Game Theory, Fall 2022
Problem Set 9

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1. ST Exercise 15.1.

   (a) **Soln:** The normal form is in Figure 1.

	L	M	R
A	$0, 4$	$0, 1$	$4, 0$
B	$4, 0$	$0, 1$	$0, 4$
C	$1, 3$	$1, 3$	$1, 3$

Figure 1: The normal form

We consider the Nash equilibria, or Bayesian Nash equilibria given that every player has only 1 type.

- Equilibrium (σ_1, σ_2) in which $\sigma_1(C) = 1$.
  Given σ_1, any mixture for player 2 is optimal. For player 1 to have no incentive to deviate, we must have $4\sigma_2(R) \leq 1$ and $4\sigma_2(L) \leq 1$. Therefore, $(C, \sigma_2(L) \circ L + \sigma_2(M) \circ M + \sigma_2(R) \circ R)$ is a Nash equilibrium for any $\sigma_2(L) \in [0, \frac{1}{4}]$, $\sigma_2(R) \in [0, \frac{1}{4}]$ and $\sigma_2(L) + \sigma_2(M) + \sigma_2(R) = 1$.
- Equilibrium (σ_1, σ_2) in which $\sigma_1(C) < 1$.
  In this case, $\sigma_1(A) + \sigma_1(B) > 0$. If $3\sigma_1(A) > \sigma_1(B)$, L for 2 is strictly better than M. If $3\sigma_1(A) \leq \sigma_1(B)$, we must have $\sigma_1(A) < 3\sigma_1(B)$, in which case R is strictly better than M. This means that $\sigma_2(M) = 0$, or equivalently $\sigma_2(L) + \sigma_2(R) = 1$. If $\sigma_2(L) \geq \frac{1}{2}$, B for player 1 is strictly better than C. If $\sigma_2(R) \geq \frac{1}{2}$, R is strictly better than C. This means that $\sigma_1(C) = 0$. Then, it is easy to see that there is a unique equilibrium in this case: $(\frac{1}{2} \circ A + \frac{1}{2} \circ B, \frac{1}{2} \circ L + \frac{1}{2} \circ R)$.

*Special thanks go to Peixuan Fu and Shuang Wu, who wrote the last version of these solutions.

In sum, Nash equilibria are: $(C, \sigma_2(L) \circ L + \sigma_2(M) \circ M + \sigma_2(R) \circ R)$ for $\sigma_2(L) \in [0, \frac{1}{4}]$, $\sigma_2(R) \in [0, \frac{1}{4}]$ and $\sigma_2(L)+\sigma_2(M)+\sigma_2(R) = 1$; and $(\frac{1}{2}\circ A+\frac{1}{2}\circ B, \frac{1}{2}\circ L+\frac{1}{2}\circ R)$.

(b) **Soln:** Consider a Nash equilibrium of the first type: $(C, \sigma_2(L) \circ L + \sigma_2(M) \circ M + \sigma_2(R) \circ R)$ for $\sigma_2(L) \in [0, \frac{1}{4}]$, $\sigma_2(R) \in [0, \frac{1}{4}]$. Clearly, $\sigma_2(M) > 0$. But this behavior is not sequentially rational for *any belief* over 2's information set. To see this, note that if $3\mu(x_2) > \mu(x_3)$, L is strictly better than M. If $3\mu(x_2) \leq \mu(x_3)$, we must have $\mu(x_2) < 3\mu(x_3)$, in which case R is strictly better than M. This implies that any sequentially rational behavior at 2's information set must plays M with zero probability. Therefore, any Nash equilibrium of this type is not a Perfect Bayesian equilibrium.

The Nash equilibrium of the second type $(\frac{1}{2} \circ A + \frac{1}{2} \circ B, \frac{1}{2} \circ L + \frac{1}{2} \circ R)$ is clearly a Perfect Bayesian equilibrium as every information set is on the path of play.

2. ST 16.3

(a) **Soln:** The game tree is in figure 2 and the corresponding matrix is in figure 3.

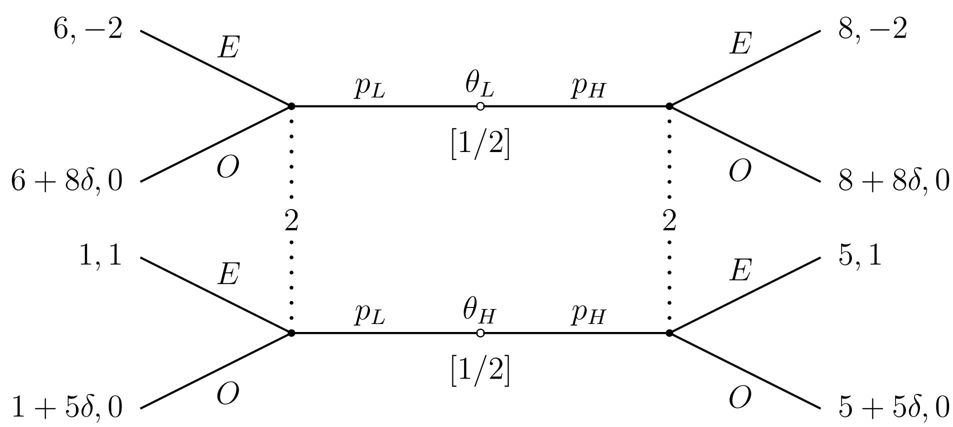

Figure 2: The extensive-form game tree for Question 2a

(b) **Soln:** Now $\delta = 1$. We denote μ_L and μ_H as player 2's beliefs that player 1's type is L after observing p_L or p_H. In the pooling PBE in which player 1 chooses p_L, the consistent beliefs for player 2 are $\mu_L = \frac{1}{2}$ and $\mu_H \in [0, 1]$. After observing p_L, the expected payoff of choosing E for player 2 is $\mathrm{E}_\mu[v_2(p_L, E)] = -2 \times \frac{1}{2} + 1 \times \frac{1}{2} < 0$. Thus, player 2 will choose O in the pooling PBE. In this case, type θ_L and θ_H will get 14 and 6 respectively.

Player 2

Player 1		EE	EO	OE	OO
	p_Lp_L	$3.5, -0.5$	$3.5, -0.5$	$3.5 + 6.5\delta, 0$	$3.5 + 6.5\delta, 0$
	p_Lp_H	$5.5, -0.5$	$5.5 + 2.5\delta, -1$	$5.5 + 4\delta, 0.5$	$5.5 + 6.5\delta, 0$
	p_Hp_L	$4.5, -0.5$	$4.5 + 4\delta, 0.5$	$4.5 + 2.5\delta, -1$	$4.5 + 6.5\delta, 0$
	p_Hp_H	$6.5, -0.5$	$6.5 + 6.5\delta, 0$	$6.5, -0.5$	$6.5 + 6.5\delta, 0$

Figure 3: The corresponding matrix for Question 2a

For player 1 not deviating to p_H, we need his payoffs no more than those when he chooses p_L for both types. If we restrict to pure strategy, player 2 must choose E after observing p_H. To rationalize player 2's action, we need $\mathrm{E}_\mu[v_2(p_H, E)] = -2 \times \mu_H + 1 \times (1 - \mu_H) \geq 0$, or equivalently, $\mu_H \in [0, \frac{1}{3}]$.

Therefore, (p_Lp_L, OE) with consistent beliefs $\mu_L = \frac{1}{2}$ and $\mu_H \in [0, \frac{1}{3}]$ is a (pure) pooling PBE.

(c) **Soln:** In a separating PBE in which θ_L chooses p_L and θ_H chooses p_H, player 2's consistent beliefs should be $\mu_L = 1$ and $\mu_H = 0$. Thus, in the PBE player 2 will choose strategy OE (we assume $\delta > 0$).

To let player 1 have no motivation to deviate in any type, following inequalities should satisfy:

$$\begin{cases} 6 + 8\delta \geq 8, \\ 5 \geq 1 + 5\delta, \end{cases}$$

or equivalently, $\delta \in [\frac{1}{4}, \frac{4}{5}]$.

3. ST 16.5

(a) **Soln:** The game tree is in figure 4.

(b) **Soln:** First, note that $-c < r - c < 0 < r$, so player 1 will always choose N as his best response when his type is L, no matter what player 2's strategy is. Then for a (pure) separating PBE, we only need to consider strategy AN for player 1. Now player 2's consistent beliefs are $\mu_A = 1$ and $\mu_N = 0$ (μ_X means the belief of type H after observing X). Then player 2's strategy in the equilibrium can only be BN. In this case, player 1 has no motivation to deviate in both types since $R - c > 0$ and $r - c < 0$. Therefore, (AN, BN) with beliefs $\mu_A = 1$ and $\mu_N = 0$ is a separating PBE.

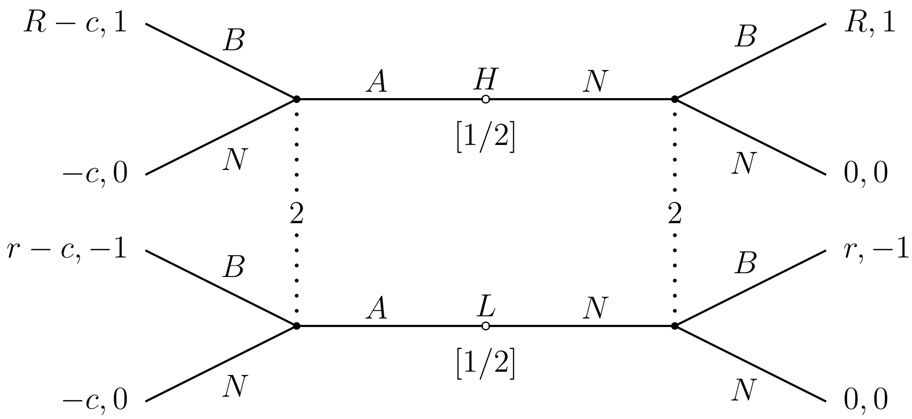

Figure 4: The extensive-form game tree for Question 3a

(c) **Soln:** As player 1 will choose N for his type L anyway, the only possible pooling PBE consists of NN for player 1.

Given any consistent belief satisfying $\mu_A \in [0,1]$, $\mu_N = \frac{1}{2}$, player 2's strategy can be $\sigma_2(B_A) \circ B + (1 - \sigma_2(B_A)) \circ N$ and $\sigma_2(B_N) \circ B + (1 - \sigma_2(B_N)) \circ N$ for any $\sigma_2(B_A) \in [0,1]$ and $\sigma_2(B_N) \in [0,1]$ after observing A and N. To satisfy player 1's incentive to choose N, we have

$$R \times \sigma_2(B_N) \geq (R - c) \times \sigma_2(B_A) + (-c) \times (1 - \sigma_2(B_A)) = R \times \sigma_2(B_A) - c,$$

We derive $\sigma_2(B_A) - \sigma_2(B_N) \leq \frac{c}{R}$ from the inequality. Thus, $(NN; \sigma_2(B_A) \circ B + (1 - \sigma_2(B_A)) \circ N, \sigma_2(B_N) \circ B + (1 - \sigma_2(B_N)) \circ N)$ can be a pooling PBE if $\sigma_2(B_A) - \sigma_2(B_N) \leq \frac{c}{R}$ with consistent belief $\mu_A = \mu_N = \frac{1}{2}$.

(d) **Soln:**

i. It is easy to derive that for player 1, separating strategies AN and NA cannot form PBE.

ii. For pooling strategy AA, to insure that player 1 has no incentive to deviate for both types, we have

$$(R - c) \times \sigma_2(B_A) + (-c) \times (1 - \sigma_2(B_A)) \geq R \times \sigma_2(B_A),$$

$$(r - c) \times \sigma_2(B_A) + (-c) \times (1 - \sigma_2(B_A)) \geq r \times \sigma_2(B_A),$$

so we can derive a new PBE: $(AA; \sigma_2(B_A) \circ B + (1 - \sigma_2(B_A)) \circ N, \sigma_2(B_N) \circ B + (1 - \sigma_2(B_N)) \circ N)$ if $\sigma_2(B_A) - \sigma_2(B_N) \geq \frac{c}{r}$ with consistent beliefs $\mu_A = \mu_N = \frac{1}{2}$.

iii. For pooling strategy NN, the PBE is the same as in 3c, because similarly we have

$$(R-c)\times\sigma_2(B_A)+(-c)\times(1-\sigma_2(B_A))\le R\times\sigma_2(B_A),$$

$$(r-c)\times\sigma_2(B_A)+(-c)\times(1-\sigma_2(B_A))\le r\times\sigma_2(B_A),$$

and $\sigma_2(AB)-\sigma_2(NB)\le\frac{c}{r}$ always satisfy if $\sigma_2(AB)-\sigma_2(NB)\le\frac{c}{R}$.

iv. For pooling mixed strategy, there exists no PBE, because for player 1 to be indifferent between A and N, we need

$$R\times\sigma_2(NB)=R\times\sigma_2(AB)-c,$$

and

$$r\times\sigma_2(NB)=r\times\sigma_2(AB)-c.$$

Obviously, these two equations cannot hold simultaneously.

The intuition is that now advertising is profitable for both types, but they have no incentive to pay for the cost if the consumer can distinguish them in PBE. One chooses to advertise only if the other chooses so.

Actually, there might exist other PBE which is neither separating PBE nor pooling PBE, like when the two types use different mixed actions.

4. ST 16.8

(a) **Soln:** The game tree is in figure 5.

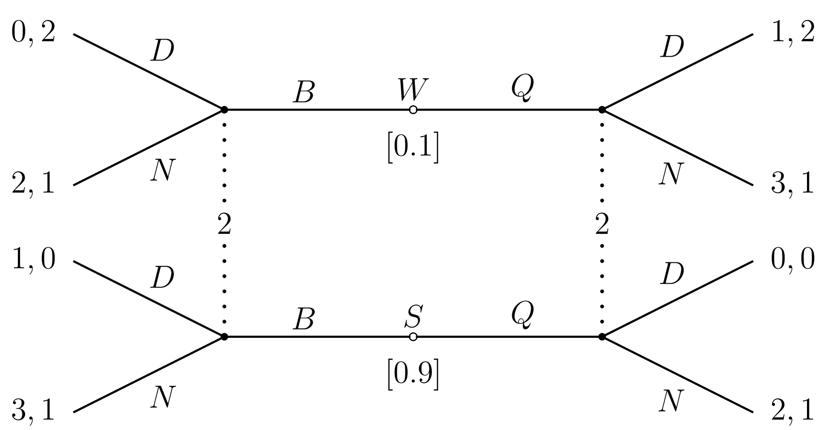

Figure 5: The extensive-form game tree for Question 4a

		Player 2			
		DD	DN	ND	NN
Player 1	BB	$0.9, 0.2$	$0.9, 0.2$	$2.9, 1$	$2.9, 1$
	BQ	$0, 0.2$	$1.8, 1.1$	$0.2, 0.1$	$2, 1$
	QB	$1, 0.2$	$1.2, 0.1$	$2.8, 1.1$	$3, 1$
	QQ	$0.1, 0.2$	$2.1, 1$	$0.1, 0.2$	$2.1, 1$

Figure 6: The corresponding matrix for Question 4b

(b) **Soln:** The corresponding matrix for Question 4a is in figure 6. We can find there are 2 pure-strategy Bayesian Nash equilibria: (BB, ND) and (QQ, DN).

We can also solve for all mixed-strategy Bayesian Nash equilibria:

$$(BB,\ \sigma_2(ND) \circ ND + (1 - \sigma_2(ND)) \circ NN),\ \sigma_2(ND) \in [\frac{1}{2}, 1),$$
$$(QQ,\ \sigma_2(DN) \circ DN + (1 - \sigma_2(DN)) \circ NN),\ \sigma_2(DN) \in [\frac{1}{2}, 1),$$
$$(\frac{1}{2} \circ BB + \frac{1}{2} \circ QQ,\ \frac{7}{10} \circ DN + \frac{3}{10} \circ ND),$$
$$(\frac{4}{9} \circ QB + \frac{5}{9} \circ QQ,\ \frac{3}{4} \circ DN + \frac{1}{4} \circ ND).$$

(c) **Soln:**

i. Firstly, we consider both types use pure strategies. It is easy to derive that both (BB, ND) and (QQ, DN) can be pooling PBEs, and there are no separating PBEs.

ii. Secondly, we consider type S uses a pure strategy and type W mixes. Let μ_X be player 2's belief of type W after observing message X. For type S choosing X, we always have $\mu_X < \frac{1}{2}$ and $\mu_Y = 1$ regardless of type W's mixed strategy (here $X, Y \in \{B, Q\}$ and $X \neq Y$). However, given player 2's strategy that $s_2(X) = N$ and $s_2(Y) = D$, type W will strictly prefer X and will not mix. Thus, there are no such PBEs.

iii. Thirdly, we consider type W uses a pure strategy and type S mixes.
If type W chooses B and type S chooses $\sigma_S(B) \circ B + (1 - \sigma_S(B)) \circ Q$, the consistent beliefs are $\mu_B = \frac{0.1}{0.1+0.9\times\sigma_S(B)}$ and $\mu_Q = 0$. After observing Q, player 2 will choose N. However, type W has the motivation to deviate to Q given this strategy of player 2. Thus, there are no such PBEs.
If type W chooses Q and type S chooses $\sigma_S(B) \circ B + (1 - \sigma_S(B)) \circ Q$, player 2 will choose N observing B and thus type S will not be indifferent.

Thus, there are no such PBEs.

iv. Lastly, we consider both types mix. Now, type X chooses $\sigma_X(B) \circ B + (1 - \sigma_X(B)) \circ Q$. Thus, the consistent beliefs are $\mu_B = \frac{0.1\times\sigma_W(B)}{0.1\times\sigma_W(B)+0.9\times\sigma_S(B)}$ and $\mu_Q = \frac{0.1\times(1-\sigma_W(B))}{0.1\times(1-\sigma_W(B))+0.9\times(1-\sigma_S(B))}$. Consider player 2's strategy $\sigma_X(D) \circ D + (1 - \sigma_X(D)) \circ N$ after observing X. To satisfy the indifference condition, we need

$$2 \times (1 - \sigma_B(D)) = 1 \times \sigma_Q(D) + 3 \times (1 - \sigma_Q(D)),$$

and

$$1 \times \sigma_B(D) + 3 \times (1 - \sigma_B(D)) = 2 \times (1 - \sigma_Q(D)).$$

Obviously, such $\sigma_B(D)$ and $\sigma_Q(D)$ do not exist. Thus, there are no such PBEs.

In sum, we only derive two pooling PBEs: (BB, ND) and (QQ, DN).

(d) **Soln:** We check whether the two pooling PBEs pass the intuitive criterion respectively.

For (BB, ND), the off-path message is Q. Then $D(Q) = S$ and $\Theta \backslash D(Q) = \{W\}$. However, $2 = \hat{u}_1(W) > \min_{r\in \mathrm{BR}(\{W\},Q)} u_1(Q, r, W) = u_1(Q, D, W) = 1$. Thus, this PBE passes the intuitive criterion.

For (QQ, DN), the off-path message is B. Then $D(B) = W$ and $\Theta \backslash D(B) = \{S\}$. Moreover, $2 = \hat{u}_1(S) < \min_{r\in \mathrm{BR}(\{S\},B)} u_1(B, r, S) = u_1(B, N, S) = 3$. Thus, this PBE fails the intuitive criterion.

5. Consider the signaling game in Figure 7. There are three types of player 1. The common prior of player 1's types is labeled in the square bracket.

(a) There is a unique pooling perfect Bayesian equilibrium. What is it?

Soln: First consider all three types choose pure strategy L. Any belief on off-path information set (i.e. after R) is consistent and the common prior is consistent after L. Thus, in PBE, player 2 will choose $s_2(L) = T_L$ and $s_2(R) = \sigma_R(T_R) \circ T_R + (1 - \sigma_R(T_R)) \circ B_R$ for any $\sigma_R(T_R) \in [0, 1]$. For player 1 not having motivation to deviate, we need:

$$3 \geq 2 \times \sigma_R(T_R) + (-1) \times (1 - \sigma_R(T_R)),$$

$$2 \geq -1 \times \sigma_R(T_R) + 2 \times (1 - \sigma_R(T_R)),$$

$$1 \geq 3 \times \sigma_R(T_R),$$

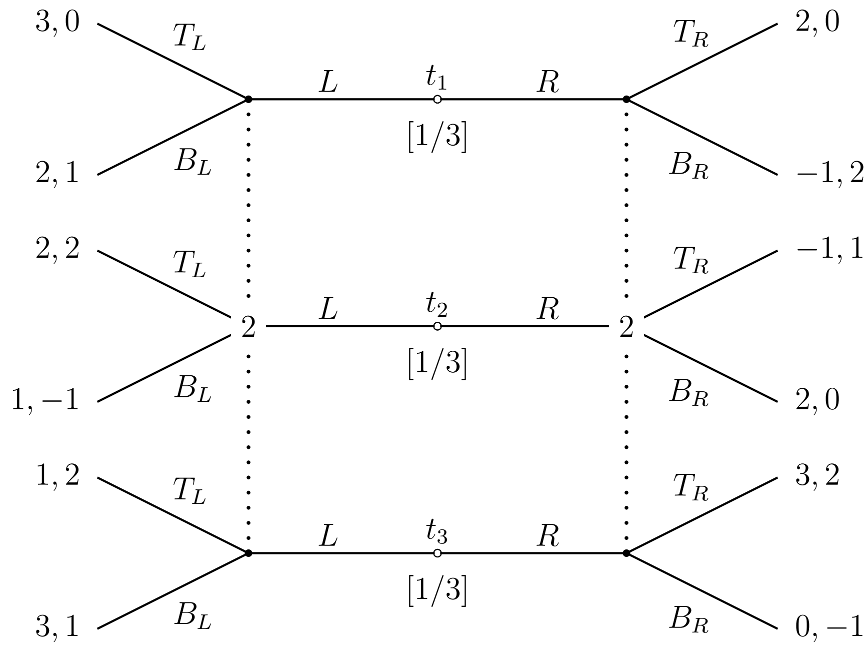

Figure 7: The signaling game for Question 5

which result in $\sigma_R(T_R) \in [0, \frac{1}{3}]$. Thus, $(LLL; T_L, \sigma_R(T_R) \circ T_R + (1 - \sigma_R(T_R))$ is a pooling PBE if $\sigma_R(T_R) \in [0, \frac{1}{3}]$.

Second consider all three types choose pure strategy R. In the same way we derive player 2's strategy: $s_2(L) = \sigma_L(T_L) \circ T_L + (1 - \sigma_L(T_L)) \circ B_L$ for any $\sigma_L(T_L) \in [0, 1]$, and $s_2(R) = T_R$. However, given any of player 2's strategies, choosing L is strictly better for type t_2 so he will deviate. Thus, we have no PBE in this case.

Last consider all three types choose the same mixed strategy $\sigma_1(L) \circ L + (1 - \sigma_1(L)) \circ R$. The consistent beliefs are the same as the common prior for player 2, and he will choose $T_L T_R$. However, no type is indifferent given this strategy. Thus, we have no PBE in this case.

In total, the only pooling PBE is $(LLL; T_L, \sigma_R(T_R) \circ T_R + (1 - \sigma_R(T_R))$ for $\sigma_R(T_R) \in [0, \frac{1}{3}]$.

(b) Does the equilibrium you find in the previous question pass or fail the intuitive criterion?

Soln: Off-path message is R. We have $D(R) = t_1$ and $\Theta \setminus D(R) = \{t_2, t_3\}$. Then $\mathrm{BR}(\Theta \setminus D(R), R) = T_R$. For type t_3, $1 = \hat{u}_1(t_3) < u_1(R, T_R, t_3) = 3$. Thus, the PBE fails the intuitive criterion.

(c) Find a pure strategy perfect Bayesian equilibrium in which two types play the same action but the other type plays a different one.

Soln: Consider strategy profile $(LLR, T_L T_R)$. The consistent beliefs are $\mu_L(t_1) = \mu_L(t_2) = \frac{1}{2}$ and $\mu_R(t_3) = 1$. For player 2, given this system of beliefs, $T_L T_R$ is the best response; given this strategy, every type chooses the best action.