Problem Set 5

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1. ST Exercise 8.4

(a)

There are infinite subgames because every quantity choice of firm 1 leads to a proper subgames.

(b)

This is a game of imperfect information because players 2 and 3 make their choices without observing each other’s choice.

(c)

There exists a unique Nash equilibrium when player 1 choose q_1:

$$q_2^*(q_1) = q_3^*(q_1) = \begin{cases} \frac{a-c-q_1}{3}, & q_1 \leq a-c, \\ 0, & \text{o.w.} \end{cases}$$

Back to the player 1's choose, consider the first order condition, we have

$$q_1 = \frac{a-c}{2}$$

Thus we find the unique SPE:

$$q_1 = \frac{a-c}{2}, q_2^*(q_1) = q_3^*(q_1) = \begin{cases} \frac{a-c-q_1}{3}, & q_1 \leq a-c, \\ 0, & \text{o.w.} \end{cases}$$

(d)

$$q_1 = \frac{a-c}{2}, q_2^*(q_1) = q_3^*(q_1) = \begin{cases} \frac{a-c-q_1}{3}, & q_1 = a-c, \\ a, & \text{o.w.} \end{cases}$$

is a Nash equilibrium but not SPE. Because when player 1 doesn't choose $\frac{a-c}{2}$, firm 2 and 3 will get negative profits.

2. ST Exercise 8.6

(a)

Consider the optimization problem of companies 1 and 2 in two cases respectively:

$$\max_{q_i} q_i(100-10-q_1-q_2)$$

which leads to $q_1=q_2=30$.The profit of firm 1 is 900.
And

$$\begin{cases}\max_{q_2} q_2(100-10-q_1-q_2)\\ \max_{q_1} q_1(100-5-q_1-q_2)-F\end{cases}$$

which leads to $q_1=\frac{100}{3}, q_2=\frac{85}{3}$. The profit of firm 1 is $\frac{10000}{9}-F$.
Thus there is a unique subgame-perfect equilibrium involving firm 1 investing $\Leftrightarrow \frac{10000}{9}-F>900$.
Hence $F^*\in(0,\frac{1900}{9})$

(b)

$$(q_1,q_2)=\begin{cases}(30,30) & \text{not investing}\\ (0,0) & \text{investing}\end{cases}$$

It is a Nash equilibrium because when (not investing,30,30) both firms would not deviate. It is not a subgame-perfect equilibrium, because $q_2=0$ is not firm 2's best response when firm 1 invests.

3. ST Exercise 8.9

(a)

In the normal form of the game, firms are represented by the set $N=\{1,2\}$. Each firm $i\in N$ has a strategy space $S_i=[0,+\infty)$. The payoff function for firm i is given by:

$$u_i(q_i, q_{-i}) = \max\{100 - q_1 - q_2, 0\}q_i - k\chi_{\{q_i>0\}}$$

where $\chi_{\{q_i>0\}}$ is the indicator function for $q_i > 0$.

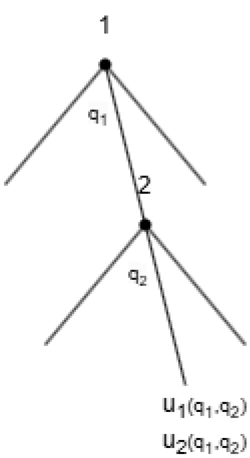
(b)

The best response of firm i is:

$$q_i^*(q_j) = \begin{cases} \frac{100-q_j}{2}, & \text{if } q_j < 100 - 20\sqrt{10} \\ \{10\sqrt{10}, 0\}, & \text{if } q_j = 100 - 20\sqrt{10} \\ 0, & \text{if } q_j > 100 - 20\sqrt{10} \end{cases}$$

This results in three pure Nash equilibria: $\left(\frac{100}{3}, \frac{100}{3}\right), (50, 0), (0, 50)$.

(c)



For $k = 25$, firm 2's best response becomes:

$$q_2^*(q_1) = \begin{cases} \frac{100-q_1}{2}, & \text{if } q_1 < 90 \\ 0, & \text{if } q_1 \geq 90 \end{cases}$$

Firm 1 then maximizes:

$$\max_{q_1<90} \left\{ \left(100 - q_1 - \frac{100 - q_1}{2}\right) q_1 \right\}$$

The optimal solution is $q_1 = 50$, leading to $q_2 = 25$, with positive payoffs $u_1 = (100 - 50 - 25) \times 50 - 25 > 0$ and $u_2 = 25 \times 25 - 25 > 0$. Choosing $q_1 \geq 90$ yields lower profits.

(d)

For $k = 725$, firm 2's best response is:

$$q_2^*(q_1) = \begin{cases} \frac{100-q_1}{2}, & \text{if } q_1 < 100 - 10\sqrt{29} \\ 0, & \text{if } q_1 \geq 100 - 10\sqrt{29} \end{cases}$$

At $q_1 \geq 100 - 10\sqrt{29}$, firm 2 exits, and firm 1 maximizes:

$$\max_{q_1 \geq 100-10\sqrt{29}} \{(100 - q_1)q_1\}$$

The solution is $q_1 = 50$, yielding $u_1 = 2500 - 725 = 1775 > 0$ and $u_2 = 0$.

If $q_1 < 100 - 10\sqrt{29}$, firm 1 solves:

$$\max_{q_1 < 100-10\sqrt{29}} \left\{ \left(100 - q_1 - \frac{100 - q_1}{2}\right) q_1 \right\}$$

Verifying the profits shows that the backward induction solution is unique, with $q_1 = 50$ and:

$$q_2 = \begin{cases} \frac{100-q_1}{2}, & \text{if } q_1 < 100 - 10\sqrt{29} \\ 0, & \text{if } q_1 \geq 100 - 10\sqrt{29} \end{cases}$$

4. ST Exercise 8.12

(a)

This game features perfect information with two players $i \in \{1, 2\}$. The strategy sets are $S_1 = X = [0, 5]$ for player 1, and $S_2 = \{A, R\}$ for player 2, where A means accepting the proposal $x \in X$, and R means rejecting it to maintain the status quo $q = 4$. The payoffs are defined as:

$$v_1(s_1, s_2) = \begin{cases} 10 - |s_1 - 1|, & \text{if } s_2 = A \\ 7, & \text{if } s_2 = R \end{cases}$$

$$v_2(s_1, s_2) = \begin{cases} 10 - |s_1 - 3|, & \text{if } s_2 = A \\ 9, & \text{if } s_2 = R \end{cases}$$

(b)

Player 2 can secure a payoff of 9 by opting for R. Thus, his best response is to choose A if $10 - |s_1 - 3| > 9$, valid for $s_1 \in (2, 4)$. For $s_1 > 4$ or $s_1 < 2$, R is preferred, and he is indifferent at $s_1 = 2$ or $s_1 = 4$.

If player 2 accepts $s_1 = 2$, it leads to subgame-perfect equilibria since player 1 can propose any $s_1 \in (2, 4)$ to maximize their payoff close to 1. Consequently, accepting $s_1 = 2$ is a valid strategy for player 2. The two subgame-perfect equilibria yield payoffs $(v_1, v_2) = (9, 9)$.

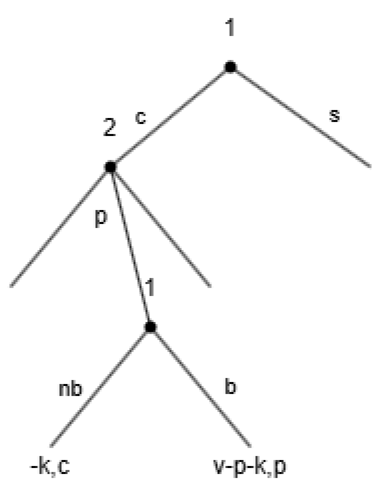
(c)

One Nash equilibrium occurs when player 2 states, "I will reject anything except $s_1 = 3$," and player 1 selects $s_1 = 3$. In this case, player 1 receives 8, while any other choice results in a payoff of 7. Player 1's best response to player 2's strategy is indeed to choose $s_1 = 3$, leading to payoffs $(v_1, v_2) = (8, 10)$.

Since player 2 can guarantee a payoff of 9, there are infinite Nash equilibria where player 2 can adopt the strategy "I will reject anything except $s_1 = x$" for $x \in [2, 4)$. Player 1 would prefer x over 4, ensuring player 2 accepts. Both players remain indifferent at $x = 4$, which also constitutes a Nash equilibrium.

5. ST Exercise 8.13

(a)



Buyer's best response to the seller's offer is:

$$BR_1(c, p) = \begin{cases} b, & \text{if } p < v \\ \{b, nb\}, & \text{if } p = v \\ nb, & \text{if } p > v \end{cases}$$

(b)

Using backward induction, the buyer should reject the offer when $p < v$ and accept when $p > v$. The buyer is indifferent at $p = v$. If the buyer rejects at $p = v$, the seller has no best response since they could instead offer $p' = \frac{p+v}{2} > p$ to improve their payoff.

Thus, in a subgame-perfect equilibrium, the buyer accepts $p = v$, and the seller offers $p = v$. In this case, the buyer only incurs a cost of $-k$ if commuting to the store, leading them to stay home. This results in a unique backward induction solution where the buyer stays home, the seller proposes $p = v$ if the buyer commutes, and the buyer accepts any $p \geq v$ while rejecting others.

The equilibrium outcome is $(0, c)$, which is Pareto dominated by $(0, v-k)$ from the scenario where the buyer commutes and accepts $p = v - k$. Thus, this outcome is not Pareto optimal.

(c)

Consider a strategy where the buyer commutes, the seller proposes $p = \frac{v-k+c}{2} > c$, and the buyer only accepts this price while rejecting others. It's clear that neither party has an incentive to deviate. The payoff for this strategy profile is:

$$\left(\frac{v-k-c}{2}, \frac{v-k+c}{2}\right)$$

This payoff is strictly higher than $(0, c)$.

(d)

If the seller sends a postcard, he sets the price before the buyer decides whether to leave home, changing the game's timing. Backward induction indicates that the buyer should leave home and purchase when $p < v - k$, stay home when $p > v - k$, and be indifferent at $p = v - k$. Similar to part (b), in a subgame-perfect equilibrium, the buyer accepts $p = v - k$, and the seller offers $p = v - k$.

The buyer's payoff remains 0, but the seller's payoff becomes $p - \epsilon = v - k - \epsilon > c$, making the seller better off than in the equilibrium of part (b) where no postcard is sent. When the seller commits to $c + \epsilon < p < v - k$, the buyer will choose to buy, resulting in both being better off. Thus, the seller has an incentive to send the postcard.