Suggested Solutions to Game Theory Midterm Exam, Fall 2021

Peixuan Fu and Shuang Wu

1. Consider the normal form game in Figure 1.

	a	b	c	d
x	$1,0$	$4,3$	$3,4$	$1,1$
y	$1,4$	$2,0$	$2,2$	$4,1$
z	$2,2$	$1,0$	$2,0$	$3,1$

Figure 1: The normal form for Question 1

(a) **(10 points)** What strategies survive iterated elimination of strictly dominated strategies? In each step of the elimination, state which strategy is strictly dominated by which (possibly mixed) strategy.

Soln: In the first round, b for player 2 is strictly dominated by $\frac{1}{5} \circ a + \frac{4}{5} \circ c$, and d for player 2 is strictly dominated by $\frac{2}{3} \circ a + \frac{1}{3} \circ c$.

In the second round, after deleting b and d, y for player 1 is strictly dominated $\frac{1}{2} \circ x + \frac{1}{2} \circ z$.

Then no more strategy is strictly dominated. Thus, $\{x, z\} \times \{a, c\}$ survive IESDS. Figure 2 illustrates the smaller game after deletion.

	a	c
x	$1,0$	$3,4$
z	$2,2$	$2,0$

Figure 2: The strategies that survive IESDS

(b) **(10 points)** Find all Nash equilibria (pure and mixed).

Soln: We can focus on the game in Figure 2. There are 2 pure strategy Nash equilibria: (z, a) and (x, c). Obviously, there is no mixed strategy equilibrium in which only one player mixes. Finally, consider a mixed strategy profile (σ_1, σ_2). If it is a Nash equilibrium, both players must be indifferent between their pure strategies. That is,

$$\sigma_2(a) + 3(1 - \sigma_2(a)) = 2,$$
$$2(1 - \sigma_1(x)) = 4\sigma_1(x).$$

We obtain, $\sigma_1(x) = \frac{1}{3}$ and $\sigma_2(a) = \frac{1}{2}$. Therefore, there is one equilibrium in mixed strategies $(\frac{1}{3} \circ x + \frac{2}{3} \circ z, \frac{1}{2} \circ a + \frac{1}{2} \circ c)$.

(c) **(10 points)** Suppose the stage game is twice repeated with common discount factor δ. Argue that, for any $\delta \in (0, 1)$, there is no pure strategy subgame perfect equilibrium in which (y, c) is played at least once on the path of play.

Soln: Suppose by contradiction that $s = (s_1, s_2)$ is a pure strategy SPE in which (y, c) is played on the path of play. Because for all history $h \in \{x, y, z\} \times \{a, b, c, d\}$ after the first period, either $s^2(h) = (z, a)$ or $s^2(h) = (x, c)$, we know $s^1 = (y, c)$. Because s is a SPE, no player has a profitable one-shot deviation in the first period. In particular, we must have

$$u_2(y, c) + \delta u_2(s^2(yc)) \geq u_2(y, a) + \delta u_2(s^2(ya)),$$

or equivalently

$$u_2(y, a) - u_2(y, c) \leq \delta \left(u_2(s^2(yc)) - u_2(s^2(ya))\right).$$

Because $s^2(yc), s^2(ya) \in \{(z, a), (x, c)\}$, we know $u_2(s^2(yc)) - u_2(s^2(ya)) \leq 2$. Therefore, the above inequality implies

$$2 \leq 2\delta.$$

This is impossible since $\delta \in (0, 1)$. Therefore, our assumption that s is a SPE is wrong. This means that there is no pure strategy SPE in which (y, c) is played at least once on the path of play.

(d) **(10 points)** Suppose the stage game is infinitely repeatedly played with common discount factor $\delta \in (0, 1)$. Construct a strategy profile which satisfies: (1) it is a subgame perfect equilibrium for some $\delta \in (0, 1)$; (2) none of the stage game Nash equilibria is ever played along the path of play.

Soln: Consider the following trigger strategy profile:

For $t = 1$, $s_1^1 = x$, and $s_2^1 = b$.

For all $t \geq 2$,

$$s_1^t(h) = \begin{cases} x, & \text{if } h = (xb, \ldots, xb) \in H^{t-1}, \\ z, & \text{otherwise,} \end{cases}$$

$$s_2^t(h) = \begin{cases} b, & \text{if } h = (xb, \dots, xb) \in H^{t-1}, \\ a, & \text{otherwise.} \end{cases}$$

First consider the initial node. For no one to have a profitable one-shot deviation, we require

$$u_1(x,b) \geq (1-\delta)u_1(y,b) + \delta u_1(z,a),$$
$$u_1(x,b) \geq (1-\delta)u_1(z,b) + \delta u_1(z,a),$$

and

$$u_2(x,b) \geq (1-\delta)u_2(x,a) + \delta u_2(z,a),$$
$$u_2(x,b) \geq (1-\delta)u_2(x,c) + \delta u_2(z,a),$$
$$u_2(x,b) \geq (1-\delta)u_2(x,d) + \delta u_2(z,a).$$

It is easy to see that all these inequalities are satisfied if $\delta \geq \frac{1}{2}$.

Next, notice that the continuation play of s after any history of the form $h = (xb, \dots, xb) \in H^{t-1}$ for any $t \geq 2$ is the same as the original s. Therefore, we know that when $\delta \geq \frac{1}{2}$, no one has a profitable one-shot deviation at these histories.

Finally, consider any history that is not all xb. By construction, the continuation play of s is simply repeated play of the same stage Nash (z,a) regardless of history. We know no one has a profitable one-shot deviation. Therefore, when $\delta \geq \frac{1}{2}$, the constructed strategy profile is a SPE. Moreover, the outcome path is $(xb, xb, \dots,)$. That is, stage Nash is never played on the path of play.

2. Two players, $i \in \{1,2\}$, involve in a contest for a single prize. Each player i chooses her effort level $e_i \geq 0$ with cost $c_i e_i$, where $c_i > 0$ is i's marginal cost of effort. If the effort levels of the two players are e_1 and e_2, and at least one of them exerts strictly positive effort, then player i wins the prize with probability $\frac{e_i}{e_1+e_2}$ and loses with probability $1 - \frac{e_i}{e_1+e_2}$. In case both players choose zero effort, they both lose. Suppose that winning the prize obtains $V > 0$ and losing yields 0. Each player's total payoff is the expected payoff of winning minus her effort cost. They choose their efforts simultaneously and we restrict attention to pure strategies.

   (a) **(10 points)** Write out player 1's payoff as a function of the strategy profile (e_1, e_2).

Soln: Given the strategy profile (e_1, e_2), player 1's payoff function can be written as:

$$u_1(e_1, e_2) = \begin{cases} \frac{e_1}{e_1+e_2}V - c_1 e_1, & \text{if } e_1 + e_2 > 0, \\ 0, & \text{if } e_1 + e_2 = 0. \end{cases}$$

(b) **(10 points)** Find a Nash equilibrium of this game.

Soln: Suppose (e_1^*, e_2^*) is a Nash equilibrium. It is impossible that $e_1^* = e_2^* = 0$, as both players have an incentive to deviate to some small but positive effort level to win the prize for sure. It is impossible that $e_1^* = 0$ and $e_2^* > 0$, as player 2 can gain by deviating to $e_2^*/2$. Similarly, it is impossible that $e_1^* > 0$ and $e_2^* = 0$.

We are left with the last possible case $e_1^* > 0$ and $e_2^* > 0$. Because $e_i^* \in \arg\max_{e_i \geq 0} u_i(e_i, e_{-i}^*)$, we know

$$\frac{\partial u_i(e_i^*, e_{-i}^*)}{\partial e_i} = 0, \ \forall i = 1, 2.$$

Equivalently,

$$\frac{e_2^* V}{(e_1^* + e_2^*)^2} = c_1,$$
$$\frac{e_1^* V}{(e_1^* + e_2^*)^2} = c_2,$$

which yields

$$\begin{cases} e_1^* = \frac{c_2}{(c_1+c_2)^2}V, \\ e_2^* = \frac{c_1}{(c_1+c_2)^2}V. \end{cases}$$

To verify that this in deed a Nash equilibrium, it is sufficient to notice that every player's expected payoff is concave in his own effort.

3. Consider the following "knowledge (or skill) transfer" game between an adviser (she) and a student (he). The adviser has in total 1 unit of knowledge that can be freely transferred to the student. The interaction lasts for at most two periods. In period $t = 1$, the adviser decides how much knowledge $x_1 \in [0, 1]$ to transfer to the student. With this amount of knowledge, the student automatically and costlessly produces value x_1 *for the adviser* in this period. In period $t = 2$, the student first decides whether to stay with his adviser or leave. If he leaves, their relationship ends. In this case, the adviser obtains 0 in this period, but the student can use the knowledge x_1 learned in period $t = 1$

to produce value x_1 *for himself* in this period. If he decides to stay, then the adviser decides how much knowledge to transfer to the student, from what she is left with, $x_2 \in [0, 1 - x_1]$. In this case, the student produces value $x_1 + x_2$ (because he has gained knowledge $x_1 + x_2$ in total now) *for the adviser* in this period. Then, their relationship ends and the student produces $x_1 + x_2$ *for himself* in period $t = 3$. (The adviser obtains nothing in period $t = 3$.)

Let $\delta \in (0, 1)$ be their common discount factor. For instance, suppose the adviser transfers x_1 to the student in period $t = 1$. If the student leaves in period $t = 2$, then their total payoffs are x_1 and δx_1 respectively. If the student stays and the adviser transfers x_2 in period $t = 2$, then their total payoffs will be $x_1 + \delta(x_1 + x_2)$ and $\delta^2(x_1 + x_2)$ respectively.

(a) **(10 points)** What is a strategy for each player?

Soln: A strategy for the student specifies, for each knowledge transfer level $x_1 \in [0, 1]$, whether he will stay or leave. Thus a strategy is $s_2 : [0, 1] \to \{S, L\}$.

A strategy for the advisor specifies how much knowledge to transfer in the first period and in the second period conditional on the student having chosen to stay. Thus, a strategy is a pair (s_1^1, s_1^2), where $s_1^1 \in [0, 1]$ and $s_1^2 : [0, 1] \to [0, 1]$ such that $s_1^2(x_1) \in [0, 1 - x_1]$ for all x_1.

(b) **(10 points)** Suppose $\delta < \frac{1}{2}$, find a subgame perfect equilibrium.

Soln: We do backward induction.

Consider first the situation (history) where the adviser has chosen x_1 in the first period and the student decides to stay. The adviser's total payoff if she transfers x_2 in this period is $x_1 + \delta(x_1 + x_2)$. To maximize her payoff, it is obvious that the adviser should transfer all that is left, i.e.,

$$s_1^2(x_1) \equiv 1 - x_1, \ \forall x_1 \in [0, 1].$$

Now consider the student's decision at history x_1. Given s_1^2, if the student stays, his payoff is δ^2. If he leaves, his payoff is δx_1. Therefore, he strictly prefers staying if $x_1 < \delta$, strictly prefers leaving if $x_1 > \delta$, and is indifferent if $x_1 = \delta$. Therefore, both

$$s_2(x_1) = \begin{cases} S, & \text{if } x_1 \leq \delta, \\ L, & \text{if } x_1 > \delta, \end{cases} \quad \text{and } \tilde{s}_2(x_1) = \begin{cases} S, & \text{if } x_1 < \delta, \\ L, & \text{if } x_1 \geq \delta, \end{cases}$$

are consistent with backward induction.

Consider the adviser's initial transfer x_1. Given s_2 and s_1^2, the adviser's payoff if she transfers x_1 in the first period will be

$$\begin{cases} x_1 + \delta, & \text{if } x_1 \leq \delta, \\ x_1, & \text{if } \delta < x_1 \leq 1. \end{cases}$$

Because $\delta < \frac{1}{2}$, $x_1 + \delta < 1$ for any $x_1 \leq \delta$, implying that the adviser's best transfer in the first period is $s_1^1 = 1$. Therefore, we find one SPE

$$s_1^1 = 1,\ s_1^2(x_1) = 1 - x_1\ \forall x_1,\ \text{and}\ s_2(x_1) = \begin{cases} S, & \text{if } x_1 \leq \delta, \\ L, & \text{if } x_1 > \delta. \end{cases}$$

If the student's strategy is $\tilde{s}_2$ instead of s_2, we can also calculate the adviser's total payoff if she transfers x_1 in the first period:

$$\begin{cases} x_1 + \delta, & \text{if } x_1 < \delta, \\ x_1, & \text{if } \delta \leq x_1 \leq 1. \end{cases}$$

But obviously, $s_1^1 = 1$ also maximizes the adviser's payoff. Therefore, we find another SPE

$$s_1^1 = 1,\ s_1^2(x_1) = 1 - x_1\ \forall x_1,\ \text{and}\ \tilde{s}_2(x_1) = \begin{cases} S, & \text{if } x_1 < \delta, \\ L, & \text{if } x_1 \geq \delta. \end{cases}$$

(c) **(10 points)** Suppose $\delta > \frac{1}{2}$, find a unique subgame perfect equilibrium.

Soln: Observe that our analysis in the previous question regarding the adviser's second period transfer and the student's decision at the beginning at the second period does not change at all.

Moreover, given s_2 and s_1^2 in the previous question, the adviser's payoff if she transfers x_1 in the first period is still

$$\begin{cases} x_1 + \delta, & \text{if } x_1 \leq \delta, \\ x_1, & \text{if } \delta < x_1 \leq 1. \end{cases}$$

Because $\delta > \frac{1}{2}$, $s_1^1 = \delta$ uniquely maximizes the adviser's payoff. Therefore, we find a SPE:

$$s_1^1 = \delta,\ s_1^2(x_1) = 1 - x_1\ \forall x_1,\ \text{and}\ s_2(x_1) = \begin{cases} S, & \text{if } x_1 \leq \delta, \\ L, & \text{if } x_1 > \delta. \end{cases}$$

To show that it is the unique SPE, it is sufficient to show that there is no SPE in which the student's strategy is $\tilde{s}_2$. For this, notice that, given $\tilde{s}_2$ and s_1^2, the adviser's payoff if she transfers x_1 in the first period is

$$\begin{cases} x_1 + \delta, & \text{if } x_1 < \delta, \\ x_1, & \text{if } \delta \leq x_1 \leq 1. \end{cases}$$

Because $\delta > \frac{1}{2}$, the adviser does not have an optimal transfer level in this first period. Therefore, $\tilde{s}_2$ can not be the student's strategy in a SPE.

Similarly, $s_1^2(x_1) = 1 - x_1$ in any SPE.

(d) **(10 points)** Suppose $\delta = \frac{1}{2}$, find a Nash equilibrium that is not subgame perfect.

Soln: Consider the following strategy profile:

$$s_1^1 = \frac{1}{2},\ s_1^2(x_1) = \min\{\frac{1}{2}, 1 - x_1\},\ s_2(x_1) = \begin{cases} S, & \text{if } x_1 \leq \frac{1}{2}, \\ L, & \text{if } x_1 > \frac{1}{2}. \end{cases}$$

It is easy to verify that this is a Nash equilibrium since nobody has a profitable deviation. But it is not a SPE. For instance, in the subgame following the adviser choosing $x_1 = \frac{1}{3}$ in the first period and the student staying in the second period, the unique Nash equilibrium (single agent decision problem) is to transfer $\frac{2}{3}$. But $s_1^2(\frac{1}{3} = \frac{1}{2}$.