# Game Theory Lecture Notes 2
# Static Games with Complete Information

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# Motivating Examples

## The prisoner's dilemma

- Two suspects are held in different rooms at the police station.
- Each of them has two choices: confess ($C$) or deny ($D$).
- If both of them deny, then both get $2$ years in prison.
- If only one of them deny, then the suspect who denies gets $5$ years in prison while the one who confesses gets $1$ year.
- If both of them confess, then both get $4$ years in prison.

# Motivating Examples

## The prisoner's dilemma

- We can represent this situation by the following $2 \times 2$ matrix:

| | | Suspect 2 | |
|---|---|---|---|
| | | $D$ | $C$ |
| Suspect 1 | $D$ | $-2, -2$ | $-5, -1$ |
| | $C$ | $-1, -5$ | $-4, -4$ |

Figure 2.1: The prisoner's dilemma

- Each row represents one of suspect $1$'s choice.
- Each column represents one of suspect $2$'s choice.
- Each entry in this matrix contains two numbers. The first one is suspect $1$'s payoff while the second one is suspect $2$'s payoff.
- Because each entry contains two numbers instead of one, such matrix is sometimes called a bi-matrix.

# Motivating Examples

## The prisoner's dilemma

- Adding each of the numbers by $4$, we obtain:

| | $E$ | $S$ |
|---|---|---|
| $E$ | $2, 2$ | $-1, 3$ |
| $S$ | $3, -1$ | $0, 0$ |

Figure 2.2: The partnership game

- $E$ represents effort while $S$ represents shirking.

# Motivating Examples

## Rock-paper-scissors

- Two players simultaneously choose among rock ($R$), paper ($P$) and scissors ($S$).
- Rock beats scissors, scissors beats paper, and paper beats rock.
- The loser pays one dollar to the winner.
- If they choose the same, it is a draw and no one pays.
- Its situation is represented by the following $3 \times 3$ matrix:

| | $R$ | $P$ | $S$ |
|---|---|---|---|
| $R$ | $0, 0$ | $-1, 1$ | $1, -1$ |
| $P$ | $1, -1$ | $0, 0$ | $-1, 1$ |
| $S$ | $-1, 1$ | $1, -1$ | $0, 0$ |

Figure 2.3: Rock-paper-scissors

# Motivating Examples

## Voting on a new agenda

- Three voters on a committee vote on whether to remain at the status quo or adopt a new policy.
- Each can vote "yes" ($Y$), "no" ($N$), or "abstain" ($A$).
- Each receives a payoff $0$ from the status quo.
- Players $1$ and $2$ receive a payoff $1$ from the new policy, while player $3$ receive $-1$ from it.
- Majority rule: the new policy is adopted only if strictly more than half of the voting voters vote $Y$.

# Motivating Examples

## Voting on a new agenda

- We can represent this situation by the following matrices:

| | $Y$ | $N$ | $A$ |
|---|---|---|---|
| $Y$ | $1, 1, -1$ | $1, 1, -1$ | $1, 1, -1$ |
| $N$ | $1, 1, -1$ | $0, 0, 0$ | $0, 0, 0$ |
| $A$ | $1, 1, -1$ | $0, 0, 0$ | $1, 1, -1$ |

$Y$

| | $Y$ | $N$ | $A$ |
|---|---|---|---|
| $Y$ | $1, 1, -1$ | $0, 0, 0$ | $0, 0, 0$ |
| $N$ | $0, 0, 0$ | $0, 0, 0$ | $0, 0, 0$ |
| $A$ | $0, 0, 0$ | $0, 0, 0$ | $0, 0, 0$ |

$N$

| | $Y$ | $N$ | $A$ |
|---|---|---|---|
| $Y$ | $1, 1, -1$ | $0, 0, 0$ | $1, 1, -1$ |
| $N$ | $0, 0, 0$ | $0, 0, 0$ | $0, 0, 0$ |
| $A$ | $1, 1, -1$ | $0, 0, 0$ | $0, 0, 0$ |

$A$

Figure 2.5: Voting on a new agenda

- Voter $1$ chooses one of the rows, voter $2$ chooses one of the columns, and voter $3$ chooses one of the tables.

# Normal Form Games

- Each of the above matrix informs us of three components:
  - The players;
  - Each player's available choices;
  - The payoff to each player as a function of all players' choices.
- In fact, these three components together define a **normal form game**.
- Matrix representation is very intuitive. But it has its obvious drawbacks: it can only represent simple games.
- Therefore, we need a more general and formal way to describe a game.

# Normal Form Games



## Definition 2.1

A **normal form game** includes three components as follows:

1. A finite **set of players**, $N = \{1, , 2, \ldots, n\}$.
2. A **strategy space** $S_i$ for each player $i \in N$.
3. A **payoff function** $v_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ for each player $i \in N$.

In summary, a normal form game is a triple $\Gamma = (N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n)$.

- Strategy space $S_i$ contains all the available strategies for player $i$.
- Each element $s_i \in S_i$ is a player $i$'s **strategy**.
- A vector of strategies $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ is a **strategy profile**.
- A normal form game $G$ is **finite** if every player's strategy space contains only finitely many strategies.

# Normal Form Games

## Definition

- A short hand notation.
- Given a strategy profile $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ and a player $i$, we usually use

$$s_{-i} \equiv (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$$

  to denote the strategies chosen by the players who are not player $i$.
- Thus, $S_{-i} \equiv S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$ is the set of all possible combinations of player $i$'s opponents' strategies.
- In this spirit, we write $s = (s_i, s_{-i})$ to denote the whole strategy profile. For example, we usually write $v_i(s_i, s_{-i})$. In this way, we can easily separate the strategy chosen by player $i$ and those chosen by other players.

# Normal Form Games

## Examples

- The prisoner's dilemma:
  - Set of players: $N = \{1, 2\}$;
  - Strategy space: $S_1 = S_2 = \{C, D\}$;
  - Payoffs:

$$v_1(D, D) = v_2(D, D) = -2,$$

$$v_1(C, C) = v_2(C, C) = -4,$$

$$v_1(D, C) = v_2(C, D) = -5,$$

$$v_1(C, D) = v_2(D, C) = -1.$$

# Normal Form Games

## Examples

- Rock-paper-scissors:
  - Set of players: $N = \{1, 2\}$;
  - Strategy space: $S_1 = S_2 = \{R, P, S\}$;
  - Payoffs:

$$v_1(R, R) = v_1(P, P) = v_1(S, S) = 0,$$
$$v_1(R, S) = v_1(S, P) = v_1(P, R) = 1,$$
$$v_1(S, R) = v_1(P, S) = v_1(R, P) = -1,$$
$$v_2(R, R) = v_2(P, P) = v_2(S, S) = 0,$$
$$v_2(S, R) = v_2(P, S) = v_2(R, P) = 1,$$
$$v_2(R, S) = v_2(S, P) = v_2(P, R) = -1.$$

# Normal Form Games

## Examples

- Voting on a new agenda:
  - $N = \{1, 2, 3\}$.
  - $S_1 = S_2 = S_3 = \{Y, N, A\}$.
  - For $i = 1, 2$,

$$v_i(s_1, s_2, s_3) = \begin{cases} 1, & \text{if } |\{j|s_j = Y\}| > |\{j|s_j \neq A\}|/2, \\ 0, & \text{otherwise.} \end{cases}$$

  - For $i = 3$,

$$v_3(s_1, s_2, s_3) = \begin{cases} -1, & \text{if } |\{j|s_j = Y\}| > |\{j|s_j \neq A\}|/2, \\ 0, & \text{otherwise.} \end{cases}$$

# Normal Form Games

## Examples

- Cournot competition.
- Two firms compete in their supplies in a market with demand curve

$$P(Q) = \max\{100 - Q,\ 0\},$$

  where $Q$ is the total supply of the products.
- Firm $i$'s cost is $q_i^2$ if $q_i$ units of the product is produced.
- Normal form game:
  - $N = \{1, 2\}$;
  - $S_1 = S_2 = \mathbb{R}_+$;
  - $v_i(q_i, q_j) = D(q_i + q_j)q_i - q_i^2$.
- What about $n > 2$ firms?

# Normal Form Games

## Examples

- Bertrand competition.
- Two firms compete in price in a market.
- The market contains a continuum of consumers of total mass $1$.
- If firm $i$'s price is lower than that of firm $j \neq i$, all consumers will buy from firm $i$.
- If two firms set the same price, then half of the buyers will buy from one firm and the others will buy from the other firm.
- Firm $i$ has a constant marginal cost $c_i$.
- Normal form game:
  - $N = \{1, 2\}$;
  - $S_1 = S_2 = \mathbb{R}_+$;
  - $$v_i(p_i, p_j) = \begin{cases} p_i - c_i, & \text{if } p_i < p_j, \\ 0, & \text{if } p_i > p_j, \\ \frac{1}{2}(p_i - c_i), & \text{if } p_i = p_j. \end{cases}$$

# Normal Form Games

## Examples

- First price auction.
- There is one good to be sold.
- There are two bidders.
- The value of the good to bidder $i$ is $v_i \geq 0$.
- The bidders simultaneously bid a price.
- The one with the higher price wins the good and pays his own bid.
- The loser gets nothing and does not pay.
- If there is a tie in bids, then the good is randomly allocated to the two bidders with equal probability.

# Normal Form Games

## Examples

- Normal form game:
  - $N = \{1, 2\}$;
  - $S_1 = S_2 = \mathbb{R}_+$;
  - $$v_i(p_i, p_j) = \begin{cases} v_i - p_i, & \text{if } p_i > p_j, \\ \frac{1}{2}(v_i - p_i), & \text{if } p_i = p_j, \\ 0, & \text{if } p_i < p_j. \end{cases}$$

# Normal Form Games

## Examples

- Second price auction.
- Everything is the same with the first price auction except the payment rule.
- The winning bidder pays the second highest bid.
- Normal form game:
  - $N = \{1, 2\}$,
  - $S_1 = S_2 = \mathbb{R}_+$,
  - $$v_i(p_i, p_j) = \begin{cases} v_i - p_j, & \text{if } p_i > p_j, \\ \frac{1}{2}(v_i - p_j), & \text{if } p_i = p_j, \\ 0, & \text{if } p_i < p_j. \end{cases}$$

# Dominance

## Dominated strategies

- Consider the prisoner's dilemma:

| | $D$ | $C$ |
|---|---|---|
| $D$ | $-2, -2$ | $-5, -1$ |
| $C$ | $-1, -5$ | $-4, -4$ |

Figure 2.6: The prisoner's dilemma

- Focus on player $1$ (row player):
  - If player $2$ chooses $D$, choosing $D$ is strictly worse than $C$ for player $1$;
  - If player $2$ chooses $C$, choosing $D$ is strictly worse than $C$ for player $1$ as well.
- In sum, choosing $D$ is strictly worse than $C$ for player $1$ *regardless of what player* $2$ *chooses*.
- A similar observation applies to player $2$ (column player) too: choosing $D$ is strictly worse than $C$ for player $2$ *regardless of what player* $1$ *chooses*.

# Dominance

## Dominated strategies

### Definition 2.2

Let $s_i \in S_i$ and $s'_i \in S_i$ be possible strategies for player $i$. We say that $s'_i$ is **strictly dominated** by $s_i$ if for any possible combination of the other players' strategies $s_{-i} \in S_{-i}$, player $i$'s payoff from $s'_i$ is strictly less than that from $s_i$. That is,

$$v_i(s_i, s_{-i}) > v_i(s'_i, s_{-i}), \ \forall s_{-i} \in S_{-i}.$$

- In the prisoner's dilemma, $D$ is strictly dominated by $C$ for both players.

# Dominance

## Dominated strategies

- A simple observation: a rational player will (should) never play a strictly dominated strategy.
- For the prisoner's dilemma, each player will never choose $D$.
- Therefore, both players will choose $C$ since each player only has two possible strategies.
- For the prisoner's dilemma, rationality alone is enough to offer a prediction about which outcome will prevail: $(C, C)$.

# Dominance

## Dominated strategies

| | $L$ | $M$ | $H$ |
|---|---|---|---|
| $L$ | $6, 8$ | $2, 6$ | $0, 4$ |
| $M$ | $8, 4$ | $4, 2$ | $1, 3$ |
| $H$ | $4, 1$ | $3, 0$ | $2, 2$ |

Figure 2.7: A $3 \times 3$ game

- For player $1$, $L$ is strictly dominated by $M$. But neither $M$ nor $H$ is strictly dominated.
- For player $2$, $M$ is strictly dominated by $L$. But neither $L$ nor $H$ is strictly dominated.
- For this game, rationality alone does not offer a unique prediction.

# Dominance

## Dominant strategies

- The following concept, loosely speaking, is the converse of strictly dominated strategies.

### Definition 2.3

$s_i \in S_i$ is a **strictly dominant strategy** for $i$ if every other strategy of $i$ is strictly dominated by it, that is, for all $s_i' \in S_i$, $s_i' \neq s_i$,

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}), \ \forall s_{-i} \in S_{-i}.$$

- In the prisoner's dilemma, $C$ is a strictly dominant strategy for every player.
- In the game in Figure 2.7, no player has a strictly dominant strategy.

- A simple observation: a rational player will (should) play his strictly dominant strategy if he has one.
- This observation provides us our first solution concept in this course.

**Definition 2.4**

The strategy profile $s \in S$ is a **strictly dominant strategy equilibrium** if $s_i \in S_i$ is a strictly dominant strategy for all $i \in N$.

- $(C, C)$ is a strictly dominant strategy equilibrium for the prisoner's dilemma. In this equilibrium, every player chooses $C$. The **equilibrium outcome** is $(C, C)$ and the **equilibrium payoff** is $(-4, -4)$.

# Dominance

## Dominant strategies

- As we have seen, players may not have strictly dominant strategies.
- But if a player has a strictly dominant strategy, it must be unique.

### Lemma 2.1

*If player $i$ has a strictly dominant strategy, it is unique.*

- Yes?
- As a corollary, strictly dominant strategy equilibrium may not exist.
- But if it exists, it is unique.

### Proposition 2.1

*If the game $\Gamma = (N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n)$ has a strictly dominant strategy equilibrium $s$, then $s$ is the unique dominant strategy equilibrium.*

- Yes?

# Dominance

## Dominant strategies

- Is the equilibrium outcome of a strictly dominant strategy equilibrium always desirable?

### Definition 2.5

A strategy profile $s \in S$ **Pareto dominates** strategy profile $s' \in S$ if $v_i(s) \geq v_i(s')$ for all $i \in N$ and $v_i(s) > v_i(s')$ for at least one $i \in N$ (in which case, we will also say that $s'$ is **Pareto dominated** by $s$). A strategy profile is **Pareto optimal** if it is not Pareto dominated by any other strategy profile.

- A strategy profile is Pareto dominated if there is another strategy profile that makes every player at least as good as before and some player strictly better.
- Pareto optimal is sometimes called **Pareto efficient**.

# Dominance

## Dominant strategies

- In the prisoner's dilemma, $(C, C)$ is the unique strictly dominant strategy equilibrium.
- It is not Pareto optimal: $(D, D)$ Pareto dominates $(C, C)$.
- Compare with competitive market equilibrium outcome.

# Iterated elimination of strictly dominated strategies

An example

Player 2

| Player 1 | | $L$ | $C$ | $R$ |
|---|---|---|---|---|
| | $U$ | $4,3$ | $5,1$ | $6,2$ |
| | $M$ | $2,1$ | $8,4$ | $3,6$ |
| | $D$ | $3,0$ | $9,6$ | $2,8$ |

Figure 2.8: An example for IESDS

- Neither player $1$ nor player $2$ has strictly dominant strategy.
- Player $1$ has no strictly dominated strategy.
- $C$ for player $2$ is strictly dominated by $R$.
- As we have noted earlier, rationality of player $2$ predicts that player $2$ will never play $C$.

# Iterated elimination of strictly dominated strategies

## An example

- Suppose player $1$ knows that player $2$ is rational.
- Then player $1$ will predict that player $2$ will never play $C$.
- Thus, when considering his own strategy, player $1$ can effectively ignore the possibility of player $2$ choosing $C$.
- In other words, player $1$ essentially views the game as follows:

| | $L$ | $R$ |
|---|---|---|
| $U$ | $4, 3$ | $6, 2$ |
| $M$ | $2, 1$ | $3, 6$ |
| $D$ | $3, 0$ | $2, 8$ |

- Observe that both $M$ and $D$ are strictly dominated by $U$ for player $1$ in this reduced game.
- Hence, rationality of player $1$ predicts that he will never play $M$ and $D$.

# Iterated elimination of strictly dominated strategies

## An example

- Let's go even further.
- Suppose player $2$ knows that
  - player $1$ is rational; and
  - player $1$ knows that player $2$ is rational.
- Then player $2$ will predict that player $1$ will never play $M$ and $D$.
- This implies that player $2$ essentially view the game as follows

| | $L$ | $R$ |
|---|---|---|
| $U$ | $4, 3$ | $6, 2$ |

- Observe that $R$ is strictly dominated by $L$ for player $2$ in this further reduced game.
- Hence, rationality of player $2$ then predicts that he will never play $R$.

# Iterated elimination of strictly dominated strategies

## An example

- Thus, we obtain the unique prediction of this game: $(U, L)$.
- Notice that rationality of both players alone does not lead to this prediction.
- We need additional assumptions on the two players' **knowledge** about rationality:
  - both players are rational;
  - both players know that their opponents are rational;
  - player $2$ knows that player $1$ knows that player $2$ is rational.

# Iterated elimination of strictly dominated strategies



### Definition 2.6

An event $E$ is **common knowledge** if (1) everyone knows $E$, (2) everyone knows that everyone knows $E$, and so on *ad infinitum*.

- Under the assumption of common knowledge of rationality, our previous arguments work.

# Iterated elimination of strictly dominated strategies

## Common knowledge

- To get more sense of common knowledge, consider the following situation.
- A boyfriend and a girlfriend want to date on Valentine's day night.
- Unfortunately, there is a chance that the boyfriend has to work overtime, in which case they can not date.
- If the boyfriend does not need to work overtime, he will send a message to his girlfriend so that they can go out and date.
- But there is possibility that the message will be lost due to some technology problems.
- Thus, they reached the agreement that once the girlfriend receives the message, she sends back a confirmation.
- But there is also possibility that the girlfriend's message is lost.

# Iterated elimination of strictly dominated strategies

## Common knowledge

- So, if the boyfriend receives the confirmation, he will send a re-confirmation.
- And it is possible that this re-confirmation is lost.
- So, if the girlfriend receives this re-confirmation, she will send a re-re-confirmation.
- ......
- Unfortunate couple: no matter how many messages they have sent and received, it is *never* common knowledge that they can date!

# Iterated elimination of strictly dominated strategies

IESDS

- The process of **iterated elimination of strictly dominated strategies** can be described inductively as follows.
  - Let $S_i^0 = S_i$.
  - Suppose we have defined $\{S_i^k\}_{i=1}^n$. If there are players who have strictly dominated strategies in the reduced game $(N, \{S_i^k\}_{i=1}^n, \{v_i^k\}_{i=1}^n)$, delete all of them to get $\{S_i^{k+1}\}_{i=1}^n$. If there is no strictly dominated strategy in $(N, \{S_i^k\}_{i=1}^n, \{v_i^k\}_{i=1}^n)$, then stop.
- Then any strategy $s_i \in \cap_{k\geq 0} S_i^k = S_i^*$ is said to **survive the process of iterated elimination of strictly dominated strategies**.
- Any strategy profile $s \in S_1^* \times \cdots \times S_n^*$ is called an **iterated-elimination equilibrium**.

# Iterated elimination of strictly dominated strategies

## IESDS

- For finite games, the above process must stop in finite steps. In fact, the resulting set of strategies that survive IESDS does not depend on the order of deletion. The only requirement is that *some* (not necessarily all) strictly dominated strategies be deleted in each round.
- For infinite games, the process may never end. Moreover, if in each round, only some strictly dominated strategies are deleted, then the result will rely on the choice of these strategies in each round. In other words, the result will depend on the order of deletion. For this reason, we always require that *all* strictly dominated strategies be deleted in each round.

# Iterated elimination of strictly dominated strategies

## IESDS

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $w$ | $6,4$ | $4,4$ | $4,5$ | $12,2$ |
| $x$ | $5,9$ | $8,7$ | $5,8$ | $10,6$ |
| $y$ | $2,10$ | $7,6$ | $4,6$ | $9,5$ |
| $z$ | $4,4$ | $5,9$ | $4,10$ | $10,9$ |

- Deleting all strictly dominated strategies in each round:
  - 1st round: $y$ for player $1$ and $d$ for player $2$;
  - 2nd round: $z$ for player $1$ and $b$ for player $2$;
  - $S_1^* = \{w, x\}$ and $S_2^* = \{a, c\}$.
- Another order of deletion:
  - 1st round: $y$ for player $1$;
  - 2nd round: $b$ and $d$ for player $2$;
  - 3rd round: $z$ for player $1$;
  - $S_1^* = \{w, x\}$ and $S_2^* = \{a, c\}$.

# Iterated elimination of strictly dominated strategies

## IESDS

- Note that there might be more than one strategies that survive IESDS.
- Hence IESDS may not provide a unique prediction about the outcome.
- But

**Proposition 2.2**

*If for a game $\Gamma = (N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n)$, $s$ is a strict dominant strategy equilibrium, then $s_i$ uniquely survives IESDS for every player $i$.*

- Reason: for every $s_i' \neq s_i$, $s_i'$ is strictly dominated by $s_i$. Hence in the first round, all strategies other than $s_i$ is deleted for player $i$. The process ends just in one round.

# Iterated elimination of strictly dominated strategies

Example: Cournot Duopoly

- Consider Cournot duopoly with constant marginal cost.
- Demand curve $P(Q) = \max\{100 - Q, 0\}$.
- Payoff function

$$\begin{aligned} v_i(q_i, q_j) &= \max\{100 - (q_i + q_j), 0\} q_i - 10 q_i \\ &= \max\{(90 - q_i - q_j) q_i, -10 q_i\}. \end{aligned}$$

- Initial round $S_1^0 = S_2^0 = \mathbb{R}_+$.

# Iterated elimination of strictly dominated strategies

## Example: Cournot Duopoly

- Focus on firm $1$.
- If $q_2 \geq 90$, then any positive $q_1$ yields negative profit for firm $1$. Thus, $q_1 = 0$ is the profit maximizing quantity for firm $1$ because it guarantees zero profit.
- If $q_2 < 90$, then firm $1$ can earn positive profit. In fact, the profit maximizing quantity solves

$$\max_{q_1 \geq 0} (90 - q_1 - q_2)q_1.$$

- The first order condition yields

$$q_1 = \frac{90 - q_2}{2}.$$

- In sum, the profit maximizing quantity for firm $1$ as a function of firm $2$'s output is

$$q_1 = \max\left\{\frac{90 - q_2}{2}, 0\right\}.$$

# Iterated elimination of strictly dominated strategies

## Example: Cournot Duopoly

- Observe that, as $q_2$ changes over $\mathbb{R}_+$, firm $1$'s profit maximizing quantity changes over $[0, 45]$.
- We thus claim that any quantity $q_1 \in [0, 45]$ for firm $1$ is *not* strictly dominated: there exists $q_2 = 90 - 2q_1 \in S_2^0$ such that

$$v_1(q_1, q_2) \geq v_1(q_1', q_2),\ \forall q_1' \in S_1^0.$$

- Therefore, in the first round, quantities in $[0, 45]$ should *not* be deleted.

# Iterated elimination of strictly dominated strategies

## Example: Cournot Duopoly

- How about quantities above $45$?
- We claim that any $q_1 > 45$ is strictly dominated by $q_1' = 45$.
- To see this, we only need to observe that for any $q_2 \in S_2^0$, the mapping

$$q_1 \mapsto \max\{(90 - q_1 - q_2)q_1, -10q_1\}$$

  is strictly decreasing over the interval $[45, +\infty)$.
- This implies that for all $q_1 > 45$,

$$v_1(45, q_2) > v_1(q_1, q_2),\ \forall q_2 \in S_2^0,$$

  meaning that $q_1$ is strictly dominated by $45$.
- Therefore, quantities in $(45, \infty)$ should be deleted in the first round.
- We obtain $S_1^1 = [0, 45]$.
- By symmetry, we have $S_2^1 = [0, 45]$, completing the first round deletion.

# Iterated elimination of strictly dominated strategies

## Example: Cournot Duopoly

- We now move the second round.
- Because $S_1^1 = S_2^1 = [0, 45]$, we can write

$$v_i(q_i, q_j) = (90 - q_i - q_j)q_i.$$

- Focus on firm $1$ again.
- For any $q_2 \in S_2^1$, the profit maximizing quantity for firm $1$ in $S_1^1$ is simply

$$q_1 = \frac{90 - q_2}{2}.$$

- As $q_2$ changes in $S_2^1$, this profit maximizing quantity changes over $[22.5, 45]$.
- Then, we know that any quantity in $[22.5, 45]$ is *not* strictly dominated for firm $1$ in the second round and should be kept.

# Iterated elimination of strictly dominated strategies

## Example: Cournot Duopoly

- How about quantities below $22.5$?
- We claim that any $q_1 < 22.5$ is strictly dominated by $q_1' = 22.5$.
- To see this, simply observe that for any $q_2 \in S_2^1$, the mapping

  $$q_1 \mapsto (90 - q_1 - q_2)q_1$$

  is strictly increasing over $[0, 22.5]$.
- This implies for all $q_1 \in [0, 22.5)$,

  $$v(22.5, q_2) > v_1(q_1, q_2), \ \forall q_2 \in S_2^1,$$

  meaning that $q_1$ is strictly dominated by $22.5$.
- Therefore, quantities in $[0, 22.5)$ should be deleted.
- We obtain $S_1^2 = [22.5, 45]$.
- By symmetry, we have $S_2^2 = [22.5, 45]$, completing the second round deletion.

# Iterated elimination of strictly dominated strategies

## Example: Cournot Duopoly

- Given $S_1^2 = S_2^2 = [22.5, 45]$ and applying a similar argument as above, we can show that $S_1^3 = S_2^3 = [22.5, 33.75]$, where $33.75$ comes from $(90 - 22.5)/2$.
- Let $S_1^k = S_2^k = [\underline{q}^k, \overline{q}^k]$ for $k \geq 1$.
- We can show

$$\underline{q}^k = \phi^{(k-1)}(0) \text{ and } \overline{q}^k = \phi^{(k)}(0) \text{ if } k \text{ is odd,}$$

and

$$\underline{q}^k = \phi^{(k)}(0) \text{ and } \overline{q}^k = \phi^{(k-1)}(0) \text{ if } k \text{ is even,}$$

where $\phi(q) = (90 - q)/2$ and $\phi^{(k)}$ is the $k$th composition of $\phi$ itself.
- Because $\lim_{k\to\infty} \phi^{(k)}(0) = 30$, we know

$$S_i^* = \bigcap_{k\geq 1} S_i^k = \{30\}.$$

- Therefore, $q = 30$ is the unique quantity that survives IESDS for both firms. This is exactly the Cournot equilibrium outcome.

# Iterated elimination of strictly dominated strategies

## Example: Cournot Duopoly

$S^0: \quad [0, \quad +\infty)$

$S^1: \quad [0, \quad 45)$

$S^2: \quad [22.5, \quad 45)$

$S^3: \quad [22.5, \quad 33.75)$

$S^4: \quad [28.125, \quad 33.75)$

$\vdots$

red arrow means
$\phi(q) = (90 - q)/2$

Figure 2.9: IESDS in the Cournot game

# Best Response

- Both strictly dominant strategy equilibrium and IESDS are based on eliminating strategies that players would never play.
- An alternative approach is to ask what possible strategies might players choose to play and under what conditions?
- When we consider eliminating strategies that no rational player would play, it was by finding some strategy that is *always* better.
- A strategy that cannot be eliminated suggests that *under some conditions* this strategy is the one that the player may choose.
- When we qualify a strategy to be the best under some conditions, these conditions must be expressed in terms that are rigorous and are related to the game that is being played.

# Best Response

## The best response

| | $O$ | $F$ |
|---|---|---|
| $O$ | $2, 1$ | $0, 0$ |
| $F$ | $0, 0$ | $1, 2$ |

Figure 2.10: Battle of the sexes

- No one has strictly dominated strategies.
- If player $2$ chooses $O$, $O$ is the best for player $1$.
- If, instead, player $2$ chooses $F$, $F$ is the best for player $1$.

# Best Response



## Definition 2.7

The strategy $s_i \in S_i$ is player $i$'s **best response** to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}), \ \forall s_i' \in S_i.$$

- This is the most central concept of game theory!
- Sometimes, it is also called a **best reply**.
- In the battle of the sexes:
  - $O$ for player $i$ is a best response to his opponent's strategy $O$;
  - $F$ for player $i$ is a best response to his opponent's strategy $F$.

# Best Response

## The best response

- Best responses may not be unique for a given opponents' strategies $s_i \in S_{-i}$.

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $U$ | $3, 3$ | $5, 1$ | $6, 2$ |
| $M$ | $4, 1$ | $8, 4$ | $3, 6$ |
| $D$ | $4, 0$ | $9, 6$ | $6, 8$ |

Figure 2.11: A $3 \times 3$ game

- Both $M$ and $D$ are player $1$'s best responses to player $2$'s strategy $L$.
- $D$ is player $1$'s unique best response to player $2$'s strategy $C$.
- Both $U$ and $D$ are player $1$'s best responses to player $2$'s strategy $R$.

# Best Response

## The best response

- An observation: a rational player who believes that his opponents are playing some $s_{-i} \in S_{-i}$ will always choose a best response to $s_{-i}$.

### Proposition 2.3

*If $s_i$ is a strictly dominated strategy for player $i$, then it cannot be a best response to any $s_{-i} \in S_{-i}$.*

### Proof of Proposition 2.3.

Suppose $s_i$ is strictly dominated by $s_i'$. Then for any $s_{-i} \in S_{-i}$, we have

$$v_i(s_i', s_{-i}) > v_i(s_i, s_{-i}),$$

implying that $s_i$ is not a best response to $s_{-i}$. □

- Contrapositive: if $s_i$ is a best response to some $s_{-i} \in S_{-i}$, then it is not strictly dominated.

# Best Response

## The best response

### Proposition 2.4

*If $s_i$ is a strictly dominant strategy for player $i$, then it is the unique best response to any $s_{-i} \in S_{-i}$.*

- This directly comes from the definition of strict dominance and best response.

# Best Response

## The best response

### Proposition 2.5

*In a finite normal-form game, if $s^*$ uniquely survives IESDS, then $s_i^*$ is the unique best response to $s_{-i}^*$ for all $i \in N$.*

### Proof of Proposition 2.5.

Consider any player $i$. Suppose, by contradiction, that $s_i^*$ is not a best response to $s_{-i}^*$. Then there exists $s_i^1 \in S_i$ such that $v_i(s_i^1, s_{-i}^*) > v_i(s_i^*, s_{-i}^*)$. Because $s_i^1$ is deleted in some round of IESDS and because $s_{-i}^*$ survives the process, there must be $s_i^2 \in S_i$ such that $v_i(s_i^2, s_{-i}^*) > v_i(s_i^1, s_{-i}^*)$. But because $s_i^2$ is deleted, we can find $s_i^3$ such that $v_i(s_i^3, s_{-i}^*) > v_i(s_i^2, s_{-i}^*)$. This argument continues and we can find infinitely many such strategies. This contradicts the finite game assumption.

## Proof of Proposition 2.5 (Cont.)

If there is a strategy $s_i$ which is also a best response to $s^*_{-i}$, then by Proposition 2.3, $s_i$ is not strictly dominated in every round of the process because $s^*_{-i}$ survives. This proves that $s^*_i$ is the unique best response to $s^*_{-i}$. $\square$

- The assumption that the game is finite is indispensable. There will be a counter example in your problem set for infinite game.

# Best Response

## Best response correspondences

- When we think of player $i$'s best responses as a function of his opponents' strategies, we obtain player $i$'s **best response correspondence.**
- Think about a mapping $f: X \to Y$ that maps every element $x \in X$ into an *element* $f(x) \in Y$. We call this mapping a *function*.
- Think about a mapping $f: X \to 2^Y$ where $2^Y$ is the set of all subsets of $Y$. That is, $f$ maps every element $x \in X$ into a *subset* $f(x) \subseteq Y$. Of course, we can all $f$ a function from $X$ to $2^Y$. But more conveniently, we call $f$ a *correspondence* from $X$ to $Y$, and usually write $f: X \rightrightarrows Y$. In words, a correspondence is just a *set-valued* function.
- If $f: X \rightrightarrows Y$ is a correspondence and for every $x \in X$, $f(x)$ only contains one element, then $f$ is essentially a function: it maps every element $x \in X$ into an element in $Y$. For example, $f: X \rightrightarrows X$ with $f(x) = \{x\}$. In this case, we write $f(x) = x$ for simplicity.

# Best Response

## Best response correspondences

- Player $i$'s best response correspondence is a correspondence $BR_i : S_{-i} \Rightarrow S_i$ such that

$$BR_i(s_{-i}) = \{s_i \in S_i \mid s_i \text{ is a best response to } s_{-i}\}.$$

- Therefore, if player $i$ believes that his opponents are playing $s_{-i}$, he should play a strategy in $BR_{-i}(s_i)$.

- In the game in Figure 2.11, we have

$$BR_1(s_2) = \begin{cases} \{M, D\}, & \text{if } s_2 = L, \\ \{D\}, & \text{if } s_2 = C, \\ \{U, D\} & \text{if } s_2 = R, \end{cases}$$

  and

$$BR_2(s_1) = \begin{cases} L, & \text{if } s_1 = U, \\ R, & \text{if } s_1 = M, \\ R, & \text{if } s_1 = D. \end{cases}$$

- Note that we directly write $BR_2$ as a function because it is always single-valued.

# Best Response

## Best response correspondences

- Recall the Cournot duopoly with constant marginal cost:

$$v_i(q_i, q_j) = \max\{100 - q_i - q_j, 0\}q_i - 10q_i.$$

- In fact, we have already figured out

$$BR_1(q_2) = \max\left\{\frac{90 - q_2}{2}, 0\right\},$$

and by symmetry

$$BR_2(q_1) = \max\left\{\frac{90 - q_1}{2}, 0\right\}.$$

# Best Response

## Best response correspondences

- Recall the Bertrand competition: $N = \{1, 2\}$, $S_1 = S_2 = \mathbb{R}_+$, and

$$v_i(p_i, p_j) = \begin{cases} p_i - c_i, & \text{if } p_i < p_j, \\ \frac{1}{2}(p_i - c_i), & \text{if } p_i = p_j, \\ 0, & \text{if } p_i > p_j. \end{cases}$$

- Then

$$BR_i(p_j) = \begin{cases} (p_j, +\infty), & \text{if } p_j < c_i, \\ [p_j, +\infty), & \text{if } p_j = c_i, \\ \emptyset, & \text{if } p_j > c_i. \end{cases}$$

- Make sure you understand (1) why some interval is open while some is closed; (2) why there is $\emptyset$.

# Nash Equilibrium

## Definition

| | $O$ | $F$ |
|---|---|---|
| $O$ | $2, 1$ | $0, 0$ |
| $F$ | $0, 0$ | $1, 2$ |

Figure 2.12: Battle of the sexes

- There is no strictly dominant strategy equilibrium.
- IESDS yields "anything is possible": $S_1^* = S_2^* = \{O, F\}$.
- John Nash provided us an answer to this situation.

# Nash Equilibrium



## Definition 2.8

The strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ is a **Nash equilibrium** if for every $i \in N$, $s_i^*$ is a best response to $s_{-i}^*$. That is, for every $i \in N$,

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s_i, s_{-i}^*), \ \forall s_i \in S_i.$$

- Mutual best response: every player is playing a best response given their opponents' strategies.
- In terms of the best response correspondence: $s_i^* \in BR_i(s_{-i}^*)$ for all $i$.

## Definition 2.9

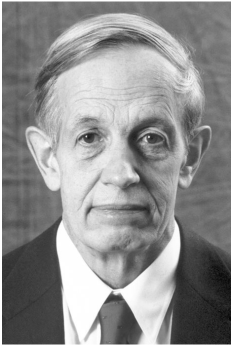
Given a strategy profile $s = (s_1, \ldots, s_n) \in S$, we say player $i$ has a **profitable deviation** if there exists $s_i'$ such that

$$v_i(s_i', s_{-i}) > v_i(s_i, s_{-i}).$$

- Nash equilibrium $\Longleftrightarrow$ no one has a profitable deviation.

# Nash Equilibrium





John F. Nash Jr.

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1994

"for their pioneering analysis of equilibria in the theroy of non-cooperative games"

Contribution: Introduced the distinction between cooperative games, in which binding agreements can be made, and non-cooperative games, where binding agreements are not feasible. Developed an equilibrium concept for non-cooperative games that now is called Nash equilibrium.

- The movie, *A Beautiful Mind*, is based the true story of John Nash.

# Nash Equilibrium



| | $O$ | $F$ |
|---|---|---|
| $O$ | $2, 1$ | $0, 0$ |
| $F$ | $0, 0$ | $1, 2$ |

- $(O, O)$ is a Nash equilibrium.
- $(O, F)$ is *not* a Nash equilibrium.
- $(F, O)$ is *not* a Nash equilibrium.
- $(F, F)$ is a Nash equilibrium.
- Notice that Nash equilibrium may not be unique. The definition of Nash equilibrium is silent about which equilibrium will/should be played.

# Nash Equilibrium

## Definition

- Nash equilibrium is about players' strategies (what they will do), not about the payoffs!

| | $O$ | $F$ |
|---|---|---|
| $O$ | $2, 1$ | $0, 0$ |
| $F$ | $2, 1$ | $1, 2$ |

- Nash equilibria: $(O, O)$ and $(F, F)$.
- Note that $(O, O)$ and $(F, O)$ yield the same payoffs to the players. However, $(F, O)$ is *not* a Nash equilibrium since player $2$ wants to deviate to $F$.

# Nash Equilibrium

## Definition

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $U$ | $7,7$ | $4,2$ | $1,8$ |
| $M$ | $2,4$ | $5,5$ | $2,3$ |
| $D$ | $8,1$ | $3,2$ | $0,0$ |

- Is there a Nash equilibrium in which player $1$ plays $U$?
  - If yes, player $2$ must play a best response to $U$. In this case, it is $R$.
  - But if player $2$ plays $R$, $U$ for player $1$ is not a best response.
  - Thus, the answer is no.
- Is there a Nash equilibrium in which player $1$ plays $M$?
  - If yes, player $2$ must play a best response to $M$. In this case, it is $C$.
  - Given player $2$ plays $C$, $M$ for player $1$ is a best response.
  - Thus, $(M, C)$ is a Nash equilibrium.
- Is there a Nash equilibrium in which player $1$ plays $D$?
  - If yes, player $2$ must play a best response to $D$. In this case, it is $C$.
  - But if player $2$ plays $C$, $D$ for player $1$ is not a best response.
  - Thus, the answer is no.

# Nash Equilibrium

## Definition

- Nash equilibrium is based on the following two principles:
  - Rationality: each player is playing a *best response* to his belief about the opponents' strategies.
  - Consistency: Players' beliefs about their opponents are *correct*.

$$\left.\begin{array}{ll} \text{player } 1 & \text{believes that player } 2 \text{ is playing } \hat{s}_2 \\ & \text{and plays a best response } s_1 \text{ to } \hat{s}_2, \\ \text{player } 2 & \text{believes that player } 1 \text{ is playing } \hat{s}_1 \\ & \text{and plays a best response } s_2 \text{ to } \hat{s}_1, \end{array}\right\} \xRightarrow[\text{requires}]{\text{consistency}} \begin{array}{l} \hat{s}_2 = s_2 \\ \hat{s}_1 = s_1 \end{array}$$

- An informal expression for consistency is that players know their opponents' strategies *in equilibrium*.

- The following proposition is a direct corollary of Propositions 2.4.

## Proposition 2.6

*If the strategy profile* $s^* = (s_1^*, \ldots, s_n^*)$ *is a strict dominant strategy equilibrium, then it is the unique Nash equilibrium.*

# Nash Equilibrium



- We have a similar result for the unique survivor of IESDS.

**Proposition 2.7**

*If the game is finite, and the strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is the unique survivor of IESDS, then it is the unique Nash equilibrium.*

- The result that $s^*$ is a Nash equilibrium is a direct corollary of Proposition 2.5.
- The uniqueness comes from the next proposition, which is a converse of Proposition 2.7.
- Note the fact that $s_i^*$ is the unique best response to $s_{-i}^*$ as claimed by Proposition 2.5 does not imply that $s^*$ is the unique Nash equilibrium. (Yes?)

# Nash Equilibrium

## Definition

### Proposition 2.8

*If $s^*$ is a Nash equilibrium, it survives IESDS.*

- It is a direct corollary of Proposition 2.3. (Yes?)

# Applications

## Two kinds of societies

| | $S$ | $H$ |
|---|---|---|
| $S$ | $5, 5$ | $0, 3$ |
| $H$ | $3, 0$ | $3, 3$ |

- Two hunters can each choose to hunt a stag ($S$) or hunt a hare ($H$).
- Hunting stags is challenging and requires mutual cooperation.
- If either hunts a stag alone, the change of success is negligible.
- Hunting hares is an individualistic enterprise that is not done in pairs.
- A stag provides a rather large and tasty meal, while a hare is much less filling.

# Applications

## Two kinds of societies

- Two Nash equilibria: $(S, S)$ and $(H, H)$.
- $(S, S)$ Pareto dominates $(H, H)$.
- Why can $(H, H)$ happen? Self-fulfilling: if everybody anticipates that the other will not join forces, then it is better for himself to hunt a hare.
- Different societies that may look very similar in their endowments, access to technology, and physical environments have very different achievements, all because of self-fulfilling beliefs or, as they are often called, norms of behavior.

# Applications

## Two kinds of societies

| | $L$ | $R$ |
|---|---|---|
| $L$ | $1, 1$ | $0, 0$ |
| $L$ | $0, 0$ | $1, 1$ |

- Two Nash equilibria: $(L, L)$ and $(R, R)$.
- Social norm: some countries drive on the right while others drive on the left.

# Applications

## Cournot duopoly

- Inverse demand curve $P(Q) = \max\{a - bQ, 0\}$.
- Constant marginal cost $c_i(q_i) = c_i q_i$, where $c_i < a$.
- Payoff function

$$\begin{aligned} v_i(q_i, q_j) &= \max\{a - b(q_i + q_j), 0\} q_i - c_i q_i \\ &= \max\{(a - bq_i - bq_j - c_i) q_i, -c_i q_i\}. \end{aligned}$$

- If $q_j < \frac{a-c_i}{b}$, the unique best response for firm $i$ is

$$BR_i(q_j) = \frac{a - bq_j - c_i}{2b}.$$

- If $q_j \geq \frac{a-c_i}{b}$, the unique best response for firm $1$ is

$$BR_i(q_j) = 0.$$

- In sum,

$$BR_i(q_j) = \max\left\{\frac{a - bq_j - c_i}{2b}, 0\right\}.$$

- Therefore, a strategy profile $(q_1^*, q_2^*)$ is a Nash equilibrium if and only if they solve the following system of equations

$$q_1^* = \mathsf{max}\left\{\frac{a - bq_2^* - c_1}{2b}, 0\right\},$$
$$q_2^* = \mathsf{max}\left\{\frac{a - bq_1^* - c_2}{2b}, 0\right\}.$$

# Applications

## Cournot duopoly

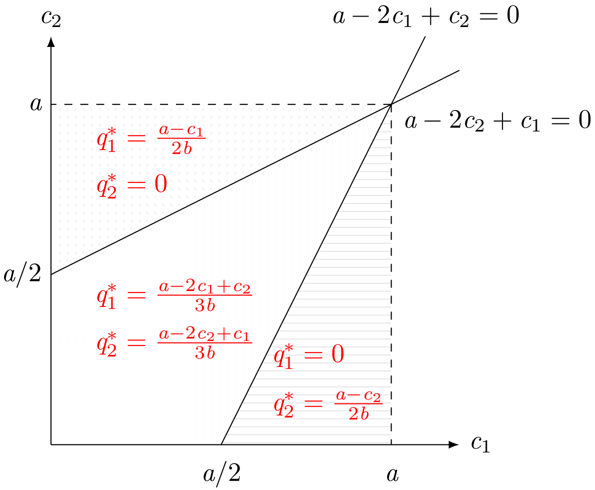



Figure 2.13: The unique Nash equilibrium in Cournot duopoly

# Applications

## Cournot duopoly

- $s^*$ is a Nash equilibrium if $s_i^* \in BR_i(s_{-i}^*)$ for all $i$.
- For a two-player game, we require $s_1^* \in BR_1(s_2^*)$ and $s_2^* \in BR_2(s_1^*)$.
- Let

$$G_1 = \{(s_1, s_2)\,|\, s_1 \in BR_1(s_2)\} \subseteq S_1 \times S_2$$

  be the graph of player $1$'s best response correspondence. Notice that if $BR_1$ is always single value, $G_1$ is simply the graph of function $BR_1$.
- Similarly, let

$$G_2 = \{(s_1, s_2)\,|\, s_2 \in BR_2(s_1)\} \subseteq S_1 \times S_2$$

  be the graph of player $2$'s best response correspondence.
- Then $(s_1^*, s_2^*)$ is a Nash equilibrium if and only if

$$(s_1^*, s_2^*) \in G_1 \cap G_2.$$

- Thus, if we draw $G_1$ and $G_2$ in the $S_1 \times S_2$ plane, their *intersections* give us all Nash equilibria.

# Applications

## Cournot duopoly

- If $a - 2c_1 + c_2 > 0$ and $a - 2c_2 + c_1 > 0$:

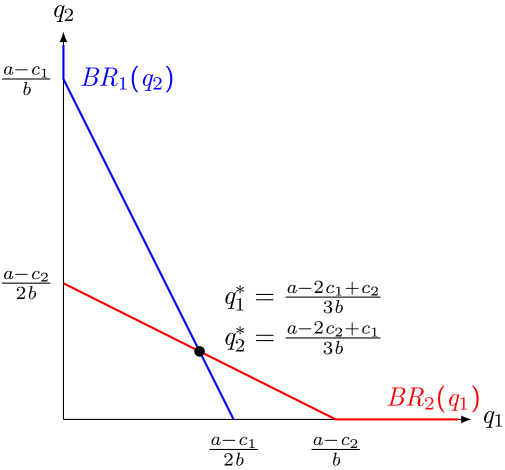



Figure 2.14: Cournot: best response functions and Nash equilibrium

# Applications

## Cournot duopoly

- Notice that if $c_1 = c_2 = c$, then the unique Nash equilibrium is

$$q_1^* = q_2^* = \frac{a - c}{3b}.$$

- When $a = 100$, $b = 1$ and $c = 10$, we have $q_1^* = q_2^* = 30$. As we have seen, this is the unique strategy profile that survives IESDS.

# Applications

## Cournot duopoly

- If $a - 2c_1 + c_2 < 0$ and $a - 2c_2 + c_1 > 0$: (the third case is symmetric)

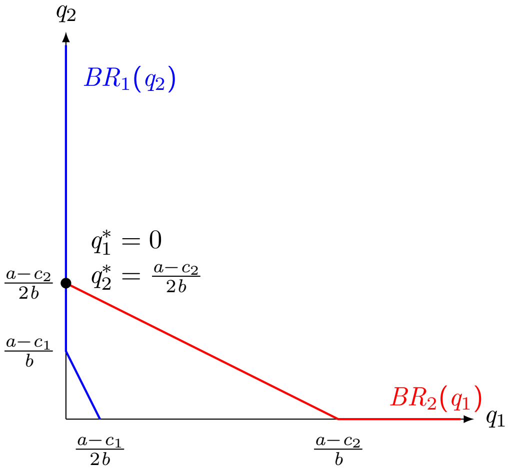



Figure 2.15: Cournot: best response functions and Nash equilibrium

# Applications

## Bertrand competition

- Price competition.
- Demand curve $Q(p) = \max\{a - p, 0\}$.
- Identical constant marginal cost $c$ with $a > c$.
- Consumers buy from the firm with lower price.
- If there is a tie in price, the two firms equally split the market.
- Therefore,

$$v_i(p_i, p_j) = \begin{cases} Q(p_i)(p_i - c), & \text{if } p_i < p_j, \\ \frac{1}{2}Q(p_i)(p_i - c), & \text{if } p_i = p_j, \\ 0, & \text{if } p_i > p_j. \end{cases}$$

- Monopoly pricing: if there were only one firm,

$$\max_{p \geq 0} Q(p)(p - c) \implies p^m = \frac{a + c}{2}.$$

# Applications

## Bertrand competition

- Nash equilibrium $(p_1^*, p_2^*)$?
- In any equilibrium, we must have $p_i^* \geq c$ for both firms; otherwise at least one firm is earning negative profit. This firm can make zero profit by deviating to $c$.
- There is no equilibrium in which one firm charges higher than $c$. To see this, $(p_1^*, p_2^*)$ is an equilibrium and $p_1^* > c$. If $p_1^* > p^m$, then we must have $p_2^* = p_m$. But then firm $1$ can deviate to any $p \in (c, p^m)$ to earn positive profit. If $p_1^* \leq p^m$, then there is no best response for firm $2$.
- We are left with $p_1^* = p_2^* = c$. This is indeed a Nash equilibrium since no firm has an incentive to deviate.
- In sum, the only Nash equilibrium of Bertrand competition is $(c, c)$, i.e., marginal cost pricing (competitive outcome?).

- Best response correspondence:

$$BR_i(p_j) = \begin{cases} (p_j, +\infty), & \text{if } p_j < c, \\ [p_j, +\infty), & \text{if } p_j = c, \\ \emptyset, & \text{if } c < p_j \leq p^m, \\ \{p^m\}, & \text{if } p_j > p^m. \end{cases}$$

- Again, make sure you understand why some interval is open while some is closed. Make sure you understand why some inequality is strict and some is weak.

# Applications

## Bertrand competition

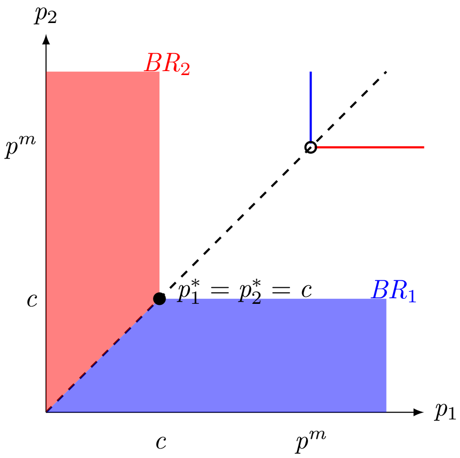



Figure 2.16: Bertrand: best response correspondences and Nash equilibrium

# Applications

## Bertrand competition

- Consider the asymmetric case: marginal cost $c_1 < c_2$.
- Notice that the best response correspondences have exactly the same form as before:

$$BR_i(p_j) = \begin{cases} (p_j, +\infty), & \text{if } p_j < c_i, \\ [p_j, +\infty), & \text{if } p_j = c_i, \\ \emptyset, & \text{if } c_i < p_j \leq p_i^m, \\ \{p_i^m\}, & \text{if } p_j > p_i^m, \end{cases}$$

where $p_i^m = (a + c_i)/2$.

# Applications

## Bertrand competition

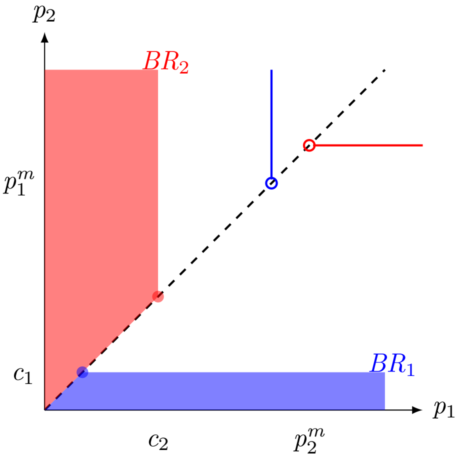



Figure 2.17: Bertrand with asymmetric marginal cost

- There is no Nash equilibrium!

# Applications

## Political ideology and electoral competition

- There are $101$ political group of citizens, each labeled by $-50, -49, \ldots, 0, \ldots, 49, 50$.
- The group's label also represents this group's ideal policy.
- Each group consists of a continuum of citizens of mass $1$.
- There are two political candidates, each caring only about being elected.
- Each candidate $i$ chooses his platform as a policy $a_i \in \{-50, -49, \ldots, 0, \ldots, 49, 50\}$.
- Each citizen votes for the candidate whose platform is closest to his ideal policy.
- If citizens in a group are indifferent between two platforms, half of them vote for one politician and the other half vote for the other.

# Applications



- The outcome is determined by majority rule.
- The candidate with more votes wins.
- If they receive the same number of votes, we say there is a tie.
- Candidates prefer winning to a tie, and a tie to losing.
- This is a discrete version of the Hotelling model.

# Applications

## Political ideology and electoral competition

- There is a unique Nash equilibrium $(0, 0)$. Why?
- First of all, $(0, 0)$ is indeed a Nash equilibrium: given $a_j = 0$, any $a'_i \neq 0$ would lose while $a_i = 0$ gets a tie.
- There is no Nash equilibrium in which one candidate, say $1$, chooses $a_1 \neq 0$. If there is such equilibrium, then candidate $2$ must win the election in equilibrium (yes?). This means candidate $1$ loses in equilibrium. But by deviating to the same platform chosen by candidate $2$, candidate $1$ can obtain a tie.

# Applications

## Political ideology and electoral competition

- Media voter theorem: if voters are different from one another along a single-dimensional “preference” line, as in Hotelling model, and if each prefers his own political location, with other platforms being less and less attractive the farther away they fall to either side of that location, then the political platform located at the median voter will defeat any other platform in a simple majority vote.

# Mixed Strategies and Expected Payoffs

## Motivating examples

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1, -1$ | $-1, 1$ |
| $T$ | $-1, 1$ | $1, -1$ |

Figure 2.18: The matching pennies

- Players $1$ and $2$ each put a penny on a table simultaneously.
- If the two pennies come up the same side (heads or tails) then player $1$ gets both; otherwise player $2$ does.
- Easy to see: no Nash equilibrium.
- Zero-sum game.

# Mixed Strategies and Expected Payoffs

## Motivating examples

| | $R$ | $P$ | $S$ |
|---|---|---|---|
| $R$ | $0, 0$ | $-1, 1$ | $1, -1$ |
| $P$ | $1, -1$ | $0, 0$ | $-1, 1$ |
| $S$ | $-1, 1$ | $1, -1$ | $0, 0$ |

Figure 2.19: Rock-paper-scissors

- No Nash equilibrium either.
- Remedy: allow players to *randomize* between several of their strategies.

# Mixed Strategies and Expected Payoffs

## Mixed strategies

### Definition 2.10

Let $S_i = \{s_{i1}, s_{i2}, \ldots, s_{im}\}$ be player $i$'s finite set of pure strategies. Define $\Delta S_i$ as the **simplex** of $S_i$, which is the set of all probability distributions over $S_i$. A **mixed strategy** for player $i$ is an element $\sigma_i \in \Delta S_i$, so that $\sigma_i = (\sigma_i(s_{i1}), \sigma_i(s_{i2}), \ldots, \sigma_i(s_{im}))$ is a probability distribution over $S_i$, where $\sigma_i(s_i)$ is the probability that player $i$ plays $s_i$.

- Equivalently, a mixed strategy is a mapping $\sigma_i : S_i \to [0,1]$ with the constraint $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.
- In the matching pennies,

$$\Delta S_i = \{(\sigma_i(H), \sigma_i(T)) \,|\, \sigma_i(H) \geq 0, \sigma_i(T) \geq 0 \text{ and } \sigma_i(H) + \sigma_i(T) = 1\}.$$

- In the rock-paper-scissors,

$$\Delta S_i = \left\{ (\sigma_i(R), \sigma_i(P), \sigma_i(S)) \;\middle|\; \begin{array}{l} \sigma_i(R) \geq 0, \sigma_i(P) \geq 0, \sigma_i(S) \geq 0, \\ \sigma_i(R) + \sigma_i(P) + \sigma_i(S) = 1. \end{array} \right\}$$

# Mixed Strategies and Expected Payoffs

## Mixed strategies

- In this course, we only consider *mixed strategies with finite strategy spaces*.
- Note that even for finite strategy space with more than two strategies, there are infinitely many mixed strategies!
- We have called each $s_i \in S_i$ as player $i$'s strategy. To distinguish it from mixed strategies, we call it a **pure strategy**.
- Observe that a pure strategy is simply a spacial case of a mixed strategy. It is a mixed strategy with a *degenerate* distribution.
- For instance, for $s_i \in S_i$, the mixed strategy $\sigma_i$ with

$$\sigma_i(s_i') = \begin{cases} 1, & \text{if } s_i' = s_i, \\ 0, & \text{if } s_i' \neq s_i. \end{cases}$$

is simply the pure strategy $s_i$, since $\sigma_i$ plays $s_i$ for sure.

# Mixed Strategies and Expected Payoffs



### Definition 2.11

Given a mixed strategy $\sigma_i$ for player $i$, we will say that a pure strategy $s_i \in S_i$ is **in the support of** $\sigma_i$ if and only if it occurs with positive probability, that is, $\sigma_i(s_i) > 0$.

- Those pure strategies that are not in the support will not be played.
- For example, the support of the degenerate mixed strategy $\sigma_i$ on the previous page contains only $s_i$.
- For another example, consider $\sigma_i(R) = \sigma_i(P) = 0.5$ and $\sigma_i(S) = 0$ in the rock-paper-scissors. Then both $R$ and $P$ are in the support while $S$ is not.
- We use the notation $\mathrm{supp}\sigma_i$ to denote the support of $\sigma_i$.

# Mixed Strategies and Expected Payoffs

## Expected payoffs

- Consider the matching pennies.
- Suppose player $1$ plays the mixed strategy $\sigma_1$ with $\sigma_1(H) = p$ and $\sigma_1(T) = 1 - p$. We also write $p \circ H + (1-p) \circ T$ to denote this strategy.
- Suppose player $2$ plays the mixed strategy $\sigma_2 = q \circ H + (1-q) \circ T$.
- We always assume that players' mixtures are independent.
- What is the outcome of this strategy profile?

| | $H$ | $T$ |
|---|---|---|
| $H$ | $p \times q$ | $p \times (1-q)$ |
| $T$ | $(1-p) \times q$ | $(1-p) \times (1-q)$ |

Figure 2.20: The outcome is a distribution over the strategy profiles

- Outcome $HH$ will occur with probability $pq$. Outcome $HT$ will occur with probability $p(1-q)$, and so on.

# Mixed Strategies and Expected Payoffs

## Expected payoffs

- We assume that players are *expected utility maximizers*.
- Thus, they evaluate their mixed strategies by their expected payoffs.
- Therefore, in this example, player $i$'s expected utility is

$$\begin{aligned} v_i(\sigma_1, \sigma_2) = {} & pqv_i(H, H) + p(1-q)v_i(H, T) \\ & + (1-p)qv_i(T, H) + (1-p)(1-q)v_i(T, T). \end{aligned}$$

# Mixed Strategies and Expected Payoffs

## Expected payoffs

- More generally, consider a $n$-player finite game.
- Consider the strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$.
- Then, the outcome $(s_1, \ldots, s_n)$ under $\sigma$ occurs with probability

$$\sigma_1(s_1) \times \sigma_2(s_2) \times \ldots \times \sigma_n(s_n) = \prod_{j=1}^{n} \sigma_j(s_j).$$

- Therefore, player $i$'s **expected payoff** under $\sigma$ is

$$\begin{aligned} v_i(\sigma) &= \sum_{s \in S} \left( \prod_{j=1}^{n} \sigma_j(s_j) \right) v_i(s) \\ &= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \left( \prod_{j=1}^{n} \sigma_j(s_j) \right) v_i(s). \end{aligned}$$

# Mixed Strategies and Expected Payoffs

## Expected payoffs

- Rearranging terms, we can also write it as

$$v_i(\sigma) = \sum_{s_i \in S_i} \left[ \sigma_i(s_i) \times \sum_{s_{-i} \in S_{-i}} \left( \prod_{j \neq i} \sigma_j(s_j) \right) v_i(s_i, s_{-i}) \right].$$

- Note that

$$\sum_{s_{-i} \in S_{-i}} \left( \prod_{j \neq i} \sigma_j(s_j) \right) v_i(s_i, s_{-i}) = v_i(s_i, \sigma_{-i})$$

  is $i$'s expected payoff from playing $s_i$ against opponents' $\sigma_{-i}$.

- Therefore, $i$'s expected payoff can be written as

$$v_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) v_i(s_i, \sigma_{-i}).$$

- Player $i$'s expected payoff from $\sigma_i$ against $\sigma_{-i}$ is the *weighted average* of his expected payoffs from playing pure strategies.

# Mixed-Strategy Nash Equilibrium

## Definition

- We are ready to extend the concept of best response and Nash equilibrium to mixed strategies.

### Definition 2.12

A mixed strategy $\sigma_i$ for player $i$ is a best response to the opponents' strategy $\sigma_{-i}$ if

$$v_i(\sigma_i, \sigma_{-i}) \geq v_i(s_i, \sigma_{-i}), \ \forall s_i \in S_i.$$

- $\sigma_i$ is a best response to $\sigma_{-i}$ if it is as good as any pure strategy $s_i \in S_i$.
- In fact, if $\sigma_i$ is a best response to $\sigma_{-i}$, then it is as good as any *mixed strategy* $\sigma_i' \in \Delta S_i$:

$$v_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i'(s_i) v_i(\sigma_i, \sigma_{-i}) \geq \sum_{s_i \in S_i} \sigma_i'(s_i) v_i(s_i, \sigma_{-i}) = v_i(\sigma_i', \sigma_{-i}).$$

- The following lemma is simple. But it plays an important (perhaps the most important) role in computing mixed strategy Nash equilibria.

### Lemma 2.2

*A mixed strategy $\sigma_i$ is a best response to $\sigma_{-i}$ if and only if every pure strategy $s_i$ in the support of $\sigma_i$ is a best response to $\sigma_{-i}$. Consequently, player $i$ is indifferent between all pure strategies in the support of $\sigma_i$.*

# Mixed-Strategy Nash Equilibrium



## Proof of Lemma 2.2.

"if" part: suppose every pure strategy $s_i$ in the support of $\sigma_i$ is a best response to $\sigma_{-i}$:

$$v_i(s_i, \sigma_{-i}) \geq v_i(s_i', \sigma_{-i}),\ \forall s_i' \in S_i.$$

Then, for any $s_i' \in S_i$, we have

$$\begin{aligned} v_i(\sigma_i, \sigma_{-i}) = & \sum_{s_i \in \mathrm{supp}\sigma_i} \sigma_i(s_i) v_i(s_i, \sigma_{-i}) \\ \geq & \sum_{s_i \in \mathrm{supp}\sigma_i} \sigma_i(s_i) v_i(s_i', \sigma_{-i}) \\ = & v_i(s_i', \sigma_{-i}). \end{aligned}$$

This proves that $\sigma_i$ is a best response to $\sigma_{-i}$.

# Mixed-Strategy Nash Equilibrium

Definition

## Proof of Lemma 2.2 (Cont.)

"only if" part: assume $\sigma_i$ is a best response to $\sigma_{-i}$. Suppose, by contradiction, that $s_i$ is in the support of $\sigma_i$ but it is not a best response to $\sigma_{-i}$. Let $\hat{s}_i \neq s_i$ be a best response to $\sigma_{-i}$. Then we have

$$v_i(\hat{s}_i, \sigma_{-i}) \geq v_i(s'_i, \sigma_{-i}), \ \forall s'_i \in S_i,$$

and

$$v_i(\hat{s}_i, \sigma_{-i}) > v_i(s_i, \sigma_{-i}).$$

## Proof of Lemma 2.2 (Cont.)

Therefore,

$$\begin{aligned} v_i(\hat{s}_i, \sigma_{-i}) = & \sum_{s_i' \in \text{supp} \sigma_i} \sigma_i(s_i') v_i(\hat{s}_i, \sigma_{-i}) \\ > & \sum_{s_i' \in \text{supp} \sigma_i} \sigma_i(s_i') v_i(s_i', \sigma_{-i}) \\ = & v_i(\sigma_i, \sigma_{-i}), \end{aligned}$$

contradicting the assumption that $\sigma_i$ is a best response. Hence, every pure strategy in the support of $\sigma_i$ must be a best response to $\sigma_{-i}$. Consequently, if $s_i$ is in the support of $\sigma_i$, we have

$$v_i(s_i, \sigma_{-i}) = \max_{s_i'} v_i(s_i', \sigma_{-i}),$$

proving the indifference condition. $\square$

# Mixed-Strategy Nash Equilibrium

## Definition

### Definition 2.13

A mixed strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a Nash equilibrium if for each player $i$, $\sigma_i^*$ is a best response to $\sigma_{-i}^*$.

- Conceptually identical to pure strategy Nash equilibrium: mutual best responses.

# Mixed-Strategy Nash Equilibrium

## Example: the matching pennies

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1,-1$ | $-1,1$ |
| $T$ | $-1,1$ | $1,-1$ |

Figure 2.21: The matching pennies

- No pure strategy Nash equilibrium.
- Suppose $\sigma_1(H) = p$ and $\sigma_2(H) = q$.
- Then

$$\begin{aligned} v_1(\sigma_1, \sigma_2) &= pq - p(1-q) - (1-p)q + (1-p)(1-q) \\ &= 2(2q-1)p + 1 - 2q. \end{aligned}$$

- Thus,

$$BR_1(q) = \begin{cases} 0, & \text{if } q < 1/2, \\ [0,1], & \text{if } q = 1/2, \\ 1, & \text{if } q > 1/2. \end{cases}$$

# Mixed-Strategy Nash Equilibrium

## Example: the matching pennies

- Similarly,

$$\begin{aligned} v_2(\sigma_1, \sigma 2) &= -pq + p(1-q) + (1-p)q - (1-p)(1-q) \\ &= 2(1-2p)q + 2p - 1. \end{aligned}$$

- Thus,

$$BR_2(p) = \begin{cases} 1, & \text{if } p < 1/2, \\ [0,1], & \text{if } p = 1/2, \\ 0, & \text{if } p > 1/2. \end{cases}$$

# Mixed-Strategy Nash Equilibrium

Example: the matching pennies

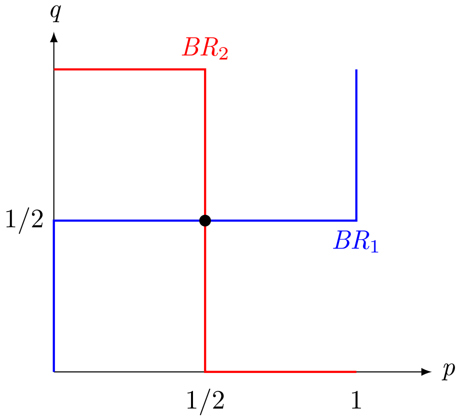



Figure 2.22: Best response correspondences in the matching pennies

- Unique Nash equilibrium: $(\frac{1}{2} \circ H + \frac{1}{2} \circ T, \frac{1}{2} \circ H + \frac{1}{2} \circ T)$.

# Mixed-Strategy Nash Equilibrium

## Example: the matching pennies

- We can also use the indifference condition to solve this game.
- Indifference condition first implies that, in this game, if one player mixes in equilibrium, the other player must mix as well. (Yes?)
- Hence, there will be no equilibrium in which only one player mixes.
- Suppose $\sigma_1(H) = p \in (0, 1)$ and $\sigma_2(H) = q \in (0, 1)$.
- Indifference condition for player $1$ requires

$$q - (1 - q) = -q + (1 - q) \implies q = 1/2.$$

- Indifference condition for player $2$ requires

$$-p + (1 - p) = p - (1 - p) \implies q = 1/2.$$

- We obtain the same result: $(\frac{1}{2} \circ H + \frac{1}{2} \circ T, \frac{1}{2} \circ H + \frac{1}{2} \circ T)$.

# Mixed-Strategy Nash Equilibrium

## Example: rock-paper-scissors

| | $R$ | $P$ | $S$ |
|---|---|---|---|
| $R$ | $0, 0$ | $-1, 1$ | $1, -1$ |
| $P$ | $1, -1$ | $0, 0$ | $-1, 1$ |
| $S$ | $-1, 1$ | $1, -1$ | $0, 0$ |

Figure 2.23: Rock-paper-scissors

- No pure strategy Nash equilibrium, as we have already known.
- Nash equilibrium in mixed strategies?
- A brave guess: $(\sigma, \sigma)$ where $\sigma = \frac{1}{3} \circ R + \frac{1}{3} \circ P + \frac{1}{3} \circ S$.
- It is indeed a Nash equilibrium

$$v_i(R, \sigma) = v_i(P, \sigma) = v_i(S, \sigma) = 0.$$

# Mixed-Strategy Nash Equilibrium

## Example: rock-paper-scissors

- It is more challenging to show that this is the *unique* equilibrium.
- The best response correspondence approach is not very useful, since we can not draw $4$-dimensional graphs. (why $4$-dimensional?)
- Therefore, we rely on the indifference condition with a careful discussion about the support of the equilibrium strategies.

# Mixed-Strategy Nash Equilibrium

## Example: rock-paper-scissors

- First observe that there is no equilibrium in which only one player mixes. (The same reason as that in the matching pennies.)
- Thus, we consider potential equilibrium in which both players mix.
- We claim that there is no equilibrium in which one player only mixes between two pure strategies.
- Consider an equilibrium $(\sigma_1, \sigma_2)$. Suppose, by contradiction, that $\sigma_1$ mixes between $R$ and $P$ only. Note that

$$v_2(P, \sigma_1) = \sigma_1(R) > -\sigma_1(P) = v_2(R, \sigma_1),$$

  implying that $R$ is not a best response for player $2$ to $\sigma_1$.
- Because $\sigma_2$ is a best response to $\sigma_1$, we know $\sigma_2(R) = 0$. Hence, $\sigma_2$ mixes between $P$ and $S$ only.

# Mixed-Strategy Nash Equilibrium



- But if $\sigma_2$ only mixes between $P$ and $S$, we know

$$v_1(S, \sigma_2) = \sigma_2(P) > -\sigma_2(S) = v_1(P, \sigma_2),$$

  implying that $P$ is not a best response to $\sigma_2$.
- This contradicts our assumption that $\sigma_1$ is a best response to $\sigma_2$ and $\sigma_1(P) > 0$.
- Therefore, there is no equilibrium in which player $1$ only mixes between $R$ and $P$.
- Applying similar arguments, we can show that there is no equilibrium in which one player only mixes between two strategies.

# Mixed-Strategy Nash Equilibrium

## Example: rock-paper-scissors

- We are left with the case where both players mix between all three pure strategies.
- Then, this requires that every player is indifferent between all three pure strategies given the opponent's strategy.
- Given $\sigma_2$, we know

$$\begin{aligned} v_1(R, \sigma_2) &= -\sigma_2(P) + \sigma_2(S), \\ v_1(P, \sigma_2) &= \sigma_2(R) - \sigma_2(S), \\ v_1(S, \sigma_2) &= -\sigma_2(R) + \sigma_2(P). \end{aligned}$$

- Indifference conditions require

$$-\sigma_2(P) + \sigma_2(S) = \sigma_2(R) - \sigma_2(S) = -\sigma_2(R) + \sigma_2(P).$$

- These two equations, together with $\sigma_2(R) + \sigma_2(P) + \sigma_2(S) = 1$, yield

$$\sigma_2(R) = \sigma_2(P) = \sigma_2(S) = \frac{1}{3}.$$

# Mixed-Strategy Nash Equilibrium

## Example: rock-paper-scissors

- Similarly, applying player $2$'s indifference conditions would yield

$$\sigma_1(R) = \sigma_1(P) = \sigma_1(S) = \frac{1}{3}.$$

- Therefore, both players mixing over all three pure strategies with equal probability is the unique Nash equilibrium.

# Mixed-Strategy Nash Equilibrium

## Example: coexistence of pure and mixed strategy Nash equilibrium

| | $O$ | $F$ |
|---|---|---|
| $O$ | $2, 1$ | $0, 0$ |
| $F$ | $0, 0$ | $1, 2$ |

Figure 2.24: Battle of the sexes

- Two pure strategy Nash equilibria: $(O, O)$ and $(F, F)$.
- Mixed strategy Nash equilibrium:
  - In this game, if in equilibrium one player mixes, the other player must mix as well. Thus, we look for equilibrium in which both players mix.
  - Suppose $\sigma_1(O) = p \in (0, 1)$ and $\sigma_2(O) = q \in (0, 1)$.
  - Player $1$'s indifference condition requires

$$2 \times q + 0 \times (1 - q) = 0 \times q + 1 \times (1 - q) \implies q = 1/3.$$

  - Player $2$'s indifference condition requires

$$1 \times p + 0 \times (1 - p) = 0 \times p + 2 \times (1 - p) \implies p = 2/3.$$

- Unique mixed strategy Nash equilibrium: $(\frac{2}{3} \circ O + \frac{1}{3} \circ F, \frac{1}{3} \circ O + \frac{2}{3} \circ F)$.

# Existence of Nash Equilibrium

- We have seen simple games in which there is no pure strategy Nash equilibrium, e.g., the matching pennies and rock-paper-scissors.
- We also know that these two games have Nash equilibrium in mixed strategies.
- In fact, if players are allowed to randomize, then every finite game has a Nash equilibrium.
- This is Nash's existence theorem.

**Theorem 2.1**

*Every finite normal form game has a Nash equilibrium (possibly in mixed strategies).*

# Dominance and IESDS Revisited

## Dominance by a mixed strategy

### Definition 2.14

Let $\sigma_i \in \Delta S_i$ and $s_i \in S_i$ be possible strategies for player $i$. We say that $s_i$ is strictly dominated by $\sigma_i$ if

$$v_i(\sigma_i, s_{-i}) > v_i(s_i, s_{-i}),\ \forall s_{-i} \in S_{-i}. \tag{2.1}$$

- That is, to consider a strategy as strictly dominated, we no longer require that some other *pure strategy* dominate it.
- We allow for mixed strategies to dominate it as well.

# Dominance and IESDS Revisited

## Dominance by a mixed strategy

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $U$ | $5, 1$ | $1, 4$ | $1, 0$ |
| $M$ | $3, 2$ | $0, 0$ | $3, 5$ |
| $D$ | $4, 3$ | $4, 4$ | $0, 3$ |

- No strategy is strictly dominated by another pure strategy for each player.
- However, $L$ is strictly dominated by some mixture of $C$ and $R$.
- To see this, consider the mixed strategy $\sigma_2$ with $\sigma_2(C) = \sigma_2(R) = 1/2$.
- Then,

$$v_2(U, \sigma_2) = 2 > 1 = v_2(U, L),$$
$$v_2(M, \sigma_2) = 5/2 > 2 = v_2(M, L),$$
$$v_2(D, \sigma_2) = 7/2 > 3 = v_2(D, L).$$

- Note that there are other mixed strategies that strictly dominate $L$.

# Dominance and IESDS Revisited

## Dominance by a mixed strategy

- It is worth having a more careful look at Definition 2.14.
- In this definition, we require that $\sigma_i$ do better than $s_i$ for every *pure strategy* $s_{-i} \in S_{-i}$ of the opponents.
- If we allow player $i$ to mix, why don't we allow the opponents to mix as well and require that $\sigma_i$ do better than $s_i$ for *every* $\sigma_{-i} \in \Delta S_{-i}$? That is, we require

$$v_i(\sigma_i, \sigma_{-i}) > v_i(s_i, \sigma_{-i}), \ \forall \sigma_{-i} \in \Delta S_{-i}. \tag{2.2}$$

- This is because (2.1) and (2.2) are actually equivalent:
  - On the one hand, it is obvious that (2.2) implies (2.1).
  - On the other hand, if (2.1) holds, then for any $\sigma_{-i} \in \Delta S_{-i}$, we have

$$v_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i}} \sigma_{-i}(s_{-i}) v_i(\sigma_i, s_{-i}) > \sum_{s_{-i}} \sigma_{-i}(s_{-i}) v_i(s_i, s_{-i}) = v_i(s_i, \sigma_{-i}),$$

  implying that (2.2) holds.

# Dominance and IESDS Revisited

## IESDS

- The idea of IESDS still applies.
- But in each round, we also delete those strategies that are dominated by a mixed strategy.

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $U$ | $5,1$ | $1,4$ | $1,0$ |
| $M$ | $3,2$ | $0,0$ | $3,5$ |
| $D$ | $4,3$ | $4,4$ | $0,3$ |

- In the first round, we delete $L$.
- In the second round, we delete $U$ because it is strictly dominated by $\frac{1}{2} \circ M + \frac{1}{2} \circ R$.
- Then $S_1^* = \{M, D\}$ and $S_2^* = \{C, R\}$ survive IESDS.

# Dominance and IESDS Revisited

## IESDS

- One of the significance of IESDS is stated in the following proposition.

### Proposition 2.9

*Consider a finite game. Suppose $S_1^* \times \ldots \times S_n^*$ survive IESDS. Then, $\sigma^*$ is a Nash equilibrium of the original game if and only if it is a Nash equilibrium of the reduced game $(N, S_1^* \times \ldots \times S_n^*, \{v_i\}_{i=1}^n)$.*

- Note that the "only if" part implicitly asserts that every pure strategy $s_i$ in the support of $\sigma_i^*$ survives IESDS. This is a mixed strategy analog of Proposition 2.8.
- For games with many pure strategies, it is usually not easy to find out all the (possibly mixed) equilibria.
- Proposition 2.9 asserts that IESDS can help us in that it may narrow down the set of pure strategies we need to consider.

# Dominance and IESDS Revisited

## IESDS

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $U$ | $5, 1$ | $1, 4$ | $1, 0$ |
| $M$ | $3, 2$ | $0, 0$ | $3, 5$ |
| $D$ | $4, 3$ | $4, 4$ | $0, 3$ |

- $S_1^* = \{M, D\}$ and $S_2^* = \{C, R\}$ survive IESDS.
- To find all Nash equilibria, we only need to find all Nash in the reduced game:

| | $C$ | $R$ |
|---|---|---|
| $M$ | $0, 0$ | $3, 5$ |
| $D$ | $4, 4$ | $0, 3$ |

- Nash: $(M, R)$, $(D, C)$ and $(\frac{1}{6} \circ M + \frac{5}{6} \circ D, \frac{3}{7} \circ C + \frac{4}{7} \circ R)$.

# Dominance and IESDS Revisited



## Proof of Proposition 2.9.

"if" part: assume $\sigma^*$ is a Nash equilibrium of the reduced game. Suppose, by contradiction, that it is not a Nash equilibrium of the original game. Then in the original game, some player, say $i$, must have a profitable deviation $s_i^1$: $v_i(s_i^1, \sigma_{-i}^*) > v_i(\sigma^*)$. Because $\sigma^*$ is a Nash in the reduced game, we must have $s_i^1 \notin S_i^*$, that is, $s_i^1$ must be deleted in the process of IESDS.

Because $s_i^1$ is deleted and because every pure strategy $s_j$ in the support of $\sigma_j^*$ always remains for $j \neq i$, there must exist some $\sigma_i^1$ such that

$$v_i(\sigma_i^1, s_{-i}) > v_i(s_i^1, s_{-i}),\ \forall s_{-i} \in \mathrm{supp}\sigma_{-i}^*.$$

This implies $v_i(\sigma_i^1, \sigma_{-i}^*) > v_i(s_i^1, \sigma_{-i}^*)$. Thus, there must exist $s_i^2$ such that $v_i(s_i^2, \sigma_{-i}^*) > v_i(s_i^1, \sigma_{-i}^*)$.

### Proof of Proposition 2.9.

Similarly as $s_i^1$, we know $s_i^2 \notin S_i^*$, that is, $s_i^2$ must be deleted in the process of IESDS as well. Applying the same arguments, we can actually find infinitely many strategies $s_i^3, \ldots, s_i^m, \ldots$. This contradicts our finite game assumption.

## Proof of Proposition 2.9 (Cont.)

"only if" part: we prove the claim that *if $\sigma_i^*$ is a Nash equilibrium, then no pure strategy $s_i$ in the support of $\sigma_i^*$ is strictly dominated.* The desired result will directly follow. (Yes?)

Suppose, by contradiction, that $s_i$ is in the support of $\sigma_i^*$ and it is strictly dominated by some $\sigma_i$. Then, we have

$$v_i(\sigma_i, s_{-i}) > v_i(s_i, s_{-i}), \ \forall s_{-i} \in S_{-i}.$$

This implies

$$\sum_{s_{-i} \in S_{-i}} \sigma_{-i}^*(s_{-i}) v_i(\sigma_i, s_{-i}) > \sum_{s_{-i} \in S_{-i}} \sigma_{-i}^*(s_{-i}) v_i(s_i, s_{-i}).$$

# Dominance and IESDS Revisited

## IESDS

### Proof of Proposition 2.9 (Cont.)

Equivalently,

$$v_i(\sigma_i, \sigma^*_{-i}) > v_i(s_i, \sigma^*_{-i}),$$

contradicting the assumptions that $\sigma^*_i$ is a best response to $\sigma^*_{-i}$ and $s_i$ is in the support of $\sigma^*_i$. $\square$