppose random variable $(X_1, \cdots, X_d) \sim \mathcal{N}(\mu, I_d)$*, then we have* $X_i \sim \mathcal{N}(\mu_i, 1)$ *and* $X_1, \cdots, X_d$ *are independent random variables.]*

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Problem 7 (10'). Read in a photochrome and convert to a matrix. Perform a singular value decomposition of the matrix. Reconstruct the photo using only 1,2,4,16 singular values.

(1) Print the reconstructed photo. How good is the quality of the reconstructed photo? Write down your observations.

(2) What percent of the Frobenius norm is captured in each case? Describe the way you calculate the captured Frobenius norm.

You need to include your original photochrome and all reconstructed photos in your answer and submit the code as an attachment.

[Hint: You may want to perform SVD on each channel.] ◀