Game Theory, Fall 2022
Problem Set 9

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1. ST 12.2

   Soln: Firm 1 and 2 have 2 types c_h and c_l. Firm 1's strategy is a pair $(q_{1,l}, q_{1,h}) \in \mathbb{R}^2_+$, and firm 2's strategy is a pair $(q_{2,l}, q_{2,h}) \in \mathbb{R}^2_+$. The price is determined by the demand function: $P(q_1, q_2) = \max\{a - b(q_1 + q_2), 0\}$. The strategy profile $(q^*_{1,l}, q^*_{1,h}, q^*_{2,l}, q^*_{2,h})$ is a Bayesian Nash equilibrium if and only if

$$q^*_{1,l} = \arg\max_{q_{1,l}} \; q_{1,l}[\mu(P(q_{1,l}, q^*_{2,l}) - c_l) + (1-\mu)(P(q_{1,l}, q^*_{2,h}) - c_l)],$$
$$q^*_{1,h} = \arg\max_{q_{1,h}} \; q_{1,h}[\mu(P(q_{1,h}, q^*_{2,l}) - c_h) + (1-\mu)(P(q_{1,h}, q^*_{2,h}) - c_h)],$$
$$q^*_{2,l} = \arg\max_{q_{2,l}} \; q_{2,l}[\mu(P(q^*_{1,l}, q_{2,l}) - c_l) + (1-\mu)(P(q^*_{1,h}, q_{2,l}) - c_l)],$$
$$q^*_{2,h} = \arg\max_{q_{2,h}} \; q_{2,h}[\mu(P(q^*_{1,l}, q_{2,h}) - c_h) + (1-\mu)(P(q^*_{1,h}, q_{2,h}) - c_h)].$$

   Assuming interier solution, we can get

$$q^*_{1,l} = \frac{1}{2}[\frac{a-c_l}{b} - (\mu q^*_{2,l} + (1-\mu)q^*_{2,h})],$$
$$q^*_{1,h} = \frac{1}{2}[\frac{a-c_h}{b} - (\mu q^*_{2,l} + (1-\mu)q^*_{2,h})],$$
$$q^*_{2,l} = \frac{1}{2}[\frac{a-c_l}{b} - (\mu q^*_{1,l} + (1-\mu)q^*_{1,h})],$$
$$q^*_{2,h} = \frac{1}{2}[\frac{a-c_h}{b} - (\mu q^*_{1,l} + (1-\mu)q^*_{1,h})].$$

   Thus, we have

$$q^*_{1,l} = q^*_{2,l} = \frac{2a - 2c_h + (3-\mu)(c_h - c_l)}{6b},$$
$$q^*_{1,h} = q^*_{2,h} = \frac{2a - 2c_h - \mu(c_h - c_l)}{6b}.$$

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2. Consider the above Cournot duopoly again. But now assume that the firms' costs are determined by

$$c_1 = \theta + \varepsilon_1,$$
$$c_2 = \theta + \varepsilon_2,$$

where θ, which can be either 3 or 5 with equal probabilities, is the common shock to both firms. And, ε_i, which can be either -1 or 1 with equal probabilities, is firm i's idiosyncratic shock. Assume θ, ε_1 and ε_2 are independent. This cost structure is common knowledge. As before, firm i only observes c_i. This means that it does not observe c_{-i}, θ, ε_1 and ε_2. Model this situation as a Bayesian game and write down each type's posterior beliefs.

Soln: There are two players, firm 1 and firm 2. Each firm can choose a production level $q_i \geq 0$. Firm i's type space is $C_i = \{2, 4, 6\}$ and $c_i \in C_i$ is the type. Firm i's payoff function is

$$v_i(q_1, q_2, c_1, c_2) = q_i(\max\{a - b(q_1 + q_2), 0\} - c_i).$$

The common prior is

$$\mathbb{P}(2,2) = \frac{1}{8},\ \mathbb{P}(2,4) = \frac{1}{8},\ \mathbb{P}(2,6) = 0,$$
$$\mathbb{P}(4,2) = \frac{1}{8},\ \mathbb{P}(4,4) = \frac{1}{4},\ \mathbb{P}(4,6) = \frac{1}{8},$$
$$\mathbb{P}(6,2) = 0,\ \mathbb{P}(6,4) = \frac{1}{8},\ \mathbb{P}(6,6) = \frac{1}{8}.$$

Firm i's posterior belief about firm $-i$ is

$$\phi_i(2|2) = \frac{1}{2},\ \phi_i(4|2) = \frac{1}{2},\ \phi_i(6|2) = 0;$$
$$\phi_i(2|4) = \frac{1}{4},\ \phi_i(4|4) = \frac{1}{2},\ \phi_i(6|4) = \frac{1}{4};$$
$$\phi_i(2|6) = 0,\ \phi_i(4|6) = \frac{1}{2},\ \phi_i(6|6) = \frac{1}{2}.$$

3. ST 12.3

   (a) **Soln:** Each of the two players has a type $\theta_i \in \{W, S\}$. The common prior is $\mathbb{P}(\theta_1, \theta_2) = \frac{1}{4}$ and their posterior belief is $\phi_i(\theta_{-i}|\theta_i) = \frac{1}{2}$. A pure strategy of agent i is a mapping $s_i : \{W, S\} \to \{A, N\}$. We adopt the convention that $s_i = XY$ means $s_i(W) = X$ and $s_i(S) = Y$. Obviously, once a player chooses not to attack, his payoff is 0 for certain (regardless of the other's strategy and type).

   First, consider $\theta_1 = W$.

- Suppose $s_2 = AA$. If player 1 chooses to attack, he can never win the territory, and then his payoff is $v_1(A, A; W, \theta_2) = -w < 0$. Thus, his best-response action is N.
- Suppose $s_2 = NN$. If player 1 chooses to attack, he wins the territory, and his payoff is $v_1(A, N; W, \theta_2) = m > 0$. Thus, his best-response action is A.
- Suppose $s_2 = AN$. Player 1's expected payoff of choosing A is $\phi_1(W|W) \times v_1(A, A; W, W) + \phi_1(S|W) \times v_1(A, N; W, S) = \frac{m-w}{2} > 0$. Thus, his best-response action is A.
- Suppose $s_2 = NA$. Player 1's expected payoff of choosing A is $\phi_1(W|W) \times v_1(A, N; W, W) + \phi_1(S|W) \times v_1(A, A; W, S) = \frac{m-w}{2} > 0$. Thus, his best-response action is A.

Next, consider $\theta_1 = S$.

- Suppose $s_2 = AA$. If player 1 chooses to attack, he wins the territory for the probability of $\frac{1}{2}$, and his expected payoff is $\phi_1(W|S) \times v_1(A, A; S, W) + \phi_1(S|S) \times v_1(A, A; S, S) = \frac{m}{2} - s > 0$. Thus, his best-response action is A.
- Suppose $s_2 = NN$. If player 1 chooses to attack, he wins the territory, and his payoff is $v_1(A, N; S, \theta_2) = m > 0$. Thus, his best-response action is A.
- Suppose $s_2 = AN$. Player 1's expected payoff of choosing A is $\phi_1(W|S) \times v_1(A, A; S, W) + \phi_1(S|S) \times v_1(A, N; S, S) = m - \frac{s}{2} > 0$. Thus, his best-response action is A.
- Suppose $s_2 = NA$. Player 1's expected payoff of choosing A is $\phi_1(W|S) \times v_1(A, N; S, W) + \phi_1(S|S) \times v_1(A, A; S, S) = \frac{m-s}{2} > 0$. Thus, his best-response action is A.

By symmetry, we can derive player 2's best response. Observe that given one's type is S, his only best response is A no matter what his opponent's strategy is. Any pure BNE must have the form: (XA, XA).

Given one's strategy is $s_i = AA$, the best response is $s_{-i} = NA$; given one's strategy is $s_i = NA$, the best response is $s_{-i} = AA$. Thus, the BNEs are (AA, NA) and (NA, AA).

Intuitively, since the costs of fighting are less than the value of the territory, both types are more willing to attack except for the condition that given one always attacking, the other will not attack.

(b) **Soln:** We just repeat the procedure in 3a.

The differences are:

- When $\theta_1 = W$, $s_2 = AN$ or NA, player 1's expected payoff of choosing A is $\frac{m-w}{2} < 0$. Thus, his best-response action becomes N.
- When $\theta_1 = S$, $s_2 = AA$, player 1's expected payoff of choosing A is $\frac{m}{2} - s < 0$. Thus, his best-response action becomes N.

By observation, we know that the weak type will choose to attack only if the other's strategy is NN. Overall, BNEs are (AA, NN), (NN, AA) and (NA, NA). Intuitively, the costs rise and now they choose to attack only when they are more "confident". In other words, given the other one's strategy, they attack when they are more sure to win.

4. ST 12.6

(a) **Soln:** The farmer knows his own type $\theta \in [0, 1]$, and for each type he proposes a wage $w \geq 0$. After his proposal, the owner chooses his action $a \in \{R, A\}$. Thus, the sets of pure strategies for the farmer and owner are

$$W = \{w | w : [0, 1] \to [0, +\infty)\},$$
$$S = \{s | s : [0, +\infty) \to \{R, A\}\}.$$

The owner knows the farmer's talent is uniformly distributed on $[0, 1]$, so his expected payoff after observing w is

$$\mathbb{E}u_o(w, a) = \begin{cases} 0, & \text{if } a = R, \\ \frac{3}{2} \times \frac{1}{2} - w, & \text{if } a = A. \end{cases}$$

Thus, the owner will reject $w > \frac{3}{4}$, accept $w < \frac{3}{4}$, and be indifferent when $w = \frac{3}{4}$. It is easy to verify that in any Bayesian Nash equilibrium he must accept $w = \frac{3}{4}$. Given the owner's strategy, the expected payoff for the farmer is

$$\mathbb{E}u_f(\theta, w, s) = \begin{cases} \theta, & \text{if } w > \frac{3}{4}, \\ w, & \text{if } w \leq \frac{3}{4}. \end{cases}$$

Thus, the strategy profile (w^*, s^*) is a Bayesian Nash equilibrium if

$$w^*(\theta) = \begin{cases} \frac{3}{4}, & \text{if } \theta \in [0, \frac{3}{4}), \\ [\frac{3}{4}, +\infty), & \text{if } \theta = \frac{3}{4}, \\ (\frac{3}{4}, +\infty), & \text{if } \theta \in (\frac{3}{4}, 1]; \end{cases}$$

$$s^*(w) = \begin{cases} A, & \text{if } w \leq \frac{3}{4}, \\ R, & \text{if } w > \frac{3}{4}. \end{cases}$$

(b) **Soln:** In the BNE, for the farmer with $\theta \in [0, \frac{3}{4})$, he proposes $w = \frac{3}{4}$ and will be hired. For the the farmer with $\theta \in (\frac{3}{4}, 1]$, he proposes $w > \frac{3}{4}$ and will not be hired. Thus, the social surplus from the above equilibrium is

$$SP = \underbrace{\frac{3}{4} \times \frac{3}{4} + \int_{\frac{3}{4}}^{1} \theta \ \mathrm{d}\theta}_{\text{sum of expected payoffs of the farmer}} + \underbrace{\int_{0}^{\frac{3}{4}} (\frac{3}{2}\theta - \frac{3}{4}) \ \mathrm{d}\theta}_{\text{sum of expected payoffs of the owner}}$$

$$= \int_{\frac{3}{4}}^{1} \theta \ \mathrm{d}\theta + \int_{0}^{\frac{3}{4}} \frac{3}{2}\theta \ \mathrm{d}\theta = \frac{7}{32} + \frac{27}{64} = \frac{41}{64}.$$

Another way to calculate the social surplus is to consider the total productivity when the farmer is hired or not:

$$SP = \int_{0}^{\frac{3}{4}} (\frac{3}{4} + \frac{3}{2}\theta - \frac{3}{4}) \ \mathrm{d}\theta + \int_{\frac{3}{4}}^{1} \theta \ \mathrm{d}\theta = \frac{41}{64}.$$

The outcomes are the same. It is easy to find that when the farmer is hired, the wage is only a transfer and does not influence the social surplus.

(c) **Soln:** In the new setting where the farmer only gets $\frac{\theta}{2}$ from farming his land, the best response of the owner remains unchanged, since the common prior stays the same. Now for the farmer, the wage he can get at most is still $\frac{3}{4}$, but the payoff from farming is at most $\frac{1}{2}$. It'll be optimal for the farmer to ask for $w = \frac{3}{4}$ no matter what his type is.

Thus, the new BNE is

$$w^*(\theta) = \frac{3}{4}, \ \forall \theta \in [0, 1];$$

$$s^*(w) = \begin{cases} A, & \text{if } w \leq \frac{3}{4}, \\ R, & \text{if } w > \frac{3}{4}. \end{cases}$$

In this case, the farmer always proposes $w = \frac{3}{4}$ and will always be hired. The social surplus changes to

$$SP' = \underbrace{1 \times \frac{3}{4}}_{farmer} + \underbrace{\int_{0}^{1} (\frac{3}{2}\theta - \frac{3}{4}) \ \mathrm{d}\theta}_{owner} = \frac{3}{4} > \frac{41}{64}.$$

Thus, using the criterion of social surplus, we should advocate for this new policy. Intuitively, the new policy forces the farmer to work in the manufacturing plant, which improves the total productivity from θ to $w + \frac{3}{2}\theta - w = \frac{3}{2}\theta$.

5. ST 12.7

(c) **Soln:** We first prove (c), and then apply it to both (a) and (b). Assume that the value of i's house to j is kv_i for $k > 0$. A strategy of agent i is a mapping $s_i : [0, 1] \to [0, 1]$, where $s_i(v_i)$ is the probability that i exchanges when his value is v_i.

Suppose (s_1, s_2) is a BNE. We consider two cases.

Case 1: Consider the case where at least one agent exchanges with positive probability. Without loss of generality, assume it is agent 1. That is, $\int_0^1 s_1(v_1)\mathrm{d}v_1 > 0$. Consider agent 2 with value v_2. If he does not exchange, his payoff is v_2. If he exchanges, his expected payoff is

$$\int_0^1 \left[kv_1 s_1(v_1) + v_2(1 - s_1(v_1))\right] \mathrm{d}v_1.$$

Then 2's best response is

$$s_2(v_2) = \begin{cases} 0, & \text{if } v_2 > \frac{k\int_0^1 v_1 s_1(v_1)\mathrm{d}v_1}{\int_0^1 s_1(v_1)\mathrm{d}v_1}, \\ 1, & \text{otherwise.} \end{cases}$$

It has two implications. First, s_2 is a threshold strategy, with the threshold $\bar{v}_2 \equiv \frac{k\int_0^1 v_1 s_1(v_1)\mathrm{d}v_1}{\int_0^1 s_1(v_1)\mathrm{d}v_1}$. Note that $\frac{\int_0^1 v_1 s_1(v_1)\mathrm{d}v_1}{\int_0^1 s_1(v_1)\mathrm{d}v_1}$ has a very intuitive interpretation. It is the expected value of 1's house given that 1 exchanges. Thus, agent 2's best response is that 2 is willing to exchange only if he thinks that the value he obtains from the trade on average exceeds his current value. Second, because $\bar{v}_2 > 0$, this means that 2 must exchange with positive probability, i.e., $\int_0^1 s_2(v_2)\mathrm{d}v_2 > 0$. Applying the same argument as above, we know that s_1 must be a threshold strategy, with threshold $\bar{v}_1 \equiv \frac{k\int_0^1 v_2 s_2(v_2)\mathrm{d}v_2}{\int_0^1 s_2(v_2)\mathrm{d}v_2}$. This implies that in this equilibrium, both agents use threshold strategies.

Case 2: Both agents exchange with probability 0, e.g., $s_i(v_i) \equiv 0$. This is clearly a BNE, since no one has profitable deviation given that the other agent never exchanges. It is also in threshold strategies with threshold $\bar{v}_i = 0$ for both $i = 1, 2$.

To summarize, any BNE must involve threshold strategies for both agents. We can further characterize the above equilibrium with trade. Given the thresholds $\bar{v}_1, \bar{v}_2 > 0$, we must have

$$\bar{v}_1 = \frac{k\int_0^{\min\{\bar{v}_2,1\}} v_2 \mathrm{d}v_2}{\int_0^{\min\{\bar{v}_2,1\}} \mathrm{d}v_2} = \frac{k\min\{\bar{v}_2, 1\}}{2},$$

$$\bar{v}_2 = \frac{k\int_0^{\min\{\bar{v}_1,1\}} v_1 \mathrm{d}v_1}{\int_0^{\min\{\bar{v}_1,1\}} \mathrm{d}v_1} = \frac{k\min\{\bar{v}_1, 1\}}{2}.$$

(Note that $\bar{v}_1 = \bar{v}_2 = 0$ is always a solution. This corresponds to the no trade equilibrium we found above.)

(a) **Soln:** When $k = \frac{3}{2}$, the unique solution is $\bar{v}_1 = \bar{v}_2 = 0$. Thus, the unique equilibrium involves no trade.

(b) **Soln:** When $k = \frac{5}{2}$, besides the solution $\bar{v}_1 = \bar{v}_2 = 0$, there is another solution $\bar{v}_1 = \bar{v}_2 = \frac{5}{4} > 1$. This corresponds to the BNE in which both agents always exchange.