Suggested Solutions to Game Theory Midterm Exam, Fall 2019

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Important: This is a closed-book exam. No books, lecture notes or calculators are permitted. You have **120 minutes** to complete the exam. Answer all questions. Write legibly. Good luck!

1. Three bidders involve in a second price auction. Bidder $i \in \{1,2,3\}$ values the good in this auction $v_i > 0$. Each bidder i makes a bid $b_i \geq 0$ and they bid simultaneously. The bidder with the highest bid wins the good and pays the second highest bid (the highest bid among all the opponents' bids). If there is a tie in bids, then a winner is randomly picked among those who bid the highest. The losers pay nothing.

   (a) **(10 points)** Show that bidding everyone's own value $v_i > 0$ is a Nash equilibrium.

   Soln: The payoff function for each bidder is as follows:

$$U_i(b_1,b_2,b_3) = \begin{cases} \frac{v_i - \max_{j\neq i} b_j}{|\{1\leq j\leq 3:\, b_j = b_i\}|}, & \text{if } b_i = \max_{1\leq j\leq 3} b_j, \\ 0, & \text{otherwise.} \end{cases}$$

   Without loss of generality, we assume that $v_1 \geq v_2 \geq v_3$.

   Consider bidder 1 first.

   - Assume $v_1 > v_2$. Then bidder 1's payoff is $v_1 - v_2 > 0$. If bidder 1 deviates to $b_1' > v_2$, his payoff is still $v_1 - v_2$. If bidder 1 deviates to $b_1' = v_2$, his payoff is $\frac{1}{2}(v_1 - v_2)$. If bidder 1 deviates to $b_1' < v_2$, his payoff is 0. Thus, bidder 1 has no incentive to deviate.
   - Assume $v_1 = v_2$. Then bidder 1's payoff is either $\frac{1}{2}(v_1 - v_2) = 0$ (if $v_2 > v_3$) or $\frac{1}{3}(v_1 - v_2) = 0$ (if $v_2 = v_3$). If bidder 1 deviates to a lower $b_1' < v_1$, his payoff is still 0 since he will lose. If he deviates to a higher $b_1' > v_1$, he will win for sure. But his payoff is still $v_1 - v_2 = 0$. Thus, bidder i has no incentive to deviate.

   Hence, in both cases, 1 has no incentive to deviate.

   Consider bidder 2 now.

- If $v_1 = v_2$, we know 2 has no incentive to deviate from the above analysis with bidders 1 and 2 interchanged.
- Assume $v_1 > v_2$. In this case, 2's payoff is 0. If 2 deviates to $b_2' = v_1$, 2 wins with half probability but the payoff is $\frac{1}{2}(v_2 - v_1) < 0$. If 2 deviates to $b_2 > v_1$, 2 wins for sure and the payoff is $v_2 - v_1 < 0$. Thus, 2 has no incentive to deviate.

Hence, in both cases, 2 has no incentive to deviate.

Finally, consider bidder 3. A similar analysis as above will show that bidder 3 will get either 0 or negative payoff if he deviates. Thus, 3 has no incentive to deviate either.

The above analysis verifies that (v_1, v_2, v_3) is indeed a Nash equilibrium.

(b) **(10 points)** Find another Nash equilibrium.

Soln: There are many other Nash equilibria. For example, consider the strategy profile $(b_1, 0, 0)$ where $b_1 > \max_i v_i$. In this case, bidder 1 wins the object and obtains a payoff $v_1 > 0$. Bidders 2 and 3 lose and obtain a payoff 0.

If bidder 1 deviates to $b_1' > (0, b_1) \cup (b_1, +\infty)$, he still obtains v_1. If he deviates to $b_1' = 0$, he only obtains $\frac{1}{3}v_1$. Thus, bidder 1 has no incentive to deviate.

If bidder 2 deviates to $b_2' \in (0, b_1)$, he still obtains 0. If he deviates to $b_2' = b_1$, he wins with half probability and obtains $\frac{1}{2}(v_2 - b_1) < 0$. If he deviates to $b_2' > b_1$, he wins for sure and obtains $v_2 - b_1 < 0$. Thus, bidder 2 has no incentive to deviate. The analysis for bidder 3 is analogous.

Therefore, no one has an incentive to deviate. It is a Nash equilibrium.

2. Consider the normal form game in Figure 1. Player 1 is the row player and player 2 is the column player, as usual. When considering dominance, we allow mixed strategies.

	a	b	c	d
x	$1,1$	$3,3$	$3,1$	$2,3$
y	$2,1$	$0,2$	$1,0$	$2,2$
z	$1,2$	$2,3$	$0,3$	$3,1$
w	$1,4$	$2,0$	$1,4$	$1,2$

Figure 1: The normal form game for Question 2

(a) **(5 points)** Does player 1 have a strictly dominated strategy? If yes, show which strategy strictly dominates which strategy. If no, explain why.

Soln: w for player 1 is strictly dominated by $\frac{4}{5} \circ x + \frac{1}{5} \circ y$.

(b) **(5 points)** Does player 2 have a strictly dominated strategy? If yes, show which strategy strictly dominates which strategy. If no, explain why.

Soln: Player 2 has no strictly dominated strategy. This is because each (pure) strategy is a best response to some of player 1's strategy.

(c) **(10 points)** What strategies survive the process of iterated elimination of strictly dominated strategies? In each step of your deletion, show which strategy is strictly dominated by which strategy.

Soln: In the first round, w for player 1 is deleted. In the second round, a for player 2 is strictly dominated by b. Thus, a is deleted. In the third round, y for player 1 is strictly dominated by $\frac{1}{2} \circ x + \frac{1}{2} \circ z$. Thus, y is deleted in the third round. We end up with $\{x, z\} \times \{b, c, d\}$. No strategy is strictly dominated now. These strategies survive iterated deletion of strictly dominated strategies.

(d) **(10 points)** Find all Nash equilibria (pure and mixed) of this game.

Soln: As we know, we only need to analyze the reduced game. It is described in Figure 2.

	b	c	d
x	$3,3$	$3,1$	$2,3$
z	$2,3$	$0,3$	$3,1$

Figure 2: The normal form game for Question 2d

We now find all Nash equilibria (σ_1, σ_2).

If $\sigma_1(x) = 1$, then 2 is indifferent between b and d. For σ_1 to be a best reply to 2's choice, we must have

$$3\sigma_2(b) + 2(1 - \sigma_2(b)) \geq 2\sigma_2(b) + 3(1 - \sigma_2(B)) \Rightarrow \sigma_2(b) \in [\frac{1}{2}, 1].$$

Thus, any strategy profile in

$$\{(x, \alpha \circ b + (1 - \alpha) \circ d) \,|\, \frac{1}{2} \leq \alpha \leq 1\} \tag{1}$$

is a Nash equilibrium.

If $\sigma_1(x) = 0$, then 2 is indifferent between b and d and strictly prefers these two to c. However, when 2 mixes between b and d, 1 strictly prefers x. Therefore, there is no Nash equilibrium in which $\sigma_1(x) = 0$.

Finally, if $\sigma_1(x) \in (0, 1)$, then 2 strictly prefers b to c and d. However, if 2 plays b, 1 is not indifferent between x and z. Therefore, there is no Nash equilibrium in which 1 mixes.

In sum, the set of all Nash equilibria is given by (1).

3. Three farmers live in a village. They simultaneously and independently decide whether or not to exert effort to build a road. Exerting effort imposes a cost of 2 on the player who exerts effort. If two or more farmers exert effort, the road is built and every farmer receives a benefit of 4 regardless of whether he himself exerted effort. Otherwise, every farmer receives zero benefit. The payoff to each farmer is the realized benefit minus the cost of his effort (if he exerted effort).

   (a) **(5 points)** Find all Pareto efficient strategy profiles.

   Soln: We use E for exerting effort and S for shirking. Any strategy profile in which the road is not built is Pareto dominated by (E, E, E), since each farmer obtains a payoff at most 0 from the former and a payoff 2 from the later. Moreover, (E, E, E) itself is Pareto dominated by (E, E, S), since farmer 3 is strictly better off under the later profile while others are not worse off. We are left with (E, E, S), (E, S, E) and (S, E, E). It is easy to see they are Pareto efficient.

   (b) **(5 points)** Does there exist a pure strategy Nash equilibrium? If yes, find all pure strategy Nash equilibria. If no, explain why.

   Soln: We can get the payoff functions directly:

$$U_i(x_i, x_j, x_k) = \begin{cases} 4, & \text{if } x_i = \text{S}, x_j = \text{E}, \text{and } x_k = \text{E}, \\ 2, & \text{if } x_i = \text{E}, x_j = \text{S}, \text{or } x_k = \text{E}, \\ 0, & \text{if } x_i = \text{S}, x_j = \text{S}, \text{or } x_k = \text{S}, \\ -2, & \text{if } x_i = \text{E}, x_j = \text{S}, \text{and } x_k = \text{S}, \end{cases}$$

   Any strategy profile in which there is one and only one farmer exerts effort is not a Nash equilibrium, since the one who exerts effort would like to deviate to shirking. The strategy profile in which all three farmers exert effort is not a Nash equilibrium either, since every farmer has an incentive to deviate to shirking.

We are left with (S, S, S), (E, E, S), (E, S, E) and (S, E, E). In the first strategy profile, the road is not built. In the other three strategy profiles, the road is built. It is easy to check that each of these four strategy profiles is a Nash equilibrium.

(c) **(10 points)** Find a mixed strategy Nash equilibrium.

Soln: We consider an symmetric equilibrium in which all farmers mix between effort and shirking with the same probability. Let $p \in (0, 1)$ be the probability of exerting effort. Given this strategy for two farmers, the other farmer is willing to mix only if

$$-2 + 4(1 - (1 - p)^2) = 4p^2,$$

where the LHS is this farmer's payoff from exerting effort and the RHS is his expected payoff from shirking. From this equation, we obtain

$$p = \frac{1}{2}.$$

Thus, there is a symmetric equilibrium $\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{2} \circ E + \frac{1}{2} \circ S$.

(d) **(10 points)** Now assume that these three farmers make decisions sequentially. Farmer 1 first chooses whether or not to exert effort. Then farmer 2 chooses after observing 1's choice. Finally, farmer 3 only observes how many farmers have exerted effort and then makes a decision. Draw the game tree for this game, including the payoffs. Find all pure strategy Nash equilibria in which the road is built.

Soln:

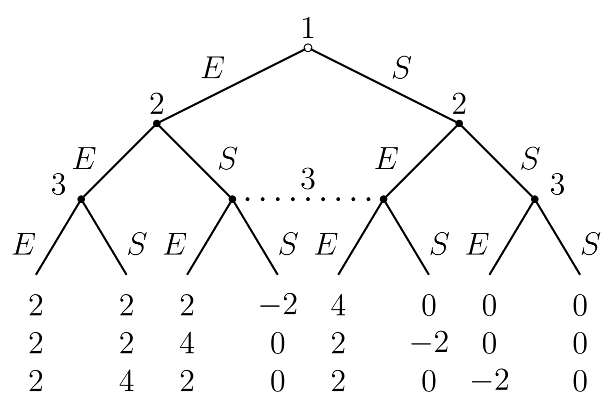

Figure 3: The game tree for Question 3d

The game tree is in Figure 3. Framer 1 has one information set, and his strategy space is $\{E, S\}$. Framer 2 has 2 information sets, and his strategy space is $\{EE, ES, SE, SS\}$, where XY means that farmer 2 chooses X if farmer 1 chooses E and Y if farmer 1 chooses S. Framer 3's strategy space is $\{EEE, EES, ESE, ESS, SEE, SES, SSE, SSS\}$ where XYZ means that farmer 3 chooses X if both farmers 1 and 2 choose E, Y if one and only one of farmers 1 and 2 chooses E, Z if both farmers 1 and 2 choose S.

We focus on the strategy profiles in which the road is built. There are four potential outcomes.

First, consider a strategy profile (s_1, s_2, s_3) whose outcome path is EEE. Regardless of what s_2 is, farmer 3 has an incentive to deviate to a s_3' where s_3' chooses S instead of E after observing that bother farmers 1 and 2 choose E. Thus, this is not a Nash equilibrium.

Second, consider a strategy profile (s_1, s_2, s_3) whose outcome path is EES. Then $s_1 = E$ and $s_2 \in \{EE, ES\}$.

- Suppose $s_2 = EE$. Farmer 3 has four best responses to $s_1 = E$ and $s_2 = EE$: SEE, SES, SSE and SSS. However, if $s_3 = SEE$ or SES, $s_2 = EE$ is not a best response for farmer 2 since choosing S after farmer 1 choosing E is profitable. Moreover, if $s_3 = SSE$ or SSS, we can easily check that no one has an incentive to deviate. Thus, we find two Nash: (E, EE, SSE) and (E, EE, SSS).
- Suppose $s_2 = ES$. Farmer 3 still has four best responses: SEE, SES, SSE and SSS. Using the same argument as above, we can find two Nash (E, ES, SSE) and (E, ES, SSS).

Third, consider a strategy profile (s_1, s_2, s_3) whose outcome path is ESE. Then $s_1 = E$ and $s_2 \in \{SE, SS\}$.

- Suppose $s_2 = SE$. Then farmer 3 has four best responses: EEE, EES, SEE and SES. But for any of these four strategies and given $s_2 = SE$, farmer 1 has an incentive to deviate to S. Thus, there is no equilibrium of this form.
- Suppose $s_2 = SS$. Then farmer 3 still has four best responses: EEE, EES, SEE and SES. It is easy to see that any of these strategies together with $s_1 = E$ and $s_2 = SS$ is a Nash equilibrium. Thus, we find four more Nash equilibria: (E, SS, EEE), (E, SS, EES), (E, SS, SEE) and (E, SS, SES).

Finally, consider a strategy profile (s_1, s_2, s_3) whose outcome path is SEE. Then $s_1 = S$, $s_2 \in \{EE, SE\}$.

- Suppose $s_2 = EE$. Then farmer 3 has four best responses: EEE, EES, SEE and SES. It is easy to see that any of these strategies together with $s_1 = E$ and $s_2 = SS$ is a Nash equilibrium. Thus, we find four more Nash equilibria: (S, EE, EEE), (S, EE, EES), (S, EE, SEE) and (S, EE, SES).
- Suppose $s_2 = SE$. Then farmer 3 has four best responses as above: EEE, EES, SEE and SES. It is easy to see that any of these strategies together with $s_1 = E$ and $s_2 = SE$ is a Nash equilibrium. Thus, we find four more Nash equilibria: (S, SE, EEE), (S, SE, EES), (S, SE, SEE) and (S, SE, SES).

  In total, there are 16 pure strategy Nash equilibria in which the road is built.

the outcome which is $\{\hat{x}_1 = \text{E}\ , \hat{x}_2 = \text{E}\ , \hat{x}_3 = \text{S}\}$. Farmer 1' stretegy must be E. When farmer 2's strategy is EE, there are 2 Nash Equilibriums (E, EE, SSE), (E, EE, SSS). When farmer 2 strategy is ES, there are 2 Nash Equilibriums (E, ES, SSE), (E, ES, SSS).

Third, we consider the outcome which is $\{\hat{x}_1 = \text{E}\ , \hat{x}_2 = \text{S}\ , \hat{x}_3 = \text{E}\}$. Farmer 1' stretegy must be E. When farmer 2's strategy is SS, there are 4 Nash Equilibriums (E, SS, EEE), (E, SS, SEE), (E, SS, EES), and (E, SS, SES). When farmer 2 strategy is SE, there doesn't exist Nash equilibrium. Fianlly, we consider the outcome which is $\{\hat{x}_1 = \text{S}\ , \hat{x}_2 = \text{E}\ , \hat{x}_3 = \text{E}\}$. Farmer 1' stretegy must be S. When farmer 2's strategy is EE, there are 4 Nash Equilibriums (S, EE, EEE), (S, EE, SEE), (S, EE, EES), and (S, EE, SES). When farmer 2 strategy is SE, there are 4 Nash Equilibriums (S, SE, EEE), (S, SE, SEE), (S, SE, EES), and (S, SE, SES).

(e) **(10 points)** We continue to assume that these three farmers make decisions sequentially in the same order as 3d. But now, we assume that farmer 3 perfectly observes the previous two farmers' choices. Find a backward induction solution and a Nash equilibrium that is not a backward induction solution.

Soln: The game tree is in Figure 4. This is a game of perfect information. It's obvious that the unique backward induction solution is $(S, SE, SEES)$. There are many Nash equilibria that are not the backward induction solution. For example, $(E, EE, SSSE)$ is a Nash equilibrium, since no one has an incentive to deviate.

(f) **(10 points)** Let's return to the case where the three farmers choose simulta-

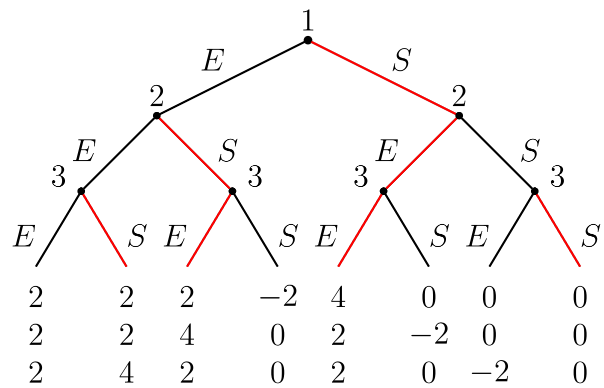

Figure 4: The game tree for Question 3e

neously, but we make the following modifications. Instead of either exerting effort or not, each farmer now can choose an effort level $e \in [0, 2]$. The cost of effort e is just e. The road is built if and only if the total effort of the three farmers is bigger than or equal to 4, i.e., $e_1 + e_2 + e_3 \geq 4$. Similarly as before, if the road is built, each obtains a benefit of 4. Find all pure strategy Nash equilibria.

Soln: We can get the payoff function directly:

$$U_i(e_1, e_2, e_3) = \begin{cases} 4 - e_i, & \text{if } e_i + e_j + e_k \geq 4, \\ -e_i, & \text{otherwise} \end{cases}$$

Consider a strategy profile (e_1, e_2, e_3) such that $e_1 + e_2 + e_3 > 4$. Then there must exist $e_i > 0$. By deviating to a slightly lower effort level $e_i' < e_i$, the road is still built but i exerts less effort. Thus, i has an incentive to deviate. This is not a Nash equilibrium.

Consider a strategy profile (e_1, e_2, e_3) such that $0 < e_1 + e_2 + e_3 < 4$. Again, there must exist $e_i > 0$. By deviating to $e_i' = 0$, i exerts less effort without affecting the outcome (the bridge is not built). Thus, i has an incentive to deviate. This is not a Nash equilibrium either.

Consider a strategy profile (e_1, e_2, e_3) such that $e_1 + e_2 + e_3 = 4$. In this case, since the road is built, no one has an incentive to increase his effort. No one has an incentive to lower his effort either. This is because everyone's payoff under (e_1, e_2, e_3) is strictly positive and he can obtain at most 0 by reducing his effort. Thus, this strategy profile is a Nash equilibrium.

Finally, $(0, 0, 0)$ is also a Nash equilibrium, because no one has an incentive to increase his effort to $e \in (0, 2]$.

In sum, the set of all pure strategy Nash equilibria is

$$\{(e_1, e_2, e_3) \in [0, 2]^2 \mid e_1 + e_2 + e_3 = 4 \text{ or } (0, 0, 0)\}.$$