Suggested Solutions to Game Theory Midterm Exam, Fall 2023

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1. There are $n > 2$ workers who are to determine how much product to produce. Simultaneously, each worker i announces a number $x_i \geq 0$. The amount of this product to be produced is then determined by

$$y = \min\{x_1, x_2, \ldots, x_n\}.$$

The payoff of worker i is then $\sqrt{y} - \frac{y}{n}$.

(a) **(10 points)** Is the strategy profile $(x_1, \ldots, x_n) = (0, 1, 1, \ldots, 1)$ a Nash equilibrium?

Soln: We argue that worker 1 has profitable deviation from $x_1 = 0$. Strategy profile $(x_1, \ldots, x_n) = (0, 1, 1, \ldots, 1)$ yields $y = \min\{0, 1, 1, \ldots, 1\} = 0$ and $v_1(0, 1, 1, \ldots, 1) = 0$. If worker 1 deviates to announce $x_1' = 1$, strategy profile $(1, 1, 1, \ldots, 1)$ yields $y = 1$ and $v_1(1, 1, 1, \ldots, 1) = 1 - \frac{1}{n} > 0$. Hence, $x_1' = 1$ is a profitable deviation for worker 1. Therefore, strategy profile $(x_1, \ldots, x_n) = (0, 1, 1, \ldots, 1)$ is not a Nash equilibrium.

(b) **(10 points)** There are many Nash equilibria in which $y = 0$. Find two of them.

Soln: For example, $(0, 0, \ldots, 0)$ and $(0, 0, 1, 1, \ldots, 1)$ are two Nash equilibria.

(c) **(10 points)** Let $x > 0$ be a number. Under what condition of x is the symmetric strategy profile $(x_1, \ldots, x_n) = (x, x, \ldots, x)$ a Nash equilibrium?

Soln: By symmetry, we can focus on worker 1. We have

$$v_1(x_1, x, \ldots, x) = \begin{cases} \sqrt{x} - \frac{x}{n}, & \text{if } x_1 \geq x, \\ \sqrt{x_1} - \frac{x_1}{n}, & \text{if } x_1 < x. \end{cases}$$

Hence, worker 1 does not have a profitable deviation if and only if

$$\sqrt{x} - \frac{x}{n} \geq \sqrt{x_1} - \frac{x_1}{n}, \ \forall x_1 < x. \tag{1}$$

Note that the function $x \mapsto \sqrt{x} - \frac{x}{n}$ is strictly increasing over $[0, \frac{n^2}{4}]$ and strictly decreasing over $[\frac{n^2}{4}, \infty)$. Hence, (1) holds if and only if $x \leq \frac{n^2}{4}$. In other words, $(x, \ldots, x)$ is a Nash equilibrium if and only if $x \leq \frac{n^2}{4}$.

2. Suppose the stage game in Figure 2 is repeatedly played with common discount factor $\delta \in (0, 1)$. We restrict attention to pure strategies.

	L	M	R
T	$2,0$	$5,0$	$2,4$
C	$2,5$	$4,4$	$0,2$
B	$1,0$	$1,1$	$5,1$

Figure 1: Stage game for Question 2

(a) **(10 points)** Suppose the stage game is infinitely repeatedly played. Does there exist a subgame perfect equilibrium for sufficiently large δ in which CM is always played on the equilibrium path?

Soln: Consider the following strategy profile. In words, they play CM at the beginning and continue to play CM if CM is always played in the past; *once* player 1 *unilaterally* deviates or both players deviate, they switch to CL *permanently*; *once* player 2 *unilaterally* deviates, they switch to BR *permanently*.

Formally, for every $h = (a^1, \ldots, a^t)$ such that $h \neq (CM, \ldots, CM)$, let $\tau(h) \equiv \{1 \leq \tilde{\tau} \leq t | a^{\tilde{\tau}} \neq CM\}$ be the first time that deviation from CM occurs. Note that, if $a_2^{\tau(h)} = M$, we know that player 1 unilaterally deviates; if $a_1^{\tau(h)} = C$, we know that player 2 unilaterally deviates; if $a_1^{\tau(h)} \neq C$ and $a_2^{\tau(h)} \neq M$, we know that both players deviate. The above strategy profile can be written as

$$s(\emptyset) = CM \text{ and } s(h) = \begin{cases} CM, & \text{if } h = (CM, CM, \ldots, CM), \\ CL, & \text{if } a_2^{\tau(h)} = M \text{ or } (a_1^{\tau(h)} \neq C \text{ and } a_2^{\tau(h)} \neq M), \\ BR, & \text{if } a_1^{\tau(h)} = C. \end{cases}$$

We verify that this strategy profile is a subgame perfect equilibrium when δ is sufficiently large.

Consider $h = \emptyset$. If no one deviates, their payoffs are 4 and 4 respectively. If player 1 deviates to T at this history, his total payoff is $5(1-\delta) + 2\delta$. If he deviates to B at this history, his total payoff is $(1-\delta) + 2\delta$. Hence, for player 1 to have no profitable one-shot deviation at this history, we must have

$$4 \geq 5(1-\delta) + 2\delta,$$
$$4 \geq (1-\delta) + 2\delta.$$

This requires $\delta \geq \frac{1}{3}$. If player 2 deviates to L at this history, his payoff is $5(1-\delta)+\delta$. If he deviates to R at this history, his payoff is $2(1-\delta)+\delta$. Hence, for player 2 to have no profitable one-shot deviation at this history, we must have

$$4 \geq 5(1-\delta) + \delta,$$
$$4 \geq 2(1-\delta) + \delta.$$

This requires $\delta \geq \frac{1}{4}$. Therefore, when $\delta \geq \max\{\frac{1}{3}, \frac{1}{4}\} = \frac{1}{3}$, no one has a profitable one-shot deviation at this history. Consider any $h = (CM, CM, \ldots, CM)$. The continuation play after this history is the same as s itself. Therefore, we know that when $\delta \geq \frac{1}{3}$, no one has a profitable one-shot deviation at this history.

Consider $h \neq (CM, \ldots, CM)$ such that $a_2^{\tau(h)} = M$ or ($a_1^{\tau(h)} \neq C$ and $a_2^{\tau(h)} \neq M$). The continuation play of s is the strategy profile in which the stage Nash equilibrium CL is always played after all histories. Because we know that it is a subgame perfect equilibrium for any δ, we know that no one has a profitable one-shot deviation at this history.

Consider $h \neq (CM, \ldots, CM)$ such that $a_1^{\tau(h)} = C$. The continuation play of s is the strategy profile in which the stage Nash equilibrium BR is always played after all histories. Because we know that it is a subgame perfect equilibrium for any δ, we know that no one has a profitable one-shot deviation at this history.

In sum, we find that when $\delta \geq \frac{1}{3}$, no one has a profitable one-shot deviation after any history. Therefore, when $\delta \geq \frac{1}{3}$, the proposed strategy profile is a subgame perfect equilibrium.

Remark: There are other subgame perfect equilibria that can support $(CM, CM, \ldots, CM)$ as the outcome path.

(b) **(10 points)** Suppose the stage game is twice repeatedly played. Does there exist a subgame perfect equilibrium for some δ in which CM is played at least once on the equilibrium path?

Soln: No. We show by contradiction. Suppose there is a subgame perfect equilibrium for some δ in which CM is played on the equilibrium path. Since either CL or BR must be played in period $t = 2$, we know that CM can only be played in period $t = 1$. Hence, the outcome path is either (CM, CL) or (CM, BR). We argue that neither is possible.

i. Consider path (CM, CL). Player 1 obtains $v_1(CM, CL) = 4 + 2\delta$. If Player 1 deviates to T in $t = 1$, payoff to Player 1 is either $v_1(TM, CL) = 5 + 2\delta$ or $v_1(TM, BR) = 5 + 5\delta$. Clearly, $5 + 5\delta > 5 + 2\delta > 4 + 2\delta$, which implies that player 1 has a profitable deviation. Hence, the equilibrium path cannot be (CM, CL).

ii. Consider path (CM, BR). Player 2 gets $v_2(CM, CL) = 4+\delta$. If Player 2 deviates to L in $t = 1$, payoff to Player 2 is either $v_2(CL, CL) = 5 + 5\delta$ or $v_2(TM, BR) = 5 + \delta$. Clearly, $5 + 5\delta > 5 + \delta > 4 + \delta$, which implies that player 2 has a profitable deviation. Hence, the equilibrium path cannot be (CM, BR).

3. There are $N > 1$ students in Game Theory. They simultaneously and independently vote whether to cancel (C) the midterm or take (T) it. If all of them vote C, the midterm will be canceled, and everyone obtains a payoff 1. Otherwise, the midterm will proceed as usual. In this case, every student who votes C incurs a loss $c > 0$, while every student who votes T obtains 0.

   (a) **(10 points)** Find all pure strategy Nash equilibria of this game.

   Soln: Clearly, there is no equilibrium in which come students vote C and some students vote T. Otherwise, those who vote C can gain by deviating to T.

   We are now left with two possibilities: either all students vote C or all students vote T. It is easy to verify that they are both Nash equilibria.

   (b) **(10 points)** Find a Nash equilibrium in mixed strategies.

   Soln: We look for an equilibrium in which all students vote C with probability $\alpha \in (0, 1)$. This strategy profile is an equilibrium if and only if everyone is indifferent between C and T given the others' strategies. That is

$$\alpha^{N-1} \times 1 + (1 - \alpha^{N-1}) \times (-c) = 0.$$

   We obtain $\alpha = (\frac{c}{1+c})^{N-1} \in (0, 1)$.

Suppose from now on that the voting is not simultaneous. Instead, it is dynamic as follows. There are in total $T > 1$ periods. In period $t = 1$, they simultaneously and independently vote C or T. After the first round of voting, they all observe who have voted C and who have voted T. The students who have voted C leave the game, meaning that voting C is irreversible. If all students have voted C in the first period, the game ends. Otherwise, the students who have voted T enter into period $t = 2$, and then they simultaneously and independently vote C or T again. After the second round of voting, they again observe who have voted C and who have voted T. As before, the students who voted C leave the game and the remaining students, if any, enter into period $t = 3$ and vote again. The game ends either because all students have voted C after some round, or they finish all T rounds of voting. The payoffs to students are the same as before: at the end of the game, if all students have voted C, the midterm is canceled and everyone obtains 1; otherwise, the midterm proceeds as usual, and the students who have once voted C incur loss $c > 0$ and the other students obtain 0.

(c) **(10 points)** Suppose $N = 2$ and $T = 2$. Draw a game tree for this game. Carefully label the student to who each information set belongs and write out the payoffs at each terminal node.

Soln: See Figure 2.

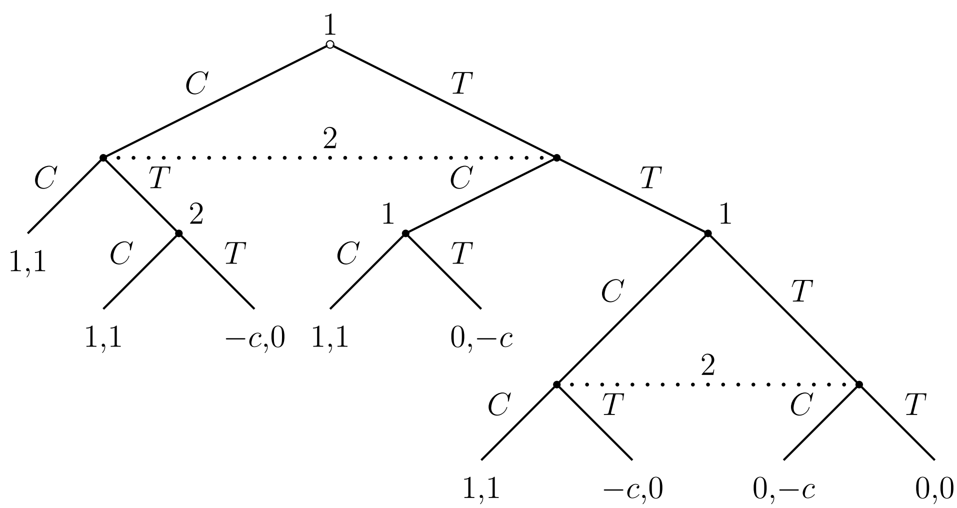

Figure 2: The game tree for Question 3(c)

(d) **(10 points)** Suppose $N = 3$ and $T = 2$. Does there exist a subgame perfect equilibrium in which the midterm is not canceled?

Soln: Consider the following strategy profile:

$$s_i^1 = T, s_i^2(h) = \begin{cases} C, & \text{if other two students have voted } C \text{ in } h, \\ T, & \text{otherwise.} \end{cases}$$

In words, all students vote T in the first round. In the second round, at any history in which there are at least two remaining students, they continue to vote T; at any history in which there is only one remaining student, he votes C.

It is easy to verify that no one has a profitable one-shot deviation. Hence, it is a subgame perfect equilibrium. The associated equilibrium path is (TTT, TTT). Therefore, midterm is not canceled.

(e) **(10 points)** Would your answer to Question 3(d) change if $T = 3$ instead of $T = 2$?

Soln: We argue that any SPE s must lead to the cancellation of midterm. For ease of exposition, we assume that s is a pure strategy profile. The result still holds if s involves mixture. We do backward induction.

Consider any history in $t = 3$ in which two students have voted C. Since s is a SPE, the only remaining student must vote C as well at this history.

Consider any history in $t = 2$ in which two students have voted C. If the remaining student votes C now, midterm will be canceled. If he votes T now, he will enter into period $t = 3$ at a history in which two students have voted C. Hence, he will vote C then. This also leads to cancellation of midterm.

Consider any history in $t = 2$ in which one student have voted C. We argue that the continuation play of s in the subgame following this history must lead to cancellation of the midterm. To see this, observe that by voting C now, every remaining student can guarantee himself payoff 1, since it must lead to cancellation of midterm regardless of what the other remaining student does now. Hence, SPE implies that every remaining student's payoff in this subgame from s must be at least 1. This then implies that midterm must be canceled in the continuation play, since this is the only outcome that gives students payoff 1. (Note that this claim does not mean they vote C now nor at least one remaining student votes C now. It is possible that they both vote T now, but vote C in $t = 3$.)

Consider $t = 1$. We can use a similar argument as above to show that the play under s must lead to cancellation of midterm. By voting C in the first period, a student can guarantee himself payoff 1 regardless of what others do now. Therefore, every student in this SPE must earn payoff at least 1. This again means that midterm in the end must be canceled. (As above, this claim does not say anything about what the students must do in the first period.)