Game Theory, Fall 2021
Problem Set 3

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1. ST Exercise 5.4.

   (a) **Soln:** If the opponent claims $s_j < 8$, the best response of player i is to order $s_i = 8 - s_j$. If the opponent claims all the 8 slices of pizza, there is no difference in how many slices of pizza player i orders since he never gets a slice. Therefore, we have

   $$BR_i(s_j) = \begin{cases} 8 - s_j, & \text{if } 0 \leq s_j < 8, \\ \{0, 1, ..., 8\}, & \text{if } s_j = 8. \end{cases}$$

   (b) **Soln:** There are multiple equilibria. In fact, any strategy profile of the form $(n, 8-n)$ for $n \in \{0, 1, ..., 8\}$ is a Nash equilibrium. Moreover, $(8, 8)$ is also a Nash equilibrium.

2. ST Exercise 5.9

   (a) **Soln:** We can rewrite roommate i's payoff function as

   $$v_i(s_i, s_{-i}) = (1 - c) \cdot s_i + \sum_{j \neq i} s_j.$$

   When $c < 1$, i's payoff function is strictly increasing in his own cleaning time, given any strategy profile of the opponents. Thus, $s_i = 5$, the maximal number of hours that i can spend, is the strictly dominant strategy for roommate i. Therefore, there is a unique Nash equilibrium. In this equilibrium, all roommates spend the maximal time, 5 hours.

   (b) **Soln:** When $c > 1$, then $s_i = 0$ is the strictly dominant strategy for roommate i. Therefore, there is also a unique Nash equilibrium. But in this Nash equilibrium, every roommate does not clean at all.

*Special thanks go to Peixuan Fu and Shuang Wu, who wrote the last version of these solutions.

(c) **Soln:** We know that in the only equilibrium, no roommate cleans and each obtains a payoff 0. This is not Pareto efficient. To see this, consider the case where all roommates clean for 5 hours. Then each roommate's payoff now is $15 > 0$. Hence, the strategy profile $(5, 5, 5, 5, 5)$ Pareto dominates the unique Nash equilibrium, implying that the Nash equilibrium is not Pareto efficient.

3. ST Exercise 5.10

(a) **Soln:** Given that player j chooses e_j, i chooses e_i to maximize

$$\max_{e_i \geq 0} (a + e_j)e_i - e_i^2.$$

The first order condition is

$$a + e_j - 2e_i = 0.$$

Thus, i's best response correspondence is

$$BR_i(e_j) = \frac{a + e_j}{2}.$$

Here, we assume that $a > 0$.

(b) **Soln:** Here the best response function of player i is *increasing* in the choice of player j. However, in the Cournot model it is decreasing in the choice of player j. This is because in this game the choices of the two players are strategic complements: i's marginal payoff from his effort e_i is increasing in j's payoff. But in the Cournot competition, they are strategic substitutes: one firm's marginal profit of its quantity is decreasing in the other firm's quantity.

(c) **Soln:** Any Nash equilibrium solves

$$e_1 = BR_1(e_2) \text{ and } e_2 = BR_2(e_1).$$

The only solution is $e_1 = e_2 = a$. So, the only Nash equilibrium is (a, a).

4. ST Exercise 5.12

(a) **Soln:** $(p_1', p_2') = (2.00, 2.01)$ is a Nash equilibrium. When frim 1 chooses 2.00 as its price, firm 2 will never choose prices below 2.00, and firm 2 is indifferent to choosing any price not less than 2.00 for getting zero profit. When firm 2 chooses 2.01, firm 1 gains the most profit by selecting 2.00 as its price.

(b) **Soln:** Any strategy profile of the form $(p_1, p_2) = (1 + m, 1.01 + m)$ for $m \in \{0.01, 0.02, ..., 1.00\}$ is a Nash equilibrium. The reason is similar as above. Therefore, there are 100 Nash equilibria in this game. For the complete procedure, consider firm 1's strategy. Facing $p_2 < 1.00$, firm 1 will give up the market and choose any price strictly bigger than p_2; when p_2 equals 1.00, firm 1 can select $p_1 \geq 1.00$; when p_2 equals 1.01, the best response for firm 1 is 1.01, since only in such case it divides the market and earns positive payoff; as p_2 increases, firm 1 will choose $p_2 - 0.01$ until p_1 reaches 50.50. For the same reason, the best response for firm 2 can be derived. The best response functions are as follows:

$$BR_1(p_2) = \begin{cases} \{p_2 + 0.01, p_2 + 0.02, ...\}, & \text{if } p_2 < 1, \\ \{1, 1.01, 1.02, ...\}, & \text{if } p_2 = 1, \\ 1.01, & \text{if } p_2 = 1.01, \\ p_2 - 0.01, & \text{if } 1.01 < p_2 < 50.51, \\ 50.5, & \text{if } p_2 \geq 50.51, \end{cases}$$

and

$$BR_2(p_1) = \begin{cases} \{p_1 + 0.01, p_1 + 0.02, ...\}, & \text{if } p_1 < 2, \\ \{2, 2.01, 2.02, ...\}, & \text{if } p_1 = 2, \\ 2.01, & \text{if } p_1 = 2.01, \\ p_1 - 0.01, & \text{if } 2.01 < p_1 < 51.01, \\ 51, & \text{if } p_1 \geq 51.01. \end{cases}$$

In this way, we can find all the Nash equilibria as shown before.

5. Consider the asymmetric Bertrand competition in ST Exercise 5.12 again. Now assume that the firms can charge any prices, as is the example we covered in class. But assume that when there is a tie in prices, the consumers buy from the firm with lower marginal cost (firm 1). Find all Nash equilibria.

Soln: In this case, the demand of the two firms are given by

$$q_1(p_1, p_2) = \begin{cases} 100 - p_1, & \text{if } p_1 \leq p_2, \\ 0, & \text{if } p_1 > p_2, \end{cases}$$

and

$$q_2(p_1, p_2) = \begin{cases} 100 - p_2, & \text{if } p_1 < p_2 \\ 0, & \text{if } p_1 \geq p_2. \end{cases}$$

We can then derive the best responses for both players.

$$BR_1(p_2) = \begin{cases} \frac{101}{2}, & \text{if } p_2 > \frac{101}{2}, \\ p_2, & \text{if } 1 < p_2 \leq \frac{101}{2}, \\ [1, +\infty), & \text{if } p_2 = 1, \\ (p_2, \infty), & \text{if } p_2 < 1, \end{cases}$$

and

$$BR_2(p_1) = \begin{cases} 51, & \text{if } p_1 > 51, \\ \emptyset, & \text{if } 2 < p_1 \leq 51, \\ [p_1, \infty), & \text{if } p_1 \leq 2. \end{cases}$$

Hence we can find all Nash equilibria

$$\{(p, p) \mid p \in [1, 2]\}.$$

6. ST Exercise 5.16

(a) **Soln:** For simplicity, in this part we assume that v is very large.

We restrict attention to non-negative prices. Suppose the prices of the two firms are p_1 and p_2 respectively. For any $p_1, p_2 \geq 0$, the consumer $x \in [0, 1]$ who will buy from firm 1 must satisfy

$$v - p_1 - x \geq 0,$$
$$v - p_1 - x \geq v - p_2 - (1 - x).$$

We have already assume that v is very large, so that all the customers will be served by at least one firm, which means the first inequality must hold.

If customer $x^* \in [0, 1]$ is indifferent between the two firms, then

$$v - p_1 - x^* = v - p_2 - (1 - x^*),$$

implying

$$x^* = \frac{1 + p_2 - p_1}{2}.$$

From the assumptions we know that such x^* exists. Therefore, the demand for firms 1 and 2 are

$$D_1(p_1, p_2) = x^* = \frac{1 + p_2 - p_1}{2},$$
$$D_2(p_1, p_2) = 1 - x^* = \frac{1 + p_1 - p_2}{2}.$$

Firm 1's maximization problem is then

$$\max_{p_1 \geq 0} \ (\frac{1 + p_2 - p_1}{2})p_1.$$

The first order condition is

$$1 + p_2 - 2p_1 = 0.$$

Thus, the best response function is

$$p_1 = \frac{1}{2} + \frac{p_2}{2}.$$

Similarly, we can derive firm 2's best response function

$$p_2 = \frac{1}{2} + \frac{p_1}{2}.$$

(b) **Soln:** We can directly calculate firm 1's demand function $D_1(p_1, p_2)$, which is illustrated in panel (a) of Figure 1. Then, for each p_2, we can also calculate firm 1's best response, which is illustrated by the red curve in panel (b). By symmetry, we can also calculate firm 2's best response. Putting them together yields the unique equilibrium $(p_1^*, p_2^*) = (\frac{1}{2}, \frac{1}{2})$. See panel (c).

(c) **Soln:** Like in part (a), assume that customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1-x^*$, and he is indifferent between buying from either, which implies that

$$v - p_1 - \frac{1}{2}x^* = v - p_2 - \frac{1}{2}(1 - x^*),$$

which gives that equation for x^*,

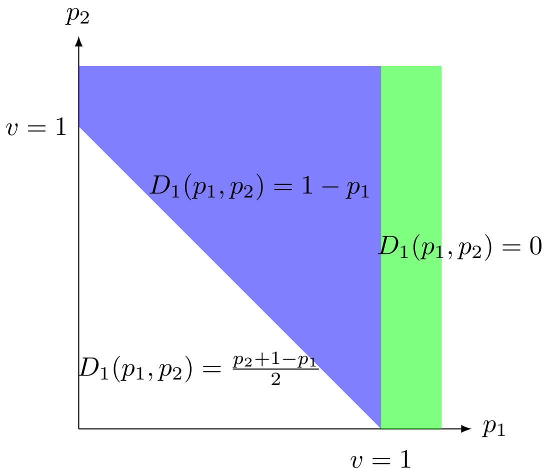
$$x^* = \frac{1}{2} + p_2 - p_1.$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

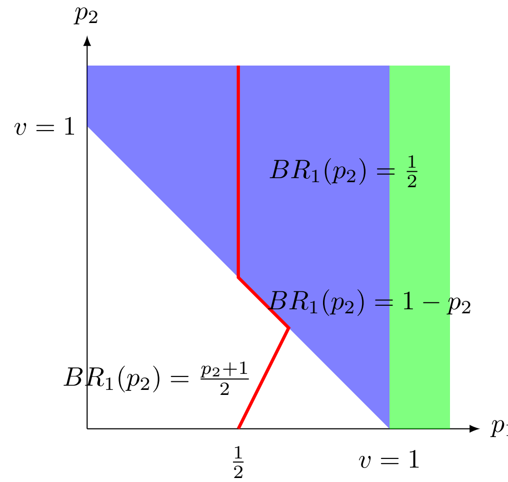
$$D_1(p_1, p_2) = x^* = \frac{1}{2} + p_2 - p_1,$$
$$D_2(p_1, p_2) = 1 - x^* = \frac{1}{2} + p_1 - p_2.$$

Firm 1's maximization problem is

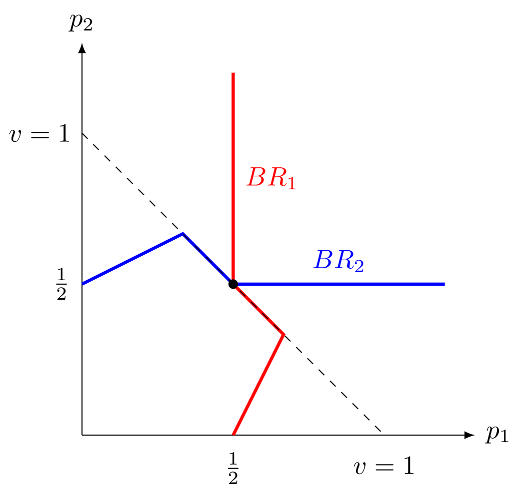
$$\max_{p_1 \geq 0} \ (\frac{1}{2} + p_2 - p_1)p_1,$$



(a) Firm 1's demand



(b) Firm 1's best response



(c) Equilibrium

Figure 1: Demand, best response and equilibrium for Question 6b

which yields the first order condition

$$\frac{1}{2} + p_2 - 2p_1 = 0,$$

implying the best response function

$$p_1 = \frac{1}{4} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{4} + \frac{p_1}{2}.$$

Thus, we have $p_1 = p_2 = \frac{1}{2}$, and $x^* = \frac{1}{2}$. We can verify that $v - p_1 - \frac{1}{2}x^* = v - p_2 - \frac{1}{2}(1 - x^*) = \frac{1}{4} > 0$, which means customer x^* will prefer to buy and so will every other customer. Then $(\frac{1}{2}, \frac{1}{2})$ is a Nash equilibrium.

(d) **Soln:** Using the assumed indifferent customer x^*, his indifference implies that

$$v - p_1 - \alpha x^* = v - p_2 - \alpha(1 - x^*),$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1).$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} D_1(p_1, p_2) &= x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1), \\ D_2(p_1, p_2) &= 1 - x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_1 - p_2). \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1 \geq 0} \ (\frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1))p_1,$$

which yields the first order condition

$$\frac{1}{2} + \frac{p_2}{2\alpha} - \frac{p_1}{\alpha} = 0,$$

implying the best response function

$$p_1 = \frac{\alpha}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{\alpha}{2} + \frac{p_1}{2}.$$

If these two best response functions hold simultaneously, we have $p_1 = p_2 = \alpha$, and $x^* = \frac{1}{2}$. In this case $v - p_1 - \alpha x^* = v - p_2 - \alpha(1 - x^*) = 1 - \frac{3\alpha}{2} > 0$, so that customer x^* will prefer to buy and so will every other customer. Then (α, α) is a Nash equilibrium.

As α decreases, the products of the two firms become more homogeneous. This intensifies the competition between the two firms, and therefore the equilibrium price decreases. In the limit as $\alpha \to 0$, the products of the two firms become perfect substitute and the market converges to Bertrand competition.

7. ST Exercise 6.5.

   (a) **Soln:** We use $\{H, P\}$ to denote $\{\text{Hanging out}, \text{Patrolling}\}$ for cops, and use $\{H, P\}$ to denote $\{\text{Staying hidden}, \text{Prowling}\}$ for robbers. The matrix is in Figure 2.

		Robber	
		H	P
Cop	H	$10, 0$	$10, 10$
	P	$0, 0$	$20, -10$

Figure 2: Normal form game for Question 7a

   (b) **Soln:** Suppose $\sigma_1(H) = p$ and $\sigma_2(H) = q$. Then players expected payoffs are

$$\begin{aligned} v_1(\sigma_1, \sigma_2) &= 10pq + 10p(1-q) + 20(1-p)(1-q) \\ &= 20 + (20q - 10)p - 20q, \\ v_2(\sigma_1, \sigma_2) &= 10p(1-q) - 10(1-p)(1-q) \\ &= (20p - 10)(1-q). \end{aligned}$$

Therefore, it is easy to see

$$BR_1(q) = \begin{cases} 0, & \text{if } q < \frac{1}{2}, \\ [0,1], & \text{if } q = \frac{1}{2}, \\ 1, & \text{if } q > \frac{1}{2}, \end{cases} \text{ and } BR_2(p) = \begin{cases} 1, & \text{if } p < \frac{1}{2}, \\ [0,1], & \text{if } p = \frac{1}{2}, \\ 0, & \text{if } p > \frac{1}{2}. \end{cases}$$

See Figure 3 for an illustration of the two best response correspondences.

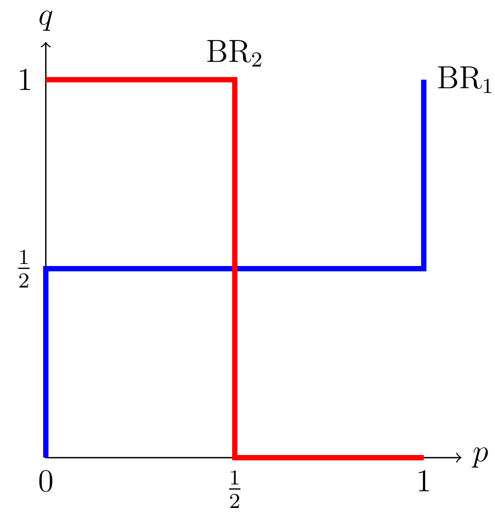


Figure 3: Best response correspondences for 7b

(c) **Soln:** From the best response function graph, we know that there is a unique Nash equilibrium $(\frac{1}{2} \circ H + \frac{1}{2} \circ P, \frac{1}{2} \circ H + \frac{1}{2} \circ P)$.

8. ST Exercise 6.8.

(a) **Soln:** We use 0 to denote staying out, and 1 to denote entry. The value function is as follows:

$$V_i(s_i, s_j, s_k) = \begin{cases} 0, & \text{if } s_i = 0, \\ -12, & \text{if } s_i = s_j = s_k = 1, \\ 13, & \text{if } s_i = 1, s_j = 0, s_k = 1 \text{ or } s_i = 1, s_j = 1, s_k = 0, \\ 88, & \text{if } s_i = 1, s_j = s_k = 0. \end{cases}$$

Then we can get the best-response function:

$$BR_i(s_j, s_k) = \begin{cases} 0, & \text{if } s_j = s_k = 1, \\ 1, & \text{if otherwise.} \end{cases}$$

That is, a firm prefers entering if and only if at most one of the opponents enters. Thus, neither all firms staying out nor all firms entering is an equilibrium. Moreover, the strategy profiles in which only one firm enters are not equilibria either, since the staying out firms have incentive to enter. Finally, the strategy profiles in which two firms enter and the other stays out are equilibria, because no firm has incentive to deviate.

(b) **Soln:** Suppose every firm enters with probability $p \in (0, 1)$. For this strategy profile to be an equilibrium, every firm must be indifferent between entering and staying out.

If a firm enters, then with probability p^2 it will face two other entrants, with probability $(1-p)^2$ it will face no other entrants, and with probability $2(1-p)p$ it will face one other entrant. Thus the expected payoff from entering is

$$(1-p)^2 \times 88 + 2(1-p)p \times 13 - p^2 \times 12.$$

Indifference condition requires

$$(1-p)^2 \times 88 + 2(1-p)p \times 13 - p^2 \times 12 = 0,$$

implying $p = \frac{4}{5}$. So, every firm entering with probability $4/5$ is a symmetric Nash equilibrium.