Suggested Solutions to Game Theory Midterm Exam, Fall 2020

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Important: This is a closed-book exam. No books, lecture notes or calculators are permitted. You have **120 minutes** to complete the exam. Answer all questions. Write legibly. Good luck!

1. Two investors $i = 1, 2$ want to start a startup. Investor $i = 1$ provides the capital, denoted by $k \in [0, 1]$. Investor $i = 2$ provides the labor, denoted by $\ell \in [0, 1]$. They will equally share the revenue, which is equal to $2k\ell$. The cost of capital for $i = 1$ is k^3, while the cost of labor for $i = 2$ is $\frac{\ell^2}{2}$. Thus, the payoffs are $v_1(k, \ell) = k\ell - k^3$ and $v_2(k, \ell) = k\ell - \frac{\ell^2}{2}$, respectively. They choose their investment simultaneously. We restrict attention to pure strategies.

   (a) **(10 points)** Find all Nash equilibria.

   Soln: We use first order condition to calculate interior optimization points first. Since $\frac{\partial v_1(k,\ell)}{\partial k} = \ell - 3k^2 = 0$, $\frac{\partial v_2(k,\ell)}{\partial \ell} = k - \ell = 0$, interior solution leads to $BR_1(\ell) = \sqrt{\frac{\ell}{3}}, BR_2(k) = k$. Second order condition guarantees above response maximizes their own utility. Hence there are two Pure NE: (0,0) and $(\frac{1}{3}, \frac{1}{3})$. For $\forall k \in [0, 1], BR_2(k) \in [0, 1]$, and for $\forall \ell \in [0, 1], \sqrt{\frac{\ell}{3}} \in [0, \sqrt{\frac{1}{3}}] \subseteq [0, 1]$, which guarantees we have found all the Nash Equilibrium.

   (b) **(10 points)** Find the set of strategies for both players that survive iterated elimination of strictly dominated strategies.

   Soln: Since $S_0^2 = [0, 1]$, for $\forall \ell \in [0, 1]$, $k' \in (\frac{1}{3}, 1]$ is dominated by $k = \sqrt{\frac{\ell}{3}}$ because $v_1(k', \ell) < v_1(k, \ell)$ from first order condition. Then $\forall k \in (\frac{1}{3}, 1]$ can be eliminated in the first round hence $S_1^1 = [0, \frac{1}{3}]$. $S_1^2 = [0, \frac{1}{3}]$ because for $\forall k \in S_1^1$ $\ell' \in (\frac{1}{3}, 1]$ is dominated by $\ell = k \in [0, \frac{1}{3}]$ which is also guaranteed by first order condition. We claim we can not further eliminate since for $\forall \ell \in S_1^2$ it's best response for $k = \ell \in S_1^1$, and for $\forall k \in S_1^1$ it's best response for $\ell = 3k^2 \in S_1^2$. Hence the answer is $S_1^1 \times S_1^2$.

2. In a finite normal form game, we say a Nash equilibrium σ^* is completely mixed if for all i and $a_i \in A_i$, we have $\sigma_i^*(a_i) > 0$.

(a) **(10 points)** Give an example of a finite normal form game which has a completely mixed Nash equilibrium.

Soln: Many games we covered in class have completely mixed Nash equilibria. For example, the Matching Pennies, the Rock-Paper-Scissors and the Battle of the Sexes.

(b) **(10 points)** Prove or disprove: there exists a two-player finite normal form game in which there are two and exactly two completely mixed Nash equilibria.

Soln: In such a game, we show that if there are more than one completely mixed Nash equilibria, there must be infinitely many. Suppose (σ_1, σ_2) and (σ_1', σ_2') are two such equilibria. Assume they are different. Because σ_1 is a best response to σ_2 and because it is completely mixed, player 1 must be indifferent between all of his strategies. That is, there exists c_1 such that

$$u_1(a_1, \sigma_2) = c_1, \ \forall a_1 \in S_1.$$

Similarly, because σ_1' is a best response to σ_2' and because it is completely mixed, there exists c_1' such that

$$u_1(a_1, \sigma_2') = c_1', \ \forall a_1 \in S_1.$$

Consider player 2's strategy σ_2^t defined by $\sigma_2^t(a_2) = t\sigma_2(a_2) + (1-t)\sigma_2'(a_2)$ for all $a_2 \in S_2$ for some $t \in (0,1)$. It is also a completely mixed strategy. We have

$$u_1(a_1, \sigma_2^t) = tu_1(a_1, \sigma_2) + (1-t)u_1(a_1, \sigma_2') = tc_1 + (1-t)c_1', \ \forall a_1 \in S_1.$$

That is, player 1 is indifferent between all of his pure strategies if player 2 plays σ_2^t. Therefore, every strategy (including completely mixed) of player 1 is a best response to σ_2^t.

Similarly, we can show that player 2 is also indifferent between all of his pure strategies if player 1 plays σ_1^t similarly defined as above. Hence, every strategy (including completely mixed) of player 2 is a best response to σ_1^t. This implies $(\sigma_1^{t_1}, \sigma_2^{t_2})$ is a Nash equilibrium. Because there are infinitely many such strategy profiles as t_1 and t_2 varies, there are infinitely many completely mixed Nash equilibria.

3. Two firms, $i = 1, 2$, involve a price competition in a market. There is a consumer with unit demand. If she buys from firm i at price p_i, her payoff is $v_i - p_i$, where

$v_i > 0$ is the value of firm i's product to her. Assume she always buys from the firm which gives her higher payoff, provided that payoff is nonnegative. If the payoffs from both firms are equal and nonnegative, she chooses firms randomly with equal probability. In case both firms provide negative payoffs, she does not buy at all. Assume there is no cost of production and the payoff to firm i is just p_i if the consumer buys from i and 0 otherwise. Suppose the firms set their prices, $p_1, p_2 \in \mathbb{R}$, simultaneously. We restrict attention to pure strategies.

(a) **(10 points)** Write down firm i's payoff function $u_i(p_i, p_j)$.

Soln: We have

$$u_1(p_1,p_2) = \begin{cases} p_1, & \text{if } v_1 - p_1 \geq 0 \text{ and } v_1 - p_1 > v_2 - p_2, \\ \frac{p_1}{2}, & \text{if } v_1 - p_1 \geq 0 \text{ and } v_1 - p_1 = v_2 - p_2 \\ 0, & \text{Otherwise,} \end{cases}$$

and similarly

$$u_2(p_1,p_2) = \begin{cases} p_2, & \text{if } v_2 - p_2 \geq 0 \text{ and } v_2 - p_2 > v_1 - p_1, \\ \frac{p_2}{2}, & \text{if } v_2 - p_2 \geq 0 \text{ and } v_2 - p_2 = v_1 - p_1, \\ 0, & \text{Otherwise.} \end{cases}$$

(b) **(10 points)** Suppose $v_1 = v_2$. Find a Nash equilibrium.

Soln: It is immediate that $(0,0)$ is a Nash Equilibrium, once you understand this is nothing but the Bertrand game we covered in class.

We can prove it by checking neither firm has an incentive to deviate. Given $(0,0)$, firm 2's profits are 0. If firm 2 deviates to $p_2' > 0$, its demand is 0 and thus profits are 0 too. If it deviates to $p_2' < 0$, its demand is 1 and profits are negative. Therefore, given $p_1 = 0$, firm 2 has no profitable deviation. Similarly firm 1 has no profitable deviation either.

(c) **(10 points)** Suppose $v_1 > v_2$. Does there exist a Nash equilibrium?

Soln: There is no pure strategy Nash equilibrium. One way to see this is to write down their best response correspondences:

$$BR_1(p_2) = \begin{cases} v_1, & \text{if } p_2 > v_2, \\ \emptyset, & \text{Otherwise,} \end{cases}$$

and

$$BR_1(p_2) = \begin{cases} v_2, & \text{if } p_1 > v_1, \\ \emptyset, & \text{if } v_1 \geq p_1 > v_1 - v_2, \\ [0, \infty), & \text{if } p_1 \leq v_1 - v_2. \end{cases}$$

We see that there is no intersection of their graphs in the $p_1 - p_2$ plane.

In particular, the Nash equilibrium $(0, 0)$ in 3b is no longer an equilibrium. This is because firm 1 can deviate to $p_1' > 0$ that satisfies $v_1 - p_1' > v_2 - 0$ to obtain positive profits.

4. There are 101 citizens, labeled by $-50, -49, \ldots, 0, \ldots, 49, 50$. There are two political candidates, taking turns to choose a policy platform in $\{-50, -49, \ldots, 0, \ldots, 49, 50\}$. Candidate 1 chooses a_1 first. Candidate 2 chooses a_2 after observing 1's choice. Citizen i's most preferred policy is i. Between a_1 and a_2, she votes for the one that is closer i. In case of indifference, she votes randomly with equal probability. (Or equivalently, she gives half vote to each of the candidates.) The rule of the voting is the simple majority rule. The candidate who gets more votes wins and the other loses. If both candidates get the same number of votes, it is a tie. Every candidate prefers winning to a tie, and prefers a tie to losing. We think about this situation as an extensive form game between the two candidates and restrict attention to pure strategies.

   (a) **(10 points)** Find two subgame perfect equilibria.

   Soln: We assume the utility of winning is 2, a tie is 1, and losing is 0. One can prove that player 2 would set $a_2 \in \{a_1 + 1, a_1 + 2, \ldots, a_1 - 1\}$ indifferently if $a_1 < 0$ to win or set $a_2 \in \{-a_1 + 1, -a_1 + 2, \ldots, a_1 - 1\}$ if $a_1 > 0$. When $a_1 = 0$ player 2 should set $a_2 = 0$ for a tie instead of losing. Hence player 1 will always set $a_1 = 0$. For player 2, you can set many strategies such as:

$$a_2(a_1) = \begin{cases} 0, & \text{if } a_1 = 0, \\ a_1 + 1, & \text{if } a_1 < 0, \\ a_1 - 1, & \text{if } a_1 > 0. \end{cases}$$

   or

$$a_2'(a_1) = \begin{cases} 0, & \text{if } a_1 \in \{0, -1, 1\}, \\ a_1 + 2, & \text{if } a_1 < -1, \\ a_1 - 2, & \text{if } a_1 > 1. \end{cases}$$

$(a_1, a_2(a_1)), (a_1, a_2'(a_1))$ are both SPE.

(b) **(10 points)** Find two Nash equilibria that are not subgame perfect.

Soln:

$a_1 = 0$ and

$$a_2(a_1) = \begin{cases} 0, & \text{if } a_1 < 50, \\ 50, & \text{if } a_1 = 50. \end{cases}$$

or

$$a_1' = 0, a_2'(a_1) = a_1, \forall a_1$$

.

We can prove they are both NE since they are best response to each other:

i. in $(a_1, a_2(a_1))$, given $a_1 = 0$, $a_2|_{a_1=0} = 0$ maximizes u_2, and given $a_2(a_1)$, player 1 gets $u_1 = 1$by setting $a_1 = 0$ or 50, gets 0 otherwise. Hence player 1 has no profitable deviation.

ii. in $(a_1', a_2'(a_1))$, given $a_1 = 0$, $a_2|_{a_1=0} = 0$ maximizes u_2, and given $a_2(a_1)$, player 1 gets $u_1 = 1$by setting $\forall a_1$. Hence player 1 has no profitable deviation.

We then prove neither of them are SPE:

i. in $(a_1, a_2(a_1))$, if firm 1 chooses 50 instead of 0, firm 2 should set $\forall a_2(50) \in \{-49, -48, \ldots 48, 49\}$ to get utility 2, which is strictly better than getting 1 by setting 50. Hence in the sub-game followed by $a_1 = 50$, the strategy profile is not NE. In other words firm 2 has profitable one-shot deviation.

ii. Totally the same. In the sub-game followed by $a_1 \neq 0$, the strategy profile is not NE

(c) **(10 points)** Show that all Nash equilibria lead to the same equilibrium outcome.

Soln: Observe that, regardless of 2's strategy, 1 can always guarantee himself at least a draw by choosing $a_1 = 0$. If there is a Nash equilibrium in which 1 chooses $a_1' \neq 0$, then we know 2 must win. But by deviating to $a_1 = 0$, 1 can get a draw, a contradiction. Therefore, in any equilibrium we must have $a_1 = 0$. This implies $a_2(0) = 0$. So there is a unique Nash equilibrium outcome $(0, 0)$.