Suggested Solutions to Game Theory Midterm Exam, Fall 2024

Zheming Liang and Yushuai Zhang

1. There are $n \geq 2$ workers in a joint project. Each i of them independently decides how much effort $e_i \geq 0$ to contribute to this project, incurring individual costs ce_i^2 where $c > 0$. The total output given their efforts is $y = n \max\{e_1, \ldots, e_n\}$, and it is shared equally amount the workers. Each worker's payoff is the amount of output he shares minus his cost of effort. We restrict attention to pure strategies unless otherwise specified.

   (a) **(10 points)** Does any worker have a strictly dominated strategy? If yes, find them all. If no, explain why.

   Soln: You can derive the results analytically. Here, we take a more intuitive approach. Figure 1 below illustrates worker i's payoff function given various levels of $\max_{j \neq i} e_j$. From these graphs, we see that for any $\max_{j \neq i} e_j \geq 0$, $u_i(\,\cdot\,, e_{-i})$ is strictly decreasing over $[\frac{1}{2c}, \infty)$. This means that $e_i \in (\frac{1}{2c}, \infty)$ is strictly dominated by $e_i' = \frac{1}{2c}$. Moreover, for any $0 \leq e_i < e_i' \leq \frac{1}{2c}$, we see from panels (a) and (c) that $u_i(e_i, e_{-i}) < u_i(e_i', e_{-i})$ if $\max_{j \neq i} e_j = 0$, while $u_i(e_i, e_{-i}) > u_i(e_i', e_{-i})$ if $\max_{j \neq i} e_j = \frac{1}{2c}$. This observation means that neither that e_i is strictly dominated by e_i' nor that e_i' is strictly dominated by e_i. Since e_i and e_i' are arbitrary, we know that no effort in $[0, \frac{1}{2c}]$ is strictly dominated.

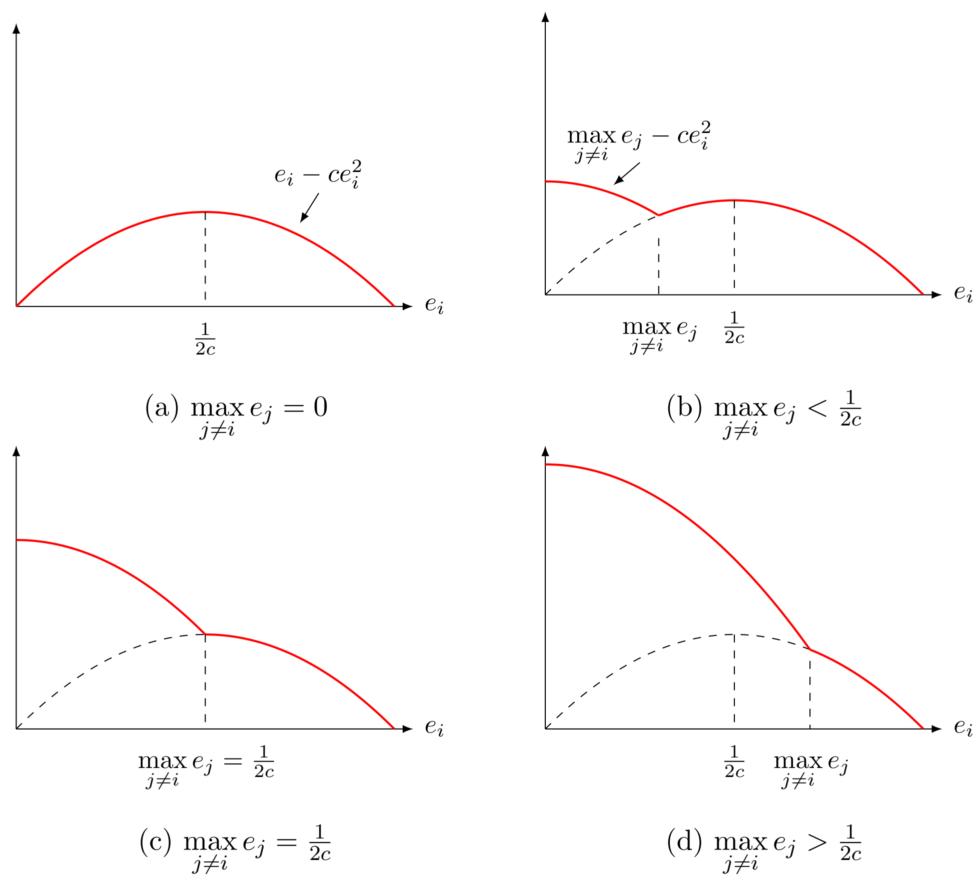
   (b) **(10 points)** Given e_{-i}, calculate i's best reply.

   Soln: i's best reply is given by

$$BR_i(e_{-i}) = \begin{cases} \frac{1}{2c}, & \text{if} \max\limits_{j \neq i} e_j < \frac{1}{4c}, \\ \{0, \frac{1}{2c}\}, & \text{if} \max\limits_{j \neq i} e_j = \frac{1}{4c}, \\ 0, & \text{if} \max\limits_{j \neq i} e_j > \frac{1}{4c}. \end{cases}$$

   (c) **(10 points)** Find all pure strategy Nash equilibria.

   Soln: Clearly, there is no equilibrium in which more than one agent exert positive effort; otherwise, someone can profitably deviate to effort 0. Combining this observation and the previous question, we know that equilibrium e^* must take the following form: there exists i such that $e_i^* = \frac{1}{2c}$



(a) $\max_{j \neq i} e_j = 0$

(b) $\max_{j \neq i} e_j < \frac{1}{2c}$

(c) $\max_{j \neq i} e_j = \frac{1}{2c}$

(d) $\max_{j \neq i} e_j > \frac{1}{2c}$

Figure 1: Payoff functions for worker i

and $e_j^* = 0$ for $j \neq i$. In other words, all pure strategy Nash equilibria are $(\frac{1}{2c}, 0, 0, \ldots, 0)$, $(0, \frac{1}{2c}, 0, \ldots, 0)$, ..., $(0, 0, 0, \ldots, 0, \frac{1}{2c})$.

(d) **(10 points)** For this question only, assume that each worker can only choose effort 0 or effort $\frac{1}{3c}$. Find a Nash equilibrium in mixed strategies.

Soln: We consider an equilibrium in symmetric strategies. Suppose every agent but i mixes between 0 with probability p and $\frac{1}{3c}$ with probability $1-p$. From i's point of view, the probability that no other agent exerts $\frac{1}{3c}$ is p^{n-1}. If he exerts 0, his payoff is

$$p^{n-1} \times 0 + (1 - p^{n-1}) \times \frac{1}{3c} = (1 - p^{n-1})\frac{1}{3c}.$$

If he exerts $\frac{1}{3c}$, his payoff is $\frac{1}{3c} - c \times \frac{1}{9c^2} = \frac{2}{9c}$. For i to be willing to mix, he must be indifferent:

$$(1 - p^{n-1})\frac{1}{3c} = \frac{2}{9c},$$

which implies

$$p = \frac{1}{3^{n-1}}.$$

(e) **(10 points)** Suppose now that $n = 3$ and the workers choose efforts sequentially in the order of worker 1, 2 and 3. Later workers observe all previous choices. Find a subgame perfect equilibrium and a Nash equilibrium that is not subgame perfect.

Soln: From our analysis of the static game, it is easy to see that the strategy profile (s_1, s_2, s_3) where

$$s_1 = 0, \; s_2(e_1) = 0, \; \forall e_1 \geq 0, \; \text{and} \; s_3(e_1, e_2) = \begin{cases} 0, & \text{if } \max\{e_1, e_2\} > \frac{1}{4c}, \\ \frac{1}{2c}, & \text{otherwise}, \end{cases}$$

is a subgame perfect equilibrium. There are many other subgame perfect equilibria. Modifying this strategy profile a little as

$$s_1' = 0, \; s_2'(e_1) = 0, \; \forall e_1 \geq 0, \; \text{and} \; s_3'(e_1, e_2) = \frac{1}{2c},$$

makes (s_1', s_2', s_3') a Nash equilibrium but not subgame perfect.

2. Suppose the following prisoners' dilemma is repeatedly played $1 < T < \infty$ times. The payoffs to the two players are simply the sum of their stage game payoffs. For simplicity, we restrict attention to pure strategies.

	E	S
E	$2,2$	$-1,3$
S	$3,-1$	$0,0$

(a) **(10 points)** Show that in any Nash equilibrium of this repeated game, the payoff to every player must be non-negative.

Soln: Let s^* be an arbitrary Nash equilibrium. Consider player 1's deviation which plays S after every history, i.e.,

$$s_1(h) \equiv S, \ \forall h \in \cup_{t=0}^{T-1} H_t.$$

Let h^T be the unique outcome path under strategy profile (s_1, s_2^*). It must take the form of $h^T = ((S, a_2^1), (S, a_2^2), \ldots, (S, a_2^T))$. Let U_1 be player 1's repeated game payoff function. We must have

$$U_1(s_1, s_2^*) = \sum_{t=1}^{T} u_1(S, a_2^t) \geq \sum_{t=1}^{T} 0 = 0.$$

Since s^* is a Nash equilibrium, we have $U_1(s^*) \geq U_1(s_1, s_2^*) \geq 0$.

(b) **(10 points)** Suppose $T = 2$. Is there a Nash equilibrium in which EE is played at least once on the path of play?

Soln: No. We show this by contradiction. Suppose s^* is a Nash equilibrium in which EE is played at least once on the path of play.

Let $h^1 \in \{E, S\}^2$ be the outcome path under s^* after the first period. For all $a_i \in \{E, S\}$, consider i's deviation s_i defined as $s_i(h) = s_i^*(h)$ if $h \neq h^1$ and $s_i(h^1) = a_i$. We have

$$\begin{aligned} U_i(s^*) &= u_i(h^1) + u_i(s_i(h^1), s_{-i}(h^1)), \\ U_i(s_i, s_{-i}^*) &= u_i(h^1) + u_i(a_i, s_{-i}(h^1)). \end{aligned}$$

Since s^* is a Nash equilibrium, we have $U_i(s^*) \geq U_i(s_i, s_{-i}^*)$, or equivalently

$$u_i(s_i^*(h^1), s_{-i}^*(h^1)) \geq u_i(a_i, s_{-i}^*(h^1)).$$

Note that the above inequality holds for all i and a_i. This implies that $s^*(h^1)$ must be a stage game Nash equilibrium, that is, $s^*(h^1) = SS$, which, by the assumption that EE is played at least once, in turn implies that $h^1 = EE$ or equivalently $s^*(\emptyset) = EE$. Now, consider player 1's deviation

$s_1(h) = S$ for all h. We have

$$U_1(s_1, s_2^*) = u_1(S,E) + u_1(S, s_2^*(SE)) \geq 3 + 0 = 3$$
$$> 2 = u_1(E,E) + u_1(S,S) = U_1(s^*).$$

This means that s_1 is a profitable deviation for player 1, contradicting the assumption that s^* is a Nash equilibrium.

Therefore, there is no Nash equilibrium in which EE is played on the path.

3. Consider the following matching pennies between players 1 and 2. Player 1 first chooses between H and T. Player 2 initially does not know player 1's choice, but he has the option to gather information about 1's choice. In particular, player 2 can choose either to pay a cost $c > 0$ to observe 1's choice, or not pay such cost and remain ignorant about 1's choice. In any case, player 2 then must choose between H and T. The payoffs to the players from the matching pennies are as usual: if they choose the same action, player 1 wins and obtains 1, while player 2 loses and obtains -1; if they choose different actions, player 1 loses and obtains -1, while player 2 wins and obtains 1. Of course, player 2's total payoff should be his payoff from the matching pennies minus c if he chooses to pay such observation cost.

   (a) **(5 points)** Draw a game tree for this game.

   Soln: See Figure 2. This is not the unique way to draw the game tree.

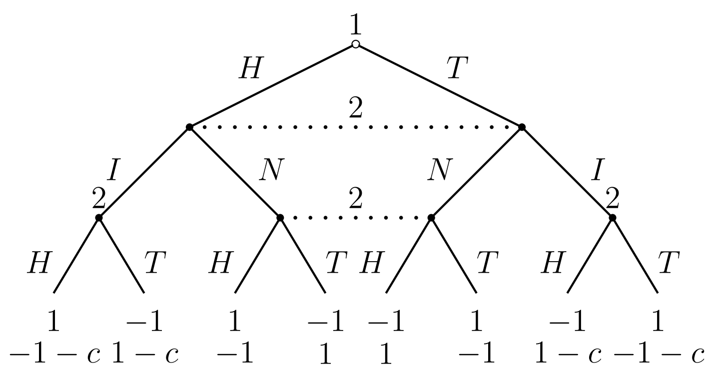


Figure 2: The game tree for Question 3

   (b) **(5 points)** Show that there is no subgame perfect equilibrium in pure strategies.

   Soln: Suppose, by contradiction, that we have a Nash equilibrium in which player 1 plays H. Then, as a best response, player 2 would not gather information, and would play T directly. But given player 2's this strategy, player 1 has an incentive to deviate to T, a contradiction. Hence, there is no Nash equilibrium in which player 1 plays H for sure. Similarly, there is no Nash equilibrium in which player 1 plays T for sure. Therefore, there is no Nash equilibrium in pure strategies; let alone subgame perfect equilibrium in pure strategies.

(c) **(10 points)** Assume $c > 1$. Show that there is a unique subgame perfect equilibrium and find it.

Soln: From now on, denote by $a_2 a_2' a_2'' a_2'''$ a pure strategy for player 2, where $a_2 \in \{I, N\}$ is 2's decision on whether to gather information or not. The rest $a_2', a_2'', a_2''' \in \{H, T\}$ respectively denote 2's action at information sets HI, $\{HN, TN\}$, and TI respectively. In other words, a_2' (resp. a_2''') is 2's choice of H or T after gathering information and observing player 1's H (resp. T); while a_2'' is 2's choice without observing player 1's choice.

Consider 2's strategy s_2 of the form $I a_2' a_2'' a_2'''$. We have

$$U_2(s_1, s_2) \leq 1 - c < 0 = U_2(s_1, \frac{1}{2} \circ N a_2' H a_2''' + \frac{1}{2} \circ N a_2' T a_2'''), \ \forall s_1.$$

This implies that such s_2 is strictly dominated by $\frac{1}{2} \circ N a_2' H a_2''' + \frac{1}{2} \circ N a_2' T a_2'''$. Thus, in any Nash equilibrium (σ_1, σ_2) of the whole game, σ_2 must put zero probability over such s_2. Moreover, if in addiction (σ_1, σ_2) is subgame perfect, we know that 2 must choose T after HI and H after TI. In other words, σ_2 only puts positive probabilities on either $NTHH$ or $NTTH$. But then, for (σ_1, σ_2) to be a Nash equilibrium, standard matching pennies game tells us that we must have $\sigma_1 = \frac{1}{2} \circ H + \frac{1}{2} \circ T$ and $\sigma_2 = \frac{1}{2} \circ NTHH + \frac{1}{2} \circ NTTH$. This is the unique subgame perfect equilibrium. In behavioral strategies, σ_2 means that 2 does not gather information for sure and mixes between H and T with equal probabilities afterwards; if he observes H he chooses T and if he observes T he chooses H.

(d) **(10 points)** Assume $c < 1$. Show that in any subgame perfect equilibrium, player 2 must gather information. Find a subgame perfect equilibrium.

Soln: Suppose (σ_1, σ_2) is a subgame perfect equilibrium. From the previous analysis, we have already known that if $\sigma_2(a_2 a_2' a_2'' a_2''') > 0$, it must take the form of $a_2 T a_2'' H$. Note that for any a_2'',

$$U_2(\sigma_1, IT a_2'' H) = 1 - c > 0.$$

We now argue that we must have $\sigma_2(NTHH) = 0$. Suppose, by contradiction, $\sigma_2(NTHH) > 0$, we must have

$$U_2(\sigma_1, NTHH) = \sigma_1(H)u_2(H,H) + \sigma_1(T)u_2(T,H) \geq 1 - c > 0.$$

This in turn implies that $U_2(\sigma_1, NTTH) < 0$, and thus $\sigma_2(NTTH) = 0$. Now consider player 1. We have

$$\begin{aligned} U_1(H, \sigma_2) &= (1 - \sigma_2(NTHH)) \times (-1) + \sigma_2(NTHH) \times 1, \\ U_1(T, \sigma_2) &= (1 - \sigma_2(NTHH)) \times (-1) + \sigma_2(NTHH) \times (-1), \end{aligned}$$

which implies that player 1 strictly prefers H to T, or equivalently $\sigma_1(H) = 1$. However, from 3b, we have already known that there is no subgame perfect equilibrium in which player 1 plays a pure strategy. Hence, this is impossible. Similar analysis shows that we must have $\sigma_2(NTTH) = 0$. Therefore, we have shown that in any subgame perfect equilibrium, player 2 must gather information for sure.

Finally, consider the strategy profile $\sigma_1 = \frac{1}{2} \circ H + \frac{1}{2} \circ T$ and $\sigma_2(ITHH) = 1$. Given σ_2, player 1 is obviously best responding because he is indifferent between H and T. Given σ_1, it is easy to see that if player 2 deviates at any $Ia_2'a_2''a_2'''$, his payoff can not be higher. If he deviates to $Na_2'a_2''a_2'''$, his payoff is

$$U_2(\sigma_1, Na_2'a_2''a_2''') = \frac{1}{2}u_2(H, a_2'') + \frac{1}{2}u_2(T, a_2'') = 0 < 1 - c.$$

Hence, such deviation is not profitable neither. Therefore, the proposed strategy profile is a subgame perfect equilibrium.