Game Theory, Fall 2022
Problem Set 6

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1. ST 8.4.

   (a) **Soln:** There are infinite proper subgames because every quantity choice of player 1 results in a proper subgame. Specifically, the set of all the subgames has the same cardinality as $\mathbb{R}$.

   (b) **Soln:** This is a game of imperfect information because players 2 and 3 make their choices without observing each other's choice first.

   (c) **Soln:** First we solve for the Nash equilibrium of the simultaneous move stage in which players 2 and 3 make their choices as a function of the choice after player 1. Fix player 1's quantity choice and player 2 maximizes

$$\max_{q_2}\{\max\{a-q_1-q_2-q_3-c,0\}q_2\},$$

which yields the best response

$$q_2=\begin{cases}\frac{a-q_1-q_3-c}{2}, & \text{if } q_3\le a-c-q_1,\\ 0, & \text{if } q_3>a-c-q_1.\end{cases}$$

Symmetrically, the best response function of player 3 is

$$q_3=\begin{cases}\frac{a-q_1-q_2-c}{2}, & \text{if } q_2\le a-c-q_1,\\ 0, & \text{if } q_2>a-c-q_1.\end{cases}$$

Hence, from the best response of player 2 and 3, it is apparent that there exists a unique Nash equilibrium of the subgame where player 1 choose q_1, which is

$$q_2^*(q_1)=q_3^*(q_1)=\begin{cases}\frac{a-q_1-c}{3}, & \text{if } q_1\le a-c,\\ 0, & \text{if } q_1>a-c.\end{cases}$$

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Back to the first node where player 1 chooses quantity, player 1 would maximize

$$\max_{q_1}\{\max\{a - q_1 - q_2^*(q_1) - q_3^*(q_1) - c, 0\}q_1\},$$

which leads to the FOC

$$\frac{a - 2q_1 - c}{3} = 0,$$

resulting in a unique solution $q_1 = \frac{a-c}{2}$. To sum up, the unique subgame-perfect equilibrium dictates that $q_1^* = \frac{a-c}{2}$ and

$$q_2^*(q_1) = q_3^*(q_1) = \begin{cases} \frac{a-q_1-c}{3}, & \text{if } q_1 \leq a - c, \\ 0, & \text{if } q_1 > a - c. \end{cases}$$

(d) **Soln:** There are many Nash equilibria in this game. We now prove that $q_1^* = \frac{a-c}{2}$ and

$$q_2^*(q_1) = q_3^*(q_1) = \begin{cases} \frac{a-q_1-c}{3}, & \text{if } q_1 = \frac{a-c}{2}, \\ a, & \text{otherwise.} \end{cases}$$

is a Nash equilibrium. When player 1 chooses $\frac{a-c}{2}$, player 2 and 3 would not deviate from their best responses. Meanwhile, player 1 would not deviate because any other choice would cause his opponents to choose a and bring player 1 negative profits. Hence the strategy profile above is a Nash equilibrium.

Apparently this strategy profile is not a subgame-perfect equilibrium, because the strategies of player 2 and 3 are not best responses in the subgames induced by a slightly different action of player 1.

2. ST 8.6.

(a) **Soln:** There are two subgames after firm 1 decides whether to invest or not. If firm 1 decides not to invest, each firm i would solve

$$\max_{q_i}\{\max\{100 - 10 - q_1 - q_2, 0\}q_i\}.$$

The solutions to the problems yield the unique Nash equilibrium $q_1 = q_2 = 30$, and the profit of firm 1 in this subgame Nash equilibrium is $(100-30-30-10)\times 30 = 900$.

If firm 1 decides to invest, firm 1 would solve a new problem, which is

$$\max_{q_1}\{\max\{100 - 5 - q_1 - q_2, 0\}q_1 - F\},$$

while firm 2 would still solve the same problem above. The solutions to the problems yield another unique Nash equilibrium $q_1 = \frac{100}{3}$ and $q_2 = \frac{85}{3}$. And the profit of firm 1 in this subgame Nash equilibrium is $(100 - \frac{100}{3} - \frac{85}{3} - 5) \times \frac{100}{3} - F = \frac{10000}{9} - F$.

Hence, if there is a unique subgame-perfect equilibrium involving firm 1 investing, it must be strictly better to invest than not to invest, which means $\frac{10000}{9} - F > 900$. Therefore, all $F^* \in (0, \frac{1900}{9})$ lead to a unique subgame-perfect equilibrium where firm 1 decides to invest.

(b) **Soln:** If $F > \frac{1900}{9}$, F would be larger than all the possible F^*, and we can construct a Nash equilibrium in which firm 1 decides not to invest, and will choose $q_1 = 30$ when not investing and $q_1 = \frac{100}{3}$ when investing, while firm 2 will choose $q_2 = 30$ when firm 1 does not invest, and $q_2 = 100$ otherwise. The outcome would be firm 1 doesn't invest and both firms produce 30. It's obvious that both firms would not deviate. Hence the strategy profile is indeed a Nash equilibrium. Moreover, this strategy profile is not a subgame-perfect equilibrium, because $q_2 = 100$ is not firm 2's best response when firm 1 invests.

3. ST 8.9.

(a) **Soln:** In the normal form representation of the game, the set of firms is $N = \{1, 2\}$. Each firm $i \in \mathcal{N}$ have a strategy space $S_i = [0, +\infty)$. The payoff function of firm i is

$$u_i(q_i, q_{-i}) = \max\{100 - q_1 - q_2, 0\}q_i - k\chi_{\{q_i>0\}}.$$

in which $\chi_{\{q_i>0\}}$ is the indicator function for the set $\{q_i > 0\}$.

(b) **Soln:** The best response of firm i is

$$q_i^*(q_j) = \begin{cases} \frac{100-q_j}{2}, & \text{if } q_j < 100 - 20\sqrt{10}, \\ \{10\sqrt{10}, 0\}, & \text{if } q_j = 100 - 20\sqrt{10}, \\ 0, & \text{if } q_j > 100 - 20\sqrt{10}. \end{cases}$$

From the best responses of the firms, we can find three pure Nash equilibria in the game, which are $(\frac{100}{3}, \frac{100}{3})$, $(50, 0)$, $(0, 50)$.

(c) **Soln:** The game tree for question 3c is depicted in Figure 1. The payoff is the same as the payoff in 3a.

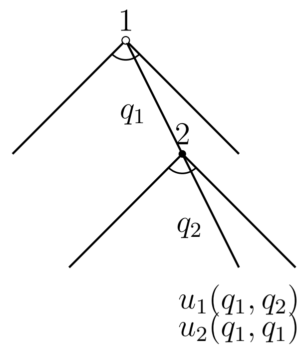

Figure 1: The game tree for Question 3c

When $k = 25$, the best response of firm 2 becomes

$$q_2^*(q_1) = \begin{cases} \frac{100-q_1}{2}, & \text{if } q_1 < 90, \\ 0, & \text{if } q_1 \geq 90. \end{cases}$$

Hence firm 1 might solve

$$\max_{q_1<90}\{(100 - q_1 - \frac{100 - q_1}{2})q_1\}$$

if the solution leads to positive profits for both firms, considering the fixed cost. The solution to the problem above is $q_1 = 50$, yielding that $q_2 = 25$, $u_1 = (100 - 50 - 25) \times 50 - 25 > 0$, and $u_2 = 25 \times 25 - 25 > 0$. Moreover, it's easy to verify that choosing $q_1 \geq 90$ is less profitable than this outcome, no matter what firm 2's strategy is. Hence there is a unique backward induction solution, in which $q_1 = 50$ and

$$q_2 = \begin{cases} \frac{100-q_1}{2}, & \text{if } q_1 < 90, \\ 0, & \text{if } q_1 \geq 90. \end{cases}$$

(d) **Soln:** When $k = 725$, the best response of firm 2 becomes

$$q_2^*(q_1) = \begin{cases} \frac{100-q_1}{2}, & \text{if } q_1 < 100 - 10\sqrt{29}, \\ 0, & \text{if } q_1 \geq 100 - 10\sqrt{29}. \end{cases}$$

For the initial node, first consider the case where $q_1 \geq 100 - 10\sqrt{29}$, firm 2 quits the market and firm 1 solves

$$\max_{q_1 \geq 100-10\sqrt{29}}\{(100 - q_1)q_1\},$$

which yields $q_1 = 50$. The corresponding payoff is $u_1 = 2500 - 725 = 1775 > 0$ and $u_2 = 0$.

Otherwise, firm 1 might solve

$$\max_{q_1 < 100 - 10\sqrt{29}} \{(100 - q_1 - \frac{100 - q_1}{2})q_1\}.$$

If the solution leads to positive profits for both firms, considering the fixed cost. However, it's obvious that

$$\begin{aligned}
\max_{100-10\sqrt{29} \le q_1 \le 100} \{(100 - q_1)q_1\} &= \max_{q_1 \le 100} \{(100 - q_1)q_1\} \\
&\ge \max_{q_1 \le 100} \{(100 - q_1 - \frac{100 - q_1}{2})q_1\} \\
&> \max_{q_1 < 100 - 10\sqrt{29}} \{(100 - q_1 - \frac{100 - q_1}{2})q_1\}.
\end{aligned}$$

Thus, there's a unique backward induction solution, in which $q_1 = 50$ and

$$q_2 = \begin{cases} \frac{100-q_1}{2}, & \text{if } q_1 < 100 - 10\sqrt{29}, \\ 0, & \text{if } q_1 \ge 100 - 10\sqrt{29}. \end{cases}$$

4. ST 8.12.

   (a) **Soln:** This is a game of perfect information. There are two players, $i \in \{1, 2\}$. The strategy sets are $S_1 = X = [0, 5]$ and $S_2 = \{A, R\}$, where A denotes accepting the proposal $x \in X$, and R means rejecting it and adopting the status quo $q = 4$. The payoffs are given by

$$v_1(s_1, s_2) = \begin{cases} 10 - |s_1 - 1|, & \text{if } s_2 = A, \\ 7, & \text{if } s_2 = R, \end{cases}$$

   and

$$v_2(s_1, s_2) = \begin{cases} 10 - |s_1 - 3|, & \text{if } s_2 = A, \\ 9, & \text{if } s_2 = R. \end{cases}$$

   (b) **Soln:** Player 2 can guarantee himself a payoff of 9 by choosing R, implying that his unique best response is to choose A if and only if $10 - |s_1 - 3| > 9$, which holds for any $s_1 \in (2, 4)$, and R for $s_1 > 4$ or $s_1 < 2$. He would be indifferent between A and R if $s_1 = 2$ or $s_1 = 4$. On the other hand, given that player 2 always accepts for any $s_1 \in (2, 4)$, player 1 would like to have an alternative proposal that is accepted by player 2 and as close to 1 as possible. If player 2 rejects $s_1 = 2$, there would be no subgame-perfect equilibria since for any $s_1 \in (2, 4]$, there always exists $2 < s_1' = \frac{2+s_1}{2} < s_1$, such that s_1' is closer to 1 than s_1, which means players

1 can always deviate to get a higher payoff. Thus, we only need to consider the case where player 2 accepts $s_1 = 2$. When $s_1 = 4$, he is indifferent between A and R, and both strategies can be equilibria. Given player 2's strategy, player 1's best response is to choose $s_1 = 2$, which is the closest to 1 for any $s_1 \in [2, 4]$ (or $s_1 \in [2, 4)$), and let player 2 accept. Thus, there are two subgame-perfect equilibria which result in the same payoffs $(v_1, v_2) = (9, 9)$.

(c) **Soln:** One Nash equilibrium is that player 2 adopts the strategy "I will reject anything except $s_1 = 3$" and that player 1 chooses 3. By doing so player 1 gets 8, while any other choice of s_1 is expected to yield a payoff of 7. Hence, player 1's best response to player 2's proposed strategy is indeed to choose $s_1 = 3$, and the payoffs from this Nash equilibrium are $(v_1, v_2) = (8, 10)$.

Since player 2 can guarantee himself a payoff of 9, there are actually infinite Nash equilibria that are not subgame-perfect following a similar logic: player 2 adopts the strategy "I will reject anything except $s_1 = x$" for some value $x \in [2, 4)$. Player 1 would strictly prefer the adoption of x over 4, and hence would indeed propose x, and player 2 would accept the proposal. For $x = 4$ both players are indifferent so it would also be supported as a Nash equilibrium.

5. ST 8.13.

(a) **Soln:** The game tree for question 5a is depicted in Figure 2.

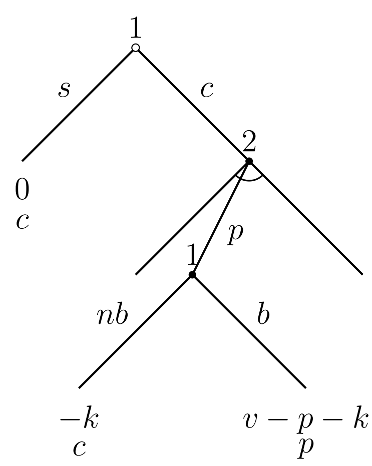

Figure 2: The game tree for Question 5a

Buyer's best response to seller's offer is

$$BR_1(c,p) = \begin{cases} b, & \text{if } p < v, \\ \{b, nb\}, & \text{if } p = v, \\ nb, & \text{if } p > v. \end{cases}$$

(b) **Soln:** Backward induction implies that buyer should reject the offer when $p < v$, and accept the offer when $p > v$. It is indifferent for the buyer between accepting $p = v$ or not. If the buyer reject the offer when $p = v$, there would be no best response for the seller because for any $p < v$ the seller could deviate to $p' = \frac{p+v}{2} > p$ to get a higher payoff than p. Hence in a subgame-perfect equilibrium, the buyer should accept $p = v$, and the seller would offer $p = v$.

In this case, the buyer could only get $-k$ if he commutes to the store, so he would stay home. This is the unique backward induction solution, in which the buyer decides to stay home, the seller would propose $p = v$ if the buyer commutes, and the buyer would accept any $p \geq v$ and reject others.

The outcome of the equilibrium is $(0, c)$, which is Pareto dominated by $(0, v - k)$ from the path that the buyer commutes and accepts the propose $p = v - k$ from the seller. Hence it is not Pareto optimal.

(c) **Soln:** Consider the strategy profile in which the buyer commutes to the store, the seller proposes $p = \frac{v-k+c}{2} > c$, and the buyer only accepts $p = \frac{v-k+c}{2}$ and rejects other cases. It's evident that nobody has the incentive to deviate. The payoff of this strategy profile is $(\frac{v-k-c}{2}, \frac{v-k+c}{2})$, which is strictly higher than $(0, c)$.

(d) **Soln:** If the seller chooses to send the postcard, he actually sets the price before the buyer decides whether to leave home, rather than after the buyer arrives. Thus, the game has a different timing. Backward induction implies that buyer should leave home and buy the good when $p < v - k$, stay home when $p > v - k$, and be indifferent when $p = v - k$. Similar to 5b, in a subgame-perfect equilibrium, the buyer should accept $p = v - k$, and the seller would offer $p = v - k$. Finally, the payoff of the buyer remains 0, but the seller gets $p - \epsilon = v - k - \epsilon > c$ and hence is strictly better off than in the subgame-perfect equilibrium of 5b, which is exactly the case where the seller doesn't send the postcard. Actually, when the seller commits $c + \epsilon < p < v - k$, the buyer would choose to buy and then both of them would be better off. Therefore, the seller would choose to send the postcard.