1. (a) $i \in N = \{1,2\}$. $S_i = \mathbb{R}_+$.

$$u_i(b_1, b_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(v_i - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

(b)

$$u_i'(b_1, b_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{1}{2}v_i - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

注意与(a)的区别. 可以尝试找这个情况的NE.

(c) **Soln:** For arbitrary v_1 and v_2, player i's best reply correspondence is

$$\phi_i(b_j) = \begin{cases} [0, b_j), & \text{if } b_j > v_i, \\ [0, b_j], & \text{if } b_j = v_i, \\ \emptyset, & \text{if } b_j < v_i. \end{cases} \tag{1}$$

Make sure you understand why some intervals are closed while some are open.

When $v \equiv v_1 = v_2$, it is easy to see that there exists a unique pure strategy Nash equilibrium (v, v). This is illustrated in Figure 2. The gray area (including the lower boundary) is the best reply area of bidder 1. The blue area (including the left boundary) is the best reply area of bidder 2. As we can easily see, the only intersection of these two areas is (v, v) which is the unique pure strategy Nash equilibrium of this game.

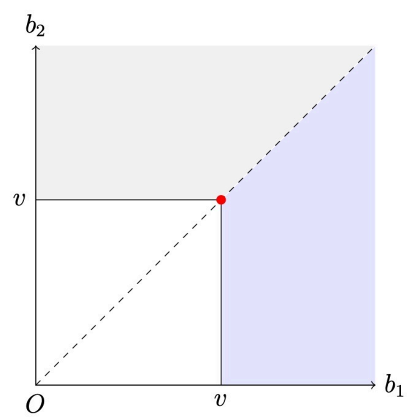
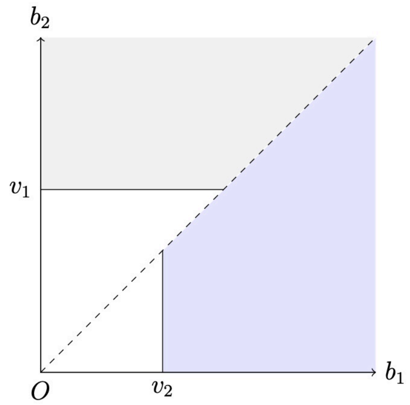


Figure 2: Best replies for first price auction with equal values

(d)

Soln: Without loss of generality, assume $v_1 > v_2$. Notice the best reply in (1) still applies. Therefore, it is easy to see that there is no pure strategey Nash equilibrium. This is illustrated in Figure 3. As above, the gray area (including the lower boundary) is the best reply area of bidder 1. The blue area (including the left boundary) is the best reply area of bidder 2. Since there is no intersection of these two area, there is no pure strategy Nash equilibrium.

Let us compare this with (~~a~~ c). From (~~a~~ c), we know (v_1, v_2) is a pure strategy Nash equilibrium if $v_1 = v_2$. But when $v_1 > v_2$, bidding v_1 is not a best reply for bidder 1 given bidder 2 bids v_2. This is because bidding v_1 gives bidder 1 zero payoff while bidding any $v_2 < b_1 < v_1$ still guarantees bidder 1 to win and give him a strictly positive payoff. Hence such b_1 is a profitable deviation. In fact, when $b_2 = v_2$, bidder 1 does not have a best reply.



2. (a)

We find all Nash equilibria by discussion.

Nash equilibria in which $\sigma_1(T) = 1$? If $\sigma_1(T) = 1$, then player 2 is indifferent between ℓ and r. However, we should make sure that given player 2's strategy, T is also 1's best reply. Hence we must have

$$-1\sigma_2(\ell) + 3(1 - \sigma_2(\ell)) \geq 2\sigma_2(\ell) + 0(1 - \sigma_2(\ell)),$$

yielding

$$\sigma_2(\ell) \leq \frac{1}{2}.$$

Therefore $(T, x \circ \ell + (1 - x) \circ r)$ is a Nash equilibrium for any $x \in [0, \frac{1}{2}]$.

Nash equilibria in which $\sigma_1(T) = 0$? If $\sigma_1(T) = 0$, then ℓ is 2's unique best reply. Given ℓ, B is also 1's best reply. So (B, ℓ) is a Nash equilibrium.

Nash equilibria in which $\sigma_1(T) \in (0, 1)$? For 1 to mix between T and B, 1 must be indifferent between T and B. Hence we must have

$$-1\sigma_2(\ell) + 3(1 - \sigma_2(\ell)) = 2\sigma_2(\ell) + 0(1 - \sigma_2(\ell)),$$

yielding

$$\sigma_2(\ell) = \frac{1}{2}.$$

Since $\sigma_2(\ell) \in (0, 1)$, 2 must be indifferent between ℓ and r. Thus we have

$$2\sigma_1(T) + 3(1 - \sigma_1(T)) = 2\sigma_1(T) + 1(1 - \sigma_1(T))$$

yielding $\sigma_1(T) = 1$, which contradicts the assumption $\sigma_1(T) \in (0, 1)$.

In sum, $(T, x \circ \ell + (1 - x) \circ r)$ for any $x \in [0, \frac{1}{2}]$ and (B, ℓ) are Nash equilibria.

(b) 难度较大. Optional.

We first do iterated deletion of strictly dominated strategies. In this game, ℓ is strictly dominated by m. After deleting ℓ, we are left with $\{T, M, B\} \times \{m, r\}$, and there is no strictly dominated strategies. To find the set of all Nash equilibria of the original game, it suffices to find all Nash in the smaller game $\{T, M, B\} \times \{m, r\}$.

Nash equilibria in which $\sigma_2(m) = 1$? Given $\sigma_2(m) = 1$, M is 1's best reply. Given M, $\sigma_2(m) = 1$ is also 2's best reply. So (M, m) is a Nash equilibrium.

Nash equilibria in which $\sigma_2(m) = 0$? Given $\sigma_2(m) = 0$, T is 1's best reply. Given T, $\sigma_2(m) = 0$ is also 2's best reply. So (T, r) is a Nash equilibrium.

Nash equilibria in which $\sigma_2(m) \in (0, 1)$? For 2 to mix between m and r in equilibrium, 2 must be indifferent between m and r. So

$$2\sigma_1(T) + 4\sigma_1(M) + 3(1 - \sigma_1(T) - \sigma_1(M)) = 3\sigma_1(T) + 4(1 - \sigma_1(T) - \sigma_1(M))$$

yielding $\sigma_1(M) = \frac{1}{5}$. Since $\sigma_1(M) \in (0, 1)$, we know either $\sigma_1(T) > 0$ or $\sigma_1(B) > 0$ or both. If $\sigma_1(T) > 0$, we must have

$$4(1 - \sigma_2(m)) = 4\sigma_2(m)$$

yielding $\sigma_2(m) = \frac{1}{2}$. In this case, player 1 is actually indifferent among all T, M and B. Hence $(x \circ T + \frac{1}{5} \circ M + (\frac{4}{5} - x) \circ B,\ \frac{1}{2} \circ m + \frac{1}{2} \circ r)$ is a Nash equilibrium for any $x \in [0, \frac{4}{5}]$.

(c) First observe that A is strictly dominated by $p \circ B + (1-p) \circ C$ for any $\frac{1}{3} < p < \frac{1}{2}$. After deleting A, x is strictly dominated by w. After deleting w, D is strictly dominated by C. So we are left with $\{B, C\} \times \{w, y, z\}$ surviving iterated deletion of strictly dominated strategies as in Figure 5. We now find all Nash equilibria (σ_1, σ_2). If $\sigma_1(B) = 1$, then 2 is indifferent between w and z. Moreover, for any mixture between w and z, $\sigma_1(B) = 1$ is a best reply. Hence $(B, x \circ w + (1-x) \circ z)$ is a Nash equilibrium for any $x \in [0, 1]$.

If $\sigma_1(B) = 0$, then 2 strictly prefers y. Moreover if 2 plays y, $\sigma_1(B) = 0$ is a best reply. So (C, y) is a Nash equilibrium.

Finally, consider $\sigma_1(B) \in (0, 1)$. For 1 to be indifferent between B and C, we must have $\sigma_2(y) = 1$. For this σ_2 be a best reply to 1's mixture, we must have

$$2\sigma_1(B) + 6(1 - \sigma_1(B)) \geq 3\sigma_1(B) + 5(1 - \sigma_1(B))$$

and

$$2\sigma_1(B) + 6(1 - \sigma_1(B)) \geq 3\sigma_1(B).$$

Hence we know $\sigma_1(B) \leq \frac{1}{2}$. Therefore $(x \circ B + (1-x) \circ C, y)$ is a Nash equilibrium for any $x \in [0, \frac{1}{2}]$.

3. simultaneously:

(a)-(c) **Soln:** There is no equilibrium (x_1^*, x_2^*) such that $0 < x_1^* + x_2^* < c$. To see this, assume without loss of generality $x_1^* > 0$. In this case, roommate 1's payoff is $-x_1^* < 0$ while he can get 0 if he deviates to 0.

There is no equilibrium (x_1^*, x_2^*) such that $x_1^* + x_2^* > c$. To see this, assume $x_1^* > 0$ again. Then roommate 1 can lower his contribution by a small $\varepsilon > 0$ so that $x_1^* + x_2^* - \varepsilon > c$.

Because $c > \max\{v_1, v_2\}$, no roommate is willing to buy the air conditioner on his own. Thus, the strategy profile $(x_1^*, x_2^*) = (0, 0)$ is a Nash equilibrium

Moreover, it is easy to see that any strategy profile in $\{(x_1, x_2) \mid x_1 + x_2 = c,\ v_i - x_i \geq 0 \text{ for } i = 1, 2\}$ is a Nash equilibrium. Because $c < v_1 + v_2$, this set is nonempty.

sequentially:

(a). $S_1 = \mathbb{R}_+$

$S_2 = \{x_2 : \mathbb{R}_+ \to \mathbb{R}_+\}$. 注意2的策略是映射.

(b). unique SPE: (x_1^*, x_2^*).

$x_1^* = c - v_2$.

$$x_2^*(x_1) = \begin{cases} 0 & x_1 < c - v_2 \\ c - x_1 & x_1 \geq c - v_2 \end{cases}$$

讨论方法与HW6一致.

(c). NE: (x_1^*, x_2^*) 不可置信的威胁.

$x_1^* = v_1$

$$x_2^*(x_1) = \begin{cases} 0 & x_1 \neq v_1 \\ c - v_1 & x_1 = v_1. \end{cases}$$