Game Theory, Fall 2022
Problem Set 7

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1. ST 10.2.

   (a) **Soln:** It's easy to find that (B, R) is the unique stage Nash equilibrium. We can use this as a punishment to construct the grim trigger strategy. Player 1's strategy is: for all $t \geq 1$,

$$s_1^t(h) = \begin{cases} M, & \text{if } h = \emptyset \text{ or } (MC, MC, ..., MC) \in H^{t-1}, \\ B, & \text{otherwise.} \end{cases}$$

Player 2's strategy is: for all $t \geq 1$,

$$s_2^t(h) = \begin{cases} C, & \text{if } h = \emptyset \text{ or } (MC, MC, ..., MC) \in H^{t-1}, \\ R, & \text{otherwise.} \end{cases}$$

There are two kinds of histories (thus two kinds of continuation plays). First, $h = \emptyset$ or $(MC, MC, ..., MC) \in H^{t-1}$. In such a continuation play, player 1's payoff from playing his strategy is

$$v_1(MC) = 4,$$

and that from deviating to B is

$$(1-\delta)v_1(BC) + \delta v_1(BR) = 5(1-\delta) + 0.$$

Thus, player 1 does not have a profitable one-shot deviation if $4 \geq 5(1-\delta)$ or equivalently $\delta \geq \frac{1}{5}$.

Second, consider any history in which there exists at least one deviation. Because in this case it's simply the repeated play of the stage Nash equilibrium, we know it is a SPE for all δ.

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By symmetry, we know the critical value is the same for player 2. In sum, when $\delta \geq \frac{1}{5}$, s is a SPE.

(b) **Soln:** The argument is totally the same as 1a. Player 1's strategy is: for all $t \geq 1$,

$$s_1^t(h) = \begin{cases} T, & \text{if } h = \emptyset \text{ or } (MC, MC, ..., MC) \in H^{t-1}, \\ B, & \text{otherwise.} \end{cases}$$

Player 2's strategy is: for all $t \geq 1$,

$$s_2^t(h) = \begin{cases} L, & \text{if } h = \emptyset \text{ or } (MC, MC, ..., MC) \in H^{t-1}, \\ R, & \text{otherwise.} \end{cases}$$

The non profitable one-shot deviation condition requires

$$v_1(TL) \geq (1-\delta)v_1(BL) + \delta v_1(BR).$$

That is

$$6 \geq 8(1-\delta).$$

To support (T, L) played in every period, we need $\delta \geq \frac{1}{4}$. The answer is different from 1a because the additional payoff for a deviation is higher.

2. ST 10.7.

(a) **Soln:** There is a unique Nash equilibrium of this game. Denote the bartender and consumer by 1 and 2. Denote the customer's action by B and N, where B is to buy and N is not to buy. If the consumer chooses B, the best response of the bartender is to choose $x = 0$. But the consumer will deviate to N if $x = 0$, so it can not construct a Nash equilibrium. If consumer chooses N, the best response of the bartender is to choose $x = 0$. Because N is also the best response for the consumer when $x = 0$, it is the unique Nash equilibrium.

(b) **Soln:** According to Thm 3.7 we proved in class, we know that if the stage game has a unique Nash equilibrium, then playing stage Nash equilibrium repeatedly after every history will be the unique SPE. So the unique SPE is

$$s_1^t(h) = 0 \quad for\ \forall 1 \leq t \leq 10\ and\ h \in H^{t-1}$$
$$s_2^t(h) = N \quad for\ \forall 1 \leq t \leq 10\ and\ h \in H^{t-1}$$

(c) **Soln:** We can use the stage Nash equilibrium as a punishment to construct a grim trigger strategy. Player 1's strategy is: for all $t \geq 1$,

$$s_1^t(h) = \begin{cases} 1, & \text{if } h = \emptyset \text{ or } h = (1B, 1B, ..., 1B) \\ 0, & \text{otherwise.} \end{cases}$$

Player 2's strategy is: for all $t \geq 1$,

$$s_2^t(h) = \begin{cases} B, & \text{if } h = \emptyset \text{ or } h = (1B, 1B, ..., 1B) \\ N, & \text{otherwise.} \end{cases}$$

We only need to consider the continuation play when $h = \emptyset$ or $h = (1B, 1B, ..., 1B)$. player 1's payoff from playing his strategy is $v_1(1B) = p - c$, and that from deviating to 0 is $(1-\delta)v_1(0B) + \delta v_1(0N) = p(1-\delta) + 0$. Thus, player 1 does not have a profitable one-shot deviation if $p - c \geq p(1-\delta)$, or equivalently $p \geq \frac{c}{\delta}$.

Player 2's deviation will lead to 0 payoff. So if $v - p \geq 0$, there will not exist any profitable deviation.

In sum, the range of price p to support such a SPE is $\frac{c}{\delta} \leq p \leq v$.

(d) **Soln:** To make the range of price p exist, we need $\frac{c}{\delta} \leq v$, that is $\delta \geq \frac{c}{v}$.

3. ST 10.8.

(a) **Soln:** In stage Cournot game, the maximization problem for firm 1 is

$$\max_{q_1 \geq 0} (200 - q_1 - q_2)q_1.$$

By F.O.C. we get $BR_1 = \max\{100 - \frac{q_2}{2}, 0\}$. Then by symmetry we know $BR_2 = \max\{100 - \frac{q_1}{2}, 0\}$. Thus $q_1^e = q_2^e = \frac{200}{3}$, $\pi_1^e = \pi_2^e = \frac{40000}{9}$.

(b) **Soln:** To achieve the monopoly profit,

$$\max_{q_1, q_2 \geq 0} (200 - q_1 - q_2)(q_1 + q_2)$$

By F.O.C we get $q_1^c + q_2^c = 100$. To split the profit equally, we have $q_1^c = q_2^c = 50$, and $\pi_1^c = \pi_2^c = 5000$. Then construct the grim trigger strategy. For $i = 1$ or 2, firm i's strategy is: for all $t \geq 1$,

$$s_i^t(h) = \begin{cases} q_i^c, & \text{if } h = \emptyset \text{ or } (q^c, q^c, ..., q^c) \\ q_i^e, & \text{otherwise.} \end{cases}$$

This strategy profile is a subgame-perfect equilibrium when there is no profitable one-shot deviation, i.e.

$$\pi_i^c \geq (1-\delta)\max_{q_i \geq 0}[(200 - q_{-i}^c - q_i)q_i] + \delta\pi_i^e.$$

That is

$$5000 \geq (1-\delta)\cdot 5625 + \delta \cdot \frac{40000}{9}$$

Thus we have $\delta \geq \frac{9}{17}$.

(c) **Soln:** Given $p_{-i}^b \geq 0$, firm i's best response is

$$BR_i(p_{-i}^b) = \begin{cases} \mathbb{R}_+, & \text{if } p_{-i}^b = 0, \\ \emptyset, & \text{if } 0 < p_{-i}^b \leq 100, \\ 100, & \text{if } p_{-i}^b > 100. \end{cases}$$

Thus, there's a unique static stage-game Bertrand-Nash equilibrium $p_1^b = p_2^b = 0$, which is the only case where the strategies are mutual best responses. And the profits will be $\pi_1^b = \pi_2^b = 0$.

(d) **Soln:** To gain the monopoly profit, we have $p_1^c = p_2^c = 100$. Then we can construct the trigger strategy. For $i = 1$ or 2, firm i's strategy is: for all $t \geq 1$,

$$s_i^t(h) = \begin{cases} p_i^c, & \text{if } h = \emptyset \text{ or } (p^c, p^c, ..., p^c), \\ p_i^b, & \text{otherwise.} \end{cases}$$

This strategy profile is a subgame-perfect equilibrium when there is no profitable one-shot deviation, i.e.

$$\pi_i^c \geq (1-\delta)\lim_{p_i \to 100^-} p_i(200 - p_i) \;+\; \delta\pi_i^b.$$

That is

$$5000 \geq (1-\delta)\cdot 10000 + \delta \cdot 0.$$

Thus we have $\delta \geq \frac{1}{2}$.

(e) **Soln:** We can define player i's strategy recursively. For $h = (a^1, \ldots, a^{t-2}, a^{t-1}) \in H^{t-1}$, define $h^{t-1} = h$, $h^{t-2} = (a^1, ..., a^{t-2})$, $h^{t-3} = (a^1, ..., a^{t-3})$. Firm 1's strategy is as follows:

For $t = 1$, $s_1^t(\emptyset) = q_i^c$.

For $t = 2$, we have

$$s_1^t(h) = \begin{cases} q_i^c, & \text{if } a^1 = q^c, \\ q_i^b, & \text{otherwise.} \end{cases}$$

For $\forall t \geq 3$, we have

$$s_1^t(h) = \begin{cases} q_i^c, & \text{if } a^{t-1} = q^c, \text{ or } a^{t-1} = q^b \text{ and } a^{t-2} = q^b, \\ q_i^b, & \text{otherwise.} \end{cases}$$

Here, q^x refers to (q_1^x, q_2^x).

When $a^{t-1} = q^c$, or $a^{t-1} = q^b$ and $a^{t-2} = q^b$ (i.e. in SPE i should choose q_i^c), there is no profitable one-shot deviation from q_i^c if

$$\pi_i^c \geq (1-\delta) \times \lim_{p_i \to 100^-} p_i(200 - p_i) + (1-\delta)\delta\pi_i^b + (1-\delta)\delta^2\pi_i^b + \delta^3\pi_i^c.$$

That is

$$5000 \geq (1-\delta)10000 + 5000\delta^3,$$

which yields $\delta \geq \frac{\sqrt{5}-1}{2}$.

Otherwise, in SPE, i should choose q_i^b, and there is no profitable one-shot deviation from q_i^b if

$$(1-\delta)(\pi_i^b + \delta\pi_i^b + \frac{\delta^2}{1-\delta}\pi_i^c) \geq (1-\delta)(\pi_i^{b'} + \delta\pi_i^b + \delta^2\pi_i^b + \frac{\delta^2}{1-\delta}\pi_i^c),$$

and

$$(1-\delta)(\pi_i^b + \frac{\delta}{1-\delta}\pi_i^c) \geq (1-\delta)(\pi_i^{b'} + \delta\pi_i^b + \delta^2\pi_i^b + \frac{\delta^2}{1-\delta}\pi_i^c).$$

Here, we use $\pi_i^{b'}$, which equals to 0, to describe the payoff after the the deviation from q_i^b in that period, since i's price is strictly larger than 0 and then loses all the market. After the deviation there will be punishment for two periods and then going back to the splitting monopoly. Note that the inequalities hold obviously because $\pi_i^c > \pi_i^b$.

Thus we have $\delta \geq \frac{\sqrt{5}-1}{2}$. The critical value is higher than that in 3d.

4. ST 10.9.

   (a) **Soln:** The profit maximization problem for firm i is

$$\max_{q_i \geq 0} (30 - q_{-i})q_i - q_i^2$$

By F.O.C. we have $BR_i(q_{-i}) = \max\{15 - \frac{q_{-i}}{2}, 0\}$. Then we have $q_1^e = q_2^e = 10$ and $\pi_1^e = \pi_2^e = 100$ in the unique stage Nash equilibrium.

To maximize the social welfare, we have

$$\max_{q_1, q_2 \geq 0} (30 - q_2)q_1 - q_1^2 + (30 - q_1)q_2 - q_2^2$$

By F.O.C. we have $q_1^p + q_1^p = 15$ which is the Pareto-optimal level. As $q_1^e + q_2^e > q_1^p + q_1^p = 15$, the Nash equilibrium is not Pareto optimal.

(b) **Soln:** In order to have a smaller critical value δ, we need to split the profit equally, i.e. $q_1^p = q_1^p = \frac{15}{2}$ and $\pi_1^p = \pi_2^p = \frac{225}{2}$. Consider the strategy profile where both firms produce q_i^p if no one ever deviates from the Pareto-optimal level and q_i^e otherwise, then the non profitable one-shot deviation condition is:

$$\pi_i^p \geq (1-\delta)\max_{q_i}[(30 - q_{-i}^p)q_i - q_i^2] + \delta\pi_i^e.$$

That is

$$\frac{225}{2} \geq (1-\delta)[(30 - \frac{15}{2})\frac{45}{4} - (\frac{45}{4})^2] + \delta \cdot 100.$$

Hence we have $\delta \geq \frac{9}{17}$.

5. Consider the infinitely repeated prisoners' dilemma with discount factor $\delta \in (0,1)$. The stage game is in Figure 1.

	E	S
E	$2,2$	$-1,3$
S	$3,-1$	$0,0$

Figure 1: The prisoners' dilemma

They play EE in the first period. At any history $h = (a^1, \ldots, a^{t-1})$, if they have played EE for all but at most one period, they continue to play EE; otherwise, they play SS. Write down this strategy profile formally and check whether it is a subgame perfect equilibrium for some $\delta \in (0,1)$?

Soln: The strategy profile S: for $i = 1,2$,

$$s_i^1 = E \text{ and } s_i^t(a^1, \ldots, a^{t-1}) = \begin{cases} E, & \text{if } |\{1 \leq \tau \leq t-1 | a^\tau \neq EE\}| \leq 1, \\ S, & \text{otherwise.} \end{cases}$$

For any $\delta \in (0,1)$, this strategy profile is not a Nash equilibrium, let alone a subgame perfect equilibrium. To see this, suppose that player 1 deviates to strategy $\tilde{s}_1'$ which differs from s_1 only at the very beginning: $\tilde{s}_1^1 = S$, and $\tilde{s}_1^t = s_1^t$ for all $t \geq 2$. Under strategy profile (s_1, s_2), the outcome path is $(EE, EE, EE, \ldots)$ which yields payoff 2 to player 1. Under strategy profile $(\tilde{s}_1, s_2)$, the outcome path is $(SE, EE, EE, \ldots)$, which yields payoff $3(1-\delta) + 2\delta$ to player 1. Clearly $3(1-\delta) + 2\delta > 2$. Hence, $\tilde{s}_1$ is a profitable deviation.