Suggested Solutions to Game Theory Final Exam, Fall 2023

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1. Consider the signaling game in Figure 1. We restrict attention to pure strategies.

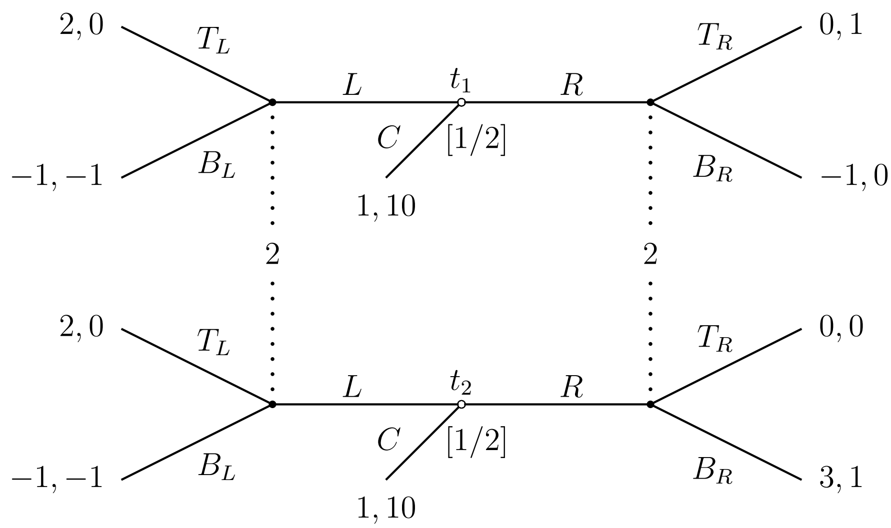

Figure 1: A signaling game

(a) **(10 points)** Find a pooling perfect Bayesian equilibrium. Does it pass or fail intuitive criterion?

Soln: In a pooling perfect Bayesian equilibrium, when player 1's strategy is LL, player 2's expected payoff is $Ev_2(T_L, L) = 2 > Ev_2(B_L, L)$ on the equilibrium path. In this case, player 2 cannot choose B_R off the path to prevent player 1 of type t_2 to deviate. Hence, we have a pooling perfect Bayesian equilibrium $(LL, T_L T_R)$.

When player 1's strategy is RR, player 2's expected payoff is $Ev_2(T_R, R) = Ev_2(B_R, R) = \frac{1}{2}$. If $s_2(R) = T_R$, then player 1 of type t_1 deviates. If $s_2(R) = B_R$, then player 2 of type t_2 deviates. Hence, $s_1 = RR$ fails to construct a perfect Bayesian equilibrium.

When player 1's strategy is CC, player 2's best response to L is T_L regardless of his beliefs. Hence, player 1 has incentive to deviate to L.

In all, the only pooling perfect Bayesian equilibrium is $(LL, T_L T_R)$. It fails intuitive criterion.

The only off-path strategy is R. $D(R) = t_1$, with $\Theta/D(R) = t_2$ and $BR_2(t_2, R) = B_R$. However, $v_1(B_R, t_2, R) = 3 > v_1(T_L, t_2, L) = 2$, which implies that this strategy fails intuitive criterion.

(b) **(10 points)** Find all other perfect Bayesian equilibria that are different from the one you find in Question 1a.

Soln: Now consider any other perfect Bayesian equilibrium where player 1 uses different strategy according to his own type.

i. If LR is chosen, $BR_2(LR) = T_L B_R$ and none has incentive to deviate. $(LR, T_L B_R)$ constructs a perfect Bayesian equilibrium.

ii. If LC is chosen, $BR_2(LC) = T_L T_R$ if belief $\mu_R(t_1) \in \left[\frac{1}{2}, 1\right]$. But player 2 of type t_2 deviates to L.

iii. If RL is chosen, $BR_2(RL) = T_L T_R$, but player 1 of type t_1 deviates to C.

iv. If RC is chosen, $BR_2(t_1, R) = T_R$ and $BR_2(L) = T_L$ even if L is off-path. Player 1 of type t_1 deviates to L.

v. If CL is chosen, $BR_2(L) = T_L$ even if L is off-path. Player 1 of type t_1 deviates to L.

vi. If CR is chosen, $BR_2(L) = T_L$ even if L is off-path. Player 1 of type t_1 deviates to L.

To sum up, there is only one perfect Bayesian equilibrium $(LR, T_L B_R)$ that is different from the one in Question 1a.

2. Consider a cheap talk game between player 1 (sender) and player 2 (receiver). Suppose player 1's type is uniformly distributed over $\Theta = [0, 1]$, and his message space is $A_1 = [0, 1]$. Suppose player 2's action space is $A_2 = \mathbb{R}$. The payoff functions for these two players are $u_1(a_2, \theta) = -(a_2 - \beta\theta)^2$ and $u_2(a_2, \theta) = -(a_2 - \theta)^2$ respectively, where $\beta \geq 0$ is some constant. We restrict attention to pure strategies.

(a) **(5 points)** Describe a babbling equilibrium, i.e., a perfect Bayesian equilibrium in which no information at all is transmitted.

Soln: In a babbling equilibrium where no information is transmitted, player 2's expected payoff from a_2 is,

$$Eu_2(a_2) = \int_0^1 -(a_2 - \theta)^2 d\theta = -a_2^2 + a_2 - \frac{1}{3}$$

By FOC, we solve $a_2^* = \frac{1}{2}$, and a babbling equilibrium:

$$\begin{cases} a_1(\theta) = a_1^* \in [0, 1] \\ a_2(a_1) = \frac{1}{2}, a_1 \in [0, 1] \end{cases}$$

.

(b) **(5 points)** Can you find a value of β under which full information is transmitted in a perfect Bayesian equilibrium?

Soln: If information is fully transmitted, for player 2, his expected payoff is given as $Eu_2(a_2, a_1) = u_2(a_2, \theta) = -(a_2 - \theta)^2$, which is maximized when $a_2^* = \theta$ since any $a_1(\theta)$ uniquely implies player 1's type θ.

For player 1, given player 2's strategy $a_2^*(a_1) = a_1$ and his own strategy $a_1(\theta) = \theta$, his payoff is $u_1(a_2^*(a_1), a_1, \theta) = -(\theta - \beta\theta)^2$ in this equilibrium. For player 1 of any type θ, it is not profitable to deviate if and only if $\beta = 1$, or player 1 of any type $\theta \in [0, 1/\beta]$ has incentive to deviate to $a_1'(\theta) = a_1(\beta\theta)$.

(c) **(10 points)** We call a perfect Bayesian equilibrium a two-message equilibrium if player 2 takes two different actions on the path of play. Show that if $\beta > 1$, there is always a two-message equilibrium.

Soln: If two messages a_1' and a_1'' is transmitted, with player 2's response $a_2(a_1') < a_2(a_1'')$. First, we claim that in the equilibrium player 1 must use a threshold strategy.

For player 1 of type θ, we have

$$v_1(a_1', a_2, \theta) = -(a_2(a_1') - \beta\theta)^2$$
$$v_1(a_1'', a_2, \theta) = -(a_2(a_1'') - \beta\theta)^2$$

$$\Delta v_1(\theta) = (a_2(a_1'') - \beta\theta)^2 - (a_2(a_1') - \beta\theta)^2$$
$$\frac{d\Delta v_1(\theta)}{d\theta} = -2(a_2(a_1'') - a_2(a_1')) < 0$$

.

Hence, a threshold θ^* exists and player 1's strategy can be rewritten as $a_1(\theta) = \begin{cases} a_1' \text{ , if } \theta < \theta^* \\ a_1'' \text{ , if } \theta \geq \theta^* \end{cases}$. For player 2, given player 1's threshold strategy, his expected payoff is as follows.

$$Eu_2(a_2, a_1', \theta) = \int_0^{\theta^*} -(a_2 - \theta)^2 d\theta$$
$$Eu_2(a_2, a_1'', \theta) = \int_{\theta^*}^{1} -(a_2 - \theta)^2 d\theta$$

By FOC, we solve player 2's best response as:

$$a_2^*(a_1) = \begin{cases} \frac{1}{2}\theta^* \text{ , if } a_1 = a_1' \\ \frac{1}{2}(\theta^* + 1) \text{ , if otherwise} \end{cases}$$

with beliefs $\mu_{a_1}(\theta^*) = 1$ for any $a_1 \notin \{a_1', a_1''\}$.

For player 1 of type θ^*, he is indifferent between a_1' and a_1'', i.e.,

$$-(\frac{1}{2}\theta^* - \beta\theta^*)^2 = -(\frac{1}{2}(\theta^* + 1) - \beta\theta^*)^2$$

which yields $\theta^* = \frac{\frac{1}{2}}{2\beta - 1}$. Thus, when $\beta > 1$ and $\theta^* < \frac{1}{2}$, there exists a perfect Bayesian equilibrium described as above.

(d) **(10 points)** Suppose now that player 1's type is uniformly distributed over $\Theta = [-1, 1]$ and $A_1 = [-1, 1]$. Show that for any β, there is a two-message equilibrium.

Soln: Now, repeat the solution in Question 2c. If two messages a_1' and a_1'' is transmitted, with player 2's response $a_2(a_1') < a_2(a_1'')$. First, we claim that in the equilibrium player 1 must use a threshold strategy.

For player 1 of type θ, we have

$$v_1(a_1', a_2, \theta) = -(a_2(a_1') - \beta\theta)^2$$
$$v_1(a_1'', a_2, \theta) = -(a_2(a_1'') - \beta\theta)^2$$

$$\Delta v_1(\theta) = (a_2(a_1'') - \beta\theta)^2 - (a_2(a_1') - \beta\theta)^2$$
$$\frac{d\Delta v_1(\theta)}{d\theta} = -2(a_2(a_1'') - a_2(a_1')) < 0$$

.

Hence, a threshold θ^* exists and player 1's strategy can be rewritten as $a_1(\theta) = \begin{cases} a_1' \text{ , if } \theta < \theta^* \\ a_1'' \text{ , if } \theta \geq \theta^* \end{cases}$. For player 2, given player 1's threshold strategy, his expected payoff is as follows.

$$Eu_2(a_2, a_1', \theta) = \int_0^{\theta^*} -(a_2 - \theta)^2 d\theta$$
$$Eu_2(a_2, a_1'', \theta) = \int_{\theta^*}^1 -(a_2 - \theta)^2 d\theta$$

By FOC, we solve player 2's best response as:

$$a_2^*(a_1) = \begin{cases} \frac{1}{2}(\theta^* - 1) \text{ , if } a_1 = a_1' \\ \frac{1}{2}(\theta^* + 1) \text{ , if otherwise} \end{cases}$$

with beliefs $\mu_{a_1}(\theta^*) = 1$ for any $a_1 \notin \{a_1', a_1''\}$.

For player 1 of type θ^*, he is indifferent between a_1' and a_1'', i.e.,

$$-(\frac{1}{2}(\theta^* - 1) - \beta\theta^*)^2 = -(\frac{1}{2}(\theta^* + 1) - \beta\theta^*)^2$$

which yields $\theta^* - 2\beta\theta^* = 0$ and $\theta^* = 0$, which does not depend on β. Hence, a two-message equilibrium is,

$$a_1(\theta) = \begin{cases} a_1' \text{ , if } \theta < 0 \\ a_1'' \text{ , if } \theta \geq 0 \end{cases}$$
$$a_2(a_1) = \begin{cases} -\frac{1}{2} \text{ , if } a_1 = a_1' \\ \frac{1}{2} \text{ , if otherwise} \end{cases}$$

with beliefs $\mu_{a_1}(0) = 1$ for any $a_1 \notin \{a_1', a_1''\}$.

3. There are two players, $i = 1, 2$, competing for a single object. The value of the object to player i is v_i, which is player i's private information. Assume that v_i is independently distributed over $[0, +\infty)$ according to a cumulative distribution function F_i with strictly positive density f_i. Time is continuous that starts at $t = 0$ and runs indefinitely. Each player i chooses time $t_i \geq 0$ to drop out, meaning that he concedes the object to the other player. The game ends once a player drops out, in which case the other player obtains the object. If both players drop out at the same time, the object is randomly allocated to either player with equal probability. Moreover, time is valuable: until the game ends,

each player loses one unit of payoff per unit of time. For example, if player i chooses to drop out at $t_i \geq 0$, the payoff to player 1 with value v_1 is

$$u_1(t_1, t_2, v_1) = \begin{cases} v_1 - t_2, & \text{if } t_1 > t_2, \\ \frac{v_1}{2} - t_2, & \text{if } t_1 = t_2, \\ -t_1, & \text{if } t_1 < t_2. \end{cases}$$

(a) **(10 points)** Assume, for this question only, that player 1's value $v_1 > 0$ and player 2's value $v_2 > 0$ are common knowledge. Find a Nash equilibrium.

Soln: The best response functions are as follows:

$$BR_1(t_2, v_1) = \begin{cases} (t_2, +\infty), & \text{if } v_1 > t_2, \\ 0, & \text{if } v_1 \leq t_2, \end{cases}$$

and

$$BR_2(t_1, v_2) = \begin{cases} (t_1, +\infty), & \text{if } v_2 > t_1, \\ 0, & \text{if } v_2 \leq t_1. \end{cases}$$

Hence there are many equilibria:

$$s_1 \geq v_2 \text{ and } s_2 = 0,$$

or

$$s_1 = 0 \text{ and } s_2 \geq v_1.$$

(b) **(10 points)** Suppose player 2's strategy s_2 is strictly increasing. If player 1's value is v_1 and chooses to drop out at t_1, what is his expected payoff.

Soln: The expected payoff of player 1 with value v_1 is

$$\begin{aligned} & \mathbb{E}u_1(t_1, s_2(v_2), v_1) \\ = & \int_{\{v_2|s_2(v_2)<t_1\}} (v_1 - s_2(v_2))\mathrm{d}F_2(v_2) + \int_{\{v_2|s_2(v_2)\geq t_1\}} (-t_1)\mathrm{d}F_2(v_2) \\ = & \int_0^{s_2^{-1}(t_1)} (v_1 - s_2(v_2))f_2(v_2)\mathrm{d}v_2 + \int_{s_2^{-1}(t_1)}^{\infty} (-t_1)f_2(v_2)\mathrm{d}v_2. \end{aligned}$$

(c) **(10 points)** Suppose, in addition, player 2's strategy is also differentiable. If t_1 is a best response to s_2 for player 1 with value v_1, write down the first order condition that t_1 must satisfy.

Soln: $t_1 = s_1(v_1)$ is a best response only if it satisfies the FOC:

$$\frac{1}{s_2'(s_2^{-1}(t_1))}(v_1 - t_1) f_2(s_2^{-1}(t_1)) - \int_{s_2^{-1}(t_1)}^{\infty} f_2(v_2)\mathrm{d}v_2 + \frac{t_1}{s_2'(s_2^{-1}(t_1))} f_2(s_2^{-1}(t_1)) = 0,$$

or equivalently

$$\frac{v_1 f_2(s_2^{-1}(t_1))}{s_2'(s_2^{-1}(t_1))} = \int_{s_2^{-1}(t_1)}^{\infty} f_2(v_2)\mathrm{d}v_2.$$

(d) **(10 points)** Show that in any equilibrium (s_1, s_2), we must have $s_1(0) = s_2(0) = 0$.

Soln: Suppose, by contraction, that $s_1(0) > 0$. Then, type $v_1 = 0$'s expected payoff is

$$\int_{\{v_2 | s_2(v_2) < s_1(0)\}} (-s_2(v_2))\mathrm{d}F_2(v_2) + \int_{\{v_2 | s_2(v_2) \geq s_1(0)\}} (-s_1(0))\mathrm{d}F_2(v_2).$$

Since $v_1 = 0$ does not want to deviate to $t_1 = 0$ in which case he can obtain 0 payoff, we have

$$\int_{\{v_2 | s_2(v_2) < s_1(0)\}} (-s_2(v_2))\mathrm{d}F_2(v_2) + \int_{\{v_2 | s_2(v_2) \geq s_1(0)\}} (-s_1(0))\mathrm{d}F_2(v_2) \geq 0.$$

Because $s_1(0) > 0$ by assumption, the above inequality can hold only if $s_2(v_2) \equiv 0$. This implies that each v_2's equilibrium payoff is

$$\frac{v_2}{2} \int_{\{v_1 | s_1(v_1) = 0\}} \mathrm{d}F_1(v_1) \leq \frac{v_2}{2}.$$

To see that this contradicts the assumption that (s_1, s_2) is a Bayesian Nash equilibrium, pick any $t_1 > 0$ such that

$$\alpha \equiv \int_{\{v_1 | s_1(v_1) < t_1\}} \mathrm{d}F_1(v_1) > \frac{1}{2}.$$

If player 2 of type v_2 deviates to t_1, his expected payoff is at least $\alpha(v_2 - t_1) - (1 - \alpha)t_1 = \alpha v_2 - t_1$. Hence, for any $v_2 > \frac{t_1}{\alpha - \frac{1}{2}}$, deviating to t_1 is profitable.

(e) **(10 points)** Suppose that F_1 and F_2 are the same exponential distribution, that is $f_1(v) = f_2(v) = e^{-v}$. Find a Bayesian Nash equilibrium of this game.

Soln: When $f_1(v) = f_2(v) = e^{-v}$, we have

$$\int_{s_2^{-1}(t_1)}^{\infty} f_2(v_2)\mathrm{d}v_2 = \int_{s_2^{-1}(t_1)}^{\infty} e^{-v_2}\mathrm{d}v_2 = e^{-s_2^{-1}(t_1)}.$$

Considering the symmetric equilibrium, i.e., $s_i = s_j = s$, by FOC we have

$$v_1 = s'(v_1).$$

Note that $s(0) = 0$ in the equilibrium, so the only solution is $s(v) = \frac{1}{2}v^2$.
It is easy to verify that given j's strategy $s_j(v_j) = \frac{1}{2}v_j^2$, $s_i(v_i) = \frac{1}{2}v_i^2$ is i's best response.
Thus, we find a symmetric BNE:

$$s_1(v) = s_2(v) = \frac{1}{2}v^2.$$