Suggested Solutions to Game Theory Final Exam, Fall 2021

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Jan 3, 2022

1. **(15 points)** Consider the following infinite horizon alternating offer bargaining over $[0, 1]$ between two agents. Agent 1 proposes in odd periods and agent 2 decides whether to accept or not. Agent 2 proposes in even periods and agent 1 decides whether to accept. Assume there is no discounting. Thus, if the proposal $(x, 1-x)$ is accepted in any period, agent 1 obtains x and agent 2 obtains $1-x$, and the game ends. If a proposal is rejected in any period, then the bargaining relationship between the two agents breakdowns with probability $\rho \in (0, 1)$, in which case the game ends and agent 1 obtains $\frac{1}{3}$ and agent 2 obtains $\frac{2}{3}$. With probability $1 - \rho$, the relationship continues and they move into next period. Find a subgame perfect equilibrium.

   Soln: We guess a stationary equilibrium: Player 1 always proposes $(x^*, 1-x^*)$ and player 2 always proposes $(y^*, 1-y^*)$; player 1 always accepts an offer $x \geq y^*$ and rejects others; player 2 always accepts an offer $1 - x \geq 1 - x^*$ and rejects others.

   For player 2 to a accept $1 - x^*$, we need

   $$1 - x^* \geq \rho \times \frac{2}{3} + (1 - \rho) \times (1 - y^*).$$

   For player 2 reject any offer $1 - x < 1 - x^*$, we need

   $$1 - x^* \leq \rho \times \frac{2}{3} + (1 - \rho) \times (1 - y^*).$$

   Thus, we have

   $$1 - x^* = \rho \times \frac{2}{3} + (1 - \rho) \times (1 - y^*).$$

   In the same way, we also have

   $$y^* = \rho \times \frac{1}{3} + (1 - \rho) \times x^*.$$

   Simultaneously, we derive

   $$x^* = y^* = \frac{1}{3}.$$

   This strategy profile is a subgame perfect equilibrium.

2. Two students, $i = 1, 2$, form a study group to complete a course project. Each of them can choose either to exert effort or to shirk. The cost of exerting effort for student i is c_i, which is i's private information. Assume c_1 and c_2 are independently and uniformly distributed over $[0, 1]$. There is no cost of shirking. If at least one student exerts effort, the project will be successful, in which case each of the students obtains $\theta \in (0, 1)$. If both shirk, the project will fail, in which case they both get zero.

   (a) **(10 points)** Show that in any Bayesian Nash equilibrium, both players play a threshold strategy (also known as a cutoff strategy).

   Soln: We assume that $(s_1^*(c_1), s_2^*(c_2))$ is a BNE. Given player 2's strategy, player 1's expected payoff of choosing E and S are

$$\mathbb{E}[u_1(E, s_2^*(c_2); c_1)] = \theta - c_1,$$

$$\mathbb{E}[u_1(S, s_2^*(c_2); c_1)] = \Pr(s_2^*(c_2) = E) \times \theta,$$

   We can observe that given $s_2^*(c_2)$, the expected payoff of choosing S should be a constant and is irrelevant to c_1. On the other hand, the expected payoff of choosing E is a linear correlation with c_1, and is thus monotonic in c_1. It is easy to derive player 1's strategy in the equilibrium:

$$s_1^*(c_1) = \begin{cases} E, & \text{if } c_1 < (1 - \Pr(s_2^*(c_2) = E)) \times \theta, \\ S, & \text{if } c_1 > (1 - \Pr(s_2^*(c_2) = E)) \times \theta. \end{cases}$$

   When $c_1 = (1 - \Pr(s_2^*(c_2) = E)) \times \theta$, player 1 is indifferent and can mix between E and S arbitrarily. This kind of strategy is a threshold (or cutoff) strategy. The same logic holds for player 2's strategy in any BNE.

   (b) **(10 points)** Find a Bayesian Nash equilibrium.

   Soln: Consider they use a symmetric cutoff strategy $\hat{s}(c) = \begin{cases} E, & \text{if } c < \bar{c}, \\ S, & \text{if } c > \bar{c}. \end{cases}$

   We still assume that when $c = \bar{c}$ they can mix arbitrarily. Given player 2's strategy $\hat{s}(c_2)$ and player 1's type c_1, player 1's expected payoff of choosing E and S are

$$\mathbb{E}[u_1(E, \hat{s}(c_2); c_1)] = \theta - c_1,$$

$$\mathbb{E}[u_1(S, \hat{s}(c_2); c_1)] = \int_0^{\bar{c}} \theta \, \mathrm{d}c_2 = \bar{c}\theta.$$

Thus, for player 1's strategy $\hat{s}(c_1)$ to maximum his expected payoff, we have

$$\theta - \bar{c} = \bar{c}\theta,$$

or equivalently,

$$\bar{c} = \frac{\theta}{\theta + 1}.$$

It is easy to verify the symmetric strategy profile $\hat{s}(c)$ is a BNE.

3. There are two bidders, $i = 1, 2$. Each bidder i receives a signal t_i. Assume that t_1 and t_2 are independently and uniformly distributed over $[1, 2]$. Bidder i's value is $\alpha t_i + (1-\alpha)t_j$ where $\alpha \in (0, 1)$ and $j \neq i$. The rule of the auction is the standard first price auction. The two bidders simultaneously bid. The one with the higher bid wins and pays his own bid. If there is a tie in bids, each of them wins with equal probability.

   (a) **(10 points)** Suppose bidder 2's strategy is σ_2 which is strictly increasing. What is player 1's expected payoff, if his signal is t_1 and he bids b_1.

   Soln: Player 1's expected payoff is

$$\begin{aligned}\mathbb{E}[u_1(b_1, \sigma_2(b_2); t_1)] =& \int_{\{t_2 | b_1 > \sigma_2(t_2)\}} \alpha t_1 + (1-\alpha)t_2 - b_1 \ \mathrm{d}F_2(t_2) \\ &+ \frac{1}{2} \int_{\{t_2 | b_1 = \sigma_2(t_2)\}} \alpha t_1 + (1-\alpha)t_2 - b_1 \ \mathrm{d}F_2(t_2) \\ =& \int_1^{\sigma_2^{-1}(b_1)} \alpha t_1 + (1-\alpha)t_2 - b_1 \ \mathrm{d}t_2 \\ =& (\alpha t_1 - b_1)[\sigma_2^{-1}(b_1) - 1] + \frac{1}{2}(1-\alpha)[\sigma_2^{-1}(b_1)^2 - 1].\end{aligned}$$

   The second equation is from the assumption that σ_2 is strictly increasing and that t_i is uniformly distributed, so that $\Pr(b_1 = \sigma_2(t_2)) = 0$ and $f_2(t_2) = 1$ for $t_2 \in [1, 2]$. We have also assumed that $\sigma_2^{-1}(b_1) \in [1, 2]$.

   (b) **(10 points)** Assume, in addition, that σ_2 is differentiable. Suppose b_1 maximizes 1's expected payoff for signal t_1. Write down the first order condition that b_1 must satisfy.

   Soln: The FOC is

$$-[\sigma_2^{-1}(b_1) - 1] + (\alpha t_1 - b_1) \cdot \frac{1}{\sigma_2'(\sigma_2^{-1}(b_1))} + (1-\alpha) \cdot \sigma_2^{-1}(b_1) \cdot \frac{1}{\sigma_2'(\sigma_2^{-1}(b_1))} = 0,$$

   or equivalently,

$$[\sigma_2^{-1}(b_1) - 1]\sigma_2'(\sigma_2^{-1}(b_1)) = \alpha t_1 - b_1 + (1-\alpha)\sigma_2^{-1}(b_1).$$

(c) **(10 points)** Find a Bayesian Nash equilibrium.

Soln: We consider a symmetric BNE in which both of them use strategy $\hat{\sigma}(t)$. Then the FOC becomes

$$(t-1)\sigma'(t) = \alpha t - \sigma(t) + (1-\alpha)t = t - \sigma(t).$$

We guess $\sigma(t)$ has the form $\sigma(t) = mt + n$. Then we get

$$(t-1)m = t - mt - n.$$

Thus, we have

$$m = n = \frac{1}{2}.$$

It is easy to verify that $\sigma_i(t_i) = \frac{1}{2}t_i + \frac{1}{2}$ for $i = 1, 2$ is indeed a symmetric BNE.

4. Consider the signaling game in Figure 1. Player 1 has two possible types t_1 and t_2. The common prior is $\frac{3}{4} \circ t_1 + \frac{1}{4} \circ t_2$.

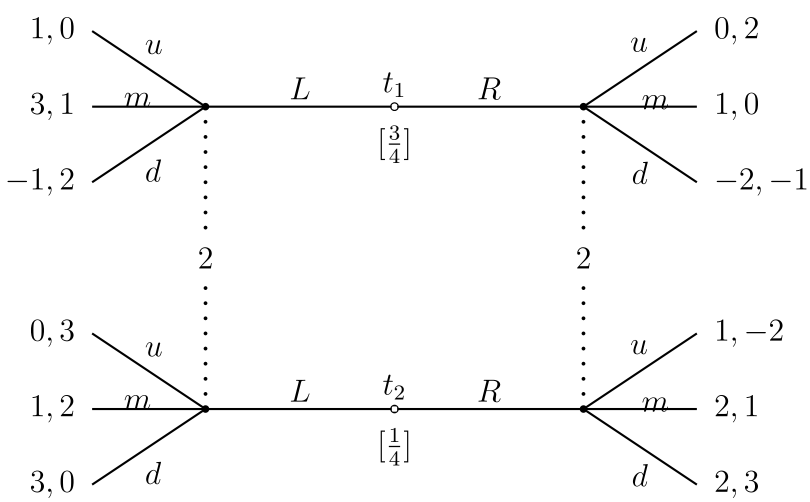

Figure 1: The signaling game for Question 4

(a) **(10 points)** Is there a separating perfect Bayesian equilibrium? Explain why or why not.

Soln: In any separating PBE, player 2 can recognize player 1's type for sure after receiving different messages. If $s_1 = (L, R)$, player 2 will choose $s_2 = (d, d)$ to maximize his payoff. But given this, type t_2 of player 1 will deviate to R. Thus, this cannot form a PBE. If $s_1 = (R, L)$, similarly we have $s_2 = (u, u)$. But given this, both types of player 1 will deviate. Thus, there is no separating PBE.

(b) **(10 points)** Find a pooling perfect Bayesian equilibrium. Does it pass or fail the intuitive criterion?

Soln: We denote μ_X as player 2's belief of the probability of type t_1 after observing X. Consider $s_1 = (L, L)$. In a pooling PBE, $\mu_L = \frac{3}{4}$, so player 2's best response is d after observing L. To ensure that player 1 do have the incentive to choose L for both types, we can construct player 2's strategy to be (d, d). This is sequential rational with the consistent belief $\mu_L = \frac{3}{4}$ and $\mu_R = 0$.

As for the intuitive criterion, note that $D(R) = \{t_2\}$, because no matter what player 2 chooses after observing R, type t_2's equilibrium payoff $(\hat{u}_1(t_2) = 3)$ is always higher. Thus, $BR_2(\Theta \setminus D(R), R) = BR_2(\{t_1\}, R) = \{u\}$. Then we have

$$-1 = \hat{u}_1(t_1) < \min_{a_2 \in BR_2(\Theta \setminus D(R), R)} v_1(R, a_2; t_1) = v_1(R, u; t_1) = 0,$$

which means this equilibrium fails the intuitive criterion.

(c) **(10 points)** Argue that, in any perfect Bayesian equilibrium, player 2 plays m with zero probability after observing R.

Soln: In a PBE, any consistent belief must satisfy $\mu_R \in [0, 1]$. Given that, player 2 will choose m with positive probability after observing R only if

$$\begin{cases} 1 - \mu_R \geq 2\mu_R - 2(1 - \mu_R), \\ 1 - \mu_R \geq -\mu_R + 3(1 - \mu_R). \end{cases}$$

Or equivalently, $\mu_R \leq \frac{3}{5}$ and $\mu_R \geq \frac{2}{3}$, which is impossible. So player 2 chooses m with zero probability after observing R.

(d) **(5 points)** Find a Bayesian Nash equilibrium that is not a perfect Bayesian equilibrium.

Soln: Following 4c, it is straightforward to construct a BNE in which player 2 chooses m with positive probability after observing R, to ensure it is not a PBE.

Consider $s_1 = (L, L)$, $s_2 = (d, \frac{1}{3} \circ m + \frac{2}{3} \circ d)$. We can easily verify that no one will deviate given the other's strategy, so it is indeed a BNE.