Game Theory Lecture Notes 3
Dynamic Games with Complete Information

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Fall 2024

The Extensive Form Game

Introduction

- Normal form games are mostly used to capture static strategic interactions: players act only once and simultaneously.
- How about dynamic environment: players act sequentially?
- Examples of such situations:
  - chess;
  - tennis;
  - bargaining;
  - chatting; etc.
- We use **extensive form games** to describe such situations.

The Extensive Form Game

Game Trees

- An extensive form game describes *who* moves *when* with *what available actions* and *information*, as well as players' payoffs as a function of outcomes.
- Similarly as matrix representation for normal form games, we can use a **game tree** to represent an extensive form game.

The Extensive Form Game

Game Trees

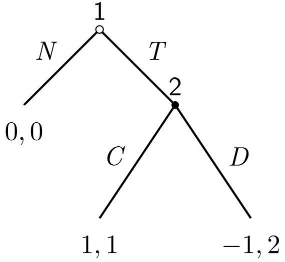


Figure 3.1: A trust game.

- Player 1 first chooses whether to ask for the services of player 2.
- He can trust player 2 (T) or not trust him (N).
- The latter choice gives both players a payoff of 0.
- If player 1 plays T, player 2 can choose to cooperate (C) or defect (D).
- If player 2 cooperates, both players get 1.
- If player 2 defects, player 1 gets -1 while player 2 gets 2.

The Extensive Form Game

Game Trees

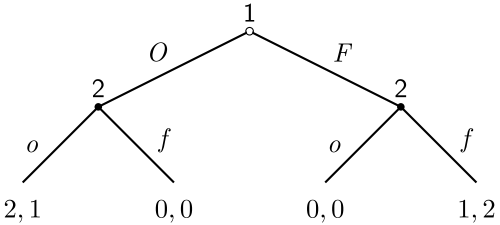


Figure 3.2: The sequential-move Battle of the Sexes game.

- A variant of the battle of the sexes game.
- Players move sequentially.
- Player 1 chooses between O or F first.
- Then, after observing player 1's choice, player 2 chooses between o and f.

The Extensive Form Game

Game Trees

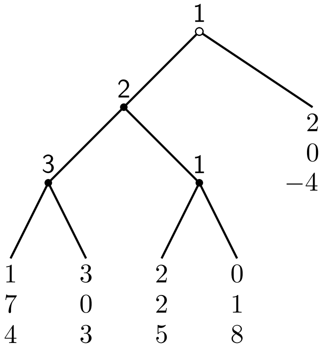


Figure 3.3: A three player game

The Extensive Form Game

Game Trees

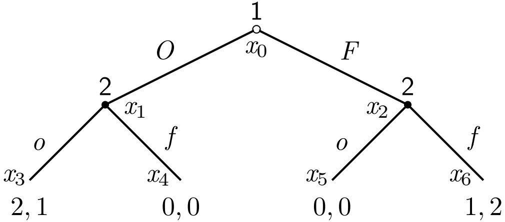
Definition 3.1

A **game tree** is a set of nodes $x \in X$ with a precedence relation $x \succ x'$, which means "x precedes x'." Every node in a game tree has only one predecessor. There is a special node called the **root** (or **initial node**) of the tree. Nodes that do not precede other nodes are called **terminal nodes**. Terminal nodes denote the final outcomes of the game with which payoffs are associated. Every node x that is not a terminal node is assigned to a player, $i(x)$, with action set $A_i(x)$.

- An extensive form game is **finite** if there are only finitely many nodes.
- How about the game tree for Tianji's horse racing?

The Extensive Form Game

Game Trees



- Nodes: $X = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$
- Root (initial node): x_0
- Terminal nodes: $Z = \{x_3, x_4, x_5, x_6\}$
- Player assignment: $i(x_0) = 1$, $i(x_1) = i(x_2) = 2$
- Available action: $A_1(x_0) = \{O, F\}$, $A_2(x_1) = A_2(x_2) = \{o, f\}$
- Payoffs: $v_1, v_2 : Z \to \mathbb{R}$

$$v_1(x_3) = 2,\ v_1(x_4) = 0,\ v_1(x_5) = 0,\ v_1(x_6) = 1,$$

$$v_2(x_3) = 1,\ v_2(x_4) = 0,\ v_2(x_5) = 0,\ v_2(x_6) = 2.$$

The Extensive Form Game

Game Trees

- So far, we have seen examples of game trees in which every player, when called on to move, perfectly knows where he is in the tree.
- In other words, every player perfectly observes what has occurred (or equivalently, what other players have chosen) previously.
- Can we use game tree to describe the situation where players may not be fully informed about what has happened before.
- Example: professor gives an exam; students prepare for that exam without knowing whether it is easy or difficult.

Definition 3.2

Every player i has a collection of information sets $h_i \in H_i$ that partition the nodes of the games at which player i moves with the following properties:

- If h_i is a singleton that includes only x when player i who moves at x knows that he is at x.
- If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$, then player i who moves at x does not know whether he is at x or x'.
- If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$, then $A_i(x) = A_i(x')$.

The Extensive Form Game

Game Trees

- The 2nd point states that player i can not distinguish nodes that are contained in the same information set.
- The 3rd point is to maintain the logic of information. If both x and x' are contained in the same information set, but $A_i(x) \neq A_i(x')$, then player i can indeed distinguish x and x' by observing what actions available to him.

The Extensive Form Game

Game Trees

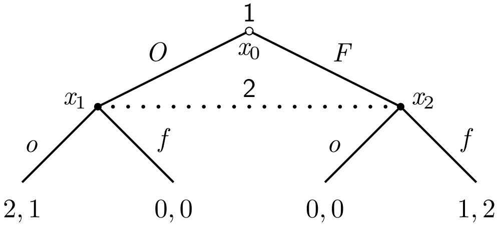


Figure 3.4: The simultaneous-move Battle of the Sexes game.

- $H_1 = \{h_1\}$ where $h_1 = \{x_0\}$.
- $H_2 = \{h_2\}$ where $h_2 = \{x_1, x_2\}$.

The Extensive Form Game

Game Trees

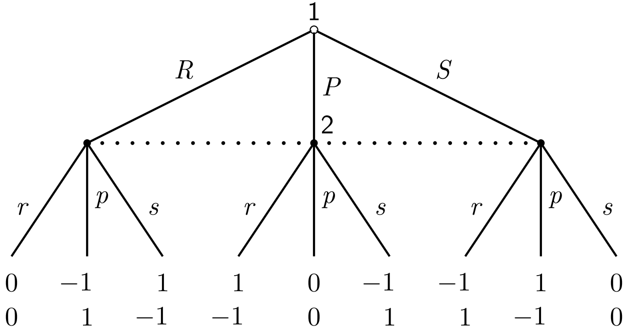
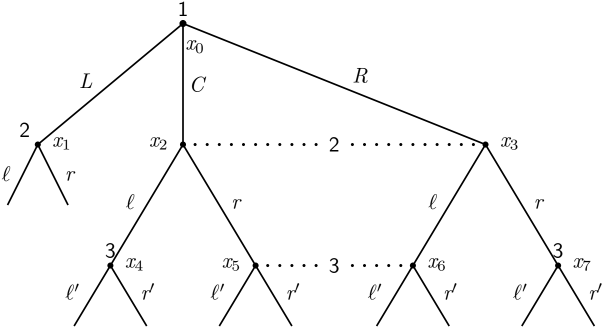


Figure 3.5: Game tree of rock-paper-scissors

- Every normal form game can be represented by an extensive form game!

The Extensive Form Game

Game Trees



- Player 1 has one information set: $h_1 = \{x_0\}$.
- Player 2 has two information sets: $h_2 = \{x_1\}$ and $h_2' = \{x_2, x_3\}$.
- Player 3 has three information sets: $h_3 = \{x_4\}$, $h_3' = \{x_5, x_6\}$ and $h_3'' = \{x_7\}$.

The Extensive Form Game

Perfect recall

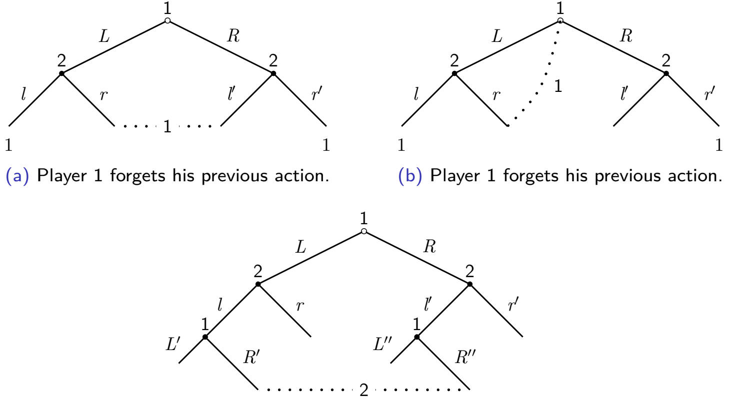
- We will only consider extensive form games in which every player knows whatever he knew previously, including his own previous actions.

Definition 3.3

A game of **perfect recall** in one in which no player ever forgets information that he previously knew.

The Extensive Form Game

Perfect recall



(a) Player 1 forgets his previous action.

(b) Player 1 forgets his previous action.

(c) Player 2 forgets what he previously knew.

Figure 3.6: Examples of imperfect recall. They are beyond the scope of this course.

The Extensive Form Game

Perfect recall

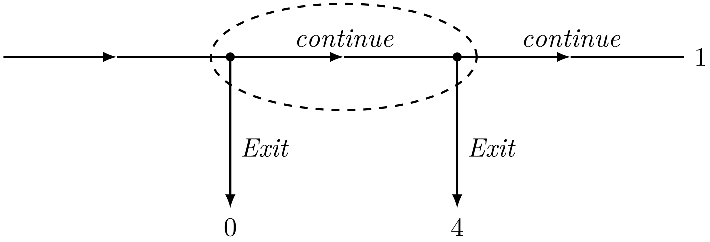


Figure 3.7: The absent-minded driver. It is a one-player game (decision problem) with imperfect recall.

The Extensive Form Game

Imperfect v.s. perfect information

Definition 3.4

An extensive form game is called a **game of perfect information** if every information set is a singleton. It is called a **game of imperfect information** if some information sets contain several nodes.

- Perfect information: every player, when called on to move, knows exactly what has happened previously.
- Imperfect information: some players do not know what some other players have chosen previously at some information sets.

Strategies in Extensive Form Games

Pure strategies

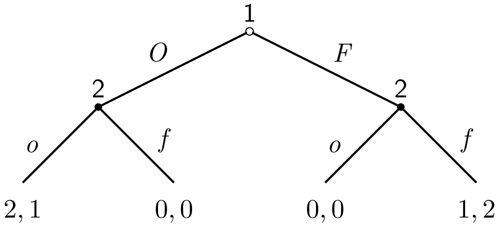


Figure 3.8: The sequential-move Battle of the Sexes game.

Strategies in Extensive Form Games

Pure strategies

- Player 1 has a single information set.
- It is then intuitive that a pure strategy for him is simply O or F.
- Strategy space for player 1: $\{O, F\}$.
- How about player 2? How does player 2 think about her situation?
- Because player 2 moves after observing player 1's choice, player 2 can make a plan for her choices:
  - if player 1 chooses O, I am going to choose $x \in \{o, f\}$;
  - if player 1 chooses F, I am going to choose $y \in \{o, f\}$.
- Thus, player 2's strategy is not simply o or f.
- Rather, it should be a **complete plan of play** that describes which pure action she will choose at each of her information sets.
- Strategy space for player 2: $\{oo, of, fo, ff\}$.
- xy means if player 1 chooses O (the left branch), player 2 will choose x; if player 1 chooses F (the right branch), player 2 will choose y.

Strategies in Extensive Form Games

Pure strategies

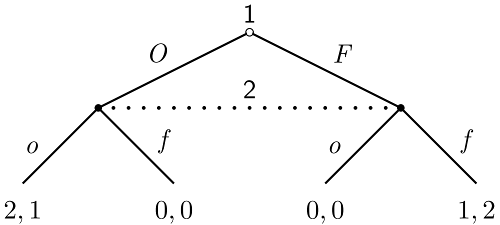


Figure 3.9: The simultaneous-move Battle of the Sexes game.

- A pure strategy for player 1 is still either O or F, as before, $S_1 = \{O, F\}$.
- Now, player 2 can not observe player 1's choice when called on to move.
- Therefore, player 2 can not condition her choice on player 1's choice.
- Consequently, player 2 can only simply chooses either o or f, $S_2 = \{o, f\}$.
- Note that this is also a complete plan of play: it fully describes player 2's behavior when called upon to move.

Strategies in Extensive Form Games

Pure strategies

- H_i: the collection of all information sets at which player i plays.
- $h_i \in H_i$: one of i's information sets.
- $A_i(h_i)$: the actions available to player i at h_i.
- $A_i \equiv \cup_{h_i \in H_i} A_i(h_i)$: the set of all actions of player i.

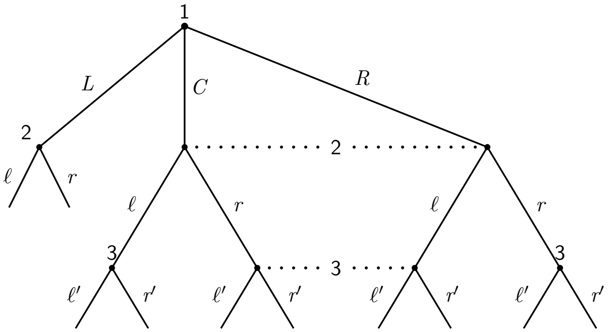
Definition 3.5

A **pure strategy** for player i is a mapping $s_i : H_i \to A_i$ that assigns an action $s_i(h_i) \in A_i(h_i)$ for every information set $h_i \in H_i$. We denote by S_i the set of all pure strategy mappings $s_i \in S_i$.

- Usually, we also say that a strategy in an extensive form game is a *complete contingent plan*: a plan of actions *conditional on this player's information about where he is in the game*.

Strategies in Extensive Form Games

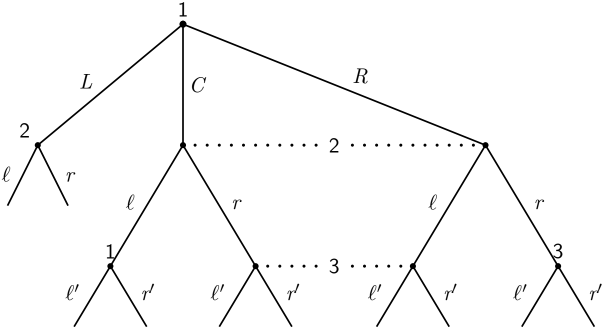
Pure strategies



- $S_1 = \{L, C, R\}$;
- $S_2 = \{\ell\ell, \ell r, r\ell, rr\} = \{\ell, r\} \times \{\ell, r\}$;
- $S_3 = \{\ell'\ell'\ell', \ell'\ell'r', \ell'r'\ell', \ell'r'r', r'\ell'\ell', r'\ell'r', r'r'\ell', r'r'r'\} = \{\ell', r'\} \times \{\ell', r'\} \times \{\ell', r'\}$.

Strategies in Extensive Form Games

Pure strategies



- Note that if player 1 plays L at his first information set, then his second information set is never reached regardless of other players' choices.
- But every strategy of player 1 must still specify player 1's plan of choice at his second information set, e.g., $L\ell'$.
- $S_1 = \{L, C, R\} \times \{\ell', r'\}$.

Strategies in Extensive Form Games

Pure strategies

- Every pure strategy profile $s = (s_1, \ldots, s_n)$ leads to a unique terminal node, denoted by $z(s)$.
- This terminal node is the **outcome** of strategy profile s.
- Note the difference between a strategy profile and the induced outcome in extensive form games.

Strategies in Extensive Form Games

Mixed and behavioral strategies

- Given a pure strategy space S_i for player i in an extensive form game, it is straightforward to define mixed strategies for player i, as we do for normal form games.

Definition 3.6

A **mixed strategy** for player i is a probability distribution over his pure strategies $s_i \in S_i$.

- Interpretation: the player selects a plan randomly *before the game is played* and then follows a *particular pure strategy*.
- For example, consider $\sigma_2 = \frac{1}{2} \circ oo + \frac{1}{2} \circ ff$ in the sequential move battle of the sexes. It can be interpreted as the following:
  - player 2 randomizes with equal probability between oo and ff before the game is played;
  - if oo is chosen, then player 2 plays o regardless of player 1's choice;
  - if ff is chose, then player 2 plays f regardless of player 1's choice.

Strategies in Extensive Form Games

Mixed and behavioral strategies

- How about the following natural description of a "random plan":
  - player 2 randomizes between o with probability $1/3$ and f with probability $2/3$ if player 1 plays O;
  - she randomizes between o with probability $1/4$ and f with probability $3/4$ if player 1 plays F.
- It is a "plan" that allows players to randomize as *the game unfolds*.

Definition 3.7

A **behavioral strategy** σ_i specifies for each information set $h_i \in H_i$ an independent probability distribution $\sigma_i(h_i) \in \Delta A_i(h_i)$. Thus, for action $a_i \in A_i(h_i)$, $\sigma_i(h_i)[a_i]$ is the probability that player i plays a_i at information set h_i.

- The above example: $\sigma_2(O)[o] = \frac{1}{3}$, $\sigma_2(O)[f] = \frac{2}{3}$, $\sigma_2(F)[o] = \frac{1}{4}$, and $\sigma_2(F)[f] = \frac{3}{4}$.

Strategies in Extensive Form Games

Mixed and behavioral strategies

- Consider the sequential move battle of the sexes again.
- Pure strategy space for player 2: $S_2 = \{oo, of, fo, ff\}$.
- Mixed strategy space:

$$\left\{ (p_{oo}, p_{of}, p_{fo}, p_{ff}) \in [0,1]^4 \mid p_{oo} + p_{of} + p_{fo} + p_{ff} = 1 \right\}.$$

  This is a 3-dimensional space. (Yes?)
- Behavior strategy space:

$$\left\{ (\sigma(O)[o], \sigma(O)[f], \sigma(F)[o], \sigma(F)[f]) \in [0,1]^4 \;\middle|\; \begin{array}{l} \sigma(O)[o] + \sigma(O)[f] = 1, \\ \sigma(F)[o] + \sigma(F)[f] = 1. \end{array} \right\}.$$

  This is a 2-dimensional space. (Yes?)
- Mixed v.s. behavioral strategy?

Strategies in Extensive Form Games

Mixed and behavioral strategies

- Every mixed/behavioral strategy profile leads to a probability distribution of the terminal nodes.
- This distribution is the **outcome** of this mix/behavioral strategy profile.
- Consider an arbitrary behavioral strategy $\sigma_2(O)[o] = q_O$ and $\sigma_2(F)[o] = q_F$.
- If player 1 plays O, the outcome is $q_O \circ Oo + (1 - q_O) \circ Of$.
- If player 1 plays F, the outcome is $q_F \circ Fo + (1 - q_F) \circ Ff$.
- Now, consider an arbitrary mixed strategy $(p_{oo}, p_{of}, p_{fo}, p_{ff})$.
- If player 1 plays O, the outcome is $(p_{oo} + p_{of}) \circ Oo + (p_{fo} + p_{ff}) \circ Of$.
- If player 1 plays F, the outcome is $(p_{oo} + p_{fo}) \circ Fo + (p_{of} + p_{ff}) \circ Ff$.

Strategies in Extensive Form Games

Mixed and behavioral strategies

- Starting from the behavioral strategy, we can construct a mixed strategy so that it induces the same outcome *regardless of player* 1*'s play.*
- To see this, consider

$$
\begin{aligned}
p_{oo} &\equiv q_O q_F \geq 0, \\
p_{of} &\equiv q_O(1 - q_F) \geq 0, \\
p_{fo} &\equiv (1 - q_O) q_F \geq 0, \\
p_{ff} &\equiv (1 - q_O)(1 - q_F) \geq 0.
\end{aligned}
$$

- It is straightforward to see $p_{oo} + p_{of} + p_{fo} + p_{ff} = 1$. So it is indeed a mixed strategy.
- Moreover, because $p_{oo} + p_{of} = q_O$ and $p_{oo} + p_{fo} = q_F$, it induces the same outcome as the given behavioral strategy *no matter whether player* 1 *plays* O *or* F.

Strategies in Extensive Form Games

- Starting from the mixed strategy, we can also construct a behavioral strategy so that it induces the same outcome *regardless of player 1's play.*
- To see this, consider

$$q_O \equiv p_{oo} + p_{of} \in [0, 1],$$
$$q_F \equiv p_{oo} + p_{fo} \in [0, 1].$$

- By construction, it induces the same outcome as the given mixed strategy *no matter whether player 1 plays O or F*.
- The above analysis will also work even if we consider player 1's randomization.

- In sum, mixed strategies and behavioral strategies are "equivalent".

Theorem 3.1 (Kuhn)

For finite games with perfect recall, every mixed strategy has a realization equivalent behavioral strategy.

- Any randomization over play can be represented by either mixed or behavioral strategies.
- We can choose the one that is more convenient.

Nash Equilibrium

Normal form representation of extensive form games

- We have already known the pure strategy space S_i for player i in an extensive form game.
- We also know that every strategy profile s leads an outcome $z(s)$.
- Thus, player i's payoff from this strategy profile is $v_i(z(s))$.
- That is, we can always transform an extensive form game into a normal form game: $(N, \{S_i\}_{i=1}^n, \{v_i(z(\cdot))\}_{i=1}^n)$.
- This is called the **normal form representation of an extensive form**.

Nash Equilibrium

Normal form representation of extensive form games

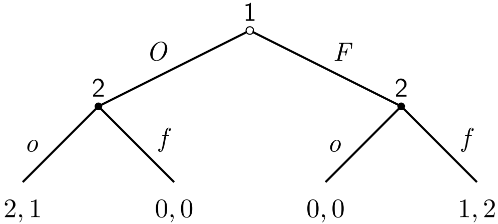


Figure 3.10: The sequential-move Battle of the Sexes game.

- Its normal form representations is

	oo	of	fo	ff
O	$2,1$	$2,1$	$0,0$	$0,0$
F	$0,0$	$1,2$	$0,0$	$1,2$

Nash Equilibrium

Normal form representation of extensive form games

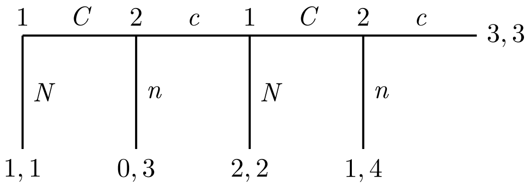


Figure 3.11: The centipede game

- Its normal form representation is:

	nn	nc	cn	cc
NN	$1, 1$	$1, 1$	$1, 1$	$1, 1$
NC	$1, 1$	$1, 1$	$1, 1$	$1, 1$
CN	$0, 3$	$0, 3$	$2, 2$	$2, 2$
CC	$0, 3$	$0, 3$	$1, 4$	$3, 3$

Nash Equilibrium

- Once we transform an extensive form game into its normal form representation, we can then consider the Nash equilibria of this normal form game.
- This is indeed how we define Nash equilibrium for an extensive form game.

Definition 3.8

A strategy profile σ^* is a **Nash equilibrium of an extensive form game** if it is a Nash equilibrium of the normal form representation.

Nash Equilibrium

Nash equilibrium

- Normal form representation of the sequential move battle of the sexes:

	oo	of	fo	ff
O	$2, 1$	$2, 1$	$0, 0$	$0, 0$
F	$0, 0$	$1, 2$	$0, 0$	$1, 2$

- Three pure strategy Nash equilibria: (O, oo), (O, of) and (F, ff).
- Nash equilibrium is about *strategy profiles*, not about the *outcomes*. For instance, (F, of) leads to the same outcome Ff as (F, ff), but (F, of) is not a Nash equilibrium.
- Mixed strategy Nash equilibria?

Nash Equilibrium

- Its normal form representation is:

	nn	nc	cn	cc
NN	$1,1$	$1,1$	$1,1$	$1,1$
NC	$1,1$	$1,1$	$1,1$	$1,1$
CN	$0,3$	$0,3$	$2,2$	$2,2$
CC	$0,3$	$0,3$	$1,4$	$3,3$

- Four pure strategy Nash equilibria: (NN, nn), (NN, nc), (NC, nn), (Nc, nc).
- Mixed strategy Nash equilibria?

Sequential Rationality

Motivating examples

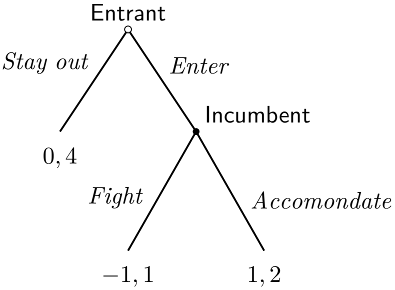


Figure 3.12: The chain-store game.

- Nash equilibria: $(Stay\ out, Fight)$ and $(Enter, Accomondate)$.
- Intuitively, $(Stay\ out, Fight)$ is not very appealing:
  - The entrant stays out because he faces the *threat* that the incumbent will fight if he enters.
  - But is it the incumbent's own interest to fight were the entrant to enter?
  - The threat to fight is *not credible*.

Sequential Rationality

Motivating examples

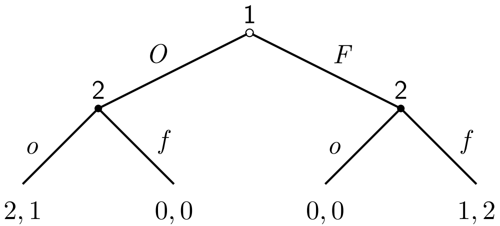


Figure 3.13: The sequential-move Battle of the Sexes game.

- A similar criticism applies to Nash equilibria (O, oo) and (F, ff).
- For example, in (F, ff), player 1 plays F rather than O because he believes that 2 will play f after O. But if 1, for whatever reasons, indeed played O, it was not optimal for 2 to play f. Again, playing f after O is *not credible*.
- Similarly, playing o after F is *not credible* either, which occurs in (O, oo).

Sequential Rationality

Motivating example

- The reason that a Nash equilibrium in an extensive form game may involve incredible threats is that its normal form representation, from which we define Nash equilibrium, ignores the dynamic nature of the game.
- As a result, Nash equilibrium only requires that players play optimally at those information sets *reached under the strategy profile*.
- For example, in the $(Stay\ out, Fight)$ equilibrium in the chain store game, the incumbent's information set is not reached under this strategy profile. Consequently, Nash equilibrium does not require that the incumbent play $Accomondate$, which is the optimal action *if his information set is reached*.

Sequential Rationality

Backward induction

Definition 3.9

Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Nash equilibrium profile of behavioral strategies in an extensive form game. We say that an information set is **on the equilibrium path** if given σ^* it is reached with positive probability. We say that an information set is **off the equilibrium path** if given σ^* it is never reached.

- Nash equilibrium only requires that players play optimally on the equilibrium path.
- Incredible threats then may occur off the equilibrium path.

Sequential Rationality

Backward induction

- To avoid those Nash equilibria that involve incredible threats, we should require that players play strategies that are **sequentially rational**.
- That is, player i's strategy is a best response to the opponents' *at each of his information sets*.
- The strategy $Accomondate$ is the only sequentially rational strategy for the incumbent in the chain store game.
- The strategy of is the only sequentially rational strategy for player 2 in the sequential move battle of the sexes game.

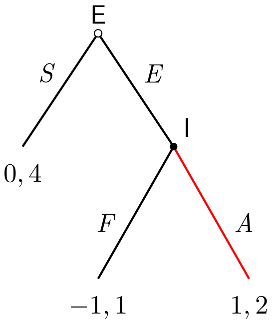
Sequential Rationality

Backward induction

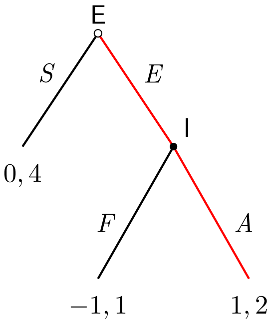
- For finite games of perfect information, there is an easy and very natural way to find all Nash equilibria that are sequentially rational / do not involve incredible threats.
- This method is called **backward induction**.
- It starts at the *bottom* of a game tree.
- It moves backwards *up* the tree, keeping at each node exactly one action.
- This action must be sequentially rational, i.e., an optimal choice conditional on
  - being at that node, and
  - the actions already specified at subsequent nodes.
- A **backward induction solution** is a strategy profile generated by this procedure.

Sequential Rationality

Backward induction



(a) The first step



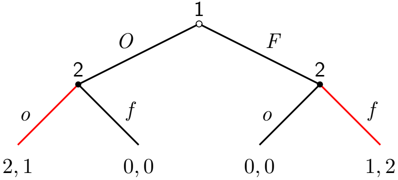
(b) The second step

Figure 3.14: Backward induction in the chain store game

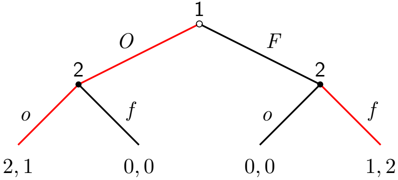
- The only backward induction solution is (E, A).

Sequential Rationality

Backward induction



(a) The first step



(b) The second step

Figure 3.15: Backward induction in the sequential move battle of the sexes

- The only backward induction solution is (O, of).

Sequential Rationality

Backward induction

Theorem 3.2

Every backward induction solution is a Nash equilibrium in pure strategies.

- This result is intuitive, but not straightforward. It is a consequence of Theorem 3.5.

Theorem 3.3 (Kuhn)

Every finite game of perfect information has a backward induction solution (and hence a Nash equilibrium in pure strategies).

- Chess is a trivial game?

Theorem 3.4

Every finite game of perfect information has a unique backward induction solution if no player is indifferent between any two terminal nodes.

Sequential Rationality

Backward induction

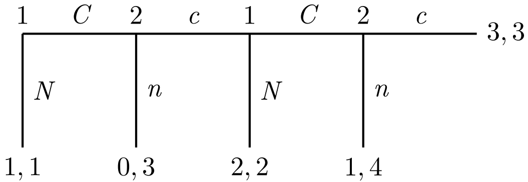


Figure 3.16: The centipede game

- Unique backward induction solution (NN, nn).

Sequential Rationality

Backward induction

- The ultimatum game.
- Two players decide how to split $\$100$.
- Player 1 makes a proposal and player 2 decides whether to accept.
- A proposal is of the form $(x, 100 - x)$ where $x = 0, 1, \ldots, 100$.
- If player 2 accepts $(x, 100 - x)$, then player 1 gets x and player 2 gets $100 - x$.
- If player 2 rejects, both get 0.
- There are two backward induction solutions:
  - Player 1 proposes $(99, 1)$; player 2 accepts all proposals but $(100, 0)$.
  - Player 1 proposes $(100, 0)$; player 2 accepts all proposals.

Subgame Perfect Equilibrium

Subgames

- How to generalize backward induction so it applies to games of imperfect information?

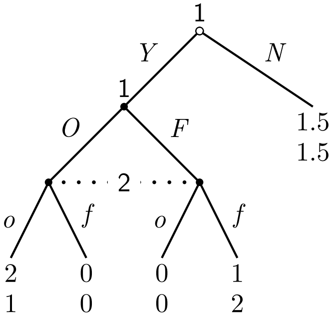


Figure 3.17: The voluntary battle of the sexes game

- Can not simply do backward induction since player 2's optimal choice depends on player 1's behavior.

Subgame Perfect Equilibrium

Subgames

Definition 3.10

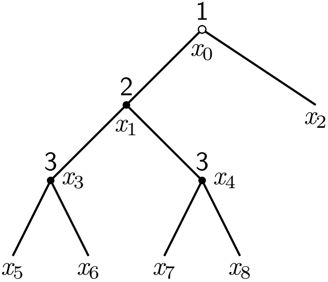
A **(proper) subgame** G of an extensive form game Γ consists of only a single non-terminal node and all its successors in Γ with the property that $x \in G$ and $x' \in h(x)$ then $x' \in G$. The subgame G is itself a *game tree* with its information sets and payoffs inherited from Γ.

- We "dissect" an extensive form game into a sequence of smaller games.
- The word "only" is the key in the definition: node x can be the root of a subgame if and only if
  - it is a singleton information set; and
  - if two nodes are in the same information set and one of them is a successor of x, then the other one must be a successor of x too.
- If the root of a subgame is node x, we say node x *induces* a subgame.
- Every subgame is an extensive form in its own right.
- By definition, the game Γ itself is a subgame.

Subgame Perfect Equilibrium

Subgames

- In a game of perfect information, every non-terminal node induces a subgame.



Subgame Perfect Equilibrium

Subgames

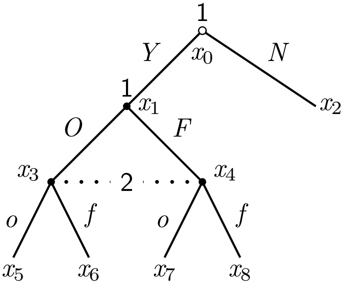
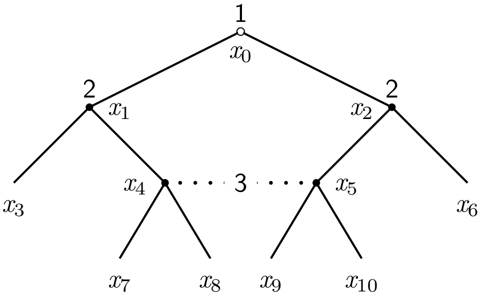


Figure 3.18: The voluntary battle of the sexes game

- Consider x_3 and all its successors, x_5 and x_6, i.e., $G = \{x_3, x_5, x_6\}$.
- They do not form a subgame, because x_3 and x_4 are in the same information set, but $x_4 \notin G$.
- Thus, x_3 does not induce a subgame. Neither does x_4.
- A node can induce a subgame only if it is a singleton information set.
- x_0 induces the original game and x_1 induces $G = \{x_1, x_3, x_4, \ldots, x_8\}$.

Subgame Perfect Equilibrium

Subgames



- x_1 does not induce a subgame. This is because x_4 is a successor of x_1 and x_4 and x_5 are in the same information set, but x_5 is not a successor of x_1.
- The only subgame of this game is the whole game itself.

Subgame Perfect Equilibrium

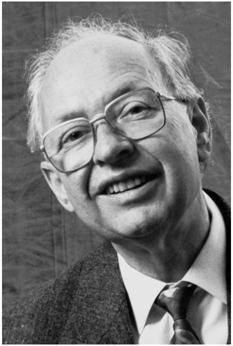
Subgame perfect equilibrium

- Every subgame itself is an extensive form game.
- Thus, we can talk about Nash equilibrium in every subgame.

Definition 3.11

Let Γ be an n-player extensive form game. A behavioral strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a **subgame perfect equilibrium** if for every subgame G of Γ the restriction of σ^* to G is a Nash equilibrium in G.

- A subgame perfect equilibrium (SPE) is a Nash equilibrium. (Yes?)
- "every subgame G" includes those subgames reached and not reached in equilibrium.
- SPE requires that the strategy profile consists of mutual best responses both *on and off the equilibrium path*.



Reinhard Selten

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1994

"for their pioneering analysis of equilibria in the theroy of non-cooperative games"

Contribution: Refined the Nash equilibrium concept for analyzing dynamic strategic interaction by getting rid of unlikely equilibria. He also applied the refined concept to analyses of oligopolistic competition.

Subgame Perfect Equilibrium

Subgame perfect equilibrium

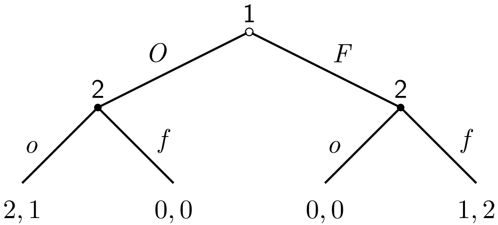


Figure 3.19: The sequential-move Battle of the Sexes game.

- Recall three Nash: (O, oo), (F, ff), (O, of).
- (O, oo) is not subgame perfect: it is not a Nash equilibrium when restricted to the subgame following player 1 choosing F.
- (F, ff) is not subgame perfect neither: it is not a Nash equilibrium when restricted to the subgame following player 1 choosing O.
- (O, of) is the only SPE, which is also the unique backward induction solution.
- Any mixed strategy SPE?

Subgame Perfect Equilibrium

- Restrict attention to finite game of perfect information for now.
- What is the difference between backward induction solution and SPE?

Theorem 3.5

For finite game of perfect information, the set of pure strategy subgame perfect equilibria coincides with the set of backward induction solutions.

- The algorithm of backward induction requires that at each node, the player's *action at this node* be optimal, given *the opponents' and this player's own* future play.
- A SPE requires that at each node, the player's *strategy in this subgame* be optimal given the opponents' future play.
- Thus, a SPE must be a backward induction solution.

Subgame Perfect Equilibrium

Subgame perfect equilibrium

- But why is a backward induction solution a SPE?

Definition 3.12

Consider strategy s_i for player i. We call a strategy s_i' is a **one-shot deviation** of s_i, if s_i' differs from s_i at one and only one node in H_i: there exists a unique $h_i \in H_i$ such that $s_i'(h_i) \neq s_i(h_i)$ and $s_i'(h_i') = s_i(h_i')$ for all $h_i' \neq h_i$.

- s_i' is a one-shot deviation of s_i if s_i' deviates from s_i' only once.
- Other names: one-step deviation / one-stage deviation.

Subgame Perfect Equilibrium

Definition 3.13

Consider a strategy profile s. We say a strategy s_i' is a **profitable one-shot deviation** if (i) s_i' is a one-shot deviation of s_i at node h_i, and (ii) s_i' is a profitable deviation from s_i in the subgame induced by h_i, given s_{-i}.

- The following theorem is usually referred to as the one-shot deviation principle for finite game of perfect information.

Theorem 3.6

Consider a finite game of perfect information. The strategy profile s is a subgame perfect equilibrium if and only if no one has profitable one-shot deviation.

- Theorem 3.6 $\Longrightarrow$ Theorem 3.5 $\Longrightarrow$ Theorem 3.2.

- In fact, the one-shot deviation principle holds for many other classes of extensive form games.
- In particular, its variations holds in all games we consider in this course.

Subgame Perfect Equilibrium

Subgame perfect equilibrium

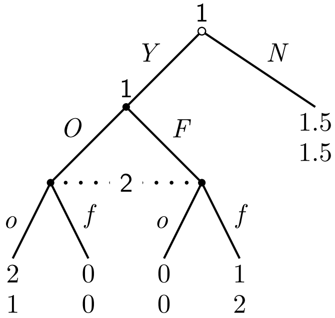


Figure 3.20: The voluntary battle of the sexes

Subgame Perfect Equilibrium

Subgame perfect equilibrium

- Its normal form representation:

	o	f
YO	$2, 1$	$0, 0$
YF	$0, 0$	$1, 2$
NO	$1.5, 1.5$	$1.5, 1.5$
NF	$1.5, 1.5$	$1.5, 1.5$

- Three pure strategy Nash: (YO, o), (NO, f) and (NF, f).
- (YO, o) and (NF, f) are SPE's, while (NO, f) is not.
- Many mixed strategy Nash: $(\alpha \circ NO + (1-\alpha) \circ NF, \beta \circ o + (1-\beta) \circ f)$ for $\alpha \in [0, 1]$ and $\beta \in [0, \frac{3}{4}]$.
- The mixed strategy $\alpha \circ NO + (1-\alpha) \circ NF$ is equivalent to the behavioral strategy that player 1 plays N for sure at his first information set and then mixes between O and F with probability α and $1-\alpha$ at his second information set.
- Only when $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$, it is a SPE.

Subgame Perfect Equilibrium

Subgame perfect equilibrium

- One-shot deviation principle also holds for this game.
- Applying one-shot deviation principle, we can find out the SPE's similarly as backward induction.
- In the subgame following player 1 choosing Y, there are three Nash: (O, o), (F, f) and $(\frac{2}{3} \circ O + \frac{1}{3} \circ F, \frac{1}{3} \circ o + \frac{2}{3} \circ f)$.
- Consider any behavioral strategy profile σ^*.
- If it is a SPE, it must play one of the three equilibria in this subgame.
- If (O, o) is played, then it is optimal for player 1 to choose Y in the first information set. This leads to (YO, o).
- If (F, f) or $(\frac{2}{3} \circ O + \frac{1}{3} \circ F, \frac{1}{3} \circ o + \frac{2}{3} \circ f)$ is played, then it is optimal for player 1 to choose N in the first information set. This leads to either (NF, f) or $(\frac{2}{3} \circ NO + \frac{1}{3} \circ NF, \frac{1}{3} \circ o + \frac{2}{3} \circ f)$.
- Note that these three strategy profiles are just the SPE's we found earlier.

SPE Examples

Stackelberg competition

- Stackelberg game: sequential move variation on the Cournot duopoly.
- Demand $P = 100 - Q$.
- Firm 1 chooses $q_1 \geq 0$ first.
- Firm 2 then chooses $q_2 \geq 0$ after observing firm 1's choice.
- Market supply $Q = q_1 + q_2$.
- Constant marginal cost $c_i(q_i) = 10q_i$.
- Payoff $(100 - q_1 - q_2)q_i - 10q_i$.
- Infinite game of perfect information.

SPE Examples

Stackelberg competition

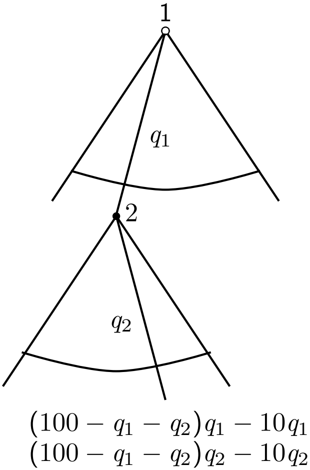


Figure 3.21: Stackelberg competition

SPE Examples

Stackelberg competition

- We restrict attention to pure strategies.
- Firm 1 only has one information set.
- Thus, a strategy for firm 1 is simply a quantity $q_1 \in \mathbb{R}_+$.
- Firm 2 has infinitely many information sets; every action q_1 of firm 1 leads to an information set for firm 2.
- Thus, a strategy for firm 2 is a function $q_2 : \mathbb{R}_+ \to \mathbb{R}_+$, where $q_2(q_1)$ means firm 2's quantity choice if firm 1 chooses q_1.

SPE Examples

Stackelberg competition

- Since it is a game of perfect information, every non-terminal node induces a subgame.
- There are infinitely many subgames.
- We can also do backward induction in this game to find out SPE.
- In the subgame after firm 1 choosing $q_1 \geq 0$, we know firm 2's optimal choice $q_2^*(q_1)$ solves

$$\max_{q_2 \geq 0} (100 - q_1 - q_2)q_2 - 10q_2,$$

  and thus

$$q_2^*(q_1) = \frac{90 - q_1}{2}.$$

- Here, we assume $q_1 \leq 90$.

SPE Examples

Stackelberg competition

- Inducting backwardly, at the initial node, firm 1's problem is

$$\max_{q_1 \geq 0} \; (100 - q_1 - q_2^*(q_1))q_1 - 10q_1. \text{ (Yes?)}$$

- Plugging in the expression for q_2^*, we have

$$\max_{q_1 \geq 0} \; \left(100 - q_1 - \frac{90 - q_1}{2}\right)q_1 - 10q_1.$$

- FOC yields

$$q_1^* = 45.$$

- Then, firm 2's equilibrium quantity choice is

$$q_2^*(q_1^*) = 22.5.$$

- Profits are $\pi_1^* = 1012.5$ and $\pi_2^* = 506.25$.

SPE Examples

Stackelberg competition

- Cournot (simultaneous move) equilibrium $q_1^* = q_2^* = 30$ with profits $\pi_1^* = \pi_2^* = 900$.
- When firm 1 moves first, it earns more.
- This is usually called the *first mover advantage*.
- Why?

SPE Examples

Stackelberg competition

- Nash equilibria?
- Consider $\hat{q}_1 \geq 0$ such that

$$(100 - \hat{q}_1 - q_2^*(\hat{q}_1))\hat{q}_1 - 10\hat{q}_1 \geq 0.$$

- Suppose

$$\hat{q}_2(q_1) = \begin{cases} q_2^*(q_1), & \text{if } q_1 = \hat{q}_1, \\ 90, & \text{if } q_1 \neq \hat{q}_1. \end{cases}$$

- Then the strategy profile $(\hat{q}_1, \hat{q}_2(\cdot))$ is a Nash equilbrium. (Yes?)
- Incredible threat. (Yes?)

SPE Examples

Mutually assured destruction

- Cuba missile crisis of 1962.
- The US (player 1) discovered Soviet (player 2) nuclear missiles in Cuba.
- Player 1 decides whether to ignore the incident (I), resulting in payoffs $(0, 0)$, or escalate the situation (E).
- Following escalation by player 1, player 2 can back down (B), causing it to lose face and resulting in payoffs of $(10, -10)$, or it can choose to proceed to a nuclear confrontation (N).
- Upon choice of N, the players play a simultaneous move game in which they can either retreat (R for player 1 and r for player 2) or choose Doomsday (D for player 1 and d for player 2), in which the world is all but destroyed.
- If both retreat then they suffer a small loss due to the mobilization process and payoffs are $(-5, -5)$, while if either party chooses Doomsday then the world destructs and payoffs are $(-100, -100)$.

SPE Examples

Mutually assured destruction

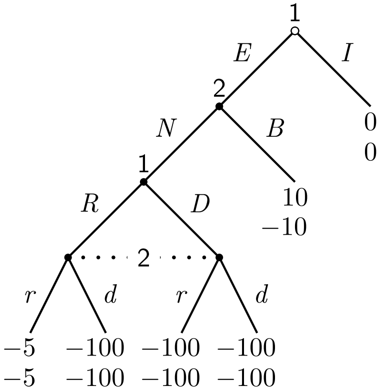


Figure 3.22: Mutually assured destruction

SPE Examples

Mutually assured destruction

- Pure strategy SPE?
- One-shot deviation principle.
- Consider the subgame after player 1 choosing E and player 2 choosing N.
- There are two Nash: (R, r) and (D, d).
- If σ^* is a SPE, it must play either (R, r) or (D, d) in this subgame.
- If it is (R, r),
  - then player 2's optimal choice at its first information set is N;
  - player 1's optimal choice at its first information is I.
- Thus, a SPE (IR, Nr).
- If it is (D, d),
  - then player 2's optimal choice at its first information set is B;
  - player 1's optimal choice at its first information set is E.
- Thus, another SPE (ED, Bd).

- Consider the partnership game (the prisoners' dilemma):

	E	S
E	$2, 2$	$-1, 3$
S	$3, -1$	$0, 0$

Figure 3.23: The partnership game

- E stands for effort; S stands for shirking.
- Unique Nash equilibrium: (S, S).
- Cooperation can not arise in equilibrium.
- Why can we observe cooperation behavior in reality?

- Consider the Cournot duopoly: $P(Q) = \max\{100 - Q, 0\}$, $c_1 = c_2 = 10$.
- Instead of operating independently, imagine the firms choose quantities jointly so as to maximize the industry profits and then divide up the profits among them. In other words, they form a cartel and collude:

$$\max_Q P(Q)Q - 10Q.$$

- $Q^* = 45$.
- If each firm produces 22.5, each earns profits $45 \times 22.5 = 1012.5$.
- But we know this is not a Nash equilibrium. In the unique Nash equilibrium, $q_1^* = q_2^* = 30$ and equilibrium profits are $30 * 30 = 900$ for each firm.
- Forming a cartel can make both firms strictly better and we do observe collusion behavior in reality.

Topics I: Repeated Games

Motivating examples

- One possible reason that what we observe in reality is not equilibrium behavior in the previous two examples is that the simultaneous move game only captures one-shot interaction.
- Intuitively, agents in reality do not interact only once. Instead, they interact through a long-run relationship.
- In such a relationship, agents not only care about their today's payoffs but also their future payoffs.
- Moreover, agents can condition their behavior on what has happened in the past.
- Thus, it is intuitive to expect different behavior than that predicted by the one-shot interaction.
- Repeated games provide us a useful tool to analyze agents' long-run interactions.

Topics I: Repeated Games

Repeated games

- Consider a normal form game G.
- Suppose this game is repeatedly played for T times.
- In period $t = 1$, players simultaneously choose actions in G.
- In period $t = 2$, they observe all players' choices in period $t = 1$ and then simultaneously choose actions in G again.
- In period $t = 3$, they again observe all players' choices in period $t = 2$, (they also know the choices in period $t = 1$ by perfect recall), and then simultaneously choose actions again.
- This repeated play lasts for T periods. T can be either finite or infinite in which case the repeated play lasts forever.
- This dynamic game is called a **repeated game** (payoffs to be specified later). The game G is called the **stage game** of this repeated game.
- If T is finite, it is a **finitely repeated game**. If $T = \infty$, it is an **infinitely repeated game**.

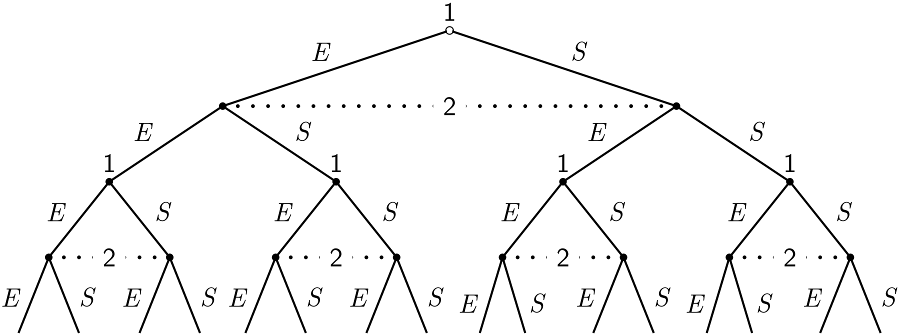


Figure 3.24: The game tree for the twice repeated partnership game

- Note that every repeated game is an extensive form game of imperfect information.

Topics I: Repeated Games

Repeated games

- In each round, players obtain payoffs from the play of the stage game.
- Player i's *total payoff* from the repeated game is the **discounted sum of his stage game payoffs**.
- Suppose T is finite. Let $\delta \in (0,1]$ be the discount factor. If player i obtains payoffs $v_i^1, \ldots, v_i^T$ in each period throughout the play, his discounted sum is

$$v_i \equiv v_i^1 + \delta v_i^2 + \ldots + \delta^{T-1} v_i^T = \sum_{t=1}^{T} \delta^{t-1} v_i^t.$$

- Suppose $T = \infty$. Let $\delta \in (0,1)$ be the discount factor. If player i obtains payoffs $v_i^1, v_i^2, v_i^3, \ldots$ in each period throughout the play, his discounted sum is

$$v_i \equiv v_i^1 + \delta v_i^2 + \delta^2 v_i^3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1} v_i^t.$$

  We require $\delta < 1$ in this case; otherwise the infinite sum may not be well defined.

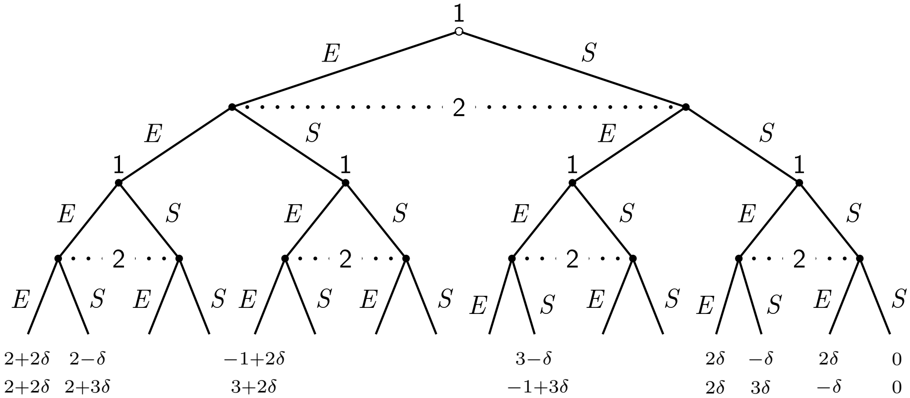


Figure 3.25: The twice repeated partnership game with some payoffs

Topics I: Repeated Games

Repeated games

- Two usual interpretations of δ:
  - it measures how *impatient* the players are (this is why we call it the discount factor).
  - it measures the *probability* that the players will meet and interact tomorrow.

- How about strategies?
- Consider the twice repeated partnership game.
- Each player has 5 information sets.
- Clearly, a strategy for player i should specify an action for each of these 5 information sets.
- Is there a convenient way to express each strategy?

Topics I: Repeated Games

Repeated games

- Let $A = A_1 \times \ldots \times A_n$ denote the set of all action profiles in the stage game.
- Suppose the players have finished the play in the first period and are at the beginning of period $t = 2$.
- Note that an action profile $a \in A$ can fully describe *what has happened in the first period*.
- For example, in the twice repeated partnership game, the action profile EE can be used to describe that both players have chosen E in the first period.
- Thus, every action profile $a \in A$ can be considered as a possible **history up to period** $t = 1$.
- We can define $H_1 = A$ to denote the set of all possible histories up to period $t = 1$.

- More generally, for $t \geq 1$, we can define $H_t = A^t$ be the set of all possible **histories up to period** t.
- Every element $h_t \in H_t$ fully describes *what has happened in the first* t *periods*, and vice versa.
- For example, consider the $T \geq 5$ times repeated partnership game.
- The history (EE, SS, SS, SS, SS) means that both players played E in the first period, but then they played S in periods 2 to 5.
- We also use $\emptyset$ to denote the "empty" history (initial node), which means "nothing has happened". Let $H_0 = \{\emptyset\}$.

Topics I: Repeated Games

Repeated games

- The nice property of histories is that *every player's every information set can be uniquely identified by a history*.
- Thus, a strategy s_i for player i in the T times repeated game is simply a mapping
$$s_i : \bigcup_{t=0}^{T-1} H_t \to A_i.$$
- If T is finite, we can explicitly write a strategy s_i as T mappings $(s_i^1, s_i^2, \dots, s_i^T)$ such that s_i^t is a mapping $s_i^t : H_{t-1} \to A_i$ specifying player i's behavior in period t.
- It has a very natural interpretation: in period $t = 1, \dots, T$, if $h \in H_{t-1}$ has happened in the past $t-1$ periods, player i is going to choose $s_i^t(h)$ in period t.
- If $T = \infty$, then a strategy s_i is in fact an infinite sequence of mappings $(s_i^1, s_i^2, \dots)$ such that s_i^t is a mapping $s_i^t : H_{t-1} \to A_i$ specifying player i's behavior in period t.

Topics I: Repeated Games

Repeated games

- Consider the twice repeated partnership game.
- Consider the strategy

$$s_1^1 = E \text{ and } s_1^2(a) = \begin{cases} E, & \text{if } a = EE, \\ S, & \text{if } a \neq EE. \end{cases}$$

- In words,
  - player 1 plays E in the first period;
  - he continues to player E in the first period if both players play E in the first period;
  - otherwise, he plays S in the second period.
- Consider another strategy: player 1 plays S in the first period; in the second period, he imitates player 2's behavior in the first period. How to write down this strategy?

Topics I: Repeated Games

Repeated games

- Consider the infinitely repeated partnership game.
- Consider the strategy s_1:

$$s_1^1 = E \text{ and for all } t \geq 2,\ s_1^t(h) = \begin{cases} E, & \text{if } h = (EE, \ldots, EE) \in H_{t-1}, \\ S, & \text{otherwise.} \end{cases}$$

- In words,
  - player 1 plays E in the first period;
  - in any period $t \geq 2$, if both players have played E in all previous periods, then he plays E again;
  - otherwise, he plays S.
- Cooperation if they have cooperated in the past; otherwise, never cooperate in the future.
- Consider another strategy s_1: player 1 plays E in the first period; in any period $t \geq 2$, if player 2 has played S in the past for at least twice, he switches to S; otherwise, he continues to play E. (Forgiving once).

Topics I: Repeated Games

Finitely repeated games

- We first consider finitely repeated games.
- We focus on SPE's.
- Every non-terminal history $h \in H_t$ for $t = 0, \ldots, T-1$ induces a subgame.
- One-shot deviation principle holds for these games.

- Consider the twice repeated partnership game.
- If s is a SPE, what would it play in the second period after history $h \in H_1 = \{EE, ES, SE, SS\}$?
- The subgame after h has the following matrix representation:

	E	S
E	$v_1(h) + 2\delta, v_2(h) + 2\delta$	$v_1(h) - \delta, v_2(h) + 3\delta$
S	$v_1(h) + 3\delta, v_2(h) - \delta$	$v_1(h) + 0\delta, v_2(h) + 0\delta$

- Observe that when player 1 compares his payoff from E and S (given player 2's action in this subgame), player 1 only needs to compare his payoff *from the second period*. The same observation also applies to player 2.

- Therefore, the subgame after h is equivalent to the following game

	E	S
E	$2,2$	$-1,3$
S	$3,-1$	$0,0$

  which is just the partnership game (the stage game) itself.
- We know there is a unique Nash equilibrium SS in this game.
- Since s is a SPE, we know $s_1^2(h) = s_2^2(h) = S$ for all $h \in H_1$.

Topics I: Repeated Games

Finitely repeated games

- We now move on to the first period play. By the one-shot deviation principle, we only need to focus on their play in the first period, given their behavior in the second period which we have known.
- They again face a simultaneously move game whose matrix is

	E	S
E	$2+\delta v_1(SS), 2+\delta v_2(SS)$	$-1+\delta v_1(SS), 3+\delta v_2(SS)$
S	$-1+\delta v_1(SS), 3+\delta v_2(SS)$	$0+\delta v_1(SS), 0+\delta v_2(SS)$

- Observe again that this game is equivalent to

	E	S
E	$2,2$	$-1,3$
S	$3,-1$	$0,0$

  which is again the stage game itself.
- This is true because their second period play (which we have figured out) is *independent of their first period play*, i.e., $s(h) = SS$ *for all* $h \in H_1$.
- Since s is a SPE, we know $s_1^1 = s_2^1 = S$.

- Therefore, there is a unique SPE in the twice repeated partnership game. In this SPE, players play S after every history. The associated outcome path is (SS, SS).
- It is disappointing! Even if the game is repeatedly played, the only equilibrium is simply a repetition of the one-shot game.
- How about $T \geq 3$?

- In the previous example, we show two results:
  - repeated play of a stage Nash equilibrium after every history is a SPE;
  - in this particular example, it is the unique SPE.
- In fact, these two results can be generalized.

Theorem 3.7

Consider a finitely repeated game. Suppose action profile $a^* = (a_1^*, \ldots, a_n^*)$ is a Nash equilibrium of the stage game. Then the strategy profile s which plays a^* after every history is a subgame perfect equilibrium of the repeated game, for any $\delta \in (0, 1]$. If, in addition, a^* is the unique Nash equilibrium in the stage game, then this strategy profile is the unique SPE in the repeated game.

Topics I: Repeated Games

Finitely repeated games

- The first part of Theorem 3.7 is encouraging.
- It states the existence of SPE in finitely repeated games.
- The second part is discouraging.
- We can expect nothing but the repetition of the stage Nash in any finitely repeated games if the stage game has only one Nash equilibrium.
- Even if T is very large, cooperation can not arise in equilibrium in the repeated partnership game.

Proof of Theorem 3.7.

Consider the strategy profile s which plays a^* after every history. We can directly verify that s is a SPE without invoking the one-shot deviation principle.

Consider any history $h = (\tilde{a}^1, \ldots, \tilde{a}^{t-1})$ at the beginning of period $t = 1, \ldots T - 1$. In the subgame following h, given the opponents' strategy, player i's payoff from any strategy can be written as

$$\sum_{\tau=1}^{t-1} \delta^{\tau-1} v_i(\tilde{a}^\tau) + \sum_{\tau=t}^{T} \delta^{\tau-1} v_i(a_i^\tau, a_{-i}^*)$$

Because a_i^* maximizes player i's stage game payoff if the opponents play a_{-i}^*, it is obvious that playing s_i is optimal for player i in this subgame.

This proves that s is a SPE.

Proof of Theorem 3.7.

Now, assume a is the unique Nash equilibrium of the stage game. We show that s is the unique SPE of the repeated game.

Consider any SPE $\tilde{s}$. Consider any history $h = (\tilde{a}^1, \ldots, \tilde{a}^{T-1})$ at the beginning of the last period. The subgame following h is a simultaneous move game with strategy space A_i for player i and payoff function $\hat{v}_i(a) = \sum_{\tau=1}^{T-1} \delta^{\tau-1} v_i(\tilde{a}^\tau) + \delta^{T-1} v_i(a)$. Because $\tilde{s}$ is a SPE, it must play a Nash equilibrium in this subgame. Because this subgame is equivalent to the stage game itself, $\tilde{s}$ in this subgame must play a^*, which is the unique stage Nash equilibrium. Thus, we have shown that $\tilde{s}_i^T(h) = a_i^*$ for all i and $h \in H_{T-1}$.

Proof of Theorem 3.7.

Suppose we have shown that $\tilde{s}$ play a^* in periods $t = k, k+1, \ldots, T-1, T$ after any history. We now show that $\tilde{s}$ must also play a^* in period $t = k-1$ after any history. Pick any $h = (\tilde{a}^1, \ldots, \tilde{a}^{t-1}) \in H_{t-1}$. Because $\tilde{s}$ is a SPE, no one has a profitable deviation after h. This means that for all i, $\tilde{s}_i^t(h)$ solves

$$\max_{a_i \in A_i} \sum_{\tau=1}^{t-1} \delta^{\tau-1} v_i(\tilde{a}^\tau) + \delta^{t-1} v_i(a_i, \tilde{s}_{-i}^t(h)) + \sum_{\tau=t+1}^{T} \delta^{\tau-1} v_i(a^*).$$

Equivalently, $\tilde{s}_i^t(h)$ solves

$$\max_{a_i \in A_i} \; v_i(a_i, \tilde{s}_{-i}^t(h)),$$

which implies that $\tilde{s}^t(h)$ is a Nash equilibrium of the stage game. Therefore, $\tilde{s}^t(h) = a^*$ since a^* is the unique stage Nash equilibrium. $\square$

- In the previous proof, we proved a result which is worth stating of itself.

Lemma 3.1

Suppose s is a subgame perfect equilibrium of a finitely repeated game. In the last period, after any history, s must play a stage Nash equilibrium.

- Note that after different histories, different Nash equilibria can be played.

	E	S	P
E	$2, 2$	$-1, 3$	$-3, -3$
S	$3, -1$	$0, 0$	$-3, -3$
P	$-3, -3$	$-3, -3$	$-2, -2$

Figure 3.26: The augmented partnership game

- There are two stage Nash equilibria: SS and PP.
- Can EE be played in the first period in a SPE? (EE can not be played in the second period after any history. Yes?)

- Consider the following strategy profile:

$$s_i^1 = E \text{ and } s_i^2(h) = \begin{cases} S, & \text{if } h = EE, \\ P, & \text{if } h \neq EE. \end{cases}$$

- In words,
  - both players play E in the first period;
  - in the second period, if both players have played E in the first period, they play S;
  - if at least one player played an action different from E in the first period, they play P.

Topics I: Repeated Games

Finitely repeated games

- Is this strategy profile a SPE?
- Their play in the second period is consistent with a SPE, by construction. (Yes?)
- So we only need to worry about their play in the first period.
- Consider player 1. Given player 2's strategy and player 1's own strategy in the second period, player 1's payoff from choosing E in the first period is

$$v_1(EE) + \delta v_1(SS) = 2;$$

that from choosing S is

$$v_1(SE) + \delta v_1(PP) = 3 - 2\delta;$$

and that from choosing P is

$$v_1(PE) + \delta v_1(PP) = -3 - 2\delta.$$

- As long as $2 \geq 3 - 2\delta$, or equivalently $\delta \geq 1/2$, choosing E in the first period is optimal.
- By symmetry, it is also optimal for player 2 to play E in the first period provided $\delta \geq 1/2$.
- Therefore, when $\delta \geq 1/2$, the above strategy profile is indeed a SPE!
- Cooperation EE, which is non stage equilibrium behavior, appears in a SPE.

- There are some general messages we can learn from the above example.
- First, when the stage game has more than one Nash equilibrium, it is possible that players play non stage Nash equilibrium in early periods of a finitely repeated game in a SPE. In other words, it is possible that players are willing to sacrifice their myopic interests in order to obtain higher long-run payoffs.

- Second, the key mechanism is the conditional play of different Nash equilibria in the second period, which promise different *continuation payoffs to the players*. This serves as the "stick and carrot" policy:
  - SS is the "carrot": if no player has deviated in the first period, they play SS then and obtain 0 in the second period;
  - PP is the "stick": if someone has deviated in the first period, they punish each other by playing P and obtain only $-2 < 0$ in the second period. (P for *punishment*)

  It is this "stick and carrot" behavior that gives players the *intertemporal incentive* to choosing E in the first period. (Their *myopic best response* in the first period is S.)

- Third, for this mechanism to work, we require that the discount factor be sufficiently large, e.g., $\delta \geq 1/2$ in the above example. This is very intuitive since δ measures how much the players care about their future payoffs. The larger δ is, the more the players care about their future payoffs. If δ is too small, it is not possible to induce a player to play a non myopic best response by promising him a high payoff in the future, since he does not care about his future payoff.

Topics I: Repeated Games

Infinitely repeated games

- We now move on to the infinitely repeated games.
- We have defined previously players' payoffs from the repeated game to be the discounted sum of stage game payoffs.
- When infinitely repeated game is considered, it is usually conventional to use the **average discounted sum of stage game payoffs** as players' payoffs:

$$v_i \equiv (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}v_i^t.$$

- It is simply a *normalization* of the discounted sum.
- For example, if the same action profile a is infinitely repeatedly played, then player i's payoff is

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}v_i(a) = v_i(a).$$

-

- We can not do backward induction now since there is no terminal node.
- But one-shot deviation principle still applies.
- However, because there are infinitely many subgames, it seems very difficult to check whether a given strategy profile is a SPE even if we apply the one-shot deviation principle.

Topics I: Repeated Games

Infinitely repeated games

- Consider an history $h = (a^1, \ldots, a^t)$ and the subgame following it.
- If some strategy is played in this subgame which leads to the outcome path $(a^{t+1}, a^{t+2}, \ldots)$, the player i's payoff is

$$
\begin{aligned}
v_i &= (1-\delta)\sum_{\tau=1}^{\infty} \delta^{\tau-1} v_i(a^\tau) \\
&= (1-\delta)\sum_{\tau=1}^{t} \delta^{\tau-1} v_i(a^\tau) + (1-\delta)\sum_{\tau=t+1}^{\infty} \delta^{\tau-1} v_i(a^\tau) \\
&= (1-\delta)\sum_{\tau=1}^{t} \delta^{\tau-1} v_i(a^\tau) + \delta^t(1-\delta)\sum_{\tau=1}^{\infty} \delta^{\tau-1} v_i(a^{\tau+t})
\end{aligned}
$$

- Thus, this subgame is in fact equivalent to the original repeated game! (This is the key difference between the finitely and infinitely repeated games.)

Topics I: Repeated Games

Infinitely repeated games

- Consider a strategy profile s.
- Given any history h, s specifies players' play in the subgame following h.
- We call this part of strategy as the **continuation play of** s in the subgame following h.
- Note that this continuation play can also be considered as a strategy of the original repeated game.
- We therefore have

Lemma 3.2

Consider a strategy profile s in an infinitely repeated game. s is a subgame perfect equilibrium if and only if the continuation play of s after any history h is a Nash equilibrium of the original repeated game.

- The following theorem is analogous to Theorem 3.7.

Theorem 3.8

Consider an infinitely repeated game. Suppose action profile $a = (a_1, \ldots, a_n)$ is a Nash equilibrium of the stage game. Then the strategy profile s which plays a after every history is a subgame perfect equilibrium of the repeated game, for any $\delta \in (0, 1)$.

Proof.

By Lemma 3.2, we only need to show that this strategy profile is a Nash equilibrium of the repeated game. Given s_{-i}, which always plays a_{-i} after any history, always playing a_i is clearly optimal for player i since it maximizes player i's stage payoff in every period. Note also that this argument is independent of δ. $\square$

Topics I: Repeated Games

Infinitely repeated games

- Other SPE?
- Consider the partnership game.
- Consider the following strategy profile:

$$\forall t \geq 1,\ s_i^t(h) = \begin{cases} E, & \text{if } h = \emptyset \text{ or } h = (EE, \ldots, EE) \in H_{t-1}, \\ S, & \text{otherwise.} \end{cases}$$

- In words,
  - players play E in the first period;
  - in any period after any history, if no one has deviated from E, they continue to play E;
  - otherwise, they play S.
- Cooperating if they all have cooperated in the past.

- Consider history $h = \emptyset$ (or equivalently period $t = 1$).
- Given player 2's strategy and player 1's own strategy from period $t = 2$ on, player 1's payoff from playing E in this period is

$$v_i(EE) = 2 \text{ (Yes?)},$$

and that from playing S is

$$(1-\delta)v_i(SE) + \delta v_i(SS) = 3(1-\delta) \text{ (Yes?)}.$$

- Thus, player 1 does not have a profitable one-shot deviation in period $t = 1$ if $2 \geq 3(1-\delta)$, or equivalently $\delta \geq 1/3$.

- Now, consider any history of the form $h = (EE, \ldots, EE)$.
- Because the continuation play of s in the subgame following h is identical to s, we have already known that no one has a profitable one-shot deviation at h provided $\delta \geq 1/3$.
- Finally, consider any history h in which S is played at least once.
- Because the continuation play of s in the subgame following h is simply the repeated play of the stage Nash equilibrium after all continuation histories, we know it is a SPE in the subgame for all δ. Thus no one has profitable one-shot deviation at h for all δ.
- In sum, when $\delta \geq 1/3$, no one has profitable one-shot deviation after every history.
- Therefore, s is a SPE provided $\delta \geq 1/3$.
- The same idea of "stick and carrot" behavior.

Topics I: Repeated Games

Infinitely repeated games

- Even if the stage game has a unique Nash equilibrium, there is a SPE whose outcome path involves repeated play of non stage Nash equilibrium, provided δ is large enough.
- This kind of strategy is usually referred to as a **grim trigger strategy**: deviating from E is a trigger that causes the players to revert to a bad stage Nash equilibrium, resulting in a very grim future.
- This kind of strategy profile is widely used to support non stage Nash equilibrium behavior as a SPE in the infinitely repeated game.

Theorem 3.9

Consider an infinitely repeated game. Suppose $a^* = (a_1^*, \ldots, a_n^*)$ *is a stage Nash equilibrium. Assume that there is an action profile* $a = (a_1, \ldots, a_n)$ *in the stage game that gives every player strictly higher stage game payoff than* a^**. Then there exists* $\underline{\delta} \in (0, 1)$ *such that when* $\delta \geq \underline{\delta}$*, there is a subgame perfect equilibrium in which the outcome path is repeated play of* a*.*

Proof of Theorem 3.9.

Define $M \equiv \max_{i,a} u_i(a)$ to be the largest stage game payoff across all players and all action profiles. Consider the trigger strategy profile

$$\forall t \geq 1,\ s_i^t(h) = \begin{cases} a_i, & \text{if } h = \emptyset \text{ or } h = (a, \dots, a) \in H_{t-1}, \\ a_i^* & \text{otherwise.} \end{cases}$$

In the firs period, if player i plays a_i, given his opponents' strategy and his own future play, his payoff is $v_i(a)$. If player i plays an action a_i' different from a_i, his payoff is

$$(1-\delta)v_i(a_i', a_{-i}) + \delta v_i(a^*) \leq (1-\delta)M + \delta v_i(a^*).$$

Proof of Theorem 3.9 (Cont.)

Thus, player i does not have a profitable one-shot deviation in period $t = 1$ if

$$v_i(a) \geq (1-\delta)M + \delta v_i(a^*),$$

or equivalently

$$\delta \geq \delta_i = \frac{M - v_i(a)}{M - v_i(a^*)} \in [0, 1).$$

Let $\underline{\delta} \equiv \max_i \delta_i$. Then, when $\delta \in [\underline{d}, 1)$, no one has a profitable one-shot deviation in the first period.

Consider any history $h = (a, \ldots, a)$. Since players' continuation play after history h is identical to s itself, we know that no one has a profitable one-shot deviation provided $\delta \in [\underline{\delta}, 1)$.

Proof of Theorem 3.9 (Cont.)

Finally, consider any history $h \neq (a, \ldots, a)$. Because the continuation play of s is simply the repeated play of the stage Nash equilibrium a^* after every history, we know the continuation play is a SPE in the subgame. Thus no one has a profitable one-shot deviation at h.

In sum, as long as $\delta \in [\underline{\delta}, 1)$, no one has a profitable one-shot deviation, implying it is a SPE. $\square$

- Let's go back to the Cournot duopoly example.
- Suppose now that the two firms involve in a long-run relationship instead of a one-shot competition.
- They compete in every period $t = 1, 2, \ldots$, with discount factor $\delta \in (0, 1)$.
- Can they collude without entering an explicit contract, which is forbidden in most countries?

- Let $q_1^c = q_2^c = 22.5$ and $q_1^e = q_2^e = 30$.
- We have calculated that each firm's collusive payoff is 1012.5 and equilibrium payoff is 900.
- Consider the trigger strategy profile

$$\forall t \geq 1,\ s_i^t(h) = \begin{cases} q_i^c, & \text{if } h = \emptyset \text{ and } h = (q^c, \ldots, q^c), \\ q_i^e, & \text{otherwise.} \end{cases}$$

- If s is played, then the firms collude all the time.
- But the question is whether s is a SPE.

- After any history in which at least one firm has deviated from the collusive quantity, the two firms play a stage Nash equilibrium independent of histories. Thus, s is a Nash in such subgames for any δ.
- Consider period $t = 1$.
- Given the opponent's strategy and firm i's own strategy from period $t = 2$ on, firm i's total payoff from choosing $q_i = q_i^c$ in period $t = 1$ is 1012.5 and that from choosing $q_i \neq q_i^c$ is

$$(1-\delta)[P(q_i + q_{-i}^c) - 10)]q_i + \delta \times 900.$$

- Thus, if

$$1012.5 \geq \max_{q_i \geq 0} (1-\delta)[P(q_i + q^c_{-i}) - 10]q_i + \delta \times 900,$$

then firm i does not have a profitable one-shot deviation in the first period.

- Easy calculation shows that the above inequality is equivalent to

$$1012.5 \geq (1-\delta) \times 1139.0625 + \delta \times 900.$$

- Thus, when

$$\delta \geq \underline{\delta} \equiv 0.53,$$

firm i has no profitable one-shot deviation in the first period.

- In any period after the history in which firms have colluded previously, the continuation play of s is identical to s itself. Thus, no fir has a profitable one-shot deviation at such history, provided $\delta \geq \underline{\delta}$.
- Therefore, as long as $\delta \in [\underline{\delta}, 1)$, s is a SPE. In this SPE, the firms collude all the time.

Topics II: Strategic Bargaining

The alternating offer bargaining game

- Two players bargain over a pie of size 1.
- In the fist round:
  - Player 1 offers shares $(x, 1-x)$ for $x \in [0,1]$.
  - Player 2 decides to accept, causing the game to end with payoffs $v_1 = x$ and $v_2 = 1-x$, or reject, causing the game to move to the second round.
- In the second round:
  - Player 2 offers shares $(x, 1-x)$ for $x \in [0,1]$.
  - Player 1 decides to accept, causing the game to end with payoffs $v_1 = \delta x$ and $v_2 = \delta(1-x)$, or reject, causing the game to move to the next round.
- $\delta \in (0,1)$ is the discount factor: players are impatient / time is money.
- In the third round and beyond:
  - The game continues in the same way, where players alternate to offer following a rejection.
  - Each period has further discounting, so that if agreement $(x, 1-x)$ is reached in round t, payoffs are $v_1 = \delta^{t-1}x$ and $v_2 = \delta^{t-1}(1-x)$.

Topics II: Strategic Bargaining

The alternating offer bargaining game

- Assume the bargaining process can not go beyond T rounds.
- T can be finite, $T = 1, 2, 3, \ldots$. In this case, T is a "hard deadline". If an agreement is not reached in the first T rounds, the pie is destroyed and each player obtains 0.
- T can be infinite, $T = \infty$. In this case, the bargaining process can last indefinitely. If an agreement is never reached, the pie is destroyed and each player obtains 0.
- This is a game of perfect information.

Topics II: Strategic Bargaining

The alternating offer bargaining game

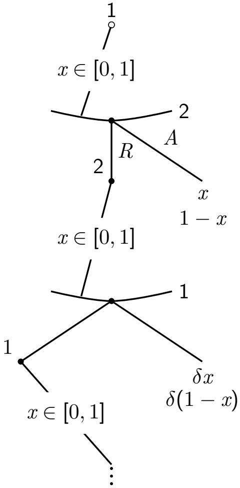


Figure 3.27: The alternating offer bargaining.

Topics II: Strategic Bargaining

One round of bargaining: the ultimatum game

- Assume $T = 1$, so there is only one round of bargaining.
- That is, if player 2 rejects player 1's offer, the game ends and each obtains 0.
- SPE by backward induction.
- If player 1 proposes $(x, 1-x)$ for $x \in [0, 1)$, the unique optimal choice for player 2 is to accept.
- If player 1 proposes $(1, 0)$, player 2 is indifferent between accepting and rejecting.
- Thus, two possibilities:
  - Player 2 accepts every proposal. The player 1's best response is to propose $(1, 0)$. This is a SPE.
  - Player 2 accepts all but $(1, 0)$ in which case he rejects. Player 1 has not best response. (Yes?)
- Unique SPE: player 1 proposes $(1, 0)$ and player 2 accepts every proposal.

- How about Nash? Many.
- Consider any $x \in [0, 1]$.
- Suppose
  - player 1 proposes $(x, 1 - x)$;
  - player 2 accepts $(x, 1 - x)$ and rejects every other proposal.
- It is a Nash. (Yes?)
- Incredible threat is involved.

Topics II: Strategic Bargaining

Finitely many rounds of bargaining

- Assume $T > 1$ but is finite.
- As an example, consider $T = 2$.

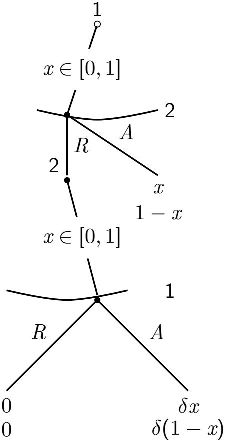


Figure 3.28: The alternating offer bargaining.

- Backward induction for SPE.
- We have already known that if player 2 rejects player 1's proposal, then 2 will get the whole pie in the second round, with payoff δ (because of discounting).
- In the first round, if player 1 proposes $(x, 1-x)$, then player 2
  - strictly prefers accepting if $1 - x > \delta$;
  - strictly prefers rejecting if $1 - x < \delta$;
  - is indifferent between accepting and rejecting if $1 - x = \delta$.

- There are two possibilities again.
- Consider first player 2 in the first round accepts any $(x, 1-x)$ such that $1 - x > \delta$ and rejects others.
- If player 1 proposes $(x, 1-x)$ in the first round, then
  - the offer will be accepted and resulting a payoff x to player 1 if $x < 1 - \delta$;
  - the offer will be rejected and resulting a payoff 0 to player 1 (in the second round) if $x \geq 1 - \delta$.
- Obviously, player 1 has no optimal choice.
- So this player 2's behavior can not be part of a SPE.

Topics II: Strategic Bargaining

Finitely many rounds of bargaining

- Consider then player 2 in the first round accepts any $(x, 1-x)$ such that $1-x \geq \delta$ and rejects others.
- If player 1 proposes $(x, 1-x)$ in the first round, then
  - the offer will be accepted and resulting a payoff x to player 1 if $x \leq 1-\delta$;
  - the offer will be rejected and resulting a payoff 0 to player 1 (in the second round) if $x > 1-\delta$.
- Thus player 1's optimal choice in the first round is $(1-\delta, \delta)$.
- Unique SPE:
  - player 1 in the first round proposes $(1-\delta, \delta)$ and accepts every proposal in the second round;
  - player 2 accepts $(x, 1-x)$ if $x \leq 1-\delta$ and rejects others in the first round. He always proposes $(0, 1)$ in the second round.
- Equilibrium payoff $(1-\delta, \delta)$. Agreement is reached in the first round.
- As $\delta \uparrow 1$, player 2 gets larger fraction of the pie.

Topics II: Strategic Bargaining

Finitely many rounds of bargaining

- How about $T = 3$?
- In the first round, player 1 proposes.
- If player 2 rejects, then they will play the unique SPE for the two-round bargaining game that we have just figured out.
- In that equilibrium, the payoffs are $(\delta^2, \delta(1-\delta))$ (one more round of discounting).
- Therefore, in the first round, player 2 must accept any proposal $(x, 1-x)$ with $1 - x \geq \delta(1-\delta)$ (can not reject when being indifferent by the same reason as above).
- Then, in the first round, player 1 proposes $(1 - \delta(1-\delta), \delta(1-\delta))$.
- Unique SPE (how to describe it?)

Topics II: Strategic Bargaining

Finitely many rounds of bargaining

- If $T = 4$, there is also a unique SPE.
- In this equilibrium, player 1 proposes $(1 - \delta(1 - \delta(1 - \delta)), \delta(1 - \delta(1 - \delta)))$.
- If $T = 5$, ...

Topics II: Strategic Bargaining

Rubinstein bargaining: the inifinite horizon bargaining

- How about $T = \infty$?
- This is usually referred to as Rubintein bargaining model.
- Backward induction does not apply due to infinite horizon.
- SPE?

Topics II: Strategic Bargaining

Rubinstein bargaining: the inifinite horizon bargaining

- The key step to derive a SPE is to observe the *stationary structure* of this game.
- Think about the whole game and a subgame starting in the 3rd round following player 1's rejection in the 2nd round.
- A simple examination reveals that these two games are the same (up to δ^2 discounting).
- More importantly, all subgames starting in odd periods following player 1's rejection in the previous round are the same.
- Similarly, all subgames starting in even periods following player 2's rejection in the previous round are the same.

Topics II: Strategic Bargaining

Rubinstein bargaining: the inifinite horizon bargaining

- With this observation, let's guess a *stationary equilibrium*.
  - player 1 always proposes $(x^*, 1-x^*)$ and 2 always proposes $(y^*, 1-y^*)$.;
  - player 1 always accepts an offer $x \geq y^*$ and rejects others;
  - player 2 always accepts an offer $1-x \geq 1-x^*$ and rejects others.

Topics II: Strategic Bargaining

Rubinstein bargaining: the inifinite horizon bargaining

- For 2 to accept $1-x^*$, $1-x^* \geq \delta(1-y^*)$.
- For 2 to reject any offer $1-x < 1-x^*$, $1-x \leq \delta(1-y^*)$.
- Hence 2 must be indifferent between accepting and rejecting $1-x^*$:

$$1-x^* = \delta(1-y^*).$$

- For 1 to accept y^*, $y^* \geq \delta x^*$.
- For 1 to reject any offer $x < y^*$, $x \leq \delta x^*$.
- Hence 1 must be indifferent between accepting and rejecting y^*:

$$y^* = \delta x^*.$$

- Therefore, we have

$$x^* = \frac{1}{1+\delta} \text{ and } y^* = \frac{\delta}{1+\delta}.$$

- This strategy profile is a SPE.

Topics II: Strategic Bargaining

Rubinstein bargaining: the inifinite horizon bargaining

- Any other SPE?
- Let G_1 be the set of all SPE payoffs in the subgames in which player 1 makes the initial proposal.
- Let $M_1 = \sup G_1$ and $m_1 = \inf G_1$.
- In words, M_1 (m_1) is the highest (lowest) SPE payoff to player 1 in the subgames in which player 1 makes the initial proposal.
- Similarly, let G_2 be the set of all SPE payoffs in the subgames in which player 2 makes the initial proposal.
- Let $M_2 = \sup G_2$ and $m_2 = \inf G_2$.

Topics II: Strategic Bargaining

Rubinstein bargaining: the inifinite horizon bargaining

- We now claim that $m_i \geq 1 - \delta M_j$ for $i \neq j$.
- For example, assume $i = 1$ and $j = 2$.
- Consider the subgame in which player 1 makes the initial proposal (e.g. the whole game).
- Observe that if player 2 rejects 1's proposal, then 2 can obtain at most δM_2.
- Thus, player 2 must be willing to accept any $1 - x > \delta M_2$.
- This implies that player 1 can get $1 - \delta M_2 - \varepsilon$ for $\varepsilon > 0$.
- Therefore, $m_1 \geq 1 - \delta M_2$.

- We also claim that $M_i \leq 1 - \delta_j m_j$ for $i \neq j$.
- For example, assume $i = 1$ and $j = 2$ again.
- Consider again the subgame in which player 1 makes the initial proposal.
- Observe that if player 2 rejects 1's proposal, then 2 can obtain at least δm_2.
- Thus player 2 must reject any $1 - x < \delta m_2$.
- If player 1's offer is accepted, 1 can at most get $1 - \delta m_2$.
- If player 1's offer is rejected, 1 can at most get $\delta(1 - m_2) \leq 1 - \delta m_2$.
- Therefore, $M_1 \leq 1 - \delta m_2$.

- Now,

$$m_i \geq 1 - \delta M_j \geq 1 - \delta(1 - \delta m_i) \Rightarrow m_i \geq \frac{1}{1+\delta},$$
$$M_i \leq 1 - \delta m_j \leq 1 - \delta(1 - \delta M_i) \Rightarrow M_i \leq \frac{1}{1+\delta}.$$

- Because $m_i \leq M_i$ by construction, we have

$$M_i = m_i = \frac{1}{1+\delta}.$$

- Up to this point, we have shown that in the subgame in which player i makes the initial proposal, the unique SPE *payoff* to player i is $1/(1+\delta)$.
- Note this is just the stationary SPE payoff (it must be).

Topics II: Strategic Bargaining

Rubinstein bargaining: the inifinite horizon bargaining

- It remains to show that the stationary SPE is the *unique SPE*.
- Consider the subgame in which 1 makes the initial proposal.
- In any SPE, if an agreement is not reached in the first round, 1 can get at most $1 - M_2 = \delta M_1 < M_1$.
- This implies that for 1 to get M_1, an agreement must be reached in the first round.
- Thus, 1 must propose $(M_1, 1 - M_1)$ in the first round and 2 accepts this proposal.
- For this behavior to be part of a SPE, 2 must reject any proposal that gives her a strictly lower payoff that $1 - M_1$; otherwise 1 can make a lower offer to 2.
- Moreover, if 1 makes a proposal that gives 2 a strictly higher payoff than $1 - M_1 = \delta M_2$, 2 must accept.

- The above arguments show that in any subgame in which 1 makes the initial proposal, any SPE prescribes that, in the initial round, 1 makes proposal $(M_1, 1 - M_1)$ and 2 accepts any offer higher than or equal to $1 - M_1$ and rejects others.
- Similar arguments apply to any SPE in the subgame in which 2 makes the initial proposal.
- We can show that 2 makes the proposal $(1 - M_2, M_2)$ and 1 accepts any offer higher than or equal to $1 - M_2$ and rejects others.
- Therefore, the stationary SPE is the unique SPE.

Topics II: Strategic Bargaining

Bonus: the pirate game

- n pirates, named $i = 1, \ldots, n$, decide how to split 100 coins.
- In the first round, $i = 1$ makes a proposal and all pirates vote about this proposal.
- If at least half of the pirates vote for this proposal, then it is accepted and the coins are split accordingly.
- Otherwise, the proposal is rejected and $i = 1$ is thrown into the sea. In this case, they move to the second round and it is $i = 2$'s turn to make a proposal.
- The game then proceed in a similar fashion until an agreement is achieved.
- A pirate's payoff is equal to the number of coins he obtains, or -1 if he is thrown into the sea.
- SPE?