Game Theory, Fall 2022
Problem Set 12

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1. ST Exercise 13.3.

   (a) **Soln:** No. Consider bidder 1 with valuation $\theta_1 = 0$. Bidding $b_1 = \theta_1 = 0$ or $b_1' = \frac{r}{2} > 0$ always yields the same expected payoff 0, regardless of bidder 2's bids. This is simply because both b_1 and b_1' are strictly below the reservation price. Thus, b_1 does not weakly dominate b_1' for $\theta_1 = 0$. Consequently, bidding one's own valuation is no longer a weakly dominant strategy when $r > 0$.

   (b) **Soln:** When $r = 0$, we know bidding one's own valuation is an equilibrium. In this equilibrium, the expected revenue of the seller is $\mathbb{E}\min\{\theta_1, \theta_2\} = \frac{1}{9}\times 2 + \frac{3}{9}\times 1 = \frac{5}{9}$.

   (c) **Soln:** When $r > 0$, we first argue that bidding one's own valuation is still an equilibrium, though it is not in dominant strategies. Consider $\theta_1 < r$ first. Bidding $b_1 = \theta_1$ yields expected payoff 0. Bidding any other $b_1' < r$ yields the same payoff. Bidding $b_1' \geq r$ can not obtain positive payoff, as bidder 1 has to pay at least r in case he wins which exceeds his valuation. Therefore, $b_1 = \theta_1$ is one of θ_1's best reply. Next, consider $\theta_1 = r$. Note that bidding $b_1 = \theta_1$ still yields expected payoff 0, even though there may be positive probability of winning. Then, we can use the previous argument to show that 1 has no incentive to deviate to $b_1' < r$ or $b_1' > r$. Finally, consider $\theta_1 > r$. We can use almost the same argument as for the case $r = 0$ to show that bidding $b_1 = \theta_1$ weakly dominates any other bids. Therefore, $b_1 = \theta_1$ is a best reply for θ_1. In sum, bidding 1's own valuation is a best reply for 1 regardless of the other one's bidding. By symmetry, the same holds true for 2. Therefore, bidding one's own valuation is a Bayesian Nash equilibrium.

   In general, when the reserve price is r, the seller's expected revenue from the above equilibrium is

$$\pi(r) \equiv r \times \mathbb{P}(\theta_1 < r, \theta_2 \geq r) + r \times \mathbb{P}(\theta_1 \geq r, \theta_2 < r) + \mathbb{E}[\min\{\theta_1, \theta_2\} 1_{\theta_1 \geq r, \theta_2 \geq r}]. \quad (1)$$

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Therefore, we can calculate $\pi(1) = 1$.

(d) **Soln:** The revenue equivalence theorem fails. This is because the condition that the bidder with the highest type wins is violated in the presence of positive reserve price. More intuitively, by setting a reserve price, the seller effectively rules out the possibility that the price of the good is too low. For example, if $r = 0$, $\theta_1 = 0$ and $\theta_2 = 1$, the price is 0. However, when $r = 1$, the price becomes 1.

(e) **Soln:** We can directly calculate from (1) that $\pi(2) = 2\times\frac{2}{3}\times\frac{1}{3}+2\times\frac{2}{3}\times\frac{1}{3}+2\times\frac{1}{3}\times\frac{1}{3} = \frac{10}{9}$. Another way to think about $\pi(2)$ is that the seller always obtains 2 whenever at least one bidder's valuation is 2. Thus, $\pi(2) = 2 \times (1 - \mathbb{P}(\theta_1 < 2, \theta_2 < 2)) = 2 \times (1 - \frac{2}{3} \times \frac{2}{3}) = \frac{10}{9}$. Thus, we have $\pi(0) < \pi(1) < \pi(2)$. It is easy to see that $\pi(r) < \pi(1)$ if $r \in (0, 1)$ and $\pi(r) < \pi(2)$ if $r \in (1, 2)$. Moreover, $\pi(r) = 0$ if $r > 2$. Therefore, the optimal reserve price for the seller is $r^* = 2$.

2. ST Exercise 13.6.

(a) **Soln:** Suppose bidder 1's valuation is θ_1. He bids b_1 and the others bid $b_{-1} = (b_2, \ldots, b_n)$. Then, bidder 1's (ex post) payoff is

$$v_1(b_1, b_{-1}; \theta_1) = \begin{cases} \frac{\theta_1 - M_3(b_1, b_{-1})}{|\{j|b_j = b_1\}|}, & \text{if } b_1 = \max_j b_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $M_3(b_1, b_{-1})$ denotes the third highest bid among $b_1, b_2, \ldots, b_n$. Given the other bidders' strategy profile $b_{-1}(\theta_{-1}) \equiv (b_2(\theta_2), \ldots, b_n(\theta_n))$, bidder 1's expected payoff from bidding b_1 is then

$$\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1)\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n,$$

where we have used the assumption that $\theta_2, \ldots, \theta_n$ are independently and uniformly distributed.

Assume $b_2, \ldots, b_n$ are strictly increasing. We can simplify

$$\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1)\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n = \int_{\theta_j < b_j^{-1}(b_1),\ j>1} [\theta_1 - M_2(b_{-1}(\theta_{-1}))]\, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n,$$

where $M_2(b_{-1}(\theta_{-1}))$ denotes the second highest bid among $b_2(\theta_2), \ldots, b_n(\theta_n)$. If bidders -1 play a symmetric strategy, i.e., $b_2 = \ldots = b_n = b$, we can further simplify

$$\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1)\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n = \int_{\theta_j < b^{-1}(b_1),\ j>1} [\theta_1 - b(M_2(\theta_2, \ldots, \theta_n))]\, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n.$$

(b) **Soln:** For simplicity, let $\beta = \frac{n-1}{n-2}$ and $b(\theta_j) = \beta\theta_j$ for $j > 1$. From the above calculation, we can write 1's expected payoff from bidding b_1 as

$$\begin{aligned}
&\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1)\mathrm{d}\theta_2 \dots \mathrm{d}\theta_n \\
=&\int_{\theta_j < b_1/\beta,\ j>1} [\theta_1 - \beta M_2(\theta_2, \dots, \theta_n)]\mathrm{d}\theta_2 \dots \mathrm{d}\theta_n \\
=&\theta_1 \left(\frac{b_1}{\beta}\right)^{n-1} - \beta \int_{\theta_j < b_1/\beta,\ j>1} M_2(\theta_2, \dots, \theta_n)\mathrm{d}\theta_2 \dots \mathrm{d}\theta_n.
\end{aligned}$$

Note

$$\begin{aligned}
&\int_{\theta_j < b_1/\beta,\ j>1} M_2(\theta_2, \dots, \theta_n)\mathrm{d}\theta_2 \dots \mathrm{d}\theta_n \\
=&\sum_{j=2}^{n} \sum_{\substack{k=2 \\ k\neq j}}^{n} \int_0^{\frac{b_1}{\beta}} \left[\theta_j \times \int_{\theta_j}^{\frac{b_1}{\beta}} \mathrm{d}\theta_k \times \prod_{i\neq j,k} \int_0^{\theta_j} \mathrm{d}\theta_i\right] \mathrm{d}\theta_j \\
=&(n-1)(n-2) \int_0^{\frac{b_1}{\beta}} \left[\theta_2 \times (\frac{b_1}{\beta} - \theta_2) \times \theta_2^{n-3}\right] \mathrm{d}\theta_2 \\
=&\frac{n-2}{n} \left(\frac{b_1}{\beta}\right)^n.
\end{aligned}$$

Therefore, 1's expected payoff is

$$\left(\frac{b_1}{\beta}\right)^{n-1} \left[\theta_1 - \frac{n-2}{n} b_1\right].$$

By the first order condition, the optimal b_1 for 1 is

$$b_1(\theta_1) = \frac{n-1}{n-2}\theta_1.$$

That is, given others' strategy profile $(b, \dots, b)$, it is also optimal for 1 to use b. Therefore b is a symmetric equilibrium.

(c) **Soln:** Intuitively, this is because the winning bidder does not need to pay his own bid. Bidding a higher price increases the probability of winning. (But clearly there is trade-off. Higher bid also increases the probability of higher price in the case of winning.)

(d) **Soln:** We first calculate the distribution of the third-order statistic $M_3(\theta_1, \dots, \theta_n)$. For $x \in (0, 1)$, the event $(M_3(\theta_1, \dots, \theta_n) \le x)$ essentially says that "there are at most two θ_i such that $\theta_i > x$." Therefore, the probability of this event is

$$F_n^{[3]}(x) = \mathbb{P}(M_3(\theta_1, \dots, \theta_n) \le x) = \sum_{j=0}^{2} \binom{n}{j} (1 - F(x))^j (F(x))^{n-j},$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ is the standard binomial coefficient. By taking derivative, we can calculate the corresponding probability density

$$f_n^{[3]}(x) = \frac{n(n-1)(n-2)}{2} f(x)(1-F(x))^2(F(x))^{n-3}.$$

Then, under uniform value distribution, the seller's expected revenue from the third price auction is

$$\begin{aligned}
&\mathbb{E}\left[\frac{n-1}{n-2} M_3(\theta_1, \ldots, \theta_n)\right] \\
=&\frac{n-1}{n-2}\int_0^1 x f_n^{[3]}(x)\,\mathrm{d}x \\
=&\frac{n-1}{n-2}\frac{n(n-1)(n-2)}{2}\int_0^1 x^{n-2}(1-x)^2\,\mathrm{d}x \\
=&\frac{n-1}{n+1}.
\end{aligned}$$

Note that this expected revenue is the same as those from the first price auction and second price aution. Not surprisingly, this is because of the revenue equilvance theorem.

3. ST Exercise 13.7.

   (a) **Soln:** In a first-price sealed-bid auction, the winner bidder pays his own bid. The (ex post) payoff for bidder 1 is

$$v_1(b_1, b_{-1}; \theta_1) = \begin{cases} \frac{\sqrt{\theta_1 - b_1}}{|\{j|b_j = b_1\}|}, & \text{if } b_1 = \max_j b_j, \\ 0, & \text{if } b_1 < \max_j b_j. \end{cases}$$

Given the other bidders' strategy profile $b_{-1}(\theta_{-1}) \equiv (b_2(\theta_2), \ldots, b_n(\theta_n))$, bidder 1's expected payoff from bidding b_1 is then

$$\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1)\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n,$$

where we have used the assumption that $\theta_2, \ldots, \theta_n$ are independently and uniformly distributed.

Assume $b_2, \ldots, b_n$ are strictly increasing. We can simplify

$$\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1)\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n = \int_{\theta_j < b_j^{-1}(b_1),\ j>1} \sqrt{\theta_1 - b_1}\ \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n,$$

If bidders -1 play a symmetric strategy, i.e., $b_2 = \ldots = b_n = b$, we can further simplify

$$\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1) \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n = \int_{\theta_j < b^{-1}(b_1),\ j>1} \sqrt{\theta_1 - b_1}\ \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n.$$

In a second-price sealed-bid auction, the winner bidder pays the second highest bid. The (ex post) payoff for bidder 1 is

$$v_1(b_1, b_{-1}; \theta_1) = \begin{cases} \frac{\sqrt{\theta_1 - \max_{j \neq 1} b_j}}{|\{j | b_j = b_1\}|}, & \text{if } b_1 = \max_j b_j, \\ 0, & \text{if } b_1 < \max_j b_j. \end{cases}$$

By making similar assumptions, we can simply the expected payoff function

$$\begin{aligned} &\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1) \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n \\ =&\int_{\theta_j < b_j^{-1}(b_1),\ j>1} \sqrt{\theta_1 - \max_{j \neq 1} b_j(\theta_j)}\ \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n, \\ =&\int_{\theta_j < b^{-1}(b_1),\ j>1} \sqrt{\theta_1 - b(\max_{j \neq 1} \theta_j)}\ \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n. \end{aligned}$$

(b) **Soln:** For simplicity, let $\beta = \frac{m(n-1)}{m(n-1)+1}$ and $b(\theta_j) = \beta\theta_j$ for $j > 1$. From the above calculation, we can write 1's expected payoff from bidding b_1 as

$$\begin{aligned} &\int_{\theta_{-1}} v_1(b_1, b_{-1}(\theta_{-1}); \theta_1)\ \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n \\ =&\int_{\theta_j < b_1/\beta,\ j>1} \sqrt{\theta_1 - b_1}\ \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n \\ =&\sqrt{\theta_1 - b_1} \left(\frac{b_1}{\beta}\right)^{n-1}. \end{aligned}$$

By the first order condition, the optimal b_1 for 1 is

$$b_1(\theta_1) = \frac{2(n-1)}{2(n-1)+1}\theta_1 = \frac{2n-2}{2n-1}\theta_1.$$

In a symmetric equilibrium $(b, \ldots, b)$, b is optimal given others' strategies. Thus, $b(\theta) = \frac{2n-2}{2n-1}\theta$ is a symmetric equilibrium since it satisfies the form $b(\theta) = \beta\theta$. Therefore, $s_i(\theta_i) = \theta_i \frac{m(n-1)}{m(n-1)+1}$ forms a symmetric BNE if $m = 2$. Note that in the slides, it is a symmetric equilibrium of the first-price auction when $m = 1$.

(c) **Soln:** As no bidder ever bids more than his valuation, $b_i \in [0, \theta_i]$. We show that $b_i = \theta_i$ is a weakly dominant strategy if: for any $b_i < \theta_i$, we have

$$v_i(\theta_i, b_{-i}; \theta_i) \geq v_i(b_i, b_{-i}; \theta_i),\ \forall b_{-i},$$

and there exists b_{-i} such that

$$v_i(\theta_i, b_{-i}; \theta_i) > v_i(b_i, b_{-i}; \theta_i).$$

It's easy to find that for b_{-i} such that $\max_{j\neq i} b_j < b_i$ or $\max_{j\neq i} b_j \geq \theta_i$,

$$v_i(\theta_i, b_{-i}; \theta_i) = v_i(b_i, b_{-i}; \theta_i);$$

for b_{-i} such that $b_i \leq \max_{j\neq i} b_j < \theta_i$,

$$v_i(\theta_i, b_{-i}; \theta_i) > v_i(b_i, b_{-i}; \theta_i).$$

Therefore, bidding one's own valuation is a weakly dominant strategy.

(d) **Soln:** For $x \in [0, 1]$, we first define $\theta_n^{[k]}$ as the kth highest value among all θ_i.

In the first-price case, we have

$$F_n^{[1]}(x) = \mathbb{P}(\theta_n^{[1]} \leq x) = F(x)^n = x^n.$$

The corresponding density is

$$f_n^{[1]}(x) = nx^{n-1}.$$

Note that the seller's revenue is $\beta\theta_n^{[1]}$, so the expected payoff of the seller is

$$\mathbb{E}(\beta\theta_n^{[1]}) = \beta \int_0^1 x f_n^{[1]}(x)\,\mathrm{d}x = \frac{n\beta}{n+1} = \frac{2(n-1)n}{(n+1)(2n-1)}.$$

In the second-price case, we have

$$\begin{aligned}
F_n^{[2]}(x) =& \mathbb{P}(\theta_n^{[2]} \leq x) \\
=& \sum_{i=1}^{n} \mathbb{P}(\theta_i > x, \theta_j \leq x, \forall j \neq i) + \mathbb{P}(x_i \leq x, \forall i) \\
=& \sum_{i=1}^{n} (1-x)x^{n-1} + x^n \\
=& nx^{n-1} - (n-1)x^n.
\end{aligned}$$

The corresponding density is

$$f_n^{[2]}(x) = n(n-1)x^{n-2}(1-x).$$

Note that the seller's revenue is $\theta_n^{[2]}$ since in an Bayesian Nash equilibrium all bidders bid their own valuations, so the expected payoff of the seller is

$$\mathbb{E}(\theta_n^{[2]}) = \int_0^1 x f_n^{[2]}(x)\,\mathrm{d}x = \frac{n-1}{n+1}.$$

Note that in the second-price auction, the symmetric equilibrium is the same as what we have learned in the slides, and the prices determined by the rules are the same. Thus, the expected revenue for the seller has the same value. However, the equilibria of the first-price auction are different and so the expected revenues.

We know that when bidders are not risk-neutral, which differs from the IPV setting, revenue equivalence theorem does not apply anymore.

4. ST Exercise 13.8.

Soln: We use the settings in the slides.

Each bidder's strategy is a mapping from types to bids, and we assume both bidders' strategies are strictly increasing and integrable. Given s_j, bidder i of type t_i wins when bidding b_i if $s_j(t_j) < b_i$, or equivalently, $t_j < s_j^{-1}(b_i)$. Thus, i's expected payoff is

$$\int_0^{s_j^{-1}(b_i)} (t_i + t_j - s_j(t_j))\ \mathrm{d}t_j = t_i \cdot s_j^{-1}(b_i) + \frac{(s_j^{-1}(b_i))^2}{2} - \int_0^{s_j^{-1}(b_i)} s_j(t_j)\ \mathrm{d}t_j,$$

where we have used the unifom distribution assumption.

We assume s_j is differentiable. $s_i(t_i)$ is a best response only if it satisfies the FOC:

$$t_i \cdot \frac{1}{s_j'(s_j^{-1}(b_i))} + s_j^{-1}(b_i) \cdot \frac{1}{s_j'(s_j^{-1}(b_i))} = s_j(s_j^{-1}(b_i)) \cdot \frac{1}{s_j'(s_j^{-1}(b_i))},$$

or equivalently

$$t_i + s_j^{-1}(b_i) = b_i.$$

In a symmetric equilibrium, i.e., $s_i = s_j = s$, we have

$$s(t) = 2t.$$

It is easy to verify that given j's strategy $s_j(t_j) = 2t_j$, $s_i(t_i) = 2t_i$ is i's best response. Thus, we find a symmetric BNE:

$$s_1(t) = s_2(t) = 2t.$$

5. ST Exercise 13.9.

(a) **Soln:** In the auction, the winner bidder pays his own bid and gets $\frac{1}{2} + \theta_i$. Given player 2's strategy $s_2(\theta_2) = a + b\theta_2$, player 1 wins if $s_i > a + b\theta_2$, i.e., $\theta_2 < \frac{s_i - a}{b}$. Thus, his expected payoff from bidding s_1 is

$$\int_{\{\theta_2 | \theta_2 < \frac{s_i - a}{b},\ 0 \le \theta_2 \le 1\}} (\frac{1}{2} + \theta_1 - s_1)\ \mathrm{d}\theta_2.$$

We assume that $0 \leq \frac{s_i - a}{b} \leq 1$, and the expected payoff is

$$\int_0^{\frac{s_1-a}{b}} (\frac{1}{2} + \theta_1 - s_1) \, \mathrm{d}\theta_2 = (\frac{1}{2} + \theta_1 - s_1) \times \frac{s_i - a}{b}.$$

By the FOC, $s_1(\theta_1)$ is the best response only if

$$s_1(\theta_1) = \frac{2a+1}{4} + \frac{1}{2}\theta_2.$$

By symmetric, we assume player 1's strategy is $s_1(\theta_1) = a + b\theta_1$ and player 2's best response must satisfy $s_2(\theta_2) = \frac{2a+1}{4} + \frac{1}{2}\theta_1$. Thus, we derive a symmetric BNE $s_i^*(\theta_i) = a + b\theta_1$ such that

$$\begin{cases} a = \frac{2a+1}{4}, \\ b = \frac{1}{2}, \end{cases}$$

which derives,

$$\begin{cases} a = \frac{1}{2}, \\ b = \frac{1}{2}. \end{cases}$$

We can verify that for $\theta_i \in [0, 1]$, $s_i(\theta_i) = \frac{1+\theta_i}{2} \in [0, 1]$. Also, given one's strategy $s_j(\theta_j) = \frac{1+\theta_j}{2}$, $s_i(\theta_i) = \frac{1+\theta_i}{2}$ is i's best response. Thus, we find a symmetric BNE

$$s_i^*(\theta_i) = \frac{1+\theta_i}{2}.$$

(b) **Soln:** As both players have the same strictly increasing strategy, player i has positive payoff if and only if $\theta_i \geq \theta_j$, so one's expected payoff is

$$\int_0^{\theta_i} (\frac{1}{2} + \theta_i - \frac{1+\theta_i}{2}) \, \mathrm{d}\theta_j = \frac{\theta_i^2}{2}.$$