Game Theory Lecture Notes 4
Static Games with Incomplete Information

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Motivating Example

Game of chicken with incomplete information

- Two teenagers drive toward each other.
- Just before impact, they must simultaneously choose whether to be chicken and swerve to the right, or continue driving head on.
- They want to show they are brave and gain the respect from their friends.
- But they also suffer a loss if a collision occurs (e.g., parents reprimand, if seriously injured).
- If everything is common knowledge, it is a complete information game that we have learned.

	c	d
C	$0, 0$	$0, R$
D	$R, 0$	$\frac{R}{2} - k, \frac{R}{2} - k$

where $R > 0$, $k \geq 0$.

Motivating Example

Game of chicken with incomplete information

- Now, assume the punishment k is each player's *private information*.
- For instance, every teenage knows whether his own parents are harsh or lenient, but does not know others' parents.
- If i's parents are harsh, then $k_i = H$ is high.
- If i's parents are lenient, then $k_i = L$ is low.
- Assume every player's parents can be harsh or lenient with equal probabilities.

Motivating Example

Game of chicken with incomplete information

- We have described a situation where not everything is common knowledge among the players.
- In particular, players have uncertainty over some characteristics of their opponents.
- In this example, each player is uncertain over the preferences of his opponent.
- This is an example of **games of incomplete information**.

Motivating Example

Game of chicken with incomplete information

- Consider player 1 who knows that his own parents are harsh.
- This means that player 1 knows if a collision happens, he will face a large punishment $k_1 = H$ from his parents.
- He does not know whether player 2's parents are harsh or lenient. Should he care about that?
- Although k_2 does not directly determine 1's payoff if a collision occurs, but it will affect 2's behavior which in turn will affect 1's own payoff too.
- Thus, when considering his own action, 1 should takes into account the fact that it is equally likely $k_2 = H$ or $k_2 = L$.
- The same intuition also applies to player 1 with $k_1 = L$ and player 2.
- But then, when considering his own actions, i should also takes into account the fact that j takes into account the fact that $k_i = H$ or $k_i = L$ equally likely.
- But then, i should also takes into account that j takes into account that i takes into account that ...

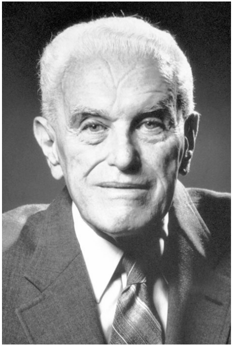
Motivating Example

Game of chicken with incomplete information

- It sounds like a very difficult and intractable situation.
- It was John Harsanyi who first realized that this difficult game of incomplete information can be transformed into an extensive form game of imperfect information which we have developed and fully understood.
- The trick is to introduce a special player, Nature, into an extensive form game, who “chooses” each player’s characteristics.

Motivating Example

Game of chicken with incomplete information



John C. Harsanyi

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1994

“for their pioneering analysis of equilibria in the theory of non-cooperative games.”

Contribution: Showed how games of incomplete information can be analyzed, thereby providing a theoretical foundation for a lively field of research - the economics of information - which focuses on strategic situations where different agents do not know each others’ objectives.

Motivating Example

Game of chicken with incomplete information

- Consider the following extensive form game.
- There are three players: Nature (player 0), players 1 and 2.
- Nature moves first. Available actions are LL, LH, HL and HH. Choosing LL means $k_1 = L$ and $k_2 = L$, and so on.
- Moreover, Nature's strategy is a fixed behavior strategy: it chooses of the four actions with probability $1/4$.
- Next, player 1 moves, observing Nature's choice k_1 but not k_2 and choosing between C and D.
- Lastly, player 2 moves, observing only Nature's choice k_2 but neither k_1 nor player 1's choice, and choosing between c and d.
- The payoff at each terminal node is properly specified.

Motivating Example

Game of chicken with incomplete information

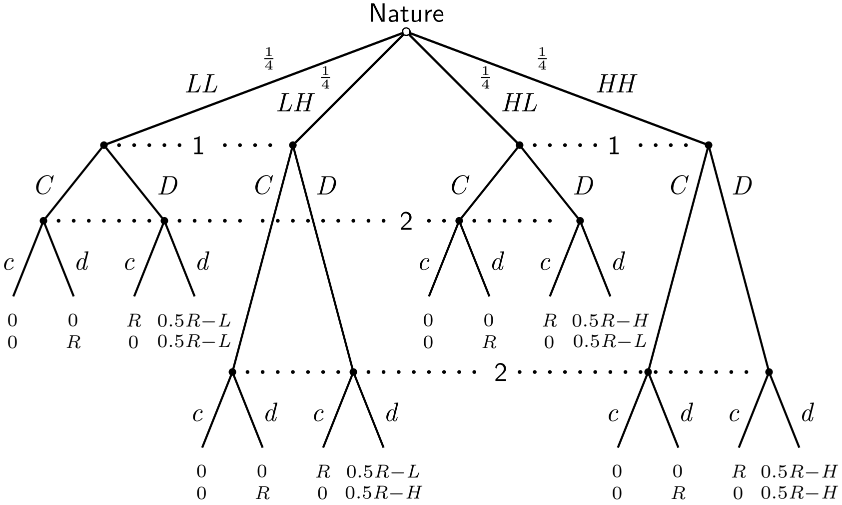


Figure 4.1: The extensive form for game of chicken with private punishment

Motivating Example

Game of chicken with incomplete information

- When player 2 chooses, he does not observe 1's choice, implying this is a simultaneous move game between players 1 and 2.
- When either player 1 or 2 chooses, he only knows his own k but not his opponent's, implying that k_i is i's private information.
- Thus, this extensive form game characterizes exactly the same strategic situation that we have described by a game of incomplete information.
- And, we know how to analyze it!

Motivating Example

Game of chicken with incomplete information

- Player 1 has two information sets. Thus, $S_1 = \{CC, CD, DC, DD\}$.
- For example, CD specifies that player 1 chooses C at his first information set and D at the second.
- In other words, CD represents the strategy that 1 chooses
  - C when $k_1 = L$ (his parents are lenient in our story), and
  - D when $k_1 = H$ (his parents are harsh).
- Similarly, player 2 has two information sets. Thus $S_2 = \{cc, cd, dc, dd\}$.
- For example, dc represents the strategy that 2 chooses
  - d when $k_2 = L$ (his parents are lenient), and
  - c when $k_2 = H$ (his parents are harsh).

Motivating Example

Game of chicken with incomplete information

- For any strategy profile $(A_L A_H, a_L a_H)$, the outcome is the distribution

$$\frac{1}{4}\circ(LL, A_L, a_L)+\frac{1}{4}\circ(LH, A_L, a_H)+\frac{1}{4}\circ(HL, A_H, a_L)+\frac{1}{4}\circ(HH, A_H, a_H).$$

- Player 1's payoff thus is

$$\begin{aligned}
\tilde{v}_1(A_L A_H, a_L a_H) \equiv & \frac{1}{4} v_1(A_L, a_L; k_1 = L) + \frac{1}{4} v_1(A_L, a_H; k_1 = L) \\
& + \frac{1}{4} v_1(A_H, a_L; k_1 = H) + \frac{1}{4} v_1(A_H, a_H; k_1 = H), \\
\tilde{v}_2(A_L A_H, a_L a_H) \equiv & \frac{1}{4} v_2(A_L, a_L; k_2 = L) + \frac{1}{4} v_2(A_L, a_H; k_2 = H) \\
& + \frac{1}{4} v_2(A_H, a_L; k_2 = L) + \frac{1}{4} v_2(A_H, a_H; k_2 = H).
\end{aligned}$$

Motivating Example

Game of chicken with incomplete information

- Assume $R = 8$, $H = 16$, and $L = 0$.
- The matrix representation is as follows:

	cc	cd	dc	dd
CC	$0, 0$	$0, 4$	$0, 4$	$0, 8$
CD	$4, 0$	$-1, -1$	$-1, 3$	$-6, 2$
DC	$4, 0$	$3, -1$	$3, 3$	$2, 2$
DD	$8, 0$	$2, -6$	$2, 2$	$-4, -4$

- Unique pure strategy Nash equilibrium: (DC, dc).
- Meaning: in this symmetric equilibrium, a teenager with harsh parents chooses to be chicken and a teenager with lenient parents choose to drive head on.
- By transforming it into a game of imperfect information, we have done analyzing a game of incomplete information without learning any new knowledge.

Bayesian Games

Players, actions, information and preferences

- To model the situation of incomplete information generally, we introduce **types** for each player.
- Each type of player i summarizes his characteristics.
- The set of all player i's possible types is his **type space**.
- For example, in the previous game of chichen, $k_1 = H$ is one of player 1's types. It summarizes player 1's preference characteristics: if a collision occurs, this type is subject to a large reprimand H.
- $k_1 = L$ is another possible type of player 1. It also summarizes player 1's preference characteristics: if a collision occurs, this type is subject to a small reprimand L.
- Hence, player 1's type space is $\{H, L\}$. So is player 2's type space.

Bayesian Games

Players, actions, information and preferences

- Uncertainty over types is described by/imagined as Nature choosing types for the different players.
- Importantly, there is common knowledge about the way in which Nature chooses between profiles of types of players.
- This is represented by a **common prior**, which is a probability distribution over type profiles that is common knowledge among the players.
- For example, in the previous game of chicken, the common prior is

$$\frac{1}{4} \circ LL + \frac{1}{4} \circ LH + \frac{1}{4} \circ HL + \frac{1}{4} \circ HH.$$

Bayesian Games

Players, actions, information and preferences

Definition 4.1

The normal-form representation of an n-player **Bayesian game** or **static game of incomplete information** is

$$(N,\ \{A_i\}_{i=1}^n,\ \{\Theta_i\}_{i=1}^n,\ \{v_i\}_{i=1}^n,\ \mathbb{P}),$$

where

- $N = \{1, 2, \ldots, n\}$ is the set of players;
- A_i is player i's action space;
- Θ_i is player i's **type space** and every element $\theta_i \in \Theta_i$ is a **type**;
- $v_i : A \times \Theta \to \mathbb{R}$ is i's payoff function; where $A = A_1 \times \ldots \times A_n$ is the set of action profiles and $\Theta = \Theta_1 \times \ldots \times \Theta_n$ is the set of type profiles;
- $\mathbb{P}$ is the **common prior**: a probability distribution over Θ.

- $v_i(a; \theta)$ is i's payoff if the action profile is a and the type profile is θ.

Bayesian Games

Deriving posteriors from a common prior: a player's beliefs

- Recall from your probability course.
- Suppose $\mathbb{P}$ is a probability distribution over some space X.
- The **conditional probability of an event** $B \subset X$ **given event** $A \subset X$ is

$$\mathbb{P}(B|A) \equiv \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)},$$

  provided the denominator is positive.
- This formula is called the **Bayes' rule**.

Bayesian Games

Deriving posteriors from a common prior: a player's beliefs

- In a Bayesian game, the common prior $\mathbb{P}$ is a probability distribution over $\Theta = \Theta_1 \times \ldots \times \Theta_n$.
- Suppose i knows that his type is $\theta_i \in \Theta_i$.
- How does he think about his opponents' types?
- In particular, how likely does player i thinks that his opponents' type is $\theta_{-i} \in \Theta_{-i}$?
- This is precisely the conditional probability of θ_{-i} given θ_i:

  $$\phi_i(\theta_{-i}|\theta_i) = \frac{\mathbb{P}(\theta_i, \theta_{-i})}{\mathbb{P}(\theta_i)},$$

  where $\mathbb{P}(\theta_i) = \sum_{\theta'_{-i} \in \Theta_{-i}} \mathbb{P}(\theta_i, \theta'_{-i})$ is the **marginal distribution of player i's types**.
- $\phi_i(\,\cdot\,|\,\theta_i)$ is called player i's **posterior belief about his opponents' types** given his own type θ_i.

Bayesian Games

Deriving posteriors from a common prior: a player's beliefs

- Consider the previous game of chicken example.
- The common prior can be represented by

	L	H
L	$1/4$	$1/4$
H	$1/4$	$1/4$

  where each row (column) represents one of player 1's (2's) types.
- Thus, $\phi_i(H|\theta_i) = \phi_i(L|\theta_i) = \frac{1}{2}$ for $i = 1, 2$ and $\theta_i \in \{H, L\}$. This is so because we have assumed that the two players' types are *independently distributed*.

Bayesian Games

Deriving posteriors from a common prior: a player's beliefs

- If, instead, the common prior is

	L	H
L	$1/3$	$1/6$
H	$1/6$	$1/3$

  then for $i = 1, 2$,

$$\phi_i(L|L) = \frac{2}{3},\ \phi_i(H|L) = \frac{1}{3},$$

  and

$$\phi_i(L|H) = \frac{1}{3},\ \phi_i(H|H) = \frac{2}{3}.$$

- Different types have different posterior beliefs.
- In particular, every type thinks that it is more likely that the opponent is of the same type.

Bayesian Games

Strategies and Bayesian Nash equilibrium

Definition 4.2

Consider a Bayesian game

$$(N,\ \{A_i\}_{i=1}^n,\ \{\Theta_i\}_{i=1}^n,\ \{v_i\}_{i=1}^n,\ \mathbb{P}).$$

A **pure strategy** for player i is a function $s_i : \Theta_i \to A_i$ that specifies a pure action $s_i(\theta_i)$ that player i will choose when his type is θ_i. A **mixed strategy** is a function $\sigma_i : \Theta_i \to \Delta(A_i)$ that specifies a mixed action $\sigma_i(\theta_i)$ for each type θ_i.

- A strategy is a complete type-contingent plan.
- Easy to understand if we think about its equivalent extensive form representation: a pure strategy is a mapping from information sets (types) to actions. A mixed strategy in the Bayesian game corresponds to a behavioral strategy in the extensive form representation.

Bayesian Games

Strategies and Bayesian Nash equilibrium

- Given a strategy profile s_{-i} of i's opponents, the expected payoff for player i if his type is θ_i and he chooses action $a_i \in A_i$ is

$$\sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(a_i, s_{-i}(\theta_{-i}); \theta_i, \theta_{-i}).$$

- Player i of type θ_i thinks that θ_{-i} is realized with probability $\phi_i(\theta_{-i}|\theta_i)$, in which case the opponents choose $s_{-i}(\theta_{-i})$ and the resulting payoff is $v_i(a_i, s_{-i}(\theta_{-i}); \theta_i, \theta_{-i})$

Definition 4.3

In the Bayesian game

$$(N,\ \{A_i\}_{i=1}^n,\ \{\Theta_i\}_{i=1}^n,\ \{v_i\}_{i=1}^n,\ \mathbb{P}),$$

a strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a **pure strategy Bayesian Nash equilibrium** if, for every player i, for each of player i's type $\theta_i \in \Theta_i$, we have

$$\begin{aligned} &\sum_{\theta_{-i}\in\Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}) \\ \geq &\sum_{\theta_{-i}\in\Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(a_i, s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}),\ \forall a_i \in A_i. \end{aligned} \tag{4.1}$$

- No type of any player wants to deviate.
- Mixed strategy equilibrium is similarly defined.

Bayesian Games

Strategies and Bayesian Nash equilibrium

- In any Bayesian game, a strategy profile is a Bayesian Nash equilibrium if and only if it is a Nash equilibrium of the equivalent extensive form game.
- To see the "only if" direction, consider the extensive form game and any strategy s_i of player i.
- (4.1) implies that for each type $\theta_i \in \Theta_i$, we have

$$\sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i})$$
$$\geq \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}).$$

Bayesian Games

Strategies and Bayesian Nash equilibrium

- Multiplying both side by $\mathbb{P}(\theta_i)$ and summing up all these inequalities over $\theta_i \in \Theta_{-i}$ yields

$$\sum_{\theta_i \in \Theta_i} \mathbb{P}(\theta_i) \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i})$$
$$\geq \sum_{\theta_i \in \Theta_i} \mathbb{P}(\theta_i) \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}).$$

- Equivalently,

$$\sum_{\theta \in \Theta} \mathbb{P}(\theta) v_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i})$$
$$\geq \sum_{\theta \in \Theta} \mathbb{P}(\theta) v_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}).$$

- The LHS is i's expected payoff in the extensive form game from the strategy profile (s_i^*, s_{-i^*}), while the RHS is that from the strategy profile (s_i, s_{-i}^*). Thus, s^* is a Nash equilibrium of the extensive form game.

Bayesian Games

Strategies and Bayesian Nash equilibrium

- To see the "if" direction, suppose (4.1) is violated for some player i of some type $\hat{\theta}_i$.
- That is, there exists $\hat{a}_i$ such that

$$\begin{aligned}&\sum_{\theta_{-i}\in\Theta_{-i}} \phi_i(\theta_{-i}|\hat{\theta}_i)v_i(\hat{a}_i, s^*_{-i}(\theta_{-i}); \hat{\theta}_i, \theta_{-i}) \\ >& \sum_{\theta_{-i}\in\Theta_{-i}} \phi_i(\theta_{-i}|\hat{\theta}_i)v_i(s^*_i(\hat{\theta}_i), s^*_{-i}(\theta_{-i}); \hat{\theta}_i, \theta_{-i}).\end{aligned}$$

- Define

$$s_i(\theta_i) = \begin{cases} \hat{a}_i, & \text{if } \theta_i = \hat{\theta}_i, \\ s^*_i(\theta_i), & \text{if } \theta_i \neq \hat{\theta}_i. \end{cases}$$

- Note that s_i differs from s^*_i only at $\hat{\theta}_i$.

Bayesian Games

Strategies and Bayesian Nash equilibrium

- Performing a similar analysis as before, we can show that

$$\sum_{\theta \in \Theta} \mathbb{P}(\theta) v_i(s_i(\theta_i), s^*_{-i}(\theta_{-i}); \theta_i, \theta_{-i})$$
$$> \sum_{\theta \in \Theta} \mathbb{P}(\theta) v_i(s^*_i(\theta_i), s^*_{-i}(\theta_{-i}); \theta_i, \theta_{-i}).$$

- This shows that s_i^* is not a best response to s^*_{-i} in the extensive form.

Bayesian Games

- A final comment about the common prior $\mathbb{P}$.
- From Definition 4.3, the common prior $\mathbb{P}$ enters the analysis only through the induced players' posterior beliefs $\{\phi_i\}_i$.
- Therefore, when writing down a Bayesian game, we sometimes directly write down the system of posterior belief $\{\phi_i\}_i$ instead of the common prior $\mathbb{P}$.
- Doing so effectively enlarges the scope of Bayesian games because it also incorporates the possibility of models *without* a common prior.
- Though it is conceptually/philosophically not easy to think about a environment where players do not share a common prior, the definition of Bayesian Nash equilibrium still applies. (But now it is impossible to transform it into an equivalent extensive form game with Nature. Yes?)

Bayesian Games

Strategies and Bayesian Nash equilibrium

- The following is a simple example of system of beliefs which does not admit a common prior:

$$
\begin{array}{cc|c|c|}
 & \multicolumn{3}{c}{\theta_2} \\
 & & H & L \\
\cline{3-4}
\theta_1 & H & 1 & 0 \\
\cline{3-4}
 & L & 0 & 1 \\
\cline{3-4}
\end{array}
\qquad
\begin{array}{cc|c|c|}
 & \multicolumn{3}{c}{\theta_1} \\
 & & H & L \\
\cline{3-4}
\theta_2 & H & 0 & 1 \\
\cline{3-4}
 & L & 1 & 0 \\
\cline{3-4}
\end{array}
$$

$$\phi_1 \qquad\qquad \phi_2$$

- Suppose, by contradiction, that a common prior $\mathbb{P}$ exists.
- ϕ_1 implies

$$\mathbb{P}(\theta_1 = H, \theta_2 = L) = 0,$$

while ϕ_2 implies

$$\mathbb{P}(\theta_1 = H, \theta_2 = L) > 0.$$

A contradiction.

Examples

Cournot duopoly with private information

- Cournot duopoly in which firm 1's marginal cost is its private information.
- In particular, firm 1's marginal cost can be either high c_h with probability $\theta \in (0, 1)$ or low c_ℓ with probability $1 - \theta$.
- Only firm 1 knows its marginal cost.
- Firm 2's marginal cost, c_2, is commonly known.
- The two firms then choose quantities simultaneously.
- Market demand is $P(q) = \max\{a - q, 0\}$.

Examples

Cournot duopoly with private information

- Firm 1 has two types c_h and c_ℓ. Thus, firm 1's strategy is a pair $(q_h, q_\ell) \in \mathbb{R}^2_+$, where q_h is its supply when its marginal cost is high and q_ℓ is its supply when its marginal cost is low.
- Firm 2's strategy is $q_2 \in \mathbb{R}_+$.
- The strategy profile (q_h^*, q_ℓ^*, q_2) is a Bayesian Nash equilibrium if

$$q_h^* \in \operatorname*{arg\,max}_{q_h \geq 0} \big[P(q_h + q_2^*) - c_h\big] q_h,$$

$$q_\ell^* \in \operatorname*{arg\,max}_{q_\ell \geq 0} \big[P(q_\ell + q_2^*) - c_\ell\big] q_\ell,$$

and

$$q_2^* \in \operatorname*{arg\,max}_{q_2 \geq 0} \theta \big[P(q_h^* + q_2) - c_2\big] q_2 + (1-\theta) \big[P(q_\ell^* + q_2) - c_2\big] q_2.$$

- The first two conditions state that each type of firm 1 maximizes its profit given firm 2's supply. The last condition states that firm 2 maximizes its profit given firm 1's strategy.

Examples

Cournot duopoly with private information

- Assume interior solution, i.e. a is large enough

$$q_h^* \in \arg\max_{q_h \geq 0} \left[a - q_h - q_2^* - c_h\right] q_h,$$

$$q_\ell^* \in \arg\max_{q_\ell \geq 0} \left[a - q_\ell - q_2^* - c_\ell\right] q_\ell,$$

$$q_2^* \in \arg\max_{q_2 \geq 0} \theta\left[a - q_h^* - q_2 - c_2\right] q_2 + (1-\theta)\left[a - q_\ell^* - q_2 - c_2\right] q_2.$$

- We have

$$q_h^* = \frac{a - q_2^* - c_h}{2},$$

$$q_l^* = \frac{a - q_2^* - c_l}{2},$$

$$q_2^* = \frac{a - \theta q_h^* - (1-\theta) q_l^* - c_2}{2}.$$

Examples

Cournot duopoly with private information

- Finally

$$q_h^* = \frac{a - 2c_h + c_2}{3} + \frac{1-\theta}{6}(c_h - c_\ell),$$
$$q_l^* = \frac{a - 2c_l + c_2}{3} - \frac{\theta}{6}(c_h - c_\ell),$$
$$q_2^* = \frac{a - 2c_2 + \theta c_h + (1-\theta)c_\ell}{3}.$$

Examples

Study groups

- Two students, who have to hand in a joint lab assignment, form a study group.
- Each student i can either put in the effort $(e_i = 1)$ or shirk $(e_i = 0)$.
- The cost of exerting effort is $0 < c < 1$, while shirking involves no cost.
- If either one or both of the students put in the effort, then the lab assignment is success.
- While if both shirk, then it is a failure.
- The value of a failure to both students is 0.
- The value of a success to player i is θ_i^2, where θ_i is player i's private information.
- Assume θ_1 and θ_2 are uniformly and independently distributed over $[0, 1]$.
- A game of voluntary contribution to public good.

Examples

Study groups

- A strategy for player i is a mapping $\sigma_i : [0,1] \to [0,1]$, where $\sigma_i(\theta_i)$ is the probability that i of type θ_i exerts effort.
- Consider an arbitrary σ_i. From student j's point view, the probability that player i exerts effort is

$$p^{\sigma_i} = \int_0^1 \sigma_i(\theta_i) f(\theta_i) \mathrm{d}\theta_i = \int_0^1 \sigma_i(\theta_i) \mathrm{d}\theta_i,$$

where $f(\theta_i) = 1$ is the density of uniform distribution.

Examples

Study groups

- Given an arbitrary σ_2, consider student 1 of type θ_1.
- If he exerts effort, the payoff is

$$\theta_1^2 - c.$$

- If he shirks, the payoff is

$$p^{\sigma_2}\theta_1^2.$$

- Therefore, if σ_1 is a best reponse to σ_2, we must have

$$\sigma_1(\theta_1) = \begin{cases} 1, & \text{if } \theta_1^2 > c/(1-p^{\sigma_2}), \\ 0, & \text{if } \theta_1^2 < c/(1-p^{\sigma_2}). \end{cases} \tag{4.2}$$

- This is an example of a **cut-off strategy** (or **threshold strategy**): there exists a cut-off type $\hat{\theta}_1 = \sqrt{c/(1-p^{\sigma_2})}$ such that student 1 exerts effort if his type is above this cut-off and shirks if his type is below this cut-off.

Examples

Study groups

- Symmetrically, given an arbitrary σ_1, if σ_2 is a best response for player 2, then we must have

$$\sigma_2(\theta_2) = \begin{cases} 1, & \text{if } \theta_2^2 > c/(1-p^{\sigma_1}), \\ 0, & \text{if } \theta_2^2 < c/(1-p^{\sigma_1}). \end{cases} \tag{4.3}$$

- Now, suppose (σ_1, σ_2) is a Bayesian Nash equilibrium. From (4.2), we know

$$p^{\sigma_1} = \int_{\min\{1,\sqrt{c/(1-p^{\sigma_2})}\}} \mathrm{d}\theta_1 = 1 - \min\{1, \sqrt{c/(1-p^{\sigma_2})}\}. \tag{4.4}$$

- Similarly, from (4.3), we know

$$p^{\sigma_2} = 1 - \min\{1, \sqrt{c/(1-p^{\sigma_1})}\}. \tag{4.5}$$

Examples

Study groups

- For $0 < c < 1$, (4.4) and (4.5) have a unique solution

$$p^{\sigma_1} = p^{\sigma_2} = 1 - c^{\frac{1}{3}}.$$

- This implies the game has an essentially unique Bayesian Nash equilibrium (σ_1^*, σ_2^*), where

$$\sigma_i^*(\theta_i) = \begin{cases} 1, & \text{if } \theta_i > c^{\frac{1}{3}}, \\ 0, & \text{if } \theta_i < c^{\frac{1}{3}}. \end{cases}$$

- The word "essentially" means that when $\theta_i = c^{\frac{1}{3}}$, any $\sigma_i(\theta_i) \in [0, 1]$ is fine.

Applications

Adverse selection

- Player 1 owns a car which he has been driving for some years and which he considers to sell.
- Player 2 is a potential buyer.
- The mechanical condition of the car is player 1's private information.
- Player 2 thinks that the condition can be poor (P), fair (F) or good (G), with equal probabilities.

Applications

Adverse selection

- Player 1's reservation value for this car is

$$v_1(\theta) = \begin{cases} 10, & \text{if } \theta = P, \\ 20, & \text{if } \theta = F, \\ 30, & \text{if } \theta = G. \end{cases}$$

- Player 2's willingness to pay for this car is

$$v_2(\theta) = \begin{cases} 14, & \text{if } \theta = P, \\ 24, & \text{if } \theta = F, \\ 34, & \text{if } \theta = G. \end{cases}$$

Applications

Adverse selection

- As a benchmark, assume now that the mechanical condition is commonly known.
- The game between 1 and 2 is an ultimatum game.
- Player 2 first announces a price p that he is willing to pay for that car.
- After observing p, player 1 decides whether to accept or not.
- If player 1 accepts, they trade at price p; otherwise, they do not trade.

Applications

Adverse selection

- If $\theta = P$, they trade at $p = 10$ in the unique SPE.
- If $\theta = F$, they trade at $p = 20$ in the unique SPE.
- If $\theta = G$, they trade at $p = 30$ in the unique SPE.
- In all cases, trade occurs and is efficient.

Applications

Adverse selection

- Now, we return to the case where the mechanical condition is player 1's private information.
- To model this situation as a Bayesian game (simultaneous move), we assume that player 1 chooses an acceptance plan which specifies for each price whether he accepts or not.
- Thus, a strategy for player 1 of type $\theta \in \{P, F, G\}$ is a mapping $s_1^\theta : \mathbb{R} \to \{a, r\}$.
- Bayesian Nash equilibria?

Applications

Adverse selection

- There is no equilibrium in which G accepts the equilibrium price.
- To see this, suppose by contradiction that G accepts the equilibrium price p in an equilibrium, i.e., $s_1^G(p) = a$.
- This implies $p \geq 30$, which in turn implies $s_1^P(p) = s_1^F(p) = a$.
- But then, player 2's expected payoff from p is

$$\frac{1}{3}v_2(P) + \frac{1}{3}v_2(F) + \frac{1}{3}v_2(G) - p = 24 - p < 0.$$

- By deviating to $p' = 0$, player 2 can obtain at least 0.
- A contradiction.

Applications

Adverse selection

- There is no equilibrium in which F accepts the equilibrium price p, i.e., $s_1^F(p) = a$, either.
- The argument is similar to the above one.
- If, by contradiction, $s_1^F(p) = a$, we know $p \geq 20$, which in turn implies $s_1^P(p) = a$.
- So, player 2's expected payoff from p is (we know $s_1^G(p) = r$ from previous analysis):

$$\frac{1}{2}v_2(P) + \frac{1}{2}v_2(F) - p = 19 - 20 < 0.$$

- Again, by deviating to $p' = 0$, player 2 can at least obtain 0.
- A contradiction.

Applications

Adverse selection

- In sum, in any Bayesian Nash equilibrium, player 1 of fair or good condition will not sell!
- There is a Bayesian Nash equilibrium in which player 1 of poor condition sells: $p = 10$ and for $\theta \in \{P, F, G\}$,

$$s_1^{\theta}(p) = \begin{cases} a, & \text{if } p \geq v_1(\theta), \\ r, & \text{if } p < v_1(\theta). \end{cases}$$

- The market consists of only the sellers with the lowest quality due to asymmetric information.
- Consequently, trade is not efficient!

Applications

Adverse selection

- This observation is first made by George Akerlof who won the Nobel Prize because of this contribution.
- The phenomenon in this example is called **adverse selection**.
- Good sellers are out of the market and bad sellers stay.
- Asymmetric information *selects* those sellers that are *adverse* to the buyer.
- More generally, in an environment with asymmetric information, the informed party usually behaves adversely to the uniformed party.

Applications

Adverse selection

George A. Akerlof

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2001

"for their analyses of markets with asymmetric information"

Contribution: Studied markets where sellers of products have more information than buyers about product quality. He showed that low-quality products may squeeze out high-quality products in such markets, and that prices of high-quality products may suffer as a result.

Applications

Committee voting

- A jury is made up of two players (jurors) who must collectively decide whether to acquit (A) or to convict (C) a defendant.
- The process calls for each player to cast a sealed vote, and the defendant is convicted only if both vote C.
- There is uncertainty about whether the defendant is guilty (G) or innocent (I).
- The prior probability that the defendant is guilty is given by $q > \frac{1}{2}$.
- Each player cares about making the right decision:

G	A	C
A	$0,0$	$0,0$
C	$0,0$	$1,1$

I	A	C
A	$1,1$	$1,1$
C	$1,1$	$0,0$

Applications

Committee voting

- If the only information available to the players is the prior probability, then it is a game with symmetric incomplete information and can be conveniently thought of as a game of complete information:

	A	C
A	$1-q, 1-q$	$1-q, 1-q$
C	$1-q, 1-q$	q, q

- Two Nash equilibria: (A, A) and (C, C).

Applications

Committee voting

- Now assume each player i receives a private signal $\theta_i \in \{\theta_G, \theta_I\}$ distributed according to

$$\mathbb{P}(\theta_i = \theta_G | G) = \mathbb{P}(\theta_i = \theta_I | I) = p > \frac{1}{2},$$

  and

$$\mathbb{P}(\theta_i = \theta_I | G) = \mathbb{P}(\theta_i = \theta_G | I) = 1 - p.$$

- So signal θ_x is more likely in state $x \in \{G, I\}$.
- The larger p is, the more informative is the signal.
- Assume also that the players' signals are conditionally independent:

$$\mathbb{P}(\theta_1 = \theta_x, \theta_2 = \theta_y | z) = \mathbb{P}(\theta_1 = \theta_x | z)\mathbb{P}(\theta_2 = \theta_y | z),$$

  for $x, y, z \in \{G, I\}$.
- This defines a Bayesian game.

Applications

Committee voting

- What would player 1 do if there is no player 2?
- After observing signal $\theta_1 = \theta_G$, player 1's posterior belief that the defendant is guilty is

$$\mathbb{P}(G|\theta_1 = \theta_G) = \frac{qp}{qp + (1-q)(1-p)} > q > \frac{1}{2}.$$

- Thus, player 1 is even more convinced that the defendant is guilty and hence will choose to convict him.

Applications

Committee voting

- After observing signal $\theta_1 = \theta_I$, player 1's posterior belief becomes

$$\mathbb{P}(G|\theta_1 = \theta_I) = \frac{q(1-p)}{q(1-p) + (1-q)p} < q.$$

- Thus, player 1 is less confident that the defendant is guilty.
- In particular,
  - if $p > q$, player 1 acquit the defendant;
  - if $p < q$, player 1 convict the defendant.
- Intuition: when p is large, the signal is very informative. Hence, player 1 is convinced that the defendant is not guilty after observing signal $\theta_1 = \theta_I$.

Applications

Committee voting

- Is voting according to one's own signal a Bayesian Nash equilibrium?
- Assume $\sigma_2(\theta_G) = c$ and $\sigma_2(\theta_I) = a$.
- Assume player 1 receives signal $\theta_1 = \theta_I$. His posterior belief is given by

	$\theta_2 = \theta_G$	$\theta_2 = \theta_I$
G	$\frac{q(1-p)p}{\mathbb{P}(\theta_1=\theta_I)}$	$\frac{q(1-p)^2}{\mathbb{P}(\theta_1=\theta_I)}$
I	$\frac{(1-q)p(1-p)}{\mathbb{P}(\theta_1=\theta_I)}$	$\frac{(1-q)p^2}{\mathbb{P}(\theta_1=\theta_I)}$

where $\mathbb{P}(\theta_1 = \theta_I) = qp^2 + qp(1-p) + (1-q)(1-p)^2 + (1-q)(1-p)p$.

Applications

Committee voting

- If player 1 chooses to acquit, his payoff is

$$\frac{(1-q)p(1-p)+(1-q)p^2}{\mathbb{P}(\theta_1=\theta_I)}.$$

- If instead, he convicts, then his payoff is

$$\frac{q(1-p)p+(1-q)p^2}{\mathbb{P}(\theta_1=\theta_I)}.$$

- Therefore, player 1 should convict, not acquit!
- Voting according to one's own signal is NOT a Bayesian Nash equilibrium.

Applications

Committee voting

- Given σ_2, player 1 is pivotal only if $\theta_2 = \theta_G$.
- Therefore, when comparing a and c, player 1 only needs to compare

$$\mathbb{P}(G, \theta_2 = \theta_G | \theta_1 = \theta_I)$$

and

$$\mathbb{P}(I, \theta_2 = \theta_G | \theta_1 = \theta_I).$$

- Although it is possible $\mathbb{P}(G | \theta_1 = \theta_I) < \mathbb{P}(I | \theta_1 = \theta_I)$ (when $p > q$), but we always have

$$\mathbb{P}(G, \theta_2 = \theta_G | \theta_1 = \theta_I) > \mathbb{P}(I, \theta_2 = \theta_G | \theta_1 = \theta_I).$$

- Another way to see the above inequality is

$$
\begin{aligned}
&\frac{\mathbb{P}(G, \theta_2 = \theta_G | \theta_1 = \theta_I)}{\mathbb{P}(I, \theta_2 = \theta_G | \theta_1 = \theta_I)} \\
=&\frac{\mathbb{P}(G | \theta_1 = \theta_I, \theta_2 = \theta_G)}{\mathbb{P}(I | \theta_1 = \theta_I, \theta_2 = \theta_G)} \\
=&\frac{\mathbb{P}(G, \theta_1 = \theta_I, \theta_2 = \theta_G)}{\mathbb{P}(I, \theta_1 = \theta_I, \theta_2 = \theta_G)} \\
=&\frac{q(1-p)p}{(1-q)p(1-p)} > 1.
\end{aligned}
$$

- That is, given both $\theta_1 = \theta_I$ and $\theta_2 = \theta_G$, player 1 thinks G is more likely.

Applications

Purification: Harsanyi's interpretation of mixed strategies

- Consider the game of Matching Pennies:

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Figure 4.2: The Matching Pennies

- Unique Nash equilibrium: $\sigma_1 = \sigma_2 = \frac{1}{2} \circ H + \frac{1}{2} \circ T$.
- In equilibrium, every player is indifferent between H and T.
- Yet, they are prescribed to randomize between these strategies in a unique, particular, and precise way, i.e., half-half, for this to be a Nash.
- Does it make sense to expect such precision when a player is indifferent?

Applications

Purification: Harsanyi's interpretation of mixed strategies

- Harsanyi's solution: players may not mix. In fact, they may always have some slight preference for choosing H over T or choosing T over H. We, as an observer, think that players are mixing because we do not have precise information about the players' preferences.

Applications

Purification: Harsanyi's interpretation of mixed strategies

- Consider the following "perturbed " Matching Pennies:

	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

Figure 4.3: The perturbed Matching Pennies

- Imagine that ε_1 and ε_2 are random variables that are independently and uniformly distributed on the interval $[-\varepsilon, \varepsilon]$ for some small $\varepsilon > 0$.
- For an example, suppose player 2 plays H with probability $1/2$.
- If player 1 plays H, his payoff is ε_1. If he plays T, his payoff is 0.
- Thus, player 1 *strictly* prefers H when $\varepsilon_1 > 0$, while he *strictly* prefers T when $\varepsilon_1 < 0$.

Applications

Purification: Harsanyi's interpretation of mixed strategies

- Suppose now that ε_i is player i's private information.
- This defines a Bayesian game.
- A strategy for player i is a mapping $\sigma_i : [-\varepsilon, \varepsilon] \to [0, 1]$, where $\sigma_i(\varepsilon_i)$ is the probability of choosing H when i's type is ε_i.
- Consider an arbitrary σ_i. From player j's point of view, the probability that player i plays H is

$$p^{\sigma_i} \equiv \int_{-\varepsilon}^{\varepsilon} \sigma_i(\varepsilon_i) f(\varepsilon_i) \mathrm{d}\varepsilon_i = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \sigma_2(\varepsilon_2) \mathrm{d}\varepsilon_i,$$

where $f(\varepsilon_i) = \frac{1}{2\varepsilon}$ is the density of uniform distribution.

Applications

Purification: Harsanyi's interpretation of mixed strategies

- Consider player 1 of type ε_1.
- If he plays H, his payoff is

$$(1+\varepsilon_1)p^{\sigma_2} + (-1+\varepsilon_1)(1-p^{\sigma_2}) = 2p^{\sigma_2} - 1 + \varepsilon_1.$$

- If he plays T, his payoff is

$$-p^{\sigma_2} + (1-p^{\sigma_2}) = -2p^{\sigma_2} + 1.$$

- Therefore, if σ_1 is a best reply, we must have

$$\sigma_1(\varepsilon_1) = \begin{cases} 1, & \text{if } \varepsilon_1 > 2 - 4p^{\sigma_2}, \\ 0, & \text{if } \varepsilon_1 < 2 - 4p^{\sigma_2}. \end{cases}$$

- Then, we can calculate

$$p^{\sigma_1} = \begin{cases} 1, & \text{if } 2 - 4p^{\sigma_2} \le -\varepsilon, \\ \frac{\varepsilon - 2 + 4p^{\sigma_2}}{2\varepsilon}, & \text{if } -\varepsilon < 2 - 4p^{\sigma_2} < \varepsilon, \\ 0, & \text{if } 2 - 4p^{\sigma_2} \ge \varepsilon. \end{cases} \tag{4.6}$$

Applications

Purification: Harsanyi's interpretation of mixed strategies

- Similarly, if σ_2 is a best reply to σ_1, we must have

$$\sigma_2(\varepsilon_2) = \begin{cases} 1, & \text{if } \varepsilon_2 > 4p^{\sigma_1} - 2, \\ 0, & \text{if } \varepsilon_2 < 4p^{\sigma_1} - 2. \end{cases}$$

- Then, we can calculate

$$p^{\sigma_2} = \begin{cases} 1, & \text{if } 4p^{\sigma_1} - 2 \leq -\varepsilon, \\ \frac{\varepsilon + 2 - 4p^{\sigma_1}}{2\varepsilon}, & \text{if } -\varepsilon < 4p^{\sigma_1} - 2 < \varepsilon, \\ 0, & \text{if } 4p^{\sigma_1} - 2 \geq \varepsilon. \end{cases} \tag{4.7}$$

Applications

Purification: Harsanyi's interpretation of mixed strategies

- Thus, if (σ_1, σ_2) is a Bayesian Nash equilibrium, p^{σ_1} and p^{σ_2} must simultaneously solve equations (4.6) and (4.7).
- When $\varepsilon < 2$, the unique solution is

$$p^{\sigma_1} = p^{\sigma_2} = \frac{1}{2}.$$

- Therefore, there is a (essentially) unique Bayesian Nash equilibrium (σ_1^*, σ_2^*) of this perturbed matching pennies:

$$\sigma_i^*(\varepsilon_i) = \begin{cases} H, & \text{if } \varepsilon_i > 0, \\ T, & \text{if } \varepsilon_i < 0. \end{cases}$$

- The word "essentially" means the fact that any arbitrary specification of $\sigma_i^*(0)$ is fine.

Applications

Purification: Harsanyi's interpretation of mixed strategies

- Notice that, in this unique equilibrium, every player "always" has strictly preference between H and T, and consequently plays a pure strategy.
- The only situation where a player is indifferent between H and T is when his type is 0. But this event occurs with probability 0.
- Moreover, since $p^{\sigma_1} = p^{\sigma_2} = 1/2$, each player plays H half of the time.
- In other words, their behavior looks *as though* they are mixing between H and T with equal probabilities, from an outside observer's point view who does not observe players' private information.
- Purification: mixed strategy Nash equilibrium of a complete information game can be purified by a nearby incomplete information game.

Auctions

Common auction formats

- English auction: open ascending price auction
- Dutch auction: open descending price auction
- First-price sealed-bid auction
- Second-price sealed bid auction
- William Vickrey was the first to analyze auctions using formal game theory tools.

Auctions

Common auction formats

William Vickrey

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1996

"for their fundamental contributions to the economic theory of incentives under asymmetric information."

Contribution: Developed methods of analyzing the problems of incomplete, or asymmetrical, information. Specialized in auction theory.

Auctions

Auctions with independent private values

- We will model auctions as Bayesian games, as Vickrey did.
- In a Bayesian game of auction, if bidders' types are distributed independently, we say it is an **independent type environment**; otherwise, we say it is a **correlated type environment**.
- Moreover, if each bidder's own type completely determines his willingness to pay, we say it is a **private value environment**; otherwise it is an **interdependent value environment**.
- We will mainly focus on the independent private value (IPV) environment in what follows, until the last subsection in which we consider **independent common value auction** which is a special case of **independent interdependent value environment**.

Auctions

Second-price sealed-bid auctions and English auctions

- There are n bidders.
- Each bidder has a valuation θ_i (willingness to pay) about the object being sold.
- Bidder i's valuation is his private information.
- Thus, each valuation is a type of bidder i.
- Assume θ_i is drawn from the interval $[\underline{\theta}_i, \overline{\theta}_i]$ with cdf F_i and density f_i.
- Valuations among the bidders are drawn independently.
- Thus we are in the IPV setting.

Auctions

Second-price sealed-bid auctions and English auctions

- In a second-price sealed-bid auction, the winning bidder pays his the second highest bid.
- Thus, the payoff for bidder i is

$$v_i(b_i, b_{-i}; \theta_i) = \begin{cases} \frac{\theta_i - \max_{j \neq i} b_j}{|\{k | b_k = \max_j b_j\}|}, & \text{if } b_i = \max_j b_j, \\ 0 & \text{if } b_i < \max_j b_j. \end{cases}$$

- In the case where several bidders bid the same highest price, the object is randomly allocated to each of them with equal probabilities.

Auctions

Second-price sealed-bid auctions and English auctions

- The second price auction has a very appealing feature, as we see now.
- Consider bidder i of type θ_i.
- For any $b_i \neq \theta_i$, we have

$$v_i(\theta_i, b_{-i}; \theta_i) \geq v_i(b_i, b_{-i}; \theta_i), \ \forall b_{-i}, \tag{4.8}$$

  and there exists b_{-i} such that

$$v_i(\theta_i, b_{-i}; \theta_i) > v_i(b_i, b_{-i}; \theta_i). \tag{4.9}$$

- Condition (4.8) says that bidding one own valuation is *always* as least as good as any other bid.
- Condition (4.9) says that for any other bid, bidding one's own valuation is sometimes strictly better.

Auctions

Second-price sealed-bid auctions and English auctions

- To see why (4.8) and (4.9) hold, consider any $b_i < \theta_i$ first.
- Then, it is easy to see that for b_{-i} such that $\max_{j \neq i} b_j < b_i$ or $\max_{j \neq i} b_j \geq \theta_i$,
$$v_i(b_i, b_{-i}; \theta_i) = v_i(\theta_i, b_{-i}; \theta_i).$$
- For b_{-i} such that $b_i \leq \max_{j \neq i} b_j < \theta_i$,
$$v_i(b_i, b_{-i}; \theta_i) < v_i(\theta_i, b_{-i}; \theta_i).$$

Auctions

Second-price sealed-bid auctions and English auctions

- Now, consider $b_i > \theta_i$.
- Then, for b_{-i} such that $\max_{j \neq i} b_j \leq \theta_i$ or $\max_{j \neq i} b_j > b_i$, we have

$$v_i(b_i, b_{-i}; \theta_i) = v_i(\theta_i, b_{-i}; \theta_i).$$

- For b_{-i} such that $\theta_i < \max_{j \neq i} b_j \leq b_i$,

$$v_i(b_i, b_{-i}; \theta_i) < v_i(\theta_i, b_{-i}; \theta_i).$$

- Therefore, (4.8 and (4.9) hold.
- We say that bidding one's own valuation is a **weakly dominant strategy**.

Auctions

Second-price sealed-bid auctions and English auctions

- It should then be easy to understand that in the Bayesian game of second price auction, every bidder biding his own valuation is a Bayesian Nash equilibrium.
- This is because $s_i(\theta_i) = \theta_i$ is a best response to any s_{-i}: for θ_i,

$$\mathbb{E}_{\theta_{-i}}(v_i(\theta_i, s_{-i}(\theta_{-i}); \theta_i)) \geq \mathbb{E}_{\theta_{-i}}(v_i(b_i, s_{-i}(\theta_{-i}); \theta_i)), \ \forall b_i.$$

Auctions

Second-price sealed-bid auctions and English auctions

- Three properties:
  - Bidders in a second price auction do not care about the probability distribution over their opponents' types. Thus, the result holds even if bidders have no idea about their opponents' valuations.
  - In the private value setting, the same result still applies even if types among bidders are correlated.
  - The equilibrium outcome is efficient: the one with the highest value wins and obtains the object.

Auctions

Second-price sealed-bid auctions and English auctions

- English auction seems more difficult to analyze.
- But we can think about it in the following way.
- At the beginning of the auction, all the bidders raise their hands.
- Then the auctioneer continuously increasing the price, starting at $p = 0$.
- If a bidder thinks the current price is too high that he is not willing to pay for the object, he simply put down his hand.
- This means "I drop out."
- As long as there are more than one bidders who still keep their hands up in the air, the auctioneer keeps raising the price.
- Once the penultimate bidder drops out, the auctioneer stops and the auction ends.
- The only bidder who still raise his hand wins the auction and pays the current price.

- The only possible difference between English auction in reality and this "hands-up" game is that in English auction there might be a minimum incremental requirement while in this game, the auction immediately ends if there is only one bidder left.
- It is also easy to see that this "hands-up" game is equivalent to the second-price auction.
- Thus, biding one's own valuation is a Bayesian Nash equilibrium.

Auctions

First-price sealed-bid auctions and Dutch auctions

- In a first-price sealed-bid auction, the winning bidder pays his own bid.
- Thus, the payoff for bidder i is

$$v_i(b_i, b_{-i}; \theta_i) = \begin{cases} \frac{\theta_i - b_i}{|\{k | b_k = \max_j b_j\}|}, & \text{if } b_i = \max_j b_j, \\ 0, & \text{if } b_i < \max_j b_j. \end{cases}$$

- As before, if there is a tie in the winning bid, the object is randomly allocated to all the winning bidders with equal probabilities.

Auctions

First-price sealed-bid auctions and Dutch auctions

- In this first price auction, bidding one's own valuation is not a dominant strategy.
- Always bidding one's own valuation yields payoff 0.
- However, if bidder bids slightly lower than his own valuation, then intuitively, there is positive probability that he would obtain this object, in which case he can get positive payoff.
- Then, can we find a Bayesian Nash equilibrium?

- Since higher type means higher willingness to pay, it is intuitive that bidder with a higher type will bid a higher price.
- Thus, let's guess that there is a Bayesian Nash equilibrium in which every bidder's strategy is strictly increasing.
- That is, in this Bayesian Nash equilibrium $(s_1, \ldots, s_n)$, for all i,

$$s_i(\theta_i) < s_i(\theta_i'),\ \forall \theta_i < \theta_i'.$$

Auctions

First-price sealed-bid auctions and Dutch auctions

- Let's now consider the best response of bidder i of type θ_i, given s_{-i}.
- If θ_i bids b_i, he is the only winner if and only if

$$s_j(\theta_j) < b_i, \ \forall j \neq i.$$

- Since s_j is strictly increasing for all j, the above condition is equivalent to

$$\theta_j < s_j^{-1}(b_i), \ \forall j \neq i,$$

where s_j^{-1} is the inverse of s_j.

- Thus, the probability that θ_i with bid b_i is the only winner is

$$\mathbb{P}(\theta_j < s_j^{-1}(b_i), \ \forall j \neq i) = \prod_{j \neq i} \mathbb{P}(\theta_j < s_j^{-1}(b_i)) = \prod_{j \neq i} F_j(s_j^{-1}(b_i)),$$

where the first equality comes from independence of valuations among the bidders.

Auctions

First-price sealed-bid auctions and Dutch auctions

- Given s_j is strictly increasing for all j, the probability that bidder i wins but is not the only winner is zero.
- Therefore, the expected payoff of θ_i with bid b_i is

$$\prod_{j \neq i} F_j(s_j^{-1}(b_i)) \times (\theta_i - b_i).$$

- Trade-off: higher bid leads to higher probability of winning but at the same time to higher price.
- To go further, let's assume that s_j is differentiable.
- Since $b_i = s_i(\theta_i)$ maximizes this expected payoff, if it is interior, it must satisfy the first order condition:

$$-\prod_{j \neq i} F_j(s_j^{-1}(b_i)) + (\theta_i - b_i) \sum_{j \neq i} \left[\frac{f_j(s_j^{-1}(b_i))}{s_j'(s_j^{-1}(b_i))} \prod_{k \neq i,j} F_k(s_k^{-1}(b_i)) \right] \Bigg|_{b_i = s_i(\theta_i)} = 0.$$

Auctions

First-price sealed-bid auctions and Dutch auctions

- Assume that the value distribution F_j is the same for all bidders.
- In particular, assume $F_j = F$ and $f_j = f$ for some cdf F and its density f over $[\underline{\theta}, \overline{\theta}]$.
- Then, it is intuitive that we should have a *symmetric equilibrium*:

$$s_1 = s_2 = \ldots = s_n = s,$$

  for some s.
- Under this assumption, $s_j^{-1}(s_i(\theta_i)) = \theta_i$.
- Then, the above first order condition is simplified to (we drop subscript i on θ_i)

$$-F^{n-1}(\theta) + (n-1)(\theta - s(\theta))\frac{f(\theta)F^{n-2}(\theta)}{s'(\theta)} = 0.$$

Auctions

First-price sealed-bid auctions and Dutch auctions

- The above formula gives us a differential equation

$$F^{n-1}(\theta)s'(\theta) + (n-1)f(\theta)F^{n-2}(\theta)s(\theta) = (n-1)\theta f(\theta)F^{n-2}(\theta).$$

- Integrating on both sides yields

$$\begin{aligned} F^{n-1}(\theta)s(\theta) &= \int_{\underline{\theta}}^{\theta} (n-1)xf(x)F^{n-2}(x)\mathrm{d}x + k \\ &= xF^{n-1}(x)\Big|_{x=\underline{\theta}}^{\theta} - \int_{\underline{\theta}}^{\theta} F^{n-1}(x)\mathrm{d}x + k \\ &= \theta F^{n-1}(\theta) - \int_{\underline{\theta}}^{\theta} F^{n-1}(x)\mathrm{d}x + k \end{aligned}$$

  where the second equality comes from integration by parts and k is some constant.

- Since $F(\underline{\theta}) = 0$, evaluating both sides at $\theta = \underline{\theta}$, we obtain $k = 0$.

Auctions

First-price sealed-bid auctions and Dutch auctions

- Therefore,
$$s(\theta) = \theta - \frac{\int_{\underline{\theta}}^{\theta} F^{n-1}(x)\mathrm{d}x}{F^{n-1}(\theta)}.$$
- Notice that it is easy to see that s is strictly increasing and differentiable.
- Thus, we found a symmetric Bayesian Nash equilibrium in this symmetric first price auction.
- Each bidder of any valuation bids strictly lower than his valuation.
- Note also that this equilibrium is also efficient: the one with the highest valuation obtains the object. This is because we focused on equilibrium with increasing strategies.

Auctions

First-price sealed-bid auctions and Dutch auctions

- For example, if F is uniform distribution over $[0,1]$, then

$$s(\theta) = \theta - \frac{\theta^n}{n\theta^{n-1}} = \frac{n-1}{n}\theta.$$

- We can also directly verify that this is a Bayesian Nash equilibrium.
- Given $s_2 = \ldots = s_n = s$, bidder 1 with valuation θ_1's expected payoff can be written as

$$\begin{cases} 0, & \text{if } b_1 \leq 0, \\ \left(\frac{nb_1}{n-1}\right)^{n-1}(\theta_1 - b_1), & \text{if } b_1 \in (0, \frac{n-1}{n}), \\ \theta_1 - b_1, & \text{if } b_1 \geq \frac{n-1}{n}. \end{cases}$$

- Then, it is optimal for θ_1 to bid $(n-1)\theta_1/n$.

Auctions

First-price sealed-bid auctions and Dutch auctions

- The Dutch and first price auctions are closely related in the same way that the English and second price auctions are.
- Each bidder in a Dutch auction thinks about at what price he will jump in and announce that he will buy, as in a first price auction each bidder considers a bid.
- Lower price leads to higher gain conditional on winning.
- But at the same time, lower price faces the risk of someone else jumping in, which reduces the probability of winning.
- Thus, the Dutch and the first price auctions are strategically equivalent. They have the same normal form and as a consequence have the same set of Bayesian Nash equilibria.

Auctions

Revenue Equivalence

- We have analyzed both first price and second price auctions from the bidder's point of view, i.e., Bayesian Nash equilibrium.
- But from the seller/auctioneer's point of view, which of these two auction formats bring higher revenue?
- To illustrate the idea, let's assume the symmetric environment where F is a uniform distribution over $[0, 1]$.

Auctions

Revenue Equivalence

- Consider the second price auction first.
- In the Bayesian Nash equilibrium, every bidder bids his own valuation and pays the second highest valuation.
- Given $(\theta_1, \ldots, \theta_n) \in [0,1]^n$, let $\theta^{[2]}$ be the second highest value among them.
- Since $\theta_1, \ldots, \theta_n$ are random variables taking values in $[0,1]$, so is $\theta_n^{[2]}$.
- It is called the **second-order statistic** of the n random variables $\theta_1, \ldots, \theta_n$.
- Note that $\theta_n^{[2]}$ is the just the seller's revenue in the second price auction.

Auctions

Revenue Equivalence

- For any $x \in [0,1]$,

$$\begin{aligned}
F_n^{[2]}(x) &= \mathbb{P}(\theta_n^{[2]} \leq x) \\
&= \mathbb{P}\left(\bigcup_{i=1}^{n} (\theta_i > x,\ \theta_j \leq x,\ \forall j \neq i)\right) + \mathbb{P}(x_i \leq x,\ \forall i) \\
&= \sum_{i=1}^{n} \mathbb{P}(\theta_i > x,\ \theta_j \leq x,\ \forall j \neq i) + \mathbb{P}(x_i \leq x,\ \forall i) \\
&= \sum_{i=1}^{n} (1 - F(x))F(x)^{n-1} + F(x)^n \\
&= nF^{n-1}(x) - (n-1)F^n(x).
\end{aligned}$$

- This is the cdf of $\theta_n^{[2]}$.

Auctions

Revenue Equivalence

- The corresponding density is

$$f_n^{[2]}(x) = n(n-1)F^{n-2}(x)f(x) - n(n-1)F^{n-1}(x)f(x).$$

- For uniform distribution, we have

$$F_n^{[2]}(x) = nx^{n-1} - (n-1)x^n,$$

  and

$$f_n^{[2]}(x) = n(n-1)x^{n-2} - n(n-1)x^{n-1}.$$

- Then, the seller's expected revenue is

$$\mathbb{E}(\theta_n^{[2]}) = \int_0^1 x f_n^{[2]}(x)\mathrm{d}x = (n-1) - \frac{n(n-1)}{n+1} = \frac{n-1}{n+1}.$$

Auctions

Revenue Equivalence

- We now turn to the first price auction.
- In the symmetric equilibrium, each bidder bids according $s(\theta) = \frac{n-1}{n}\theta$.
- Note that each s is a random variable with uniform distribution over $[0, \frac{n-1}{n}]$.
- For $(b_1, \ldots, b_n) \in [0, \frac{n-1}{n}]^n$, let $b_n^{[1]}$ be the highest bid among them

$$b_n^{[1]} = \max\{b_1, \ldots, b_n\}.$$

- Note that $b_n^{[1]}$ is the seller's revenue in the first price auction.

Auctions

Revenue Equivalence

- For any $x \in [0, \frac{n-1}{n}]$,

$$F_n^{[1]}(x) = \mathbb{P}(b_n^{[1]} \leq x) = \mathbb{P}(b_i \leq x,\ \forall i) = \left(\frac{nx}{n-1}\right)^n.$$

- This is the cdf of $b_n^{[1]}$.
- The corresponding density is

$$f_n^{[1]}(x) = \frac{n^2}{n-1}\left(\frac{nx}{n-1}\right)^{n-1}$$

Auctions

Revenue Equivalence

- Then, the seller's expected revenue from the first price auction is

$$\mathbb{E}\left(b_n^{[1]}\right) = \int_0^{\frac{n-1}{n}} x f_n^{[1]}(x)\mathrm{d}x = \frac{n-1}{n+1}.$$

- This is exactly the same as that in the second price auction!

Auctions

Revenue Equivalence

- We have demonstrated one of the fundamental results in auction theory: the **revenue equivalence theorem**.
- In the IPV setting, any auction game that satisfies the following four conditions will yield the seller the same expected revenue:
  - each bidder's type is drawn from a "well-behaved" distribution;
  - bidders are risk neutral;
  - the bidder with the highest type wins; and
  - the bidder with the lowest possible type has an expected payoff of zero.
- It was first found by Vikerey and then generalized by Roger Myerson.

Roger B. Myerson

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2007

“for having laid the foundations of mechanism design theory.”

Contribution: In the 1970s, the formulation of the ”revelation principle” (a way of simplifying the search for a feasible mechanism) and implementation theory led to great advances of mechanism design. He developed this principle to perfection and pioneered its application to economic problems such as auctions and regulations.

Auctions

Common value auction

- Two bidders in a first price auction.
- The object has the same value to both of them.
- However, neither of them knows the precise value.
- In fact, each of them receives a signal about the true value.
- The signals are uniformly and independently distributed over $[0, 1]$.
- Each bidder's signal is his private information.
- The value of the object to each of the bidder is

  $$v = t_1 + t_2,$$

  where t_i is i's signal (type).
- This is an independent common value setting.

Auctions

Common value auction

- Each bidder's strategy is a mapping from types to bids, as always.
- Again, we look for Bayesian Nash equilibrium in which both bidders' strategies are strictly increasing.
- Given s_j, consider bidder i of type t_i.
- If i bids b_i, he wins if j's type satisfies $s_j(t_j) < b_i$
- Or equivalently, if $t_j < s_j^{-1}(b_i)$, in which case i's value is $t_i + t_j$.
- Thus, i's expected payoff is

$$\int_0^{s_j^{-1}(b_i)} (t_i + t_j - b_i)\mathrm{d}t_j,$$

where we have used the uniform distribution assumption.

Auctions

Common value auction

- It is then easy to calculate

$$\int_0^{s_j^{-1}(b_i)} (t_i + t_j - b_i)\mathrm{d}t_j = (t_i - b_i)s_j^{-1}(b_i) + \frac{1}{2}\left[s_j^{-1}(b_i)\right]^2.$$

- Assume s_j is differentiable.
- If $s_i(t_i)$ is a best response, it must satisfy the first order condition:

$$-s_j^{-1}(b_i) + \frac{t_i - b_i}{s_j'\big(s_j^{-1}(b_i)\big)} + \frac{s_j^{-1}(b_i)}{s_j'\big(s_j^{-1}(b_i)\big)}\Bigg|_{b_i = s_i(t_i)} = 0.$$

- If it is a symmetric equilibrium, i.e., $s_i = s_j = s$, we have

$$-t + \frac{t - s(t)}{s'(t)} + \frac{t}{s'(t)} = 0,$$

or equivalently

$$ts'(t) + s(t) = 2t.$$

- Integrating on both sides yields

$$ts(t) = t^2 + k,$$

  where k is a constant.
- Evaluating both sides at $t = 0$ implies $k = 0$.
- Therefore, we have found a symmetric Bayesian Nash equilibrium:

$$s_1(t) = s_2(t) = t.$$

Auctions

Common value auction

- In this equilibrium, bidder 1 of type t_1 estimates that the value of the object is
$$t_1 + \mathbb{E}t_2 = t_1 + \frac{1}{2},$$
and bids t_1.
- Now suppose this bidder indeed wins the auction.
- Then, he should understand that he wins the auction because his opponent's type is lower than t_1.
- With this additional information, he should revise his estimate about the value
$$t_1 + \mathbb{E}(t_2 | t_2 < t_1) = t_1 + \frac{\int_0^{t_1} t_2 \mathrm{d}t_2}{\mathbb{P}(t_2 < t_1)} = t_1 + \frac{t_1}{2} < t_1 + \frac{1}{2}!$$

Auctions

Common value auction

- Winning is “bad news”!
- The fact that one wins tells himself that he has overestimated the value of the object.
- This phenomenon, which occurs in common-value settings, is known as the **winner’s curse**.

Auctions

Common value auction

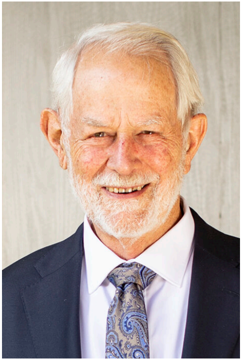
Paul R. Milgrom

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2020

“for improvements to auction theory and inventions of new auction formats.”

Work: Robert Wilson and Paul Milgrom have studied how auctions work. They have also translated their theories into practice and created new auctions for goods and services that are difficult to sell through traditional means, such as radio frequencies. Their discoveries have benefited sellers, buyers and taxpayers around the world.

Robert B. Wilson



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