Suggested Solutions to Game Theory Final Exam, Fall 2020

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1. Consider the following all-pay auction with two bidders. The rule of the auction is as follows. Bid must be nonnegative. The bidder with the higher bid wins the object. If there is a tie in bids, the object is randomly allocated to one of the bidders with equal probabilities. *Every bidder pays his own bid, regardless of whether he wins or not.* This is why it is called *all-pay* auction. We restrict attention to pure strategies.

   (a) **(10 points)** Assume it is common knowledge that bidder 1's valuation is $v_1 > 0$ and bidder 2's valuation is $v_2 > 0$. Does there exist a Nash equilibrium?

   Soln: Consider 1's payoff against b_2 under value $v_1 > 0$.

$$u_1(b_1, v_1; b_2) = \begin{cases} -b_1 & \text{if } b_1 < b_2; \\ \frac{1}{2}v_1 - b_1 & \text{if } b_1 = b_2; \\ v_1 - b_1 & \text{if } b_1 > b_2. \end{cases}$$

   Note that for any $b_1 \in (0, b_2)$, bidder 1 gets $-b_1 < 0$ derived by bidding 0. And for any $b_1 \in (b_2, +\infty)$, it is profitable for 1 to deviate to $\frac{b_1+b_2}{2}$. Finally, for $b_1 = b_2$, it is profitable foe 1 to deviate to slightly larger than b_2 since $v_1 > 0$. Hence, the best reply of 1 to 2's pure strategy exists only when $b_1 = 0$ is optimal, and so does 2's best reply. It indicates that the only possible pure Nash equilibrium is $b_1 = b_2 = 0$. However, $b_1 = 0$ is optimal only when $0 \geq u_1(b_2, v_1; b_2) = \frac{1}{2}v_1 - b_2$, implying that b_2 must be positive. Hence there is no pure Nash equilibrium.

   (b) **(5 points)** From now on, assume each bidder's valuation v_i is his private information. Assume v_i is drawn from the cumulative distribution function F_i over $[\underline{v}_i, \overline{v}_i]$ with a continuous and positive density f_i. Values are independently distributed across the bidders. Suppose bidder 2's bidding strategy is σ_2. Write down bidder 1's expected payoff if his valuation is v_1 and he bids b_1.

Soln:

$$u_1(b_1, v_1; \sigma_2) = -b_1 + v_1 \left(\mathbb{P}_2\{\sigma_2(v_2) < b_1\} + \frac{1}{2}\mathbb{P}_2\{\sigma_2(v_2) = b_1\} \right)$$

$$= -b_1 + v_1 \int_{\underline{v}_2}^{\overline{v}_2} \left(\mathbb{I}_{\{\sigma_2(v_2)<b_1\}} + \frac{1}{2}\mathbb{I}_{\{\sigma_2(v_2)=b_1\}} \right) f_2(v_2)\mathrm{d}v_2{}^1.$$

($\mathbb{I}_S$ is the indicator function of its subscript set S — it has value 1 for any member in the subscript set S and 0 otherwise.)

(c) **(5 points)** Assume σ_2 is strictly increasing and differentiable. Suppose b_1 maximizes bidder 1's expected payoff for valuation v_1. Write down the first order condition that b_1 must satisfy.

Soln: Since σ_2 is strictly increasing, we have $\{v_2 : \sigma_2(v_2) < b_1\} = \{v_2 : v_2 < \sigma_2^{-1}(b_1)\}$, and there is only one possible type v_2 s.t. $\sigma_2(v_2) = b_1$, which will be assigned with zero probability because F_2 is with a density point-wise. Hence we have

$$u_1(b_1, v_1; \sigma_2) = -b_1 + v_1 \int_{\underline{v}_2}^{\overline{v}_2} \mathbb{I}_{\{v_2<\sigma_2^{-1}(b_1)\}} f_2(v_2)\mathrm{d}v_2 = -b_1 + v_1 F_2(\sigma_2^{-1}(b_1)).$$

The optimality of b_1 requires the F.O.C. as follows.

$$\frac{\partial u_1}{\partial b_1} = -1 + v_1 f_2(\sigma_2^{-1}(b_1)) \frac{1}{\sigma_2'(\sigma_2^{-1}(b_1))} = 0.$$

(d) **(10 points)** Now assume $F_1 = F_2$ are uniform distributions over $[0, 1]$. Find a symmetric Bayesian Nash equilibrium.

Soln: Because $F_1 = F_2 \sim U[0, 1]$, immediately we have $f_1(v) = f_2(v) = 1$ for any $v \in [0, 1]$. Consider a BNE denoted as (σ, σ). Then the F.O.C. above requires that

$$-1 + v\frac{1}{\sigma'(\sigma^{-1}(b))} = 0.$$

Since $\sigma^{-1}(b) = v$, we have

$$v = \sigma'(v).$$

[1] Note that the term with $\mathbb{I}_{\{\sigma_2(v_2)=b_1\}}$ cannot be omitted here since it is possible that σ_2 assigns same action for a set of types with positive probability. Furthermore, in b) it has not been confirmed that σ is strictly increasing. Hence it is not proper to use notations such as σ_i^{-1} here.

Solve the equation and we know $\sigma(v)$ satisfying that $\sigma(v) = \frac{1}{2}v^2 + C$, where C is an undetermined constant.

If $v_1 = 0$, bid more than 0 will get a negative payoff, while bid 0 is always nonnegative. Hence $\sigma(0) = 0$ and therefore $C = 0$.

To verify that $\sigma_1 = \sigma_2 = \frac{1}{2}v^2$ is a BNE, consider 1's interim payoff.

$$u_1(b_1, v_1; \sigma(v_2)) = -b_1 + v_1 \int_0^1 \mathbb{I}_{\{v_2 < \sigma_2^{-1}(b_1)\}} \mathrm{d}v_2 = -b_1 + v_1\sqrt{2b_1}.$$

It is easy to check that $b_1 = \frac{1}{2}v_1^2$ maximizes $u_1(b_1, v_1; \sigma(v_2))$[2].

2. There is a coronavirus disease outbreak. There are two citizens, $i = 1$ and 2. Each of them observes his own symptom, which is a clue of whether he is infected or not, e.g., fever or no fever. After observing his symptom, agent i thinks that he is infected with probability $q_i \in [0, 1]$. Assume q_i is uniformly distributed over $[0, 1]$ and is i's private information. q_1 and q_2 are independent. The two citizens then simultaneously decide whether to visit a hospital. The cost of visiting the hospital is $c > 0$, e.g., transportation cost.

   There is only one slot for testing and treatment at the hospital. If only one citizen visits the hospital, he will be tested. In case the result is positive, he will be treated; otherwise, no treatment is needed. If two citizens both visit the hospital, a hospital congestion occurs. The hospital will randomly choose, with equal probability, only one citizen to test and treat (if needed). In this case, the unchosen citizen can not get treatment even if he is indeed infected. Of course, if a citizen does not visit the hospital, he will not get tested and treated. Suppose each citizen obtains a payoff 0 if he is not infected or he is infected and treated. Suppose also that each citizen obtains a payoff $-d$ if he is infected and not treated. Assume $d > c$.

   (a) **(5 points)** Suppose citizen 2's mixed strategy is σ_2. Assume citizen 1's type is $q_1 \in [0, 1]$. Write down citizen 1's expected payoff if he does not visit the hospital and if he visits the hospital, respectively.

   Soln: Denote the action of visiting hospital by V and not visiting hospital by N respectively. Then we have

$$u_1(N, q_1; \sigma_2) = -q_1 d;$$

[2]The procedure of verification must be finished since it is not sufficient to show the profile is an equilibrium by solving the F.O.C equation only.

and

$$u_1(V, q_1; \sigma_2) = -c - \frac{1}{2} q_1 \mathbb{P}_2\{\sigma_2(q_2) = V\} d = -c - \frac{1}{2} q_1 d \int_0^1 p_2(q_2) \mathrm{d}q_2,$$

where $p_2(q_2)$ is the probability that 2 choose V under type q_2.

(b) **(10 points)** Show that there is an essentially unique Bayesian Nash equilibrium of this game. Find it.

Soln: Compare the interim payoffs above and we know that the best reply of 1 to σ_2 is as follows.

$$BR_1(\sigma_2) = \begin{cases} V, & \text{if } q_1 > \frac{c}{d(1-\frac{1}{2}\mathbb{P}_2\{\sigma_2(q_2)=V\})}; \\ N, & \text{if } q_1 < \frac{c}{d(1-\frac{1}{2}\mathbb{P}_2\{\sigma_2(q_2)=V\})}. \end{cases}$$

The defined action in the case $q_1 = \frac{c}{d(1-\frac{1}{2}\mathbb{P}_2\{\sigma_2(q_2)=V\})}$ is indifferent to 1, hence the best reply above is essentially unique.

Because the best reply to any strategy is a cut-off strategy, the BNE must be a cut-off strategy profile since they are best reply to each other.

Note that $\mathbb{P}_1\{BR_1(\sigma_2)(q_1) = V\} = 1 - \frac{c}{d(1-\frac{1}{2}\mathbb{P}_2\{\sigma_2(q_2)=V\})}$, and meanwhile $\mathbb{P}_2\{BR_2(BR_1(\sigma_2))(q_2) = V\} = 1 - \frac{c}{d(1-\frac{1}{2}\mathbb{P}_1\{BR_1(\sigma_2)(q_1)=V\})}$. Since $(BR_1(\sigma_2), \sigma_2)$ is a BNE iff $\sigma_2 = BR_2(BR_1(\sigma_2))$ essentially, we know that σ_2 must satisfy that

$$\mathbb{P}_2\{\sigma_2(q_2) = V\} = 1 - \frac{c}{d(1 - \frac{1}{2}(1 - \frac{c}{d(1-\frac{1}{2}\mathbb{P}_2\{\sigma_2(q_2)=V\})})}.$$

Solve the equation and we have a unique solution[3] in $[0, 1]$,

$$\frac{c}{d(1 - \frac{1}{2}\mathbb{P}_2\{\sigma_2(q_2) = V\})} = \frac{\sqrt{1 + \frac{8c}{d}} - 1}{2}.$$

As a result, the game has a essentially unique BNE (σ_1, σ_2), where

$$\sigma_1(q) = \sigma_2(q) = \begin{cases} V, & \text{if } q > \frac{\sqrt{1+\frac{8c}{d}}-1}{2}; \\ N, & \text{if } q < \frac{\sqrt{1+\frac{8c}{d}}-1}{2}. \end{cases}$$

3. Players 1 and 2 share a joint project and interact over time. In every period, the value of the project depends on the effort that they each exert in that

[3] Although omitted here, it is necessary to check that here is no corner solution, such as the case where $\frac{\sqrt{1+\frac{8c}{d}}-1}{2} \geq 1$.

period. In particular, player i's payoff is $e_j^2 + e_j - e_i e_j$, where $e_i \geq 0$ is player i's effort and $e_j \geq 0$ is player j's effort. In every period, they decide their effort independently and simultaneously, and they both observe their past effort choices. Let $\delta \in (0,1)$ denote the common discount factor of the players. Their lifetime payoffs are the standard average discounted sum of their payoffs in every period. We restrict attention to pure strategies.

(a) **(5 points)** Show that in any Nash equilibrium of this repeated game, both players' payoffs must be nonnegative.

Soln: In any Nash equilibrium, there is a deviation that 1 (or 2, similarly) chooses $e_1 = 0$ [4] after any history. From the deviation 1 will get a lifetime payoff 0 because the deviation derives 0 in all periods. Hence, in any Nash equilibrium the lifetime payoff to 1 will be nonnegative, or the deviation above will be profitable.

(b) **(10 points)** Show that for every $\delta \in (0,1)$, there exists a subgame perfect equilibrium in which both players' payoffs are strictly positive.

Soln: Consider the following grim trigger strategy profile.[5]

$$s_1(h_t) = s_2(h_t) = \begin{cases} e > 0, & \text{if } h_t = (ee, ee, ee, \cdots, ee) \text{ or } \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $e_1 = e_2 = 0$ is a stage Nash equilibrium (and it is unique). Hence from one-shot deviation principle, we only need to check that the payoff from the path of the strategy profile above is no less than any one-shot deviation, which is

$$u_i(e,e) = e^2 + e - e^2 = e \geq (1-\delta)\left(\max_{e' \geq 0}\{e^2 + e - e'e\} + \frac{\delta}{1-\delta}0\right) = (1-\delta)(e^2+e).$$

Hence, the strategy profile above, which derives positive payoffs (e,e) for both player, is a SPE when $\delta \geq 1 - \frac{e}{e^2+e} = \frac{e^2}{e^2+e}$.

Note that $\lim_{e \to 0+0} \frac{e^2}{e^2+e} = 0$. Hence for any $\delta \in (0,1)$, there exists $\delta_e = \frac{e^2}{e^2+e} < \delta$, s.t. the strategy profile above is a SPE deriving strictly positive (e,e).

[4] Another alternative deviation is 1 chooses $e_1 = 1$ after any history.

[5] It is possible to show that for any $e_1 > 0, e_2 > 0$, a corresponding grim trigger strategy can be supported with sufficient large δ. And the lower bound of the δ is related to e_1 and e_2 and it will go through all $\delta \in (0,1)$ while e_1 and e_2 are going through $(0, +\infty)$.

Another interesting alternative is the grim trigger strategy profile with $((x,0),(0,x),\cdots)$ on path, where $x > 0$. It can be supported by any $\delta \in (0,1)$.

(c) **(10 points)** Construct a strategy profile in which (i) $e_1 = 1$ and $e_2 = 3$ is played in the first period, and (ii) it is a SPE when δ is sufficiently large.

Soln: Consider the following strategy profile.[6]

$$s_1(\varnothing) = 1, s_2(\varnothing) = 3;$$

$$s_1(h_t) = s_2(h_t) = \begin{cases} 1, & \text{if } t \geq 1, h_t = ((1,3),(1,1),\cdots,(1,1)); \\ 0, & \text{otherwise.} \end{cases}$$

As a grim trigger strategy, it has been shown by b) that any subgame starting after the first period is subgame perfect when $\delta \leq \frac{1}{2}$. We only need to show that there is no profitable deviation in the first period, which means

$$u_1 = (1-\delta)\left[(3^2+3-3)+\delta+\delta^2+\cdots\right] \geq (1-\delta)\left[(3^2+3+0)+0\right];$$

$$u_2 = (1-\delta)\left[(1^2+1-3)+\delta+\delta^2+\cdots\right] \geq (1-\delta)\left[(1^2+1+0)+0\right].$$

Obviously the inequities above hold when δ is sufficiently large (and to be specific, $\delta \geq \frac{3}{4}$).

4. Consider the following infinite horizon bargaining game among three players, $i = 1, 2, 3$. The total amount of value to be divided among the players is 1. The game begins in period $t = 1$. In odd periods $t = 1, 3, 5, \ldots$, player 1 offers a proposal $(x, 1-x)$, where x is the amount that player 1 will receive and $1-x$ is the amount that player 3 will receive. Player 3 then decides whether to accept or not. If he accepts, the game ends and the offer is implemented. If he rejects, the game continues into the next period. Note that in odd periods, player 2 is inactive. In even periods $t = 2, 4, 6, \ldots$, player 2 proposes $(x, 1-x)$ for him and player 3. It is again player 3 to decide whether to accept or reject. Hence, player 1 is inactive in even periods. If the game ends without an offer being accepted, all players get zero. Let $\delta \in (0,1)$ be the common discount factor of the three players.

   The following questions help you find the unique equilibrium of this game step by step. For later steps, you can use the results of earlier steps even if you can not show them.

[6] There are many other alternatives satisfying the requirements above. A popular one is the grim trigger with $((1,3),(3,1),\cdots)$ on path.

It is allowed to show the $\delta \to 1$ case only without specifying the lower bound.

(a) **(5 points)** Show that in any subgame perfect equilibrium, player 1's payoff must be strictly positive.

Soln: If the game goes to the second period, the total amount of value will discounted to be $\delta < 1$. Hence player 3 cannot get more than δ if the game goes to the second period. Then, if 1 proposes $(x, 1-x)$ where $1 - x > \delta$, it will be accepted by 3 immediately, indicating that 1's payoff from any SPE will be no less than any $x < 1-\delta$, which is strictly positive.

(b) **(5 points)** Show that in any subgame perfect equilibrium, agreement must be achieved, and it must be achieved in some odd period.

Soln: From above we know that in any pure SPE, 1 will get strict positive payoff. Note that in any pure strategy profile of this game, there will be at most one agreement happened on the path. If there is no agreement on the path of the strategy profile, or the agreement is achieved at an even period, the payoff to 1 will be 0, contradicting to the previous result. Hence, for any SPE, the game will end with an agreement at an odd period. Note that it implies that 2 will get 0 in any SPE.

(c) **(5 points)** Show that in any subgame perfect equilibrium, agreement must be achieved in period $t = 1$.

Soln: Consider the subgame starting at the second period. Similarly we can find that, in any SPE of this subgame, 2's payoff must be positive (see proof in a)), and hence an agreement must be achieved in one of the even periods of the origin game (see proof in b)). Therefore, if the game goes into the second period, 1 will get 0 since the game will end in an even period, contradicting that 1 gets positive payoff. Hence, the agreement must be reached at the first period.[7]

(d) **(10 points)** Show that player 3's payoff in any subgame perfect equilibrium can not be positive.

[7] A common mistake is to claim that since the realized proposal $(x, 1-x)$ in period $2k+1 > 1$ will only brings $(\delta^{2k+1}x, \delta^{2k+1}(1-x))$ to player 1 and 3, it is possible for 1 to propose the same proposal before (when $t = 1$ or $t = 2k-1$) and 3 will accept it. It is not true since in a SPE, we have no idea what is the continuation play after 3 rejects $(x, 1-x)$ in a previous period. It might not be the same result on path! The statement in fact clarifies that $(x, 1-x)$ must be rejected before for some reason, contradicting to 3's acceptance as claimed. Same mistake rises in a) and b) either. The key to understand the issue is to notice that for different histories, especially rejections after different proposals, there might be **different** continuation plays.

Soln: [8] Note that the origin game and its subgame starting at the second period are just the same to the player 3 — a profile $(\sigma_1, \sigma_2, \sigma_3)$ is a SPE in the origin game iff $(\sigma_2, \sigma_1, \sigma_3)$ is a SPE in the subgame starting at the second period.

Suppose that the supremum of 3's payoff deriving from the SPE is M. Then immediately we know that the supremum of 3's payoff deriving from the SPE in the subgame starting at the second period is also M. Since in any SPE the agreement is achieved at the first period, we know that the infimum of 1's payoff is $1 - M$.

Since from the second period 3 will get at most δM, if 1 proposes $(x, 1-x)$ where $1 - x$ is more than δM, it will be accepted by 3. Hence, 1's payoff is no less than $1 - \delta M$ since any $x > 1 - \delta M$ will be accepted. Hence we have $1 - M \geq 1 - \delta M$, which means M could only be 0.

As a result, in any SPE, the payoff to 3 must be 0.[9]

(e) **(5 points)** Show that there is a unique subgame perfect equilibrium and find it.

Soln: From the results above, we know that a pure SPE must satisfy that i) the agreement is reached in the first period, and ii) the 3 gets 0 from the SPE. Then in the first period, it must be 1 proposes $(1, 0)$ and 3 accepts it. Same logic applies to all the subgames and we can characterize the unique pure SPE as follows: **player 1 and 2 always propose $(1, 0)$, and player 3 accepts any proposal.** [10]

[8] The result from a), b) and c) only implies that the SPE of the game is with immediate acceptance at $(1, 0)$. It does not mean that it can be treated as a one-period ultimatum game. They are utterly different.

It also not proper to claim that after 2's deviation 3 will deviate. It might be true, but it is meaningless here because we only care about unilateral deviation in Nash equilibrium (and hence SPE). Furthermore, 2's deviation may not be credible.

[9] Another way to show the same result is to study the relationship of the infimum of 1's expected payoff at $t = 1$ and 2's expected payoff at $t = 2$. No other reasonable method has been discovered for now.

[10] From a) to d) we only know that the SPE is with immediate acceptance with proposal $(1, 0)$. It is necessary to apply the result to all the subgames to finish proof.