Game Theory, Fall 2022
Problem Set 2

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1. ST Exercise 4.6.

   (a) **Soln:** Given player j's strategy t_j, player i solves

   $$\max_{t_i \geq 0} (10 - t_j)t_i - t_i^2$$

   When $t_j \geq 10$, it is obvious that i's best response is $t_i = 0$. When $t_j < 10$, the first order condition is $10 - t_j - 2t_i = 0$, implying that the best response is $t_i = \frac{10-t_j}{2}$. In sum, i's best response is

   $$t_i = \max\left\{\frac{10 - t_j}{2}, 0\right\}.$$

   (b) **Soln:** Clearly, any $t_i \in [0, 5]$ is not strictly dominated because it is a best response to $t_j = 10 - 2t_i$. Moreover, consider any strategy t_i of the form $t_i = 5 + k$ for some $k > 0$. We have, for all $t_j \geq 0$,

   $$v_i(5+k, t_j) = (10-t_j)(5+k) - (5+k)^2 = 25 - 5t_j - k^2 - t_j k < 25 - 5t_j = v_i(5, t_j).$$

   Therefore, any strategy $t_i > 5$ is strictly dominated by $t_i = 5$. Thus all strategies in $(5, +\infty)$ should be deleted in the first round. We have $S_1^1 = S_2^1 = [0, 5]$.

   (c) **Soln:** In the second round of elimination, any strategy $t_i \in [\frac{5}{2}, 5]$ should be kept because it is a best response to $t_j = 10 - 2t_i \in [0, 5]$. Moreover, any strategy $t_i < \frac{5}{2}$ is strictly dominated by $t_i = \frac{5}{2}$. To see this, consider a strategy of the form $t_i = \frac{5}{2} - k$ for some $k > 0$. Then

   $$v_i(\frac{5}{2} - k, t_j) = \frac{75}{4} - \frac{5}{2}t_j - (5 - t_j)k - k^2.$$

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When $t_j \in [0,5]$, we have

$$v_i(\frac{5}{2}-k,t_j) < \frac{75}{4} - \frac{5}{2}t_j = v_i(\frac{5}{2},t_j).$$

Thus, $t_i = \frac{5}{2} - k$ is strictly dominated by $t_i = \frac{5}{2}$. Therefore, $S_1^2 = S_2^2 = [\frac{5}{2}, 5]$.

We can continue the above arguments in a similar fashion. Define function $\phi(x) = (10-x)/2$. In fact, we can show after the kth round of deletion, $S_1^k = S_2^k = [\underline{q}^k, \overline{q}^k]$ where

$$\underline{q}^k = \phi^{(k-1)}(0) \text{ and } \overline{q}^k = \phi^{(k)}(0) \text{ if k is odd,}$$
$$\underline{q}^k = \phi^{(k)}(0) \text{ and } \overline{q}^{(k)} = \phi^{(k-1)}(0) \text{ if k is even.}$$

and $\phi^{(k)}$ is the kth composition of ϕ itself. Because $\lim_{k\to\infty} \phi^k(0) = \frac{10}{3}$, we know that $S_j^* = S_i^* = \bigcap_{k\geq 1} S_i^k = \{\frac{10}{3}\}$. Hence, the unique pair of strategies that survive IESDS for this game are $t_1 = t_2 = \frac{10}{3}$.

2. ST Exercise 4.7.

(a) **Soln:** The set of players is $\{1,2\}$. The strategy spaces of players are $S_1 = \{P,B,N\}$ and $S_2 = \{P,B,N\}$. The payoffs to player 1 are

$$\begin{aligned} v_1(P,P) &= 0.5, & v_1(B,B) &= 0.5, & v_1(N,N) &= 0.5,\\ v_1(P,B) &= 0, & v_1(P,N) &= 0.3, & v_1(N,B) &= 0.6,\\ v_1(B,P) &= 1, & v_1(N,P) &= 0.7, & v_1(B,N) &= 0.4. \end{aligned}$$

And, $v_2(s) = 1 - v_1(s)$ for all $s \in \{P,B,N\} \times \{P,B,N\}$.

(b) **Soln:** The matrix is in Figure 1.

		Player 2		
		P	B	N
Player 1	P	$0.5, 0.5$	$0, 1$	$0.3, 0.7$
	B	$1, 0$	$0.5, 0.5$	$0.4, 0.6$
	N	$0.7, 0.3$	$0.6, 0.4$	$0.5, 0.5$

Figure 1: The normal form game for Question 2

(c) **Soln:** We delete all strictly dominated strategies in each round. In the first round, P for both players is strictly dominated by B. In the second round, B for both players is strictly dominated by N. Hence $\{N\} \times \{N\}$ survives iterated deletion of strictly dominated strategies.

3. ST Exercise 4.8.

   (a) **Soln:** Consider that player i chooses either 19 or 18. In both cases the average is less than 20 regardless of the number of players, and $\frac{3}{4}$ of the average is less than 15. This means that either 19 or 18 makes i the only winner. Therefore, both 19 and 18 are best responses.

   (b) **Soln:** Obviously $x = 20$ is not one of the best responses because there will be many winners. Consider any $x < 20$. Because the average is $\frac{20(n-1)+x}{n}$, player i is the only winner if and only if

   $$20 - \frac{3}{4} \cdot \frac{20(n-1)+x}{n} > \frac{3}{4} \cdot \frac{20(n-1)+x}{n} - x,$$

   or equivalently

   $$x > 10 - \frac{30}{2n-3}.$$

   Therefore the set of best-responses is

   $$\left\{ x \in \mathbb{Z} \,\middle|\, 0 \leq x < 20, x > 10 - \frac{30}{2n-3} \right\},$$

   which clearly depends on the number of players n. For instance, if $n = 2$, this set is $\{0, 1, \ldots, 19\}$. If n is sufficiently large, this set is $\{10, 11, \ldots, 19\}$.

4. Consider the following two-player game. Each player announces a nonnegative real number. The payoffs are

   $$v_i(x_i, x_j) = \begin{cases} 2, & \text{if } x_i = 0, x_j = 1, \\ \arctan x_i, & \text{if otherwise.} \end{cases}$$

   (a) Argue that every positive announcement is strictly dominated.

   Soln: Suppose $x_i > 0$. Then

   $$v_i(x_i, x_j) = \arctan x_i < \arctan(x_i + 1) = v_i(x_i + 1, x_j), \ \forall x_j.$$

   Therefore, x_i is strictly dominated.

   (b) Argue that announcement 0 is not strictly dominated.

   Soln: Because

   $$v_i(0, 1) = 2 > \arctan x_i = v_i(x_i, 1), \ \forall x_i > 0,$$

   we know 0 for player i is a best response to $x_j = 1$. Therefore, it is not strictly dominated.

(c) From the above two questions, we know only 0 survives IESDS for both players. Are they mutual best responses?

Soln: Obviously not. This is because

$$v_i(0,0) = \arctan 0 < \arctan 1 = v_i(1,0).$$

5. Consider the n-firm Cournot competition. The demand curve is still

$$D(Q) = \max\{100 - Q, 0\}.$$

If each firm i supplies q_i, the total supply is $\sum_{i=1}^{n} q_i$. Suppose each firm's marginal cost is 10.

(a) Write down its normal form game.

Soln: There are n firms, $N = \{1, 2, \ldots, n\}$. Firm i can choose any nonnegative quantities, so $S_i = \mathbb{R}_+$. Given the strategy profile $(q_1, \ldots, q_n)$, the payoff to firm i is

$$v_i(q_1, \ldots, q_n) = \big[\max\{100 - \sum_{j \neq i} q_j - q_i, 0\} - 10\big] q_i.$$

(b) For each firm, what are the strategies that survive IESDS?

Soln: We only consider $n \geq 3$. As in the duopoly case, we can write firm i's payoff function as

$$v_i(q_i, q_{-i}) = \max\{(90 - \sum_{j \neq i} q_j - q_i) q_i, -10 q_i\}.$$

Let $S_i^0 = \mathbb{R}_+$ for all i. It is easy to see that given the opponents' strategy profile $q_{-i} \in \mathbb{R}_+^{n-1}$, firm i's best response is

$$q_i = \max\left\{\frac{90 - \sum_{j \neq i} q_j}{2}, 0\right\}.$$

Thus, as $\sum_{j \neq i} q_j$ increases over the range $[0, +\infty)$, firm i's best response decreases from 45 to 0. We immediately know that any $q_i \in [0, 45]$ is not strictly dominated. Moreover, as in the duopoly case, it is also easy to see that $v_i(\,\cdot\,, q_{-i})$ is strictly decreasing over the interval $[45, +\infty)$ for any $q_{-i} \in \mathbb{R}_+^{n-1}$. This in turn implies that any quantity $q_i > 45$ is strictly dominated by 45. Therefore, in the first round, we delete all quantities above 45, and $S_i^1 = [0, 45]$ for all i.

We move to the second round. Notice $\sum_{j \neq i} S_j^1 = [0, 45(n-1)] \supset [0, 90]$ for $n \geq 3$. Moreover, as $\sum_{j \neq i} q_j$ increases over the range $[0, 90]$, firm i's best response decreases from 45 to 0. This implies that no quantity $q_i \in [0, 45] = S_i^1$ is strictly dominated. Therefore, the process of IESDS must stop here. For each i, $S_i^* = [0, 45]$.

6. ST Exercise 5.5.

   (a) **Soln:** Let s_i be the value that player i contributes, which can only be 0 or 1. Denote the value that the other two players contribute as s_{-i}. If $\sum_{j \neq i} s_j = 0$, the streetlamp would not be erected regardless of whether player i contributes or not. Hence it is optimal for player i not to contribute. If $\sum_{j \neq i} s_j = 1$, player i could obtain $3 - 1 = 2$ from the streetlamp if he contributes and 0 if he does not. Hence, it is optimal for player i to contribute. Finally, if $\sum_{j \neq i} s_j = 2$, the streetlamp would be erected regardless of whether player i contributes or not. Hence it is optimal for player i not to contribute. As a result, the best response of player i is

$$BR_i(s_{-i}) = \begin{cases} 0, & \text{if } \sum_{j \neq i} s_j = 0, \\ 1, & \text{if } \sum_{j \neq i} s_j = 1, \\ 0, & \text{if } \sum_{j \neq i} s_j = 2. \end{cases}$$

   (b) **Soln:** When no player contributes, none of them could change the result by deviating to contributing. Hence no one would deviate, and $(0, 0, 0)$ is a Nash equilibrium.

   When there is only one contributing player, the streetlamp would not be erected. Therefore, the player who contributes would deviate from contributing to not contributing. This means that any strategy profile in which there is only one contributor is not a Nash equilibrium.

   When there are two contributing players, the streetlamp would be erected. For the contributors, they would not deviate since the deviation will make them lose the streetlamp. The player who does not contribute would not deviate either since the streetlamp has been ensured. Hence any strategy profile in which there are two contributors is a Nash equilibrium.

   When all the players contribute, every player would like to deviate because the deviation could save money without losing the streetlamp. Hence it is not a Nash equilibrium.

   In sum, $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ are the Nash equilibria.