Game Theory, Fall 2022
Problem Set 5

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1. ST 7.2

   (a) **Soln:** The game tree is depicted in Figure 1, and it is a game of perfect information.

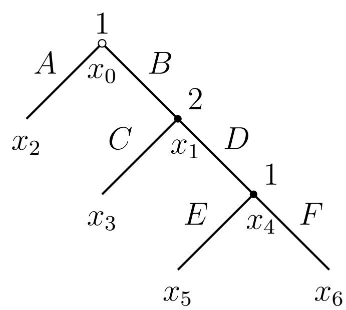

Figure 1: The game tree for Question 1

   (b) **Soln:** There are 4 terminal nodes, x_2, x_3, x_5, and x_6. There are 3 information sets, $\{x_0\}$, $\{x_1\}$, and $\{x_4\}$.

   (c) **Soln:** Player 1 has 4 pure strategies AE, AF, BE, and BF. Player 2 has 2 pure strategies C and D.

   (d) **Soln:** The extensive form of the question is depicted in Figure 2. And the normal form representation is depicted in Figure 3.

   The pure strategy Nash equilibria of the game are (AE, D), (AF, D), and (BE, C). We now consider mixed strategy Nash equilibria.

   - Equilibria in which player 2 plays C.

*Special thanks go to Peixuan Fu and Shuang Wu, who wrote the last version of these solutions.

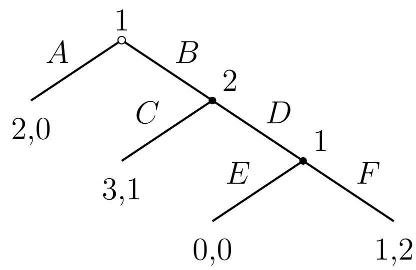

Figure 2: The extensive form for Question 1

Player 1 \ Player 2	C	D
AE	2,0	2,0
AF	2,0	2,0
BE	3,1	0,0
BF	3,1	1,2

Figure 3: The normal form representation for Question 1

Given that player 2 plays C for sure, player 1 will only mix between BE and BF. For player 2 to have no incentive to deviate to D, we must have

$$1 \times \sigma_1(BE) + 1 \times (1 - \sigma_1(BE)) \geq 2 \times (1 - \sigma_1(BE)),$$

or equivalently

$$\sigma_1(BE) \geq \frac{1}{2}.$$

Thus, $(\sigma_1(BE) \circ BE + (1 - \sigma_1(BE)) \circ BF\ ,\ C)$ is a mixed Nash equilibrium for any $\sigma_1(BE) \in [\frac{1}{2}, 1)$.

- Equilibria in which player 2 plays D.
  Given that player 2 plays D for sure, player 1 will only mix between AE and AF. Clearly, given any such mixture, player 2 has no incentive to deviate to C. Therefore, $(\sigma_1(AE) \circ AE + (1 - \sigma_1(AE)) \circ AF\ , D)$ is a Nash equilibrium for any $\sigma_1(AE) \in (0, 1)$.
- Equilibria in which player 2 mixes.
  Given that player 2 mixes between C and D, BF is strictly better than BE for player 1. Hence, in such an equilibrium, $\sigma_1(BE) = 0$. In addition, if $\sigma_1(BE) = 0$ and $\sigma_1(BF) > 0$, then D is strictly better than C for player 2. Hence, if player 2 mixes between C and D, $\sigma_1(BF) = 0$ as well. Therefore, player 1 can only mix between AE and AF. Given such mixture, player 2 is

obviously indifferent. To guarantee that player 1 has no incentive to deviate, we must have

$$2 \times \sigma_2(C) + 2 \times (1 - \sigma_2(C)) \geq 3 \times \sigma_2(C) + 0 \times (1 - \sigma_2(C)),$$
$$2 \times \sigma_2(C) + 2 \times (1 - \sigma_2(C)) \geq 3 \times \sigma_2(C) + 1 \times (1 - \sigma_2(C)),$$

or equivalently,

$$\sigma_2(C) \leq \frac{1}{2}.$$

Thus, $(\sigma_1(AE) \circ AE + (1 - \sigma_1(AE)) \circ AF\ ,\ \sigma_2(C) \circ C + (1 - \sigma_2(C)) \circ D)$ can be a mixed Nash equilibrium for $\sigma_1(AE) \in [0, 1]$ and $\sigma_2(C) \in (0, \frac{1}{2}]$.

2. ST 7.3

Let's assume that if player 1 choose X or O in the first step, he can only choose the same mark in what follows and player 2 can only choose the other mark. (The case where players can choose each mark at each step is even more complicated, but not conceptually different.)

(a) **Soln:** This game is a game of perfect information, because every player, whenever called upon to move, perfectly observes what has happened previously.

(b) **Soln:** Player 2 has 18 information sets after the player 1's first move. To see this, we only need to consider how many choices player 1 has at the initial node. Player 1 chooses from 2 kinds of symbols and then chooses from 9 blanks in the 3×3 matrix and for each blank. In total, there are 18 different choices thus 18 different histories.

(c) **Soln:** Consider how many different histories are there after player 2's moves for the first time. Player 2 has 18 information sets after player 1's move, and for each information set player 2 has 8 different choices (there are 8 blanks left). In total, there are $18 \times 8 = 144$ different histories. Hence, player 1 has 144 information sets after each of player 2's moves for the first time.

(d) **Soln:** Player 1 has $1 + 18 \times 8 + 18 \times 8 \times 7 \times 6 + 18 \times 8 \times 7 \times 6 \times 5 \times 4 + 18 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 852,913$ information sets. Player 2 has $18 + 18 \times 8 \times 7 + 18 \times 8 \times 7 \times 6 \times 5 + 18 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 394,146$.

(e) **Soln:** The game has $2 \times 9! = 2 \times 2 \times 3 \times \ldots \times 8 \times 9 = 725,760$ terminal nodes.

3. ST 7.5

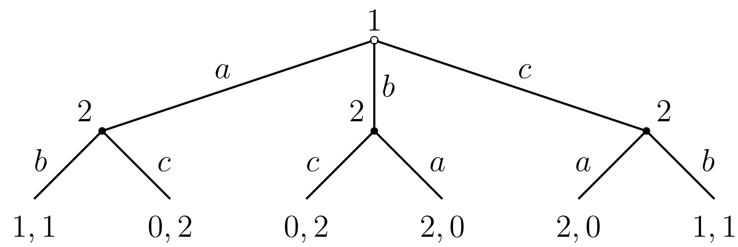

Figure 4: The game tree for Question 3

(a) **Soln:** Let $v_1(a) = v_2(c) = 2$, $v_1(b) = v_2(b) = 1$, $v_1(c) = v_2(a) = 0$. The game tree is depicted in Figure 4.

(b) **Soln:** Player 1 has 3 pure strategies, $S_1 = \{a, b, c\}$. For player 2, $S_2 = \{bca, bcb, baa, bab, cca, ccb, caa, cab\}$ where $s_2 = xyz$ means that player 2 chooses $x \in \{b, c\}$ after player 1 vetoes a, $y \in \{c, a\}$ after player 1 vetoes b, and $z \in \{a, b\}$ after player 1 vetoes c. There are 8 pure strategies for player 2.

(c) **Soln:** We do not consider mixed strategy Nash equilibria here since we only know the players' ordinal preferences.

It's easy to get the best response functions:

$$BR_1(s_2) = \begin{cases} c, & \text{if } s_2 = bca, \\ \{a, c\}, & \text{if } s_2 = bcb, \\ \{b, c\}, & \text{if } s_2 = baa, \\ b, & \text{if } s_2 = bab, \\ c, & \text{if } s_2 = cca, \\ c, & \text{if } s_2 = ccb, \\ \{b, c\}, & \text{if } s_2 = caa, \\ b, & \text{if } s_2 = cab, \end{cases}$$

$$BR_2(s_1) = \begin{cases} \{cca, ccb, caa, cab\}, & \text{if } s_1 = a, \\ \{bca, bcb, cca, ccb\}, & \text{if } s_1 = b, \\ \{bcb, bab, ccb, cab\}, & \text{if } s_1 = c. \end{cases}$$

Then we can get all pure strategy Nash equilibria. That is (c, bcb) and (c, ccb).

4. ST 7.8

(a) **Soln:** Let S denote splitting $10 - 10$ and G denote giving 20 in the first stage. The game tree is in Figure 5.

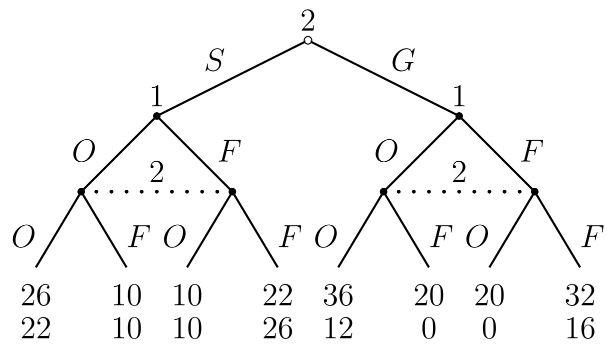

Figure 5: The game tree for Question 4

(b) **Soln:** Both players can condition their choices in the Battle of the Sexes game on the initial choice of player 2. For player 1, $S_1 = \{OO, OF, FO, FF\}$ where $s_1 = xy$ means that player 1 chooses $x \in \{O, F\}$ after player 2 chooses S while player 2 chooses $y \in \{O, F\}$ after player 1 chooses G. For player 2, however, even though he chooses first between S or G, he must specify his action for each information set even if he knows it will not happen (e.g. what he will do following S even when he plans to play G). Hence he has 8 pure strategies, $S_2 = \{SOO, SOF, SFO, SFF, GOO, GOF, GFO, GFF\}$ where $s_2 = xyz$ means that player 2 first chooses $x \in \{S, G\}$ and then chooses $y \in \{O, F\}$ if he played S and $z \in \{O, F\}$ if he played G.

(c) **Soln:** This will be a 4×8 matrix. See Figure 6.

		Player 2							
		SOO	SOF	SFO	SFF	GOO	GOF	GFO	GFF
Player 1	OO	26, 22	26, 22	10, 10	10, 10	36, 12	20, 0	36, 12	20, 0
	OF	26, 22	26, 22	10, 10	10, 10	20, 0	32, 16	20, 0	32, 16
	FO	10, 10	10, 10	22, 26	22, 26	36, 12	20, 0	36, 12	20, 0
	FF	10, 10	10, 10	22, 26	22, 26	20, 0	32, 16	20, 0	32, 16

Figure 6: The normal form for Question 4

(d) **Soln:** Firstly we note that for Player 2, GOO, GOF, GFO and GFF are all strictly dominated by $(0.51 \circ SOO + 0.49 \circ SFO)$. As a result, both players can determine their strategies in equilibrium without considering all strategies of player 2 with G as the first choice. In this case, we could focus on a partly reduced form of the game, depicted in Figure 7.

		Player 2	
		SO	SF
Player 1	O	26,22	10,10
	F	10,10	22,26

Figure 7: The reduced normal form game for Question 4

Apparently, the Nash equilibria of the reduced form are (O, SO), (F, SF), and $(\frac{4}{7} \circ O + \frac{3}{7} \circ F, \frac{3}{7} \circ SO + \frac{4}{7} \circ SF)$. Then we can get all the Nash equilibria of the original form game, which form the set:

$$\{(p \circ OO + (1-p) \circ OF\ ,\ q \circ SOO + (1-q) \circ SOF) :\ p,\ q \in [0, 1]\}$$

$$\cup\{(p \circ FO + (1-p) \circ FF\ ,\ q \circ SFO + (1-q) \circ SFF) :\ p,\ q \in [0, 1]\}$$

$$\cup\{(p_1 \circ OO + (\frac{4}{7} - p_1) \circ OF + p_2 \circ FO + (\frac{3}{7} - p_2) \circ FF\ ,$$

$$q_1 \circ SOO + (\frac{3}{7} - q_1) \circ SOF + q_2 \circ SFO + (\frac{4}{7} - q_2) \circ SFF) :$$

$$p_1, q_2 \in [0, \frac{4}{7}],\ p_2, q_1 \in [0, \frac{3}{7}]\}.$$

5. ST 7.9

(a) **Soln:** The game tree is depicted in Figure 8.

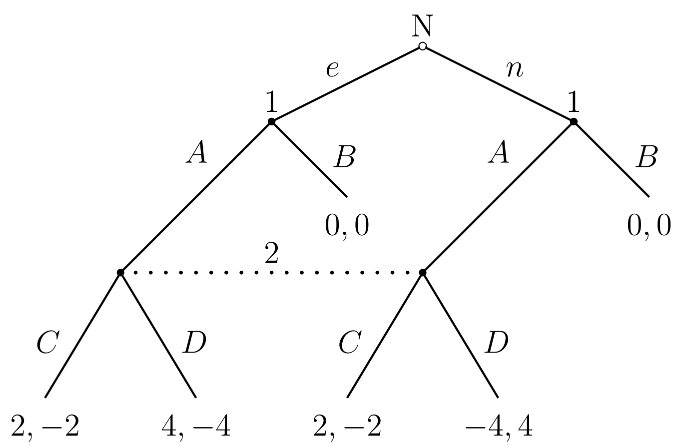

Figure 8: The extensive form for Question 5

(b) **Soln:** For player 1, $S_1 = \{AA, AB, BA, BB\}$, where $s_1 = mn$ means that player 1 chooses $m \in \{A, B\}$ if he has evidence and chooses $n \in \{A, B\}$ if he has no evidence. For player 2, $S_2 = \{C, D\}$. The normal form is in Figure 9.

		Player 2	
		C	D
Player 1	AA	2,-2	0,0
	AB	1,-1	2,-2
	BA	1,-1	-2,2
	BB	0,0	0,0

Figure 9: The normal form game for Question 5

(c) **Soln:** Note first that BB for player 1 is strictly dominated by AB, and BA is strictly dominated by $\frac{1}{2} \circ AA + \frac{1}{2} \circ AB$. Therefore, to consider the Nash equilibria, we only need to consider player 1's pure strategies AA and AB.

Clearly, there is no pure strategy Nash equilibrium. It is also easy to see that there is no equilibrium in which only one player mixes. Therefore, the only candidate equilibrium is the one that both players mix. For player 1 to be indifferent between AA and AB, we must have

$$2\sigma_2(C) = \sigma_2(C) + 2(1 - \sigma_2(C)),$$

implying

$$\sigma_2(C) = \frac{2}{3}.$$

For player 2 to be indifferent between C and D, we must have

$$-2\sigma_1(AA) - (1 - \sigma_1(AA)) = -2(1 - \sigma_1(AA)),$$

implying

$$\sigma_1(AA) = \frac{1}{3}.$$

In conclusion, there's only one Nash equilibrium: $(\frac{1}{3} \circ AA + \frac{2}{3} \circ AB, \frac{2}{3} \circ C + \frac{1}{3} \circ D)$.