Problem Set 1

2024/9/18

1

(a)

Answer:
There are n bidders, $N = [n]$. Every bidder bids a price $b_i \geq 0$. Thus the strategy is $S_i = [0, +\infty)$
Then the payoff is

$$v_i(b_i, b_{-i}) = \begin{cases} \frac{v_i - b_i}{|\{j|b_i = b_j\}|}, & \forall j,\ b_i \geq b_j \\ 0, & o.w. \end{cases}$$

(b)

For the second price auction, we have,

$$v_i(b_i, b_{-i}) = \begin{cases} \frac{v_i - \max_{j<i} b_j}{|\{j|b_i = b_i\}|}, & \forall j,\ b_i \geq b_j \\ 0, & o.w. \end{cases}$$

2. ST Exercise 4.5

Answer:
In the first round, U for player 1 is strictly dominated by M.
In the first round, C for player 2 is strictly dominated by H.
Hence $\{M, D\} \times \{L, R\}$ survives.

1\2	L	C	R
U	6,8	2.6	8,2
M	8,2	4,4	9,5
D	8,10	4,6	6,7

3. ST Exercise 4.6

Answer:

(a)

For i, when $t_j \geq 10$, his best reponse is $t_i = 0$. When $t_j < 10$, the first order condition is $10 - t_j - 2t_i = 0$. Then we find,

$$t_i = \max\{\frac{10 - t_i}{2}, 0\}$$

(b)

- $\forall t_i \in [0, 5]$ is not strictly dominated because it is a best response to $t_j = 10 - 2t_i$.
- For $t_i > 5$, assume $k = t_i - 5 > 0$,then we have

  $$\begin{aligned} v_i(5 + k, t_j) = (10 - t_j)(5 + k) - (5 + k)^2 = 25 - 5t_i - k^2 - t_j k \\ < 25 - 5t_j = v_i(5, t_j) \end{aligned}$$

  It means strategy $t_i > 5$ is strictly dominated by $t_i = 5$.
- We have $S_1^1 = S_2^2 = [0, 5]$

(c)

- Define $f(x) = 5 - x/2$.
- Assume $S_1^* = S_2^* \subset [l, u]$
  - First, $S_1^* = S_2^* \subset [0, 5]$
  - Second we find, $[f(u), f(l)] \subset [l, u], \forall l, u > 0, l + u \leq 5$
  - Similar to (b), we can find $t_i \in [f(u), f(l)]$ is not strictly dominated
  - For $t_i < f(u)$, assume $k = f(u) - t_i > 0$,similarly,

    $$\begin{aligned} v(f(u), t_j) - v(f(u) - k, t_j) = k(10 - t_j - 2f(u) + k) \\ \geq k(10 - 5 - 2f(5) + k) > 0 \end{aligned}$$

    Thus t_i is strictly dominated by $t_i = f(u)$.
  - For $t_i > f(l)$, similarly, is strictly dominated by $t_i = f(l)$.
- By induction, we can get closed interval $T_n \subset T_{n-1} \subset \cdots \subset T_1 = [0, 5]$, s.t.

$$T_i = [l_i, u_i] \Rightarrow T_{i+1} = [f(u_i), f(l_i)], i \in \mathbb{N}^+$$

and

$$S_1^* = S_2^* \subset T_n$$

- Given that the measure $|f(u) - f(l)| = |u - l|/2$, we can get

$$\lim_{n \to +\infty} |u_n - l_n| = 0$$

  By the closed interval theorem, $T_{+\infty} = \{c\}$. And from the method of generating T_n, we can know $f(c) = c$, which means $c = 10/3$.
- In conclusion , $S_1^* = S_2^* = \bigcap_{n=1}^{+\infty} T_n = T_{+\infty} = \{10/3\}$. The unique pair of strategies that survive IESDS for this game are $t_1 = t_2 = 10/3$

4. ST Exercise 4.7

Answer:

(a)

The set of the player is $\{1, 2\}$. The strategy spaces are $S_1 = S_2 = \{P, B, N\}$. The payoffs to player 1 are:

$$\begin{aligned} &v_1(P,P) = 0.5, \quad v_1(P,B) = 0, \qquad v_1(P,N) = 0.3, \\ &v_1(B,P) = 1, \qquad v_1(B,B) = 0.5, \quad v_1(B,N) = 0.4, \\ &v_1(N,P) = 0.7, \quad v_1(N,B) = 0.6, \quad v_1(N,N) = 0.5 \end{aligned}$$

And, $\forall s \in \{P, B, N\} \times \{P, B, N\}, v_2(s) = 1 - v_1(s)$.

(b)

The matrix is following.

		Player 2		
		P	**B**	**N**
Player 1	**P**	0.5, 0.5	0, 1	0.3, 0.7
	B	1, 0	0.5, 0.5	0.4, 0.6

N	0.7, 0.3	0.6, 0.4	0.5, 0.5

(c)

In the first round, P for both players is strictly dominated by B.
In the second round, B for both players is strictly dominated by N.
Thus $\{N\} \times \{N\}$ survives, which leads to a clear prediction.

5. ST Exercise 4.8

Answer:

(a)

The average is less than 20 regardless of the number of players, and $3/4$ of the average is less than 15, This means 18 also makes i the only winner.

(b)

- $x = 20$ is not one of the best responses because there will be many winners.
- For $x < 20$, the average is $\frac{20(n-1)+x}{n}$, thus player i is the only winner $\Leftrightarrow$

$$|20 - \frac{20(n-1)+x}{n}| > |\frac{20(n-1)+x}{n}|$$

  When $x < 20$, it is equivalent to $x > 10 - \frac{30}{2n-3}$
- Therefore the set of best responses is

$$\left\{x \in \mathbb{Z} \middle| 10 - \frac{30}{2n-3} < x < 20\right\}, n \geq 2,$$

  which depends on the number of players .

6.

Answer:

(a)

$$v_i(x_i, x_j) = \arctan x_i < \arctan(x_i + 1) = v_i(x_i + 1, x_j)$$

Thus x_i is strictly dominated.

(b)

$$\forall x_i > 0,\ v_i(0, 1) = 2 > \arctan x_i = v_i(x_i, 1)$$

Thus 0 for player i is a best response to $x_j = 1$, which is not strictly dominated.

(c)

$$v_i(0, 0) = \arctan 0 < \arctan 1 = v_i(1, 0)$$

They are obviously not mutual best responses.

7.

Answer:

(a)

n firms, $N = [n]$. $S_i = \mathbb{R}_+$. The payoff is:

$$v_i(q_1, q_2, \ldots, q_n) = (\max\{0, 100 - \sum_{j \in N} q_j\} - 10)q_i$$

(b)

Maximize the above payoff function,we get,

$$q_i = \max\left\{0, \frac{90 - \sum_{j \neq i} q_j}{2}\right\}$$

- Given that $q_j > 0$, when $90 - \sum_{j \neq i} q_j = c > 0$, $q_i = c$ is the best response , which traverses $[0, 45]$ when $n \geq 3$

    - On the contrary, when $n \geq 3$, we can always construct a q_{-1} such that $90-\sum_{j \neq i} q_j$ can traverse $[0,45]$, that means $[0,45]$ survives IESDS.
- for $q_j > 45$,
    - if $90-\sum_{j \neq i} q_j \geq q_i$,

$$v_i(0, q_{-i}) - v_i(q_i, q_{-i}) = 0 - (-10q_i) > 0$$

    - if $90-\sum_{j \neq i} q_j < q_i$,

$$v_i(45, q_{-i}) - v_i(q_i, q_{-i}) = (q_i - 45)(q_i - 45 + \sum_{j \neq i} q_j) > 0$$

- That means $(45, +\infty)$ is strictly dominated. And $[0,45]$ survives IESDS when $n \geq 3$
- When $n = 2$, similar to 4.6(c) , $S_1^* = S_2^* = \{\text{Fixed point of function } f(x) = 45 - x/2\} = \{30\}$
- In conclusion,

$$S_i^* = \begin{cases} \{30\}, & n = 2, \\ [0,45], & n \geq 3 \end{cases}$$