Suggested Solutions to Game Theory Midterm Exam, Fall 2022

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1. Two workers, $i = 1, 2$, operates together a machine to produce a product. If they make efforts $e_1 \geq 0$ and $e_2 \geq 0$ respectively, the total outputs are $e_1^{\alpha_1} e_2^{\alpha_2}$, where $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 + \alpha_2 < 1$. Worker i's cost of making effort e_i is $c_i e_i$, where $c_i > 0$.

   (a) **(10 points)** A pair of efforts (e_1, e_2) is said to be efficient, if it maximizes the total surplus: total outputs minus total costs. There is a unique efficient pair. Find it.

   Soln: We only need to consider the following optimization problem

   $$\max_{e_1, e_2 \geq 0} e_1^{\alpha_1} e_2^{\alpha_2} - c_1 e_1 - c_2 e_c.$$

   The first order conditions are

   $$\alpha_1 e_1^{\alpha_1 - 1} e_2^{\alpha_2} - c_1 = 0,$$
   $$\alpha_2 e_1^{\alpha_1} e_2^{\alpha_2 - 1} - c_2 = 0.$$

   Hence, we have

   $$\hat{e}_1 = (\alpha_1^{1-\alpha_2} \alpha_2^{\alpha_2} c_1^{\alpha_2 - 1} c_2^{-\alpha_2})^{\frac{1}{1-\alpha_1-\alpha_2}},$$
   $$\hat{e}_2 = (\alpha_1^{\alpha_1} \alpha_2^{1-\alpha_1} c_1^{-\alpha_1} c_2^{\alpha_1 - 1})^{\frac{1}{1-\alpha_1-\alpha_2}}.$$

   This is the unique efficient pair.

   (b) **(5 points)** Suppose the two workers choose their own efforts simultaneously. Suppose also that they always share the total outputs with equal shares. That is, if the total output is y, each worker obtains $\frac{1}{2}y$. Is $(e_1, e_2) = (0, 0)$ a Nash equilibrium of this game?

   Soln: It is a Nash equilibrium. For example, given $e_2 = 0$, agent 1's payoff from $e_1 > 0$ is $-c_1 e_1 < 0$. By choosing $e_1 = 0$, agent 1 can obtain 0.

   (c) **(10 points)** Find a Nash equilibrium $(e_1, e_2) \neq (0, 0)$. Is it efficient?

   Soln: We look for an equilibrium (e_1^*, e_2^*) such that $e_1^* > 0$ and $e_2^* > 0$. Such a pair is an equilibrium if and only if, for $i = 1, 2$,

   $$e_i^* \in \arg\max_{e_i \geq 0} \frac{1}{2} e_i^{\alpha_i} (e_{-i}^*)^{\alpha_{-i}} - c_i e_i.$$

The first order conditions are

$$\frac{1}{2}\alpha_1(e_1^*)^{\alpha_1-1}(e_2^*)^{\alpha_2} = c_1,$$
$$\frac{1}{2}\alpha_2(e_1^*)^{\alpha_1}(e_2^*)^{\alpha_2-1} = c_2.$$

We can obtain

$$e_1^* = (\frac{1}{2}\alpha_1^{1-\alpha_2}\alpha_2^{\alpha_2}c_1^{\alpha_2-1}c_2^{-\alpha_2})^{\frac{1}{1-\alpha_1-\alpha_2}},$$
$$e_2^* = (\frac{1}{2}\alpha_1^{\alpha_1}\alpha_2^{1-\alpha_1}c_1^{-\alpha_1}c_2^{\alpha_1-1})^{\frac{1}{1-\alpha_1-\alpha_2}}.$$

(d) **(10 points)** An allocation rule is a pair of functions $p_1, p_2 : \mathbb{R}_+ \to \mathbb{R}$ such that $p_1(y) + p_2(y) = y$ for all $y \geq 0$. Every allocation rule p_1, p_2 defines a simultaneously move game between the two workers: they simultaneously choose their own efforts, and if the total output is y, worker i obtains $p_i(y)$. For instance, the situation in Question 1(b) corresponds to allocation rule $p_1(y) = p_2(y) = \frac{1}{2}y$. The requirement $p_1(y) + p_2(y) = y$ for all $y \geq 0$ is the budget balance condition: the sum of what each worker obtains is equal to the total output. Given any allocation rule p_1, p_2, show that if p_1 and p_2 are differentiable, then any pure strategy Nash equilibrium (if it exists) in the induced simultaneous effort choice game is not efficient.

Soln: Consider an arbitrary budget balanced, differentiable allocation rule (p_1, p_2). Let $(\tilde{e}_1, \tilde{e}_2)$ be a Nash equilibrium in the induced game. Suppose, by contradiction, it is efficient. From 1(a), we then know that

$$\frac{\partial f(\tilde{e}_1, \tilde{e}_2)}{\partial e_1} = c_1,$$
$$\frac{\partial f(\tilde{e}_1, \tilde{e}_2)}{\partial e_2} = c_2,$$

where $f(e_1, e_2) \equiv e_1^{\alpha_1}e_2^{\alpha_2}$. But because it is a Nash equilibrium, we also know that

$$p_1'(f(\tilde{e}_1, \tilde{e}_2))\frac{\partial f(\tilde{e}_1, \tilde{e}_2)}{\partial e_1} = c_1,$$
$$p_2'(f(\tilde{e}_1, \tilde{e}_2))\frac{\partial f(\tilde{e}_1, \tilde{e}_2)}{\partial e_2} = c_2.$$

Combining these four equations yields

$$p_1'(f(\tilde{e}_1, \tilde{e}_2)) = p_2'(f(\tilde{e}_1, \tilde{e}_2)) = 1.$$

But this is impossible, since $p_1(y)+p_2(y) = y$ for all y implies $p_1'(y)+p_2'(y) = 1$ for all y. Therefore, $(\tilde{e}_1, \tilde{e}_2)$ is not efficient.

2. Consider the following two player simultaneous move game. The strategy spaces are $S_1 = S_2 = \{1, \ldots, 100\}$ and the payoff functions are

$$u_1(s_1, s_2) = \begin{cases} -s_1, & \text{if } s_1 \leq s_2, \\ 0, & \text{if } s_1 > s_2, \end{cases} \text{ and } u_2(s_1, s_2) = \begin{cases} -s_2, & \text{if } s_2 \leq s_1, \\ 0, & \text{if } s_2 > s_1. \end{cases}$$

(a) **(5 points)** Find agent 1's set of pure strategy best responses to $s_2 = 100$.

Soln: Clearly, $BR_1(s_2 = 100) = \{1\}$.

(b) **(10 points)** Is there any strategy in this game that is strictly dominated by a pure strategy?

Soln: No. Consider player 1. We have already know that $s_1 = 1$ is a best response to $s_2 = 100$. For any $s_1 > 1$, it is a best response to any $s_2 < s_1$. Hence, player 1's every strategy is a best response to some player 2's strategy. We then know that 1 has no strictly dominated strategy.

(c) **(10 points)** There are two pure strategy Nash equilibria. Find them.

Soln: We first find a Nash equilibrium in which $s_1 \leq s_2$. Note that we can not have $s_1 \leq s_2 < 100$. This is because player 1 has incentive to deviate to $s_2 + 1$ to obtain payoff 0 instead of payoff $-s_1$. Therefore, we are left with $s_1 \leq s_2 = 100$. Since s_1 is a best response, we know $s_1 = 1$ by the previous question. Note also that given $s_1 = 1$, $s_2 = 100$ is a best response. Therefore $(1, 100)$ is indeed an equilibrium. By symmetry, the other equilibrium is $(100, 1)$.

(d) **(10 points)** Find a Nash equilibrium in mixed strategies.

Soln: We guess a completely mixed strategy profile (σ_1, σ_2). That is, $\sigma_i(s_i) > 0$ for all i and s_i. It is an equilibrium if and only if every player is indifferent among all pure strategies.

Note that $u_i(1, \sigma_j) = -1$. Thus, indifference conditions require $u_i(s_i, \sigma_j) = -1$ for all s_i. That is, for all s_i,

$$-s_i \sum_{s_j = s_i}^{100} \sigma_j(s_j) = -1,$$

or equivalently

$$\sum_{s_j = s_i}^{100} \sigma_j(s_j) = \frac{1}{s_i}, \ \forall s_i = 1, \ldots, 100.$$

Therefore,

$$\sigma_j(100) = \frac{1}{100}$$

and

$$\sigma_j(s_j) = \sum_{s'_j=s_j}^{100} \sigma_j(s'_j) - \sum_{s'_j=s_j+1}^{100} \sigma_j(s'_j) = \frac{1}{s_j} - \frac{1}{s_j+1}, \ \forall s_j = 1, \ldots, 99.$$

3. A scandal of a pop star is leaked to a media outlet. If the media outlet reports to the public, the pop star will lose $L > 0$. To avoid this happening, the pop star first makes an offer $p \geq 0$ to the media outlet. After observing the offer, the media outlet decides whether to accept it or reject it. If the offer is accepted, the media outlet obtains p and does not report the scandal to the public, in which case the pop star obtains $-p$. If the offer is rejected, the media outlet reports to the public, in which case the media outlet obtains $M > 0$ and the pop star obtains $-L$. We assume that $M < L < 2M$.

   (a) **(10 points)** Find a subgame perfect equilibrium of this game.

   Soln: The unique subgame perfect equilibrium is

$$s_1 = M \text{ and } s_2(p) = \begin{cases} r, & \text{if } p < M, \\ a, & \text{if } p \geq M, \end{cases}$$

   where r denotes rejection while a denotes acceptance.

   (b) **(10 points)** Find a Nash equilibrium of this game that is not subgame perfect.

   Soln: For instance,

$$s_1 = M \text{ and } s_2(p) = \begin{cases} r, & \text{if } p \neq M, \\ a, & \text{if } p = M. \end{cases}$$

   (c) **(10 points)** Now, assume that there are two, instead of one, media outlets, $i = 1, 2$, who know this scandal. The interaction between the pop star and the two media outlets becomes the following. The pop star first make a pair of offers (p_1, p_2) to these two media outlets, where $p_i \geq 0$ is the money that the pop star is willing to pay to media outlet i for not reporting. We assume that every media outlet observes both p_1 and p_2. After both observing the pair of offers, the two media outlets simultaneously and independently decide whether to accept or reject. If i accepts, it obtains p_i and does not report. If i rejects, it reports and obtains M if it is the only one who reports, or $\frac{M}{2}$ if the other one also reports. The pop star obtains $-L$ if at least one media outlet reports. Therefore, his total payoff is $-L$ if both rejects, or $-L - p_i$ if i

accepts but $j \neq i$ rejects, or $-p_1 - p_2$ if both accepts. Find a subgame perfect equilibrium.

Soln: Denote by $\Gamma(p_1, p_2)$ the subgame folloing the pop star's offer (p_1, p_2). It is a simultaneous move game and its game matrix is given in Figure 1. If (a, a) is an equilibrium in this subgame, we have $p_1 \geq M$ and $p_2 \geq M$, in which case the pop star's payoff is $-p_1 - p_2 \leq -2M < -L$. If (a, r) is an equilibrium in this subgame, we have $p_1 \geq \frac{M}{2}$, in which case the pop star's payoff is $-p_1 - L < -L$. Similarly, if (r, a) is an equilibrium, the pop star's payoff is $-p_2 - L < -L$. If (r, r) is an equilibrium, the pop star's payoff is $-L$. Therefore, in any subgame perfect equilibrium, the pop star's offer (p_1^*, p_2^*) must lead to Nash equilibrium (r, r) in $\Gamma(p_1^*, p_2^*)$. This can be done by offering $p_1^* = p_2^* = 0$.

		Media 2	
		a	r
Media 1	a	p_1, p_2	p_1, M
	r	M, p_2	$M/2, M/2$

Figure 1: Subgame $\Gamma(p_1, p_2)$ for Question 3c

Therefore, it is now easy to construct a SPE. The pop star offers $(p_1^*, p_2^*) = (0, 0)$. In $\Gamma(p_1, p_2)$, the two media outlets play

$$\begin{cases} (a, a), & \text{if } p_1 \geq M, p_2 \geq M, \\ (r, r), & \text{if } p_1 < \frac{M}{2}, p_2 < \frac{M}{2}, \\ (a, r), & \text{if } p_1 \geq \frac{M}{2}, p_2 < M, \\ (r, a), & \text{if } p_1 < \frac{M}{2}, p_2 \geq \frac{M}{2} \text{ or } \frac{M}{2} \leq p_1 < M, p_2 \geq M. \end{cases}$$

It is easy to verify that in $\Gamma(p_1, p_2)$, the constructed strategy profile is a Nash equilibrium. Moreover, the pop star has no incentive to deviate as we have already known.

Note that this is not the unique SPE in this game. For example, if the pop star offer $(\frac{M}{4}, \frac{M}{4})$, it is still a SPE. For another example, for some offer (p_1, p_2), there may be multiple equilibria in $\Gamma(p_1, p_2)$. We can choose an arbitrary one in this case.