Game Theory, Fall 2022
Problem Set 10

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1. ST 12.8

   (a) **Soln:** We proceed to find out all pure strategy BNEs'. We adopt the convention that $s_i = XY$ means that $s_i(\theta_G) = X$ and $s_i(\theta_I) = Y$. Given s_{-i}, let $u_i(X; \theta_i, s_{-i})$ be i's expected payoff from choosing $X \in \{A, C\}$ when his signal is $\theta \in \{\theta_G, \theta_I\}$. Consider $\theta_i = \theta_G$. If i acquits, the defendant is not convicted regardless of $-i$'s vote. Therefore, the decision is right if and only if the defendant is innocent. The expected payoff is then

$$u_i(A; \theta_G, s_{-i}) = \mathbb{P}(I|\theta_i = \theta_G) = \frac{(1-q)(1-p)}{\mathbb{P}(\theta_i = \theta_G)}, \ \forall s_{-i}.$$

If i convicts, then the decision is right if (i) $-i$ convicts and the defendant is guilty, or (ii) $-i$ acquits and the defendant is innocent. Therefore, we can express i's expected payoff as

$$\begin{aligned} &u_i(C; \theta_G, s_{-i}) \\ =& \sum_{\theta_{-i} \in \{\theta_G, \theta_I\}} \left[\mathbb{P}(I, \theta_{-i}|\theta_i = \theta_G)\mathbb{1}_{(s_{-i}(\theta_{-i})=A)} + \mathbb{P}(G, \theta_{-i}|\theta_i = \theta_G)\mathbb{1}_{(s_{-i}(\theta_{-i})=C)}\right], \end{aligned}$$

where $\mathbb{1}$ is the indicator function, e.g., $\mathbb{1}_{(s_{-i}(\theta_{-i})=A)} = 1$ if $s_{-i}(\theta_{-i}) = A$; otherwise, it is 0. Then, for $s_{-i} \in \{AA, AC, CA, CC\}$, we can calculate

$$u_i(C; \theta_G, s_{-i}) = \begin{cases} \frac{(1-q)(1-p)}{\mathbb{P}(\theta_i = \theta_G)}, & \text{if } s_{-i} = AA, \\ \frac{qp(1-p)+(1-q)(1-p)^2}{\mathbb{P}(\theta_i=\theta_G)}, & \text{if } s_{-i} = AC, \\ \frac{qp^2+(1-q)p(1-p)}{\mathbb{P}(\theta_i=\theta_G)}, & \text{if } s_{-i} = CA, \\ \frac{qp}{\mathbb{P}(\theta_i=\theta_G)}, & \text{if } s_{-i} = CC, \end{cases}$$

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Recall that $p > \frac{1}{2}$ and $q > \frac{1}{2}$. Comparing $u_i(A;\theta_G, s_{-i})$ and $u_i(C;\theta_G, s_{-i})$ yields i's best reply when his signal is θ_G, given $-i$'s strategy s_{-i}:

$$BR_i(\theta_G, s_{-i}) = \begin{cases} \{A, C\}, & \text{if } s_{-i} = AA, \\ \{C\}, & \text{if } s_{-i} \in \{AC, CA, CC\}. \end{cases}$$

Although we have only analyzed i's best reply when his signal is θ_G, we have already known some property about a BNE. In particular, if (s_1, s_2) is a BNE, then $s_1 \neq AC$ and $s_2 \neq AC$. To see this, suppose by contradiction, $s_1 = AC$. Since $s_2(\theta_G) \in BR_2(\theta_G, s_1)$, we know $s_2(\theta_G) = C$. This in turn implies $s_2 \in \{CA, CC\}$, and $BR_1(\theta_G, s_2) = \{C\}$. This contradicts the assumption that $s_1(\theta_G) = A$.

Next, consider $\theta_i = \theta_I$. The expected payoff from acquitting is then

$$u_i(A;\theta_I, s_{-i}) = \mathbb{P}(I|\theta_i = \theta_I) = \frac{(1-q)p}{\mathbb{P}(\theta_i = \theta_I)}, \ \forall s_{-i}.$$

Ignoring the case $s_{-i} = AC$, the expected payoff from convicting is

$$\begin{aligned} & u_i(C;\theta_I, s_{-i}) \\ = & \sum_{\theta_{-i} \in \{\theta_G, \theta_I\}} \left[\mathbb{P}(I, \theta_{-i}|\theta_i = \theta_I) \mathbb{1}_{(s_{-i}(\theta_{-i})=A)} + \mathbb{P}(G, \theta_{-i}|\theta_i = \theta_I) \mathbb{1}_{(s_{-i}(\theta_{-i})=C)} \right], \\ = & \begin{cases} \frac{(1-q)p}{\mathbb{P}(\theta_i=\theta_I)}, & \text{if } s_{-i} = AA, \\ \frac{qp(1-p)+(1-q)p^2}{\mathbb{P}(\theta_i=\theta_I)}, & \text{if } s_{-i} = CA, \\ \frac{q(1-p)}{\mathbb{P}(\theta_i=\theta_I)}, & \text{if } s_{-i} = CC, \end{cases} \end{aligned}$$

Comparing $u_i(A;\theta_I, s_{-i})$ and $u_i(C;\theta_I, s_{-i})$, we can obtain

$$BR_i(\theta_I, s_{-i}) = \begin{cases} \{A, C\}, & \text{if } s_{-i} = AA, \\ \{C\}, & \text{if } s_{-i} = CA, \end{cases}$$

$$BR_i(\theta_I, CC) = \begin{cases} \{A\}, & \text{if } q < p, \\ \{A, C\}, & \text{if } q = p, \\ \{C\}, & \text{if } q > p. \end{cases}$$

Dividing the parameter space $q \in (\frac{1}{2}, 1)$ and $p \in (\frac{1}{2}, 1)$ into three regions, we can then discuss the set of BNE's. Let $BR_i(s_{-i}) \equiv BR_i(\theta_G, s_{-i}) \times BR_i(\theta_I, s_{-i})$.

- $q > p$. We have

$$\begin{aligned} BR_i(AA) &= \{AA, AC, CA, CC\}, \\ BR_i(CA) &= \{CC\}, \\ BR_i(CC) &= \{CC\}. \end{aligned}$$

There are two BNE's: (AA, AA) and (CC, CC).

- $q = p$. We have

$$\begin{aligned} BR_i(AA) &= \{AA, AC, CA, CC\}, \\ BR_i(CA) &= \{CC\}, \\ BR_i(CC) &= \{CC, CA\}. \end{aligned}$$

There are four BNE's: (AA, AA), (CC, CC), (CA, CC) and (CC, CA).

- $q < p$. We have

$$\begin{aligned} BR_i(AA) &= \{AA, AC, CA, CC\}, \\ BR_i(CA) &= \{CC\}, \\ BR_i(CC) &= \{CA\}. \end{aligned}$$

There are three BNE's: (AA, AA), (CA, CC) and (CC, CA).

(b) **Soln:** The optimal strategy for single-player is CC if $q > p$ and CA if $q < p$. In the former case, the decision is correct if the defendant is guilty. Thus, the expected payoff to the players is q. In the later case, the decision is correct if one of the two events occurs: (G, θ_G) and (I, θ_I). The expected payoff is then $qp + (1-q)p = p$, i.e., the total probability of "correct" signal.

- Consider (AA, AA) first. In this equilibrium, the defendant is always acquitted. Each player's expected payoff is thus $1-q$. Regardless of the relationship between p and q, it is strictly worse than the single-player payoff.
- Consider (CC, CC) when $q > p$. In equilibrium, the outcome is the same as the single-player case. Thus, it yields the same payoff to the players as in the single-player case.
- Consider (CA, CC) and (CC, CA) when $q < p$. Take (CA, CC) as an example. In equilibrium, the decision is correct if the event is $(G, \theta_1 = \theta_G)$ or $(I, \theta_1 = \theta_I)$. Therefore, the payoff to the players is the same as the single-player case.

(c) **Soln:** Consider $\theta_1 = \theta_G$. Given $s_2 = s_3 = CA$, player 1 can obtain payoff 1 from acquitting if one of the following states are realized $(G, \theta_2 = \theta_G, \theta_3 = \theta_G)$, $(I, \theta_2 = \theta_G, \theta_3 = \theta_I)$, $(I, \theta_2 = \theta_I, \theta_3 = \theta_G)$ and $(I, \theta_2 = \theta_I, \theta_3 = \theta_I)$. Hence, 1's expected payoff from acquitting is

$$
\begin{aligned}
&\mathbb{P}(G,\theta_2=\theta_G,\theta_3=\theta_G|\theta_1=\theta_G)+\mathbb{P}(I,\theta_2=\theta_G,\theta_3=\theta_I|\theta_1=\theta_G)\\
&+\mathbb{P}(I,\theta_2=\theta_I,\theta_3=\theta_G|\theta_1=\theta_G)+\mathbb{P}(I,\theta_2=\theta_I,\theta_3=\theta_I|\theta_1=\theta_G)\\
&=\frac{qp^3+(1-q)(1-p)(p+(1-p)p)}{\mathbb{P}(\theta_1=\theta_G)}.
\end{aligned}
$$

Player 1 can obtain payoff 1 from convicting, if one of the following states are realized $(G,\theta_2=\theta_G,\theta_3=\theta_G)$, $(G,\theta_2=\theta_G,\theta_3=\theta_I)$, $(G,\theta_2=\theta_I,\theta_3=\theta_G)$ and $(I,\theta_2=\theta_I,\theta_3=\theta_I)$. Hence, 1's expected payoff from convicting is

$$
\begin{aligned}
&\mathbb{P}(G,\theta_2=\theta_G,\theta_3=\theta_G|\theta_1=\theta_G)+\mathbb{P}(G,\theta_2=\theta_G,\theta_3=\theta_I|\theta_1=\theta_G)\\
&+\mathbb{P}(G,\theta_2=\theta_I,\theta_3=\theta_G|\theta_1=\theta_G)+\mathbb{P}(I,\theta_2=\theta_I,\theta_3=\theta_I|\theta_1=\theta_G)\\
&=\frac{qp^2+qp^2(1-p)+(1-q)(1-p)p^2}{\mathbb{P}(\theta_1=\theta_G)}.
\end{aligned}
$$

Using the fact that $q>\frac{1}{2}$ and $p>\frac{1}{2}$, we know that convicting is strictly better than acquitting.

Next, consider $\theta_1=\theta_I$. Similarly as above, we can calculate player 1's expected payoff from acquitting

$$
\begin{aligned}
&\mathbb{P}(G,\theta_2=\theta_G,\theta_3=\theta_G|\theta_1=\theta_I)+\mathbb{P}(I,\theta_2=\theta_G,\theta_3=\theta_I|\theta_1=\theta_I)\\
&+\mathbb{P}(I,\theta_2=\theta_I,\theta_3=\theta_G|\theta_1=\theta_I)+\mathbb{P}(I,\theta_2=\theta_I,\theta_3=\theta_I|\theta_1=\theta_I)\\
&=\frac{qp^2(1-p)+(1-q)p(p+(1-p)p)}{\mathbb{P}(\theta_1=\theta_I)}.
\end{aligned}
$$

His expected payoff from convicting is

$$
\begin{aligned}
&\mathbb{P}(G,\theta_2=\theta_G,\theta_3=\theta_G|\theta_1=\theta_I)+\mathbb{P}(G,\theta_2=\theta_G,\theta_3=\theta_I|\theta_1=\theta_I)\\
&+\mathbb{P}(G,\theta_2=\theta_I,\theta_3=\theta_G|\theta_1=\theta_I)+\mathbb{P}(I,\theta_2=\theta_I,\theta_3=\theta_I|\theta_1=\theta_I)\\
&=\frac{qp(1-p)+qp(1-p)^2+(1-q)p^3}{\mathbb{P}(\theta_1=\theta_G)}.
\end{aligned}
$$

It is easy to verify that acquitting is no worse than convicting if and only if $p\geq q$. Therefore, CA is a best reply for player 1 given $s_2=s_3=CA$ if and only if $p\geq q$. By symmetry, $s_1=s_2=s_3=CA$ is a BNE if and only if $p\geq q$.

2. ST 12.9

   (a) **Soln:** The normal form game is in Figure 1.

	G	N
G	$-1,-1$	$1,0$
N	$0,1$	$0,0$

Figure 1: The normal form game for Question 2(a)

Clearly, there are three Nash equilibria. Two of them are pure strategy equilibria: (G, N) and (N, G). The last one is an equilibrium in mixed strategies $(\frac{1}{2} \circ G + \frac{1}{2} \circ N,\ \frac{1}{2} \circ G + \frac{1}{2} \circ N)$.

(b) **Soln:** The perturbed game can be represented by the normal form in Figure 2.

	G	N
G	$-1,-1$	$1+\theta_1, 0$
N	$0, 1+\theta_2$	$0,0$

Figure 2: The perturbed game for Question 2(b)

Suppose (s_1, s_2) is a BNE. Consider agent 1 of type θ_1. If he chooses G, his expected payoff is

$$-1 \times \mathbb{P}(\theta_2 | s_2(\theta_2) = G) + (1 + \theta_1) \times \mathbb{P}(\theta_2 | s_2(\theta_2) = N).$$

If, instead, he chooses N, his expected payoff is 0. Therefore, in equilibrium,

$$s_1(\theta_1) = \begin{cases} N, & \text{if } \theta_1 < \frac{1-2p_{s_2}}{p_{s_2}}, \\ G, & \text{if } \theta_1 > \frac{1-2p_{s_2}}{p_{s_2}}, \end{cases}$$

where $p_{s_2} \equiv \mathbb{P}(\theta_2 | s_2(\theta_2) = N)$. Similarly, we know in equilibrium that

$$s_2(\theta_2) = \begin{cases} N, & \text{if } \theta_2 < \frac{1-2p_{s_1}}{p_{s_1}}, \\ G, & \text{if } \theta_2 > \frac{1-2p_{s_1}}{p_{s_1}}, \end{cases}$$

where $p_{s_1} \equiv \mathbb{P}(\theta_1 | s_1(\theta_1) = N)$. For this strategy profile to be a BNE, we must have

$$p_{s_1} = \mathbb{P}\left(\theta_1 < \frac{1-2p_{s_2}}{p_{s_2}}\right),$$
$$p_{s_2} = \mathbb{P}\left(\theta_2 < \frac{1-2p_{s_1}}{p_{s_1}}\right).$$

Using the fact that θ_1 and θ_2 are uniformly distributed over $[-\varepsilon, \varepsilon]$ with $\varepsilon < 1$, it is easy to see that this system of equations have three solutions. First, $(p_{s_1}, p_{s_2}) =$

$(1,0)$. This corresponds to the BNE in which $s_1(\theta_1) \equiv N$ and $s_2(\theta_2) \equiv G$. Second, $(p_{s_1}, p_{s_2}) = (0,1)$. This corresponds to the BNE in which $s_1(\theta_1) \equiv G$ and $s_2(\theta_2) \equiv N$. Finally, $(p_{s_1}, p_{s_2}) = (\frac{1}{2}, \frac{1}{2})$. This corresponds to the BNE in which, for $i = 1, 2$,

$$s_i(\theta_i) = \begin{cases} N, & \text{if } \theta_i < 0, \\ G, & \text{if } \theta_i > 0. \end{cases}$$

(c) **Soln:** Consider the third BNE we find in 2(b). In equilibrium, the probability that agent i plays N is $\mathbb{P}(\theta_i < 0) = \frac{1}{2}$. This exactly equals the probability that agent i plays N in the mixed strategy NE in the complete information (unperturbed) game.