Game Theory, Fall 2022
Problem Set 8

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1. ST 11.1

   Soln: Consider the strategy profile where player 1 proposes $(1, 0)$ and player 2 rejects every offer. Obviously, no one has a profitable deviation. Therefore, it is a Nash equilibrium of the ultimatum game.

   Consider T-round bargaining for $T = 2, \ldots, \infty$. We verify the following strategy profile (σ_1^*, σ_2^*) is a Nash equilibrium: player 1 always proposes $(1, 0)$ and rejects any offer from player 2; player 2 always proposes $(0, 1)$ and rejects any offer from player 1. Clearly, this strategy profile reaches disagreement for any T and yields payoff 0 to each player. If σ_1 is a profitable deviation for player 1, then (σ_1, σ_2^*) must yield positive payoff to player 1. This implies that agreement must be reached under (σ_1, σ_2^*). This is possible only if σ_1 sometimes accepts player 2's offer, because σ_2^* rejects every offer from player 1. However, because player 2 always proposes $(0, 1)$, the payoff to player 1 must be 0 when the agreement is reached under (σ_1, σ_2^*). This is a contradiction, as this deviation is not profitable.

2. ST 11.2

   (a) **Soln:** The subgame following player 2's investment L resembles the standard ultimatum game. Player 1 can propose $(x, x_L - x)$ for any $x \in [0, x_L]$ and player 2 decides whether to accept. If $(x, x_L - x)$ is accepted, the players' payoffs are x and $x_L - x - c_L$ respectively; otherwise their payoffs are 0 and $-c_L$. Then, it is straightforward to see that there is a unique subgame perfect equilibrium in this subgame: player 1 proposes $(x_L, 0)$ and player 2 accepts all. The payoffs are x_L and $-c_L$ respectively.

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Similarly, there is a unique subgame perfect equilibrium in the subgame following player 2's H choice. In this equilibrium, player 1 proposes $(x_H, 0)$ and player 2 accepts all. The payoffs are x_H and $-c_H$ respectively.

It is then clear that player 2's initial investment choice should be L. We find a unique subgame perfect equilibrium. Clearly, it is not Pareto optimal. If player 2 invests H, the total surplus of these two players is $x_H - c_H$. But the equilibrium surplus is just $x_L - c_L$.

(b) **Soln:** Pick y such that $x_L < y < x_H$ and $x_H - y - c_H > -c_L$. Because $x_H - c_H > x_L - c_L$, such y exists. Consider the following strategy profile. Player 2 chooses H initially. Player 1 proposes $(x_L, 0)$ after L and proposes $(y, x_H - y)$ after H. Player 2 only accepts when he initially chooses H and the offer is $(y, x_H - y)$. It is easy to verify that no player has a profitable deviation. The equilibrium payoffs are y and $x_H - y - c_H$ respectively. For both players, the payoffs are higher than the unique SPE payoffs.

3. ST 11.4

(a) **Soln:** Consider the subgame following player 2's rejection of some offer in the first period. It is clear that there is a unique SPE: player 2 proposes $(0, 1)$ and player 1 accepts all. The resulting payoffs are $-c_1$ and $1 - c_2$, respectively.

We now consider the first period. Facing an offer $(x, 1 - x)$, player 2 strictly prefers accepting if $1 - x > 1 - c_2$ and strictly prefers rejecting if $1 - x < 1 - c_2$. It is standard to see that if player 2 rejects the offer when $1 - x = 1 - c_2$, player 1 would have no best response. Therefore, in equilibrium player 2 must also accept the offer when $1 - x = 1 - c_2$. It is then clear that player 1 should propose $(\min\{1, c_2\}, 1 - \min\{1, c_2\})$. That is, $(1, 0)$ if $c_2 \geq 1$ and $(c_2, 1 - c_2)$ if $c_2 < 1$.

In sum, there is a unique SPE. When $c_2 \geq 1$, the equilibrium is: player 1 proposes $(1, 0)$ in the first period and accepts all offers in the second period regardless of history; player 2 accepts all offers in the first period and always propose $(0, 1)$ in the second period. When $c_2 < 1$, the equilibrium is: player 1 proposes $(c_2, 1 - c_2)$ in the first period and accepts all offers in the second period regardless of history; player 2 accepts off $(x, 1 - x)$ in the first period if and only if $1 - x \geq 1 - c_2$ and always proposes $(0, 1)$ in the second period.

(b) **Soln:** For instance, consider the following strategy profile. Player 1 proposes $(x, 1 - x)$ in the first period and rejects all in the second period. Player 2 only accepts $(x, 1 - x)$ in the first period and always proposes $(0, 1)$ in the second period.

Under this profile, the payoffs are x and $1-x$ respectively. If player 1 deviates to a strategy that changes his payoff, his initial offer must be rejected. Since player 2 always proposes $(0,1)$ in the second period, player 1's payoff must be $-c_1$ regardless of how he plays in the second period. Thus, such deviation is not profitable. On the other hand, any strategy for player 2 that changes his payoff must involve rejecting offer $(x, 1-x)$ in the first period. But since player 1 rejects all in the second period, agreement is not achieved in the second period regardless of player 2's proposal. Therefore, in this case, player 2 would obtain $-c_2 < 0$. Such deviation is not profitable. Therefore, it is a Nash equilibrium.

(c) **Soln:** From (3a), we have already known what will happen after player 2's rejection in the first period. We discuss two cases.

- $c_1 \geq 1$. If player 2 rejects player 1's first period offer, their payoffs from the following subgame will be $(-c_1, 1-c_2)$ (evaluated at $t = 1$.) Therefore, player 2 must accept any offer $(x, 1-x)$ such that $1-x \geq 1-c_2$ in the first period.
  - If $c_2 \geq 1$, it is optimal for player 1 to propose $(1,0)$ in the first period.
  - If $c_2 < 1$, it is optimal for player 1 to propose $(c_2, 1-c_2)$ in the first period.
- $c_1 < 1$. If player 2 rejects player 1's first period offer, their payoffs from the following subgame will be $(1-2c_1, c_1-c_2)$ (evaluated at $t = 1$). Therefore, player 2 must accept any offer $(x, 1-x)$ such that $1-x \geq c_1 - c_2$ in the first period.
  - If $c_2 \geq c_1$, it is optimal for plyaer 1 to propose $(1,0)$ in the first period.
  - If $c_2 < c_1$, it is optimal for player 1 to propose $(1-c_1+c_2, c_1-c_2)$ in the first period.

We can then carefully describe the unique SPE, which we omit here. Instead, let's consider their equilibrium payoffs. The above four cases are depicted in Figure 1. It is easy to observe that every player's payoff is decreasing in his own cost, while increasing in his opponent's cost.

4. ST 11.6

Soln: We use the same way to analyze as we did in the slides. First, we guess a *stationary equilibrium*: player 1 always proposes $(x^*, 1-x^*)$ and 2 always proposes $(y*, 1-y^*)$; player 1 always accepts an offer $x \geq y^*$ and rejects others; player 2 always accepts an offer $1-x \geq 1-x^*$ and rejects others.

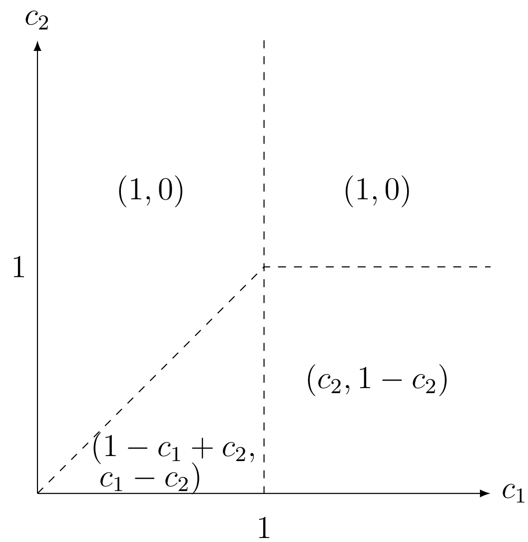

Figure 1: SPE payoffs of the bargaining game with $T = 3$ and constant cost

For 2 to accept $1-x^*$, $1-x^* \geq \delta_2(1-y^*)$. For 2 to reject any offer $1-x < 1-x^*$, $1-x \leq \delta_2(1-y^*)$. Hence 2 must be indifferent between accepting and rejecting $1-x^*$:

$$1-x^* = \delta_2(1-y^*).$$

In the same way, we can derive

$$y^* = \delta_1 x^*.$$

Therefore, we have

$$x^* = \frac{1-\delta_2}{1-\delta_1\delta_2} \text{ and } y^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}.$$

This strategy profile is a SPE.

We now prove that it is the unique SPE. Let G_i be the set of all SPE payoffs in the subgames in which player i makes the initial proposal, and let $M_i = \sup G_i$ and $m_i = \inf G_i$.

Consider the subgame in which player 1 makes the initial proposal. Observe that if player 2 rejects 1's proposal, then 2 can obtain at most $\delta_2 M_2$. Thus, player 2 must be willing to accept any $1-x > \delta_2 M_2$. This implies that player 1 can get $1-\delta_2 M_2 - \epsilon$ for $\epsilon > 0$. Therefore, $m_1 \geq 1-\delta_2 M_2$.

Observe that if player 2 rejects 1's proposal, then 2 can obtain at least $\delta_2 m_2$. Thus, player 2 must reject any $1-x < \delta_2 m_2$. If player 1's offer is accepted, 1 can at most get $1-\delta_2 m_2$. If player 1's offer is rejected, 1 can at most get $\delta_1(1-m_2) \leq 1-\delta_2 m_2$. Therefore, $M_1 \leq 1-\delta_2 m_2$.

Similarly, we can get $m_2 \geq 1 - \delta_1 M_1$ and $M_2 \leq 1 - \delta_1 m_1$. Now,

$$m_i \geq 1 - \delta_j M_j \geq 1 - \delta_j(1 - \delta_i m_i) \Rightarrow m_i \geq \frac{1 - \delta_j}{1 - \delta_1 \delta_2},$$

$$M_i \leq 1 - \delta_j M_j \leq 1 - \delta_j(1 - \delta_i M_i) \Rightarrow M_i \geq \frac{1 - \delta_j}{1 - \delta_1 \delta_2}.$$

Because $m_i \leq M_i$ by construction, we have

$$M_i = m_i = \frac{1 - \delta_j}{1 - \delta_1 \delta_2}.$$

It is just the stationary SPE payoff as $x^* = M_1 = m_1$ and $1 - y^* = M_2 = m_2$.

It remains to show that the stationary SPE is the *unique* SPE. Consider the subgame in which 1 makes the initial proposal. In any SPE, if an agreement is not reached in the first round, 1 can get at most $1 - M_2 = \delta_1 M_1 < M_1$. This implies that for 1 to get M_1, an agreement must be reached in the first round. Thus, 1 must propose $(M_1, 1 - M_1)$ in the first round and 2 accepts this proposal. For this behavior to be part of a SPE, 2 must reject any proposal that gives her a strictly lower payoff than $1 - M_1$; otherwise 1 can make a lower offer to 2. Moreover, if 1 makes a proposal that gives 2 a strictly higher payoff than $1 - M_1 = \delta_2 M_2$, 2 must accept.

The above arguments show that in any subgame in which 1 makes the initial proposal, any SPE prescribes that, in the initial round, 1 makes proposal $(M_1, 1 - M_1)$ and 2 accepts any offer higher than or equal to $1 - M_1$ and rejects others. Similar arguments apply to any SPE in the subgame in which 2 makes the initial proposal.

Therefore, the stationary SPE is the unique SPE.

5. Consider the following infinite horizon alternating offer bargaining over $[0, 1]$ between two agents. Agent 1 proposes in odd periods and agent 2 decides whether to accept or not. Agent 2 proposes in even periods and agent 1 decides whether to accept. Assume there is no discounting. Thus, if the proposal $(x, 1-x)$ is accepted in any period, agent 1 obtains x and agent 2 obtains $1 - x$, and the game ends. If a proposal is rejected in any period, then the bargaining relationship between the two agents breakdowns with probability $\rho \in (0, 1)$, in which case the game ends and agent 1 obtains $\frac{1}{3}$ and agent 2 obtains $\frac{2}{3}$. With probability $1 - \rho$, the relationship continues and they move into next period. Find a subgame perfect equilibrium.

   Soln: We guess a stationary equilibrium: Player 1 always proposes $(x^*, 1 - x^*)$ and player 2 always proposes $(y^*, 1-y^*)$; player 1 always accepts an offer $x \geq y^*$ and rejects others; player 2 always accepts an offer $1 - x \geq 1 - x^*$ and rejects others.

For player 2 to a accept $1 - x^*$, we need

$$1 - x^* \geq \rho \times \frac{2}{3} + (1 - \rho) \times (1 - y^*).$$

For player 2 reject any offer $1 - x < 1 - x^*$, we need

$$1 - x^* \leq \rho \times \frac{2}{3} + (1 - \rho) \times (1 - y^*).$$

Thus, we have

$$1 - x^* = \rho \times \frac{2}{3} + (1 - \rho) \times (1 - y^*).$$

In the same way, we also have

$$y^* = \rho \times \frac{1}{3} + (1 - \rho) \times x^*.$$

Simultaneously, we derive

$$x^* = y^* = \frac{1}{3}.$$

This strategy profile is a subgame perfect equilibrium.