Suggested Solutions to Game Theory Final Exam, Fall 2022

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1. Two investors $i = 1, 2$ want to start a startup. Investor $i = 1$ provides the capital, denoted by $k \in [0, 1]$. Investor $i = 2$ provides the labor, denoted by $\ell \in [0, 1]$. They will equally share the revenue, which is equal to $2k\ell$. The cost of capital for $i = 1$ is k^3, while the cost of labor for $i = 2$ is $\frac{\ell^2}{2}$. Thus, the payoffs are $v_1(k, \ell) = k\ell - k^3$ and $v_2(k, \ell) = k\ell - \frac{\ell^2}{2}$, respectively. They choose their investment simultaneously. We restrict attention to pure strategies.

   (a) **(10 points)** Find all Nash equilibria.

   Soln: Using first order conditions, it is easy to show that the best response correspondences are

$$BR_1(\ell) = \sqrt{\frac{\ell}{3}} \text{ and } BR_2(k) = k.$$

   Then, strategy profile (k, ℓ) is a Nash equilibrium if and only if

$$k = \sqrt{\frac{\ell}{3}} \text{ and } \ell = k.$$

   Hence there are two pure strategy Nash equilibria: $(0, 0)$ and $(\frac{1}{3}, \frac{1}{3})$.

   (b) **(10 points)** Suppose this stage game is repeatedly played twice with common discount factor $\delta = 1$. Does there exist a subgame perfect equilibrium in which they play $(k, \ell) = (\frac{1}{4}, \frac{1}{4})$ in the first period?

   Soln: The following strategy profile is a subgame perfect equilibrium:

$$s_1^1 = \frac{1}{4} \text{ and } s_1^2(h) = \begin{cases} \frac{1}{3}, & \text{if } h = (\frac{1}{4}, \frac{1}{4}), \\ 0, & \text{otherwise,} \end{cases}$$

   and

$$s_2^1 = \frac{1}{4} \text{ and } s_2^2(h) = \begin{cases} \frac{1}{3}, & \text{if } h = (\frac{1}{4}, \frac{1}{4}), \\ 0, & \text{otherwise,} \end{cases}$$

   In the second period, they play a stage Nash equilibrium after any history, no one has a profitable deviation. Consider the first period. If $i = 1$ does not deviate, his total payoff is

$$\frac{1}{4} \times \frac{1}{4} - \frac{1}{4^3} + \frac{1}{3} \times \frac{1}{3} - \frac{1}{3^3} > \frac{1}{3} \times \frac{1}{3} - \frac{1}{3^3} = \frac{1}{9} \times \frac{2}{3} > \frac{1}{18}.$$

If he deviates, his total payoff is at most

$$\max_k k \times \frac{1}{4} - k^3 + 0 = \frac{1}{6\sqrt{12}} < \frac{1}{18}.$$

Therefore, there is no profitable one-shot deviation in the first period for $i = 1$. For $i = 2$, observe that $\ell = \frac{1}{4}$ is a best response to $k = \frac{1}{4}$ in the stage game. Then, it is clear that $i = 2$ has no one-shot profitable deviation in the first period either. This means that the constructed strategy profile is a subgame perfect equilibrium.

(c) **(10 points)** Suppose this stage game is infinitely repeated with common discount factor $\delta \in (0,1)$. For any $(k,\ell) \in [0,1]$ such that $k^2 < \ell < 2k$, show that when δ is sufficiently large, there is a subgame perfect equilibrium in which (k,ℓ) is always played on the equilibrium path.

Soln: Consider the following trigger strategy profile: $s_1^1 = k$, $s_2^1 = \ell$, and for any $t \geq 2$ and $h = (a^1, \ldots, a^{t-1}) \in H_{t-1}$,

$$s_1^t(h) = \begin{cases} k, & \text{if } a^s = (k,\ell) \text{ for all } 1 \leq s \leq t-1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$s_2^t(h) = \begin{cases} \ell, & \text{if } a^s = (k,\ell) \text{ for all } 1 \leq s \leq t-1, \\ 0, & \text{otherwise.} \end{cases}$$

That is, they play the proposed (k,ℓ) repeatedly on path, and any deviation leads to playing $(0,0)$ permanently.

For any history h that is not all (k,ℓ), the continuation play $s|_h$ is simply playing the same stage Nash equilibrium $(0,0)$ after any history. We know it is a subgame perfect equilibrium for any δ. Hence, no one has a profitable deviation at such h.

Consider a history h that is either $\emptyset$ or consists of all (k,ℓ). If $i = 1$ follows $s|_h$, his payoff is $k\ell - k^3 > 0$. If he deviates in the current period only, his payoff is at most

$$(1-\delta) \max_{k' \geq 0} (k'\ell - k'^3) + \delta \times 0.$$

Thus, there exists δ_1 sufficiently close to 1 such that

$$k\ell - k^3 > (1-\delta) \max_{k' \geq 0} (k'\ell - k'^3) + \delta \times 0, \ \forall \delta \in (\delta_1, 1).$$

Similarly, if $i = 2$ follows $s|_h$, his payoff is $k\ell - \ell^2 > 0$. If he deviates in the current period only, his payoff is at most

$$(1-\delta)\max_{\ell' \geq 0}(k\ell' - \ell'^2) + \delta \times 0.$$

Thus, there exists δ_2 sufficiently close to 1 such that

$$k\ell - \ell^2 > (1-\delta)\max_{\ell' \geq 0}(k\ell' - \ell'^2) + \delta \times 0, \ \forall \delta \in (\delta_2, 1).$$

Let $\underline{\delta} = \max\{\delta_1, \delta_2\}$. Then, we know that no one has a profitable deviation at such h when $\delta(\underline{\delta}, 1)$. This means that the constructed strategy profile is a subgame perfect equilibrium.

2. Two students, $i = 1, 2$, are taking an online exam. A TA is proctoring. Each student can choose whether to cheat or not. For simplicity, we assume cheating is purely an individual behavior. Thus, we rule out the possibility of communication between students. The TA can monitor the students by looking at their live videos. Unfortunately, due to technical reasons, the TA can monitor at most one student. Hence, the TA must choose who to monitor (if any). In other words, the TA has three pure strategies: not to monitor at all, monitor student $i = 1$, and monitor student $i = 2$. If a student cheats and is monitored, he will be caught, in which case he obtains $-\frac{1}{2}$. If a student cheats and is not monitored, he will not be caught, in which case he obtains 1. If student i does not cheat, he obtains x_i regardless of whether he is monitored. Assume x_i is i's private information. Assume that x_1 is uniformly distributed over $[0, 1]$, and x_2 is distributed according to cdf $F(x) = x^\alpha$ over $[0, 1]$ for some $\alpha > 0$. x_1 and x_2 are independent. The TA's payoff is the total number of cheating she catches. Students and the TA move simultaneously.

   (a) **(10 points)** Show that there is no Bayesian Nash equilibrium in which the TA plays a pure strategy.

   Soln: Because $1 > x_i$, if i is not monitored at all, he will cheat for sure. Because $x_i > -\frac{1}{2}$, if i is monitored for sure, he will not cheat at all. Therefore, in any equilibrium, if the TA plays a pure strategy, regardless of what this strategy is, her payoff must be 0 and there must be a student who cheats for sure. If the TA deviates to monitor this particular student, she can obtain 1.

(b) **(10 points)** Show that there is an essentially unique Bayesian Nash equilibrium and characterize it.

Soln: Let p_0 denote the probability that the TA does not monitor at all. Let p_i denote the probability that the TA monitors $i = 1, 2$. Clearly $p_0 + p_1 + p_2 = 1$. Consider student $i = 1, 2$ of type x_i. If he cheats, his expected payoff is $(1 - p_i) + p_i(-\frac{1}{2}) = 1 - \frac{3p_i}{2}$. If he does not cheat, his payoff is x_i. Therefore, he cheats if $x_i < 1 - \frac{3p_i}{2}$ and does not cheat if $x_i > 1 - \frac{3p_i}{2}$.

Note that i never cheats if $1 - \frac{3p_i}{2} \le 0$, or equivalently $p_i \ge \frac{2}{3}$, and cheats with positive probability if $p_i < \frac{2}{3}$. Because we can not have $p_1 \ge \frac{2}{3}$ and $p_2 \ge \frac{2}{3}$ simultaneously, in equilibrium at least one student must be cheating with positive probability. This implies that $p_0 = 0$ because not monitoring obtains payoff 0 while monitoring this particular student obtains positive payoff. Therefore, in equilibrium, we must have $p_1 \in (0, \frac{2}{3})$, $p_2 \in (0, \frac{2}{3})$, and $p_1 + p_2 = 1$. This requires that the TA be indifferent between monitoring $i = 1$ and $i = 2$:

$$\left(1 - \frac{3p_1}{2}\right) = \left(1 - \frac{3p_2}{2}\right)^\alpha. \tag{1}$$

Using $p_1 + p_2 = 1$, we can rewrite (1) as

$$\left(1 - \frac{3p_1}{2}\right) = \left(-\frac{1}{2} + \frac{3p_1}{2}\right)^\alpha.$$

Over $[\frac{1}{3}, \frac{2}{3}]$ (the domain of p_1), the left hand side is strictly decreasing while the right hand side is strictly increasing in p_1. At $p_1 = \frac{1}{3}$, the left hand side is positive while the right hand side is zero. This relationship is reversed at $p_1 = \frac{2}{3}$. Therefore, there is a unique $p_1 \in (\frac{1}{3}, \frac{2}{3})$ such that (1) holds, and we can obtain p_2 by $p_2 = 1 - p_1$. This (p_1, p_2) together with the above students' strategies gives an essentially unique equilibrium.

(c) **(10 points)** Show that the TA monitors $i = 1$ more than $i = 2$ if and only if $\alpha > 1$. Briefly explain the intuition.

Soln: Note that $(1 - \frac{3p_2}{2})^\alpha < 1 - \frac{3p_2}{2}$ if and only if $\alpha > 1$. From (1), we then know that $1 - \frac{3p_1}{2} < 1 - \frac{3p_2}{2}$ if and only if $\alpha > 1$. Equivalently, $p_1 > p_2$ if and only if $\alpha > 1$.

When $\alpha > 1$, for any $x \in (0, 1)$, $F(x) = x^\alpha < x$, or equivalently $1 - x^\alpha > 1 - x$. This means that for any level of x, the probability that student 2 obtains a payoff higher than x from not cheating is strictly higher than student 1 does. In other words, student 2 is a better student than student 1. Therefore, from the TA's point view, student 1 has larger incentive to cheat. Consequently, she monitors student 1 more.

3. This question asks you to think about who choose *pass-fail grading system* and who choose the usual *hundred-mark system* for your final in a simplified model.

   There is a student who can be one of the four types: $t \in T = \{65, 75, 85, 95\}$. Type is his private information. He will take an exam. For now, we assume that there is no uncertainty in his exam score. More specifically, his exam score s will just equal his type t for sure. Before the exam, the student can choose either pass-fail grading system, which only print letter P on his transcript and does not reveal his score, or hundred-mark system, which print his score on the transcript. Notice that we have assumed $t \geq 60$. Thus, the student will pass for sure and we do not need to consider grade F in pass-fail system. There is a firm who hires this student and observes his transcript. The firm must decide the wage $w \in \mathbb{R}$ for the student. Its payoff function is $-(w - \sqrt{s})^2$. That is, the firm wants to choose a wage that matches the square root of the student's score. The payoff to the student is just the wage w received from the firm. Let $\mu \in \Delta(T)$ be the prior distribution of the student type. Assume $\mu(t) > 0$ for all $t \in T$.

   (a) **(5 points)** Draw a game tree for this game.

   Soln: See Figure 1.

   (b) **(5 points)** Suppose the student chooses pass-fail system if and only if his type is $t = 75$ or $t = 85$. After observing P on the transcript, what is the firm's consistent belief about the score? Which wage is sequentially rational given this belief?

   Soln: Because P is on the path of play, consistent belief after observing P must be obtained via Bayes' rule. Therefore, after observing P, the firm believes that it is $t = 75$ with probability $\frac{\mu(75)}{\mu(75)+\mu(85)}$, and it is $t = 85$ with probability $\frac{\mu(85)}{\mu(75)+\mu(85)}$.
   Given this belief, the firm chooses w to maximize

$$-\frac{\mu(75)}{\mu(75) + \mu(85)}(w - \sqrt{75})^2 - \frac{\mu(85)}{\mu(75) + \mu(85)}(w - \sqrt{85})^2.$$

   The sequentially rational wage is then

$$w = \frac{\mu(75)}{\mu(75) + \mu(85)}\sqrt{75} + \frac{\mu(85)}{\mu(75) + \mu(85)}\sqrt{85}.$$

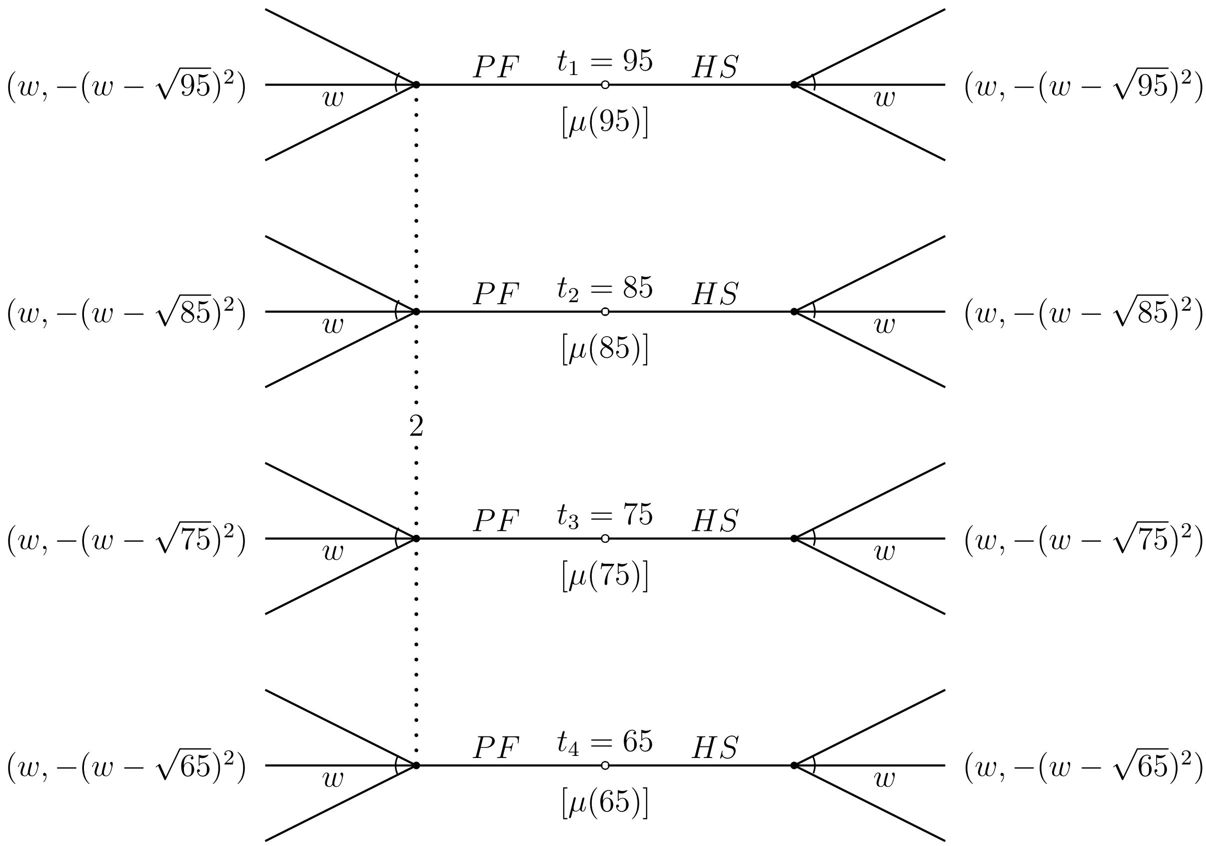

Figure 1: The signaling game for Question 3(a)

(c) **(15 points)** Show that in all perfect Bayesian equilibria, the firm's belief after observing P is the same. What is this belief?

Soln: Consider an arbitrary PBE. Let $\nu \in \Delta(T)$ be the firm's belief after observing P. Sequential rationality then requires that the wage after P, denoted by w_P, maximizes

$$-\sum_{t \in T} \nu(t)(w - \sqrt{t})^2,$$

which implies that $w_P = \sum_{t \in T} \nu(t)\sqrt{t}$. We discuss two cases.

First, suppose P is off path. This implies that $t = 65$ weakly prefers hundred-mark system than pass-fail system: $\sqrt{65} \geq w_P = \sum_{t \in T} \nu(t)\sqrt{t}$. Because $65 = \min T$, this inequality can hold only if $\nu(65) = 1$. In words, the firm thinks that it is type $t = 65$ for sure after observing P.

Next, suppose P is on path. We first argue that one and only one type chooses P with positive probability. Suppose, by contradiction, that there are at least two types who choose P with positive probability. Let $\bar{t} \in T$ be the largest type who chooses P with positive probability. Then, Bayes' rule implies that $\nu(\bar{t}) < 1$. Therefore, $w_P = \sum_{t \in T} \nu(t)\sqrt{t} < \sum_{t \in T} \nu(t)\sqrt{\bar{t}} = \sqrt{\bar{t}}$. This implies that choosing the hundred-mark system would give $\bar{t}$ strictly higher payoff than the pass-fail system. This contradicts the assumption that $\bar{t}$ chooses P with positive probability. Hence, there is only one type who chooses P with positive probability. We then argue that this only type must be $t = 65$. Suppose again, by contradiction, that $\hat{t} \neq 65$ is the only type who chooses P. Then $w_P = \sqrt{\hat{t}} > \sqrt{65}$. This implies that the pass-fail system is strictly better than the hundred-mark system for $t = 65$, which contradicts the assumption that $t = 65$ chooses the hundred-mark system. Therefore, we have shown that only $t = 65$ chooses P with positive probability. Consistency then requires that $\nu(65) = 1$.

In summary, in any PBE, the firm believes for sure that it is $t = 65$ after observing P.

(d) **(15 points)** Suppose now that the exam score is random and so the student faces uncertainty before the exam. In particular, the score for type $t \in T$ student can be $t - 5$, t, or $t + 5$ with equal probabilities. Everything else is the same as before: the student knows his type and chooses either pass-fail system or hundred-mark system before the exam; the firm observes either P if the student chooses pass-fail system, or the realized score if the student chooses hundred-mark system. How would your answer to Question 3(b) change? How would your answer to Question 3(c) change?

Soln: Consider Question 3(b) first. Clearly, the firm's consistent belief about *types* after observing P does not change. Its belief about *scores* clearly changes due to the introduction of randomness in the student's score. In particular, this belief is

$$\frac{\mu(75)}{3(\mu(75)+\mu(85))} \circ 70 + \frac{\mu(75)}{3(\mu(75)+\mu(85))} \circ 75 + \frac{\mu(75)}{3(\mu(75)+\mu(85))} \circ 80$$
$$+\frac{\mu(85)}{3(\mu(75)+\mu(85))} \circ 80 + \frac{\mu(85)}{3(\mu(75)+\mu(85))} \circ 85 + \frac{\mu(85)}{3(\mu(75)+\mu(85))} \circ 90.$$

The sequentially rational wage should be then

$$\frac{\mu(75)}{3(\mu(75)+\mu(85))}\sqrt{70}+\frac{\mu(75)}{3(\mu(75)+\mu(85))}\sqrt{75}+\frac{\mu(75)}{3(\mu(75)+\mu(85))}\sqrt{80}$$
$$+\frac{\mu(85)}{3(\mu(75)+\mu(85))}\sqrt{80}+\frac{\mu(85)}{3(\mu(75)+\mu(85))}\sqrt{85}+\frac{\mu(85)}{3(\mu(75)+\mu(85))}\sqrt{90}.$$

Consider Question 3(c) next. We argue that in any PBE, the firm must believe that it is type $t = 65$ for sure after observing P, as in Question 3(c). Equivalently, the firm must believes that the score is 60, 65, and 70 with equal probabilities.

In fact, the argument is essentially the same as above. Consider any PBE and let ν be the firm's belief about types after observing P. Then the sequentially rational wage w_P is

$$w_P = \sum_{t\in T}\nu(t)\frac{\sqrt{t-5}+\sqrt{t}+\sqrt{t+5}}{3} = \sum_{t\in T}\nu(t)w_t,$$

where $w_t \equiv \frac{\sqrt{t-5}+\sqrt{t}+\sqrt{t+5}}{3}$ is type t's expected payoff if he chooses the hundred-mark system. Note that w_t is strictly increasing in t.

If P is off path, we must have $w_{65} \geq w_P$, which in turn implies that $\nu(65) = 1$ as before. If P is on path, then we can show as before that there is one and only one type who chooses P with positive probability and this type must be $t = 65$. In other words, $\nu(65) = 1$ as well.