Game Theory Final Exam, Fall 2019

Jan 6, 2020

Important: This is a closed-book exam. No books, lecture notes or calculators are permitted. You have **120 minutes** to complete the exam. Answer all questions. Write legibly. Good luck and enjoy your winter break!

1. Consider the following situation between a worker and a firm. The worker works for the firm and decides how much effort $e \geq 0$ to exert. The cost of exerting effort e is $\frac{1}{2}e^2$ and the output is e. The firm decides how much wage $w \geq 0$ to pay to the worker. The payoffs to the worker and the firm are thus

$$v_1(e, w) = w - \frac{1}{2}e^2,$$
$$v_2(e, w) = e - w,$$

respectively. The decisions of the worker and the firm are made simultaneously.

(a) **(5 points)** Suppose this is a one-shot relationship between the worker and the firm. Find out the unique Nash equilibrium.

Soln: The worker is player 1 and the firm is player 2. It's easy to find that $(e^*, w^*) = (0, 0)$ is the unique Nash equilibrium. In fact, $e^* = 0$ is the strictly dominant strategy for the worker and $w^* = 0$ is the strictly dominant strategy for the firm.

(b) **(10 points)** Suppose the worker and the firm interact in every period $t = 1, 2, \ldots$ with common discount factor $\delta \in (0, 1)$. In every period, both observe what has happened in the previous periods. For every $e \in [0, 2)$, show that when δ is sufficiently large, there is a subgame perfect equilibrium in which the worker exerts effort e in every period.

Soln: For $e = 0$, we know the strategy profile in which $e = 0$ and $w = 0$ is played after every history is a subgame perfect equilibrium for any $\delta \in (0, 1)$. This is because $(0, 0)$ is a stage Nash equilibrium.

Now, Consider any $e \in (0, 2)$. Pick an arbitrary $w \in (\frac{1}{2}e^2, e)$. Consider the following trigger strategy profile:

$$s_1^1 = e \text{ and for } t \geq 2,\ s_1^t(h^{t-1}) = \begin{cases} e, & \text{if } h^{t-1} = (ew, ew, \ldots, ew), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$s_2^1 = w, \text{ and for } t \geq 2, \; s_2^t(h^{t-1}) = \begin{cases} w, & \text{if } h^{t-1} = (ew, ew, \ldots, ew), \\ 0, & \text{otherwise.} \end{cases}$$

In words, the worker exerts e and the firm pays w as long as no one deviates. If a deviation occurs, they switch back to the stage Nash permanently.

For this strategy profile to be a subgame perfect equilibrium, on one should have one-shot profitable deviation after any history. At a history in which a deviation has occurred previously, no one has a one-shot profitable deviation for any $\delta \in (0,1)$ since the continuation play is simply repeated play of stage Nash for any continuation history. For a history of the form $h = \emptyset$ or $h = (ew, ew, \ldots, ew)$, the relevant incentive constraints are

$$w - \frac{1}{2}e^2 \geq \max_{e \geq 0}(1-\delta)(w - \frac{1}{2}e^2) + \delta \times 0 = (1-\delta)w,$$

and

$$e - w \geq \max_{w \geq 0}(1-\delta)(e - w) + \delta \times 0 = (1-\delta)e.$$

It is easy to see that when $\delta > \underline{\delta} = \max\{\frac{e^2}{2w}, \frac{w}{2}\} \in (0,1)$, the trigger strategy profile is a subgame perfect equilibrium.

(c) **(10 points)** In the environment of 1b, for every $e > 2$, show that there is no subgame perfect equilibrium in which the worker exerts $e > 2$ in every period, regardless of what δ is.

Soln: It suffices to show such strategy profile is not a Nash equilibrium for any δ. To see this, consider any such strategy profile. If $e > 2$ is repeated played on the path, then in every period on the path of play, the sum of the worker and the firm's payoffs is

$$w_t - \frac{1}{2}e^2 + e - w_t = e - \frac{1}{2}e^2,$$

where w_t is the wage paid by the firm in period t on the path of play. This implies that the sum of the worker and the firm's total payoffs is also $e - \frac{1}{2}e^2 < 0$. This implies that at least one player obtains negative total payoff from the repeated game. If the worker obtains negative total payoff, he can deviate to the strategy s_1' which plays $e = 0$ *after every history.* By doing this, he can guarantee at least 0. If the firm obtains negative total payoff, he can deviate to the strategy s_2' which pays $w = 0$ *after every history.* By doing this, he can guarantee at least 0 as well.

2. **(15 points)** Consider a two player bargaining game that can last infinitely many periods. There are two players who have to split one dollar. Let δ be their common discount factor. Suppose player 1 proposes in periods $t = 1, 2, 4, 5, 7, 8, \ldots$ in which case player 2 decides whether to accept or not. Player 2 proposes in periods $t = 3, 6, 9, \ldots$ in which case player 1 decides whether to accept or not. Find a subgame perfect equilibrium of this game. What is the outcome of your equilibrium?

   Soln: Let's guess a stationary equilibrium. Consider the following strategy profile. Player 1 always proposes $(x^*, 1 - x^*)$ in periods $t = 1, 4, 7, \ldots$ and proposes $(y^*, 1-y^*)$ in periods $t = 2, 5, 8, \ldots$. In periods $t = 3, 6, 9, \ldots$, he always accepts a proposal $(z, 1 - z)$ with $z \geq z^*$ and rejects otherwise. Player 2 always proposes $(z^*, 1 - z^*)$ in periods $t = 3, 6, 9, \ldots$. In periods $t = 1, 4, 7, \ldots$, she always accepts a proposal $(x, 1-x)$ with $1-x \geq 1-x^*$ and rejects otherwise. In periods $t = 2, 5, 8, \ldots$, she always accepts a proposal $(y, 1-y)$ with $1-y \geq 1-y^*$ and rejects otherwise.

   We proceed to see that this strategy profile is a subgame perfect equilibrium for some x^*, y^* and z^*. Consider player 2's acceptance decision in periods $t = 1, 4, 7, \ldots$. She is willing to reject any proposal $(x, 1-x)$ with $1-x < 1-x^*$ because

$$1 - x \leq \delta(1 - y^*).$$

   Letting $x \downarrow x^*$, we have

$$1 - x^* \leq \delta(1 - y^*).$$

   She is willing to accept $(x^*, 1 - x^*)$ because

$$1 - x^* \geq \delta(1 - y^*).$$

   Therefore, we have

$$1 - x^* = \delta(1 - y^*). \tag{1}$$

   Similarly, analyzing 2's acceptance behavior in periods $t = 2, 5, 8, \ldots$, we can obtain

$$1 - y^* = \delta(1 - z^*). \tag{2}$$

   Consider player 1's acceptance decision in periods $t = 3, 6, 9, \ldots$. He is willing to reject any proposal $(z, 1 - z)$ with $z < z^*$ because

$$z \leq \delta x^*.$$

Letting $z \uparrow z^*$, we have

$$z^* \leq \delta x^*.$$

He is willing to accept $(z^*, 1 - z^*)$ because

$$z^* \geq \delta x^*.$$

Therefore, we have

$$z^* = \delta x^*. \tag{3}$$

Combining equations (1) - (3), we obtain

$$\begin{aligned} x^* &= \frac{1+\delta}{1+\delta+\delta^2}, \\ y^* &= \frac{1+\delta^2}{1+\delta+\delta^2}, \\ z* &= \frac{\delta+\delta^2}{1+\delta+\delta^2}. \end{aligned}$$

It is then easy to verify that the stationary strategy profile we constructed above is indeed a subgame perfect equilibrium.

3. **(20 points)** The drunk man decides whether to drive home or not. If he does not drive, his payoff is 0. If he drives, his payoff then depends on whether he is caught by the police officer or not. If he is not caught, his payoff is u; otherwise, his payoff is -1. The police officer decides whether to patrol the street or not. If she does not patrol, her payoff is 0. If she patrols, she pays a cost c and obtains a benefit that depends on whether she catches the driving drunk man. If she does not catches the drunk man, her benefit is 0; otherwise it is 1. Assume u is the drunk man's private information and c is the police officer's private information. Assume u and c are independently and uniformly distributed over $[0, 1]$. Suppose the drunk man is caught for sure as long as he drives and the police officer patrols. Find a Bayesian Nash equilibrium of this game.

   Soln: A strategy for the drunk man is a mapping $\sigma_1 : [0, 1] \to [0, 1]$. For type $u \in [0, 1]$, $\sigma_1(u)$ is the probability of driving. A strategy for the police officer is a mapping $\sigma_2 : [0, 1] \to [0, 1]$. For type $c \in [0, 1]$, $\sigma_2(c)$ is the probability of patrolling.

   Suppose (σ_1^*, σ_2^*) is a Bayesian Nash equilibrium. In equilibrium, the total probability that the drunk man drives is

$$p_1^* = \int_0^1 \sigma_1^*(u)\mathrm{d}u,$$

and the total probability that the police officer patrols is

$$p_2^* = \int_0^1 \sigma_2^*(c)\mathrm{d}c.$$

Consider the drunk man of type u. If he drives, his expected payoff is $(1-p_2^*)u - p_2^*$. If he does not drive, his payoff is 0. Thus, he drives if $(1-p_2^*)u - p_2^* > 0$, or equivalently $u > p_2^*/(1-p_2^*)$. He does not drive if $u < p_2^*/(1-p_2^*)$. Since σ_1^* is a best reply to σ_2^*, we know

$$\sigma_1^*(u) = \begin{cases} 1, & \text{if } u > \frac{p_2^*}{1-p_2^*}, \\ 0, & \text{if } u < \frac{p_2^*}{1-p_2^*}. \end{cases}$$

This implies

$$p_1^* = \int_{\min\{p_2^*/(1-p_2^*),1\}}^1 \mathrm{d}u = 1 - \min\{p_2^*/(1-p_2^*), 1\}. \tag{4}$$

Consider the police officer of type $c \in [0,1]$. Given σ_1^*, if she patrols, her expected payoff is $p_1^* - c$. If she does not patrol, her payoff is 0. Thus, she patrols if $c < p_1^*$. Since σ_2^* is a best reply, we know

$$\sigma_2^*(c) = \begin{cases} 1, & \text{if } c < p_1^*, \\ 0, & \text{if } c > p_1^*. \end{cases}$$

This implies

$$p_2^* = \int_0^{p_1^*} \mathrm{d}c = p_1^*. \tag{5}$$

Combining equations (4) and (5), we obtain

$$p_1^* = p_2^* = \frac{3-\sqrt{5}}{2}.$$

Thus, we find a Bayesian Nash equilibrium

$$\sigma_1^*(u) = \begin{cases} 1, & \text{if } u > \frac{\sqrt{5}-1}{2}, \\ 0, & \text{if } u < \frac{\sqrt{5}-1}{2}, \end{cases} \text{ and } \sigma_2^*(c) = \begin{cases} 1, & \text{if } c < \frac{3-\sqrt{5}}{2}, \\ 0, & \text{if } c > \frac{3-\sqrt{5}}{2}. \end{cases}$$

The behavior at the cut-offs $\hat{u} = \frac{\sqrt{5}-1}{2}$ and $\hat{c} = \frac{3-\sqrt{5}}{2}$ can be arbitrary.

4. Consider the Cournot duopoly. There are two firms $i = 1, 2$ simultaneously choosing quantity in a market with demand curve $P(Q) = a - Q$. If each firm i chooses $q_i \geq 0$, the total supply is $Q = q_1 + q_2$. Firm 2's marginal cost is

$c_2 > 0$ and is commonly known. Firm 1's marginal cost may be $c_h > 0$ with probability $\theta \in (0,1)$ and $c_\ell > 0$ with probability $1-\theta$. Firm 1 always knows its marginal cost. With probability $\gamma \in (0,1)$, firm 2 also knows firm 1's marginal cost. With probability $1-\gamma$, firm 2 does not know firm 1's marginal cost.

For Questions 4a - 4b, suppose that firm 1 knows whether firm 2 knows its (firm 1's) marginal cost.

(a) **(10 points)** Model this situation as a Bayesian game.

Soln:

Players: $N = \{1, 2\}$.

Action Spaces: $A_1 = A_2 = \mathbb{R}_+$.

Type Space: $\Theta_1 = \{c_h^1, c_\ell^1, c_h^0, c_\ell^0\}$ and $\Theta_2 = \{h, \ell, 0\}$.

Beliefs/Common Prior: we can either specify players' posterior beliefs or their common prior. Their posterior beliefs are given by the two matrices in Figure 1.

	h	ℓ	0
c_h^1	1	0	0
c_ℓ^1	0	1	0
c_h^0	0	0	1
c_ℓ^0	0	0	1

Firm 1's posterior beliefs

	c_h^1	c_ℓ^1	c_h^0	c_ℓ^0
h	1	0	0	0
ℓ	0	1	0	0
0	0	0	θ	$1-\theta$

Firm 2's posterior beliefs

Figure 1: Firms' posterior beliefs

Equivalently, the common prior is given by the matrix in Figure 2.

	h	ℓ	0
c_h^1	$\theta\gamma$	0	0
c_ℓ^1	0	$(1-\theta)\gamma$	0
c_h^0	0	0	$\theta(1-\gamma)$
c_ℓ^0	0	0	$(1-\theta)(1-\gamma)$

Figure 2: Firms' common prior

In words, we use c_h^1 to denote firm 1's type whose cost is c_h and who knows that firm 2 knows this. We use c_h^0 to denote firm 1's type whose cost is c_h

and who knows that firm 2 does not know this. We use h to denote firm 2's type who knows that firm 1's cost is c_h. We use 0 to denote firm 2's type who does not know firm 1's cost.

Payoff Functions: for all $t_2 \in \Theta_2$,

$$v_1(q_1, q_2; c_h^1, t_2) = v_1(q_1, q_2; c_h^0, t_2) = (a - q_1 - q_2)q_1 - c_h q_1,$$
$$v_1(q_1, q_2; c_\ell^1, t_2) = v_1(q_1, q_2; c_\ell^0, t_2) = (a - q_1 - q_2)q_1 - c_\ell q_1.$$

For all $t_1 \in \Theta_1$ and $t_2 \in \Theta_2$,

$$v_2(q_1, q_2; t_1, t_2) = (a - q_1 - q_2)q_2 - c_2 q_2.$$

(b) **(10 points)** Characterize a Bayesian Nash equilibrium by writing out every type's maximization problem. (You do NOT need to solve the equilibrium.)

Soln: A strategy for firm i is a mapping $s_i : \Theta_i \to A_i$. If (s_1, s_2) is a Bayesian Nash equilibrium, the following conditions must be satisfied.

Firm 1:

$$s_1(c_h^1) \in \arg\max_{q_1 \geq 0} (a - q_1 - s_2(h))q_1 - c_h q_1, \tag{c_h^1}$$
$$s_1(c_\ell^1) \in \arg\max_{q_1 \geq 0} (a - q_1 - s_2(\ell))q_1 - c_\ell q_1, \tag{c_ℓ^1}$$
$$s_1(c_h^0) \in \arg\max_{q_1 \geq 0} (a - q_1 - s_2(0))q_1 - c_h q_1, \tag{c_h^0}$$
$$s_1(c_h^0) \in \arg\max_{q_1 \geq 0} (a - q_1 - s_2(0))q_1 - c_\ell q_1. \tag{c_ℓ^0}$$

Firm 2:

$$s_2(h) \in \arg\max_{q_2 \geq 0} (a - s_1(c_h^1) - q_2)q_2 - c_2 q_2, \tag{h}$$
$$s_2(\ell) \in \arg\max_{q_2 \geq 0} (a - s_1(c_\ell^1) - q_2)q_2 - c_2 q_2, \tag{ℓ}$$
$$s_2(0) \in \arg\max_{q_2 \geq 0} \theta\big[(a - s_1(c_h^0) - q_2)q_2 - c_2 q_2\big] + (1-\theta)\big[(a - s_1(c_\ell^0) - q_2)q_2 - c_2 q_2\big]. \tag{0}$$

For Questions 4c - 4d, suppose that firm 1 does not know whether firm 2 knows its (firm 1's) marginal cost.

(c) **(10 points)** Model this situation as a Bayesian game.

Soln:

Player: $N = 1, 2$.

Action Space: $a_1 = (q_{1h}, q_{1\ell}) \in \mathbb{R}^2_+$, in which subscripted $1h$ denotes that 1's cost is c_h (and ℓ is c_ℓ). $a_2 = (q_{2h}, q_{2\ell}, q_2) \in \mathbb{R}^3_+$.

Type Space: We denote player 1's type space by (c_h, c_ℓ) and player 2's type space by $(h, \ell, 0)$.

Then we need to show different hierarchies lying behind each type. We only write the first order belief here. For player 1: c_h believes $1 \circ (c_h, c_2)$; c_ℓ believes $1 \circ (c_\ell, c_2)$. For player 2: h believes $1 \circ (c_h, c_2)$; ℓ believes $1 \circ (c_\ell, c_2)$; 0 believes $\theta \circ (c_h, c_2) + (1-\theta) \circ (c_\ell, c_2)$.

Payoff Function:

$$
\begin{aligned}
v_1(a_1, a_2, c_h, h) &= (a - c_h - (q_{1h} + q_{2h}))q_{1h} \\
v_1(a_1, a_2, c_\ell, \ell) &= (a - c_\ell - (q_{1\ell} + q_{2\ell}))q_{1\ell} \\
v_1(a_1, a_2, c_h, 0) &= (a - c_h - (q_{1h} + q_2))q_{1h} \\
v_1(a_1, a_2, c_\ell, 0) &= (a - c_\ell - (q_{1\ell} + q_2))q_{1\ell}
\end{aligned}
$$

and

$$
\begin{aligned}
v_2(a_1, a_2, c_h, h) &= (a - c_2 - (q_{1h} + q_{2h}))q_{2h} \\
v_2(a_1, a_2, c_\ell, \ell) &= (a - c_2 - (q_{1\ell} + q_{2\ell}))q_{2\ell} \\
v_2(a_1, a_2, c_h, 0) &= (a - c_2 - (q_{1h} + q_2))q_2 \\
v_2(a_1, a_2, c_\ell, 0) &= (a - c_2 - (q_{1\ell} + q_2))q_2
\end{aligned}
$$

Common Prior:

$$Pr(c_h, h) = \theta\gamma Pr(c_\ell, \ell) = (1-\theta)\gamma Pr(c_h, 0) = \theta(1-\gamma) Pr(c_\ell, 0) = (1-\theta)(1-\gamma)$$

(d) **(10 points)** Characterize a Bayesian Nash equilibrium by writing out every type's maximization problem. (You do NOT need to solve the equilibrium.)

Soln:

For player 1:

$$
\begin{aligned}
c_h: &\quad \max_{q_{1h}} \, (a - c_h - (q_{1h} + (\gamma q_{2h} + (1-\gamma)q_2)))q_{1h} \\
c_\ell: &\quad \max_{q_{1\ell}} \, (a - c_\ell - (q_{1\ell} + (\gamma q_{2\ell} + (1-\gamma)q_2))q_{1\ell}
\end{aligned}
$$

For player 2:

$$
\begin{aligned}
h: &\quad \max_{q_{2h}} (a - c_2 - (q_{1h} + q_{2h}))q_{2h} \\
l: &\quad \max_{q_{2\ell}} (a - c_2 - (q_{1\ell} + q_{2\ell}))q_{2\ell} \\
0: &\quad \max_{q_2} (a - c_2 - ((\theta q_{1h} + (1-\theta)q_{1\ell}) + q_2))q_2
\end{aligned}
$$